$K3$ SURFACES AND LOG DEL PEZZO SURFACES OF INDEX THREE

HISANORI OHASHI AND SHINGO TAKI

Abstract. We use classification of non-symplectic automorphisms of $K3$ surfaces to obtain a partial classification of log del Pezzo surfaces of index three. We can classify those with "Multiple Smooth Divisor Property", whose definition we will give. Our methods include the definition of right resolutions of quotient singularities of index three and some analysis of automorphism-stable elliptic fibrations on $K3$ surfaces. In particular we find several log del Pezzo surfaces of Picard number one with non-toric singularities of index three.

1. Introduction

We work over the complex numbers $\mathbb{C}$. A normal complete surface Z is called a *log del Pezzo surface* if it has only log terminal singularities and the anticanonical divisor $-K_Z$ is ample. Log del Pezzo surfaces constitute one of the most interesting classes of rational surfaces; they naturally appear in the outputs of the (log) minimal model program and their classification is an interesting problem. The *index* k of Z is the least positive integer such that kK_Z is a Cartier divisor.

Log del Pezzo surfaces with index $k = 1$ have only Du Val singularities and together with smooth cases, their classification is a classical topic. In the index $k = 2$ case Alexeev and Nikulin [AN] (over $\mathbb{C}$) and Nakayama [Na] (char. $p \geq 0$ and also for log pairs) gave complete classifications, whose methods are independent in nature. For index $k = 3$ only partial results are available so far. We mention that Dais [Da] classified toric log del Pezzo surfaces of index three with Picard number one.

In this paper we want to discuss a possible generalization of the idea of [AN] to the cases of index 3. Their idea was to relate log del Pezzos to $K3$ surfaces. We explain this in some detail below. Using this idea,

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we will obtain a partial classification of log del Pezzo surfaces of index three and among them we will find some examples of non-toric log del Pezzo surfaces with Picard number one, see Remark 4.8.

We can give a rough sketch of [AN] as follows. A log del Pezzo surface Z of index $k \leq 2$ always has a smooth element $C \in |-2K_Z|$ disjoint from singularities. This fact is called the *smooth divisor theorem*. Also they define the "right" resolution of singularities of index two and using them, they construct from Z a $K3$ surface X with a non-symplectic involution of elliptic type. Here a $K3$ *surface* X is a smooth projective surface with $K_X \sim 0$ and $H^1(X, \mathcal{O}_X) = 0$. An automorphism of X is *non-symplectic* if it acts on $H^{2,0}(X)$ nontrivially. An involution φ of X is *of elliptic type* if the fixed locus $X^\varphi = \{x \in X \mid \varphi(x) = x\}$ contains a smooth curve of genus $g \geq 2$. Conversely, from a pair (X, φ) consisting of a $K3$ surface and a non-symplectic involution of elliptic type, they construct a log del Pezzo surface Z of index $k \leq 2$. Thus the classification of Z reduces to that of (X, φ). Next, using a sophisticated argument of reflection groups, they define the *root invariant* of (X, φ) which describes the set of negative curves (in other words, *roots*) on the right resolution Z_r. This in turn determines the set of singularities of Z. Compared to root invariants, the lattice $H^2(X, \mathbb{Z})^\varphi$ is called the *main invariant*. The resulting table of main invariants and (extremal) root invariants is huge, but they contain not only the information of $\mathrm{Sing}(Z)$ but the set of negative curves on the right resolution Z_r.

To generalize their result, we define analogues of non-symplectic involutions of elliptic type as follows. We call a non-symplectic automorphism φ of a $K3$ surface X *of elliptic type* if X^φ contains a curve of genus $g \geq 2$. Our Proposition 4.1 shows that we can construct a log del Pezzo surface of index $k = 3$ (or 1) from a non-symplectic automorphism φ of order three and of elliptic type. This construction is canonical so that it works for any automorphism φ with analogous appropriate conditions. At the same time we can obtain a necessary condition for a log del Pezzo surface Z to arise from the pair (X, φ):

- $|-3K_Z|$ must contain a divisor $2C$, where C is a smooth curve which does not meet singularities (When $\mathrm{order}(\varphi) = 3$).

We call this property the *multiple smooth divisor property*. See the sentence after Proposition 4.1. Note that it is analogous to the smooth divisor theorem in the case of involutions. In the case of order three the multiple smooth divisor property is no longer true in general and there are many log del Pezzo surfaces which do not correspond to $K3$ surfaces, see Section 6. Hence we naturally restrict ourselves to those Z satisfying this property •. Our Theorem 5.1 will show that conversely this condition is also sufficient for Z to come from a $K3$ surface and a non-symplectic automorphism of order three of elliptic type.

We find a difficulty also in obtaining the list of singularities $\mathrm{Sing}(Z)$, because in our case X/φ has some singularities in general and the method of reflection group is difficult to generalize. Here instead pursuing sophisticated method as in [AN], we make use of the existence of φ-stable elliptic fibrations to obtain $\mathrm{Sing}(Z)$. Our starting point is the general Lemma 3.2 together with the classification of generic non-symplectic automorphisms of order three on $K3$ surfaces by Artebani, Sarti [AS] and the second author [Ta]. We note that the description of elliptic fibrations in [AS], [Ta] is not enough for our purpose of determining *all* possible $\mathrm{Sing}(Z)$. This is because their descriptions depend on the genericity assumption on Néron-Severi lattice $S_X = S_X^{\varphi}$. We can easily construct examples of X, φ without this condition, see Examples 3.3, 3.4. In this paper, we use their description of X^{φ} which holds in general, together with Proposition 3.6 that describes the fiberwise information of X^{φ}. These local and global data explicitly describe the elliptic fibration of *arbitrary* (X, φ) with φ of elliptic type, see Theorems 3.11, 3.12. After preparing Lemmas 4.2, 4.3 and 4.5, we can obtain the final list of $\mathrm{Sing}(Z)$.

Main Theorem. (1) Let φ be a non-symplectic automorphism of order three on a $K3$ surface X of elliptic type, i.e., X^{φ} contains a curve of genus $g \geq 2$. Then there exists a canonical contraction of X/φ onto a log del Pezzo surface Z. Z has index three except when $S_X \simeq U(3)$.

(2) Conversely if Z is a log del Pezzo surface of index three which satisfies the multiple smooth divisor property (• above), then Z can be obtained by the way of (1).

(3) The possible singularity $\mathrm{Sing}(Z)$ and the Picard number $\rho(Z)$ for log del Pezzo surfaces of index three with multiple smooth divisor property are as in the lists of Theorems 3.11 and 4.6.

Although the methods are different, we can observe some similarity between lists of index $k = 2$ and $k = 3$. The singularities of index three on Z depend only on the "main invariant" $H^2(X, \mathbb{Z})^{\varphi}$. As to rational double points, within each main invariant there exists a maximal one and the other possibilities are obtained as the Dynkin subdiagram of that.

In Section 2 we explain log terminal singularities of index three. The notation of singularities is fixed and explained in this section. In Section 3 we study non-symplectic automorphisms of order three of elliptic type. We classify the singular fibers of φ-stable elliptic fibrations in Theorems 3.11 and 3.12. In Section 4 we discuss the construction of log del Pezzo surfaces from (X, φ) and obtain the list of $\mathrm{Sing}(Z)$, Theorems 3.11 and 4.6. In Section 5 we show conversely that the multiple smooth divisor property implies the existence of (X, φ) for index $k = 3$. In Section 6 we discuss some examples.

Notation and Conventions. The symbols A_l, D_l, E_l are used to denote negative-definite even lattices defined by the Dynkin diagrams of each type. The same symbols also denote Du Val singularities on log del Pezzo surfaces; from the context, it will be clear which object they actually denote. The notation $A_l(*), D_l(*)$ will be used to denote singularities of index three. Their definitions are in Section 2.

A hyperbolic lattice of rank r is a lattice with signature $(1, r-1)$. We use U to denote the even unimodular hyperbolic lattice of rank 2. For a lattice L, $L(n)$ is the lattice whose bilinear form is multiplied with n. A nondegenerate lattice L is p-elementary if for the natural inclusion $L \subset L^*$ L^*/L is a p-elementary abelian group.

We denote by S_X the Néron-Severi lattice of the $K3$ surface X. Since the algebraic equivalence and linear equivalence coincide on X, we often denote the equality in S_X by $\sim$. We use $=$ to emphasize the *equality of divisors*. We say that φ *preserves* or *stabilizes* a curve C if $\varphi(C) = C$. We say C is *fixed* if moreover $\varphi|_C = id_C$, namely $C \subset X^\varphi$. An elliptic fibration on X is φ*-stable* if for the general fiber F we have $\varphi(F) \sim F$.

On a log del Pezzo surface Z, we use $\equiv$ to denote the numerical equivalence.

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2. Log terminal singularities of index three

In this section we explain log terminal singularities of index three and fix their notation.

Two-dimensional log terminal singularities are quotient singularities [Kaw] and they are classified in [Br]. Recall that a subgroup G of $\mathrm{GL}(2, \mathbb{C})$ is small if it does not contain any reflections. Let G be a finite small subgroup of $\mathrm{GL}(2, \mathbb{C})$ and $\mathbb{C}^2/G$ be the quotient singularity. Then it has index three if and only if $[G \colon G \cap \mathrm{SL}(2, \mathbb{C})] = 3$. It is easy to choose such G from [Br, Satz 2.9 and 2.11]:

G	Γ	conditions
$C_{n,q}$	$\langle n, q \rangle$	$0 < q < n, (n, q) = 1$ and $\frac{1+q}{n} \in \frac{1}{3}\mathbb{Z} \setminus \mathbb{Z}$
$G_0 := (\mathbb{Z}_6, \mathbb{Z}_6; \mathbb{D}_2, \mathbb{D}_2)$	$\langle 3; 2, 1; 2, 1; 2, 1 \rangle$	
$G_n := (\mathbb{Z}_6, \mathbb{Z}_6; \mathbb{D}_n, \mathbb{D}_n)$	$\langle 2; 2, 1; 2, 1; n, n-3 \rangle$	$n \not\equiv 0 \ (3)$, $n \geqq 4$

Here Γ denotes the dual graph of configuration of exceptional curves of minimal resolution together with the data of self-intersection numbers. $G = C_{n,q}$ denotes the group of cyclic quotient singularity $\frac{1}{n}(1, q)$. The notation for G_i and Γ is defined in [Br] and we omit it; anyway we reproduce Γ in Table 2.

In each case, by using the recursive relation of continued fractions

$$\frac{n}{n-3} = 2 - \frac{1}{\frac{n-3}{n-6}},$$

we can describe Γ more explicitly. Moreover given Γ, we can compute the discrepancies of exceptional curves. These are the contents of the next table. l denotes the number of vertices of Γ.

symbol	Γ	G
$A_1(1)$	-3 $-\frac{1}{3}$	$C_{3,1}$
$A_1(2)$	-6 $-\frac{2}{3}$	$C_{6,1}$
$A_2(12)$	$-2 \ — \ -5$ $-\frac{1}{3} \quad -\frac{2}{3}$	$C_{9,5}$
$A_2(22)$	$-4 \ — \ -4$ $-\frac{2}{3} \quad -\frac{2}{3}$	$C_{15,4}$
$A_3(11)$	$-2 \ — \ -4 \ — \ -2$ $-\frac{1}{3} \quad -\frac{2}{3} \quad -\frac{1}{3}$	$C_{12,7}$

$A_3(12)$	$\underset{-\frac{1}{3}}{(-2)} - \underset{-\frac{2}{3}}{(-3)} - \underset{-\frac{2}{3}}{(-4)}$	$C_{18,11}$
$A_3(22)$	$\underset{-\frac{2}{3}}{(-4)} - \underset{-\frac{2}{3}}{(-2)} - \underset{-\frac{2}{3}}{(-4)}$	$C_{24,7}$
$A_l(11)$	$\underset{-\frac{1}{3}}{(-2)} - \underset{-\frac{2}{3}}{(-3)} - \underset{-\frac{2}{3}}{(-2)} - \cdots - \underset{-\frac{2}{3}}{(-2)} - \underset{-\frac{2}{3}}{(-3)} - \underset{-\frac{1}{3}}{(-2)}$	$C_{9l-15,6l-11}$
$A_l(12)$	$\underset{-\frac{1}{3}}{(-2)} - \underset{-\frac{2}{3}}{(-3)} - \underset{-\frac{2}{3}}{(-2)} - \cdots - \underset{-\frac{2}{3}}{(-2)} - \underset{-\frac{2}{3}}{(-4)}$	$C_{9l-9,6l-7}$
$A_l(22)$	$\underset{-\frac{2}{3}}{(-4)} - \underset{-\frac{2}{3}}{(-2)} - \cdots - \underset{-\frac{2}{3}}{(-2)} - \underset{-\frac{2}{3}}{(-4)}$	$C_{9l-3,3l-2}$
$D_4(1)$	$\underset{-\frac{1}{3}}{(-2)} - \underset{-\frac{2}{3}}{(-3)} < \begin{matrix} \underset{-\frac{1}{3}}{(-2)} \\ \underset{-\frac{1}{3}}{(-2)} \end{matrix}$	G_0
$D_4(2)$	$\underset{-\frac{2}{3}}{(-4)} - \underset{-\frac{2}{3}}{(-2)} < \begin{matrix} \underset{-\frac{1}{3}}{(-2)} \\ \underset{-\frac{1}{3}}{(-2)} \end{matrix}$	G_4
$D_l(1)$	$\underset{-\frac{1}{3}}{(-2)} - \underset{-\frac{2}{3}}{(-3)} - \underset{-\frac{2}{3}}{(-2)} - \cdots - \underset{-\frac{2}{3}}{(-2)} < \begin{matrix} \underset{-\frac{1}{3}}{(-2)} \\ \underset{-\frac{1}{3}}{(-2)} \end{matrix}$	G_{3l-10}
$D_l(2)$	$\underset{-\frac{2}{3}}{(-4)} - \underset{-\frac{2}{3}}{(-2)} - \cdots - \underset{-\frac{2}{3}}{(-2)} < \begin{matrix} \underset{-\frac{1}{3}}{(-2)} \\ \underset{-\frac{1}{3}}{(-2)} \end{matrix}$	G_{3l-8}

Table 2: log terminal singularities of index three

One observation is the following: within the same type of Dynkin diagram, the discrepancy can vary only at the end of the diagram. This fact in mind, we introduce the following symbols for singularities. We denote the singularity by $A_n(\alpha\beta)$ (resp. $D_n(\alpha)$) if the dual graph of the exceptional curves is of type A_n (resp. D_n) and the discrepancies at the ends are $-\alpha/3$ and $-\beta/3$ (resp. the discrepancy at the end of the longest branch is $-\alpha/3$). The meaning of $A_1(\alpha)$ will be obvious.

The importance of discrepancies in defining right resolutions will show the advantage of these new symbols. See Section 5. In general they are adequate in numerical computations of singularities.

3. Non-symplectic automorphisms of order three on $K3$ surfaces

Let X be a $K3$ surface and φ a non-symplectic automorphism of order three on X. In this section we extend some part of the results of [AS] and [Ta] so that we will describe *arbitrary* non-symplectic automorphisms of order three *of elliptic type*. Here we follow [AN] to say that φ is *of elliptic type* if the fixed locus X^φ contains a curve $C^{(g)}$ of genus $g \geq 2$. Although our main interest is in the case φ is of elliptic type, let us begin with generalities.

Let φ be an arbitrary non-symplectic automorphism of order three. Since φ is non-symplectic, the fixed sublattice of $H^2(X, \mathbb{Z})$ as to the action of φ coincides with that of the Néron-Severi lattice S_X by [Ni]. Therefore we denote it by S_X^φ. This lattice is easily seen to be 3-elementary and hyperbolic, hence S_X^φ is one of the 24 lattices of [Ta, Lemma 2.3]. In particular S_X^φ represents zero in every case. The next lemma is crucial.

Lemma 3.1. Let X be a projective $K3$ surface, φ be an automorphism of X of order $n \geq 2$. Assume S_X^φ represents zero, i.e., there exists D such that

$$0 \not\sim D \in S_X^\varphi, \quad (D^2) = 0.$$

Then we can find another D' such that

$$0 \not\sim D' \in S_X^\varphi, ((D')^2) = 0 \text{ and } D' \text{ is nef.}$$

Proof. First by the Riemann-Roch theorem we can assume that D is effective. We fix an ample divisor H on X and use the descending induction on the value (H, D), noting that it is a positive integer. In this setting our lemma is reduced to the following claim.

If D is not nef, then we can find D' such that

$0 \not\sim D' \in S_X^\varphi$, $((D')^2) = 0$, D' *is effective and* $(H, D) > (H, D')$.

Since D is effective, we can make use of Zariski decomposition $D = P + N$. Here P is a nef $\mathbb{Q}$-divisor, N is an effective $\mathbb{Q}$-divisor whose prime components $\{N_i\}$ of N have negative-definite intersection matrix and $(P, N) = 0$. By uniqueness, N is determined by the numerical class of D, hence $\varphi(D) \sim D$ implies $\varphi(N) = N$. We also note that every prime component of N is a (-2)-curve on X.

Suppose D is not nef. Then for some (-2)-curve $l \in \{N_i\}$ we have $(D, l) < 0$. Let p be the least positive integer such that $\varphi^p(l) = l$ and let $E = l + \cdots + \varphi^{p-1}(l)$. The negativity of N implies $(E^2) < 0$. From this inequality, we can classify the possible configuration of the divisor E as follows.

(I) When $p = 1$ then $E = l$.

If $p \geq 2$ we put $k := (l, \varphi(l) + \cdots + \varphi^{p-1}(l))$. Then since

$$0 > (E^2) = \sum_{i=0}^{p-1} \big((\varphi^i(l), E - \varphi^i(l)) + (\varphi^i(l), \varphi^i(l)) \big) = p(k-2),$$

we obtain $k = 0$ or $k = 1$.

(II) If $k = 0$ then E is a disjoint union of p (-2)-curves.

(III) If $k = 1$ then the dual graph of E is a disjoint union of the Dynkin diagram of type A_2. Consequently p is even.

For a (-2)-element f in S_X we denote by s_f the Picard-Lefschetz reflection

$$s_f \colon x \mapsto x + (x, f)f$$

which is an isometry of S_X that preserves the positive cone. In case (I) we put $D' \sim s_l(D) = D + (D, l)l$. Then D' is an effective divisor (up to linear equivalence) whose degree as to H is less than that of D. It is easy to see that $D' \in S_X^{\varphi}$ by using $\varphi(l) = l$. Thus the claim holds. Similarly in case (II) we put

$$\begin{aligned} D' &\sim s_l s_{\varphi(l)} \cdots s_{\varphi^{p-1}(l)}(D) \\ &= D + (D, l)(l + \varphi(l) + \cdots + \varphi^{p-1}(l)). \end{aligned}$$

In case (III) we relabel $\{l, \cdots, \varphi^{p-1}(l)\} = \{a_1, b_1, \cdots, a_{p/2}, b_{p/2}\}$ so that $(a_i, b_i) = 1$ holds for all i. Then we put

$$\begin{aligned} D' &\sim s_{a_1+b_1} \cdots s_{a_{p/2}+b_{p/2}}(D) \\ &= D + 2(D, l)(a_1 + b_1 + \cdots + a_{p/2} + b_{p/2}). \end{aligned}$$

In this way we obtain the claim and hence Lemma 3.1. □

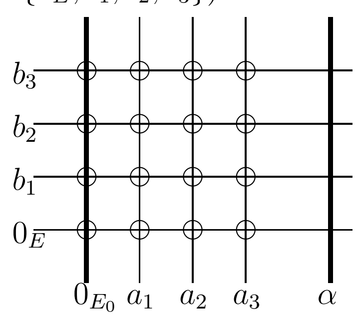
Corollary 3.2. If φ is a non-symplectic automorphism of order three of a $K3$ surface X, then there exists a nef divisor $D \not\sim 0$ such that $(D^2) = 0$ and $\varphi(D) \sim D$. In particular, there is a φ-stable elliptic pencil $f \colon X \to \mathbb{P}^1$.

[AS, Ta] also use elliptic fibrations to describe generic X. To clarify the difference from them we note that under the genericity assumption

$S_X^\varphi = S_X$, φ preserves every (-2)-curve on X. But there are many examples where this is not the case.

Example 3.3. Let E_0 be the elliptic curve with period $\zeta_3 = e^{2\pi i/3}$, the cubic root of unity, and let φ be the automorphism of E_0 of order three given by multiplication of ζ_3. Let E be another elliptic curve not isogenous to E_0. We put $X = Km(E_0 \times E)$, namely the minimal desingularization of the quotient surface $E_0 \times E/(-1)$. From the theory of Kummer surfaces [BHPV], we see that X is a $K3$ surface of Picard number $\rho = 18$; its transcendental lattice is isomorphic to $U(2) \oplus U(2)$.

The automorphism $(\varphi, 1)$ of $E_0 \times E$ clearly commutes with (-1), hence it descends to an automorphism of $E_0 \times E/(-1)$. Its fixed loci consist of the rational curve $\{0_{E_0}\} \times (E/(-1))$ and the elliptic curve $\{\alpha\} \times E$, where we put $E_0^\varphi = \{0_{E_0}, \alpha, -\alpha\}$. See the picture below. In the picture, thick lines are fixed loci, circles represent singularities of $E_0 \times E/(-1)$ and the 2-torsion points of E_0 (resp. E) are $\{0_{E_0}, a_1, a_2, a_3\}$ (resp. $\{0_E, b_1, b_2, b_3\}$).



The automorphism induced on X is denoted by the same φ. On X the sixteen circles are replaced by (-2)-curves. Since φ permutes circles on lines $\{a_i\} \times (E/(-1))$ in the picture, we see that not all (-2)-curves are preserved. It is easy to see (see below) that the fixed points of φ on X consist of

$$\{4 \text{ points}\} \cup \{\text{a rational curve}\} \cup \{\text{an elliptic curve}\}.$$

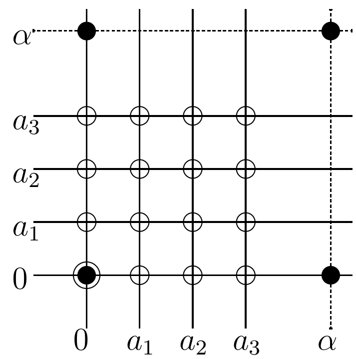
By [AS, Table 2] we obtain $S_X^\varphi \simeq U \oplus A_2^{\oplus 4}$. Thus in fact $S_X \neq S_X^\varphi$. Note that by [O] X does not have any Jacobian elliptic fibration with the root lattice of reducible fibers isomorphic to $A_2^{\oplus 4}$.

The two projections of $E_0 \times E$ induce two φ-stable elliptic fibrations on X

$$f\colon X \to E/(-1) \simeq \mathbb{P}^1,\ g\colon X \to E_0/(-1) \simeq \mathbb{P}^1.$$

(In the picture, the fibers of f are horizontal and that of g are vertical.) We see that the j-invariant of g is nonzero constant. φ acts on the base $E_0/(-1)$ non-trivially. g has the singular fiber of type I_0^*-(i) in the notation of Proposition 3.6.

Example 3.4. Let E_0 be as in the previous example. Let us consider $X = Km(E_0 \times E_0)$ and its automorphism induced from (φ, φ). In this case the picture becomes



Here the dashed lines are elliptic curves. The sixteen circles are singularities and four black circles are fixed points. The leftmost bottom one has both properties and it gives a fixed (-2)-curve. The rightmost upside black circle indicates two fixed points. Hence the fixed points on X consist of

$$\{4 \text{ points}\} \cup \{\text{a rational curve}\}$$

and $S_X^\varphi \simeq U(3) \oplus A_2^{\oplus 4}$, which is a proper sublattice of S_X which has rank 20. The elliptic fibration induced from the projection has the singular fiber of type I_0^*-(ii) in the notation of Proposition 3.6. It acts on the base non-trivially.

In the rest of this section we restrict ourselves to the case of elliptic type and give a description of arbitrary φ.

Proposition 3.5. Assumptions and $f\colon X \to \mathbb{P}^1$ as in Corollary 3.2. Assume φ is of elliptic type. Then the following holds.

(1) The j-invariant of f is identically zero. Hence every smooth fiber is isomorphic to the elliptic curve with the period ζ_3.
(2) φ acts on the base trivially.
(3) The map $X^\varphi \to \mathbb{P}^1$ is generically 3 to 1.
(4) Every fiber of f is of type either I_0, I_0^*, II, II^*, IV or IV^*.

Proof. Since the fixed curve $C^{(g)}$ cannot be located inside a fiber, it intersects with every fiber. Hence (2) follows. Also every smooth fiber has an automorphism of order three with fixed points, hence (1) and (3). (4) is also clear from the classification. □

Thus φ acts on each fiber F, $\varphi(F) = F$. In the next subsection we study the fixed points of this fiberwise action.

3.1. **Fixed points on Kodaira fibers.** We keep the elliptic fibration f of Proposition 3.5. We can normalize the action of φ on $H^{2,0}(X)$ as multiplication by ζ_3, the primitive cubic root of unity, without loss of generality. Then recall [Ta, Lemma 3.1] that the local action of φ on X around the fixed point is either

$$A = \begin{pmatrix} \zeta_3^2 & 0 \\ 0 & \zeta_3^2 \end{pmatrix} \text{ (isolated) or } B = \begin{pmatrix} 1 & 0 \\ 0 & \zeta_3 \end{pmatrix} \text{ (fixed curve).} \tag{1}$$

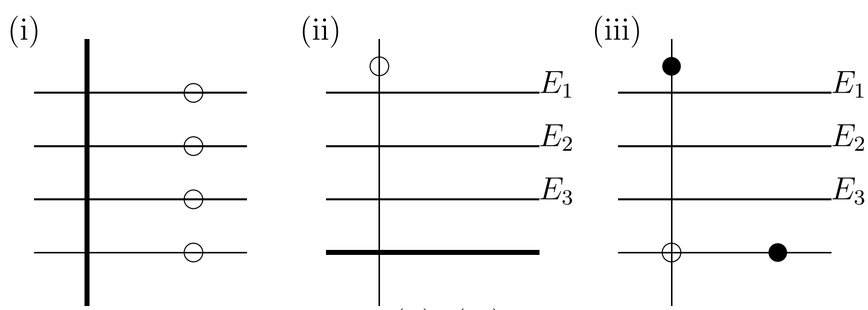
In the following we focus on the relationship between action *on* X and *on* F. We say that a fixed point P of the action of φ on F is

- isolated, if the local action on X is given by the matrix A,
- on a fixed curve, if the local action is given by B and the fixed curve is inside F,
- intermediate, if the local action is given by B and the fixed curve is outside F and intersects F at P.

The point is that we can distinguish these three types by considering only the action on F. We are going to prove the following

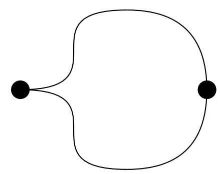
Proposition 3.6. Let F be a singular fiber of f, see Proposition 3.5. Then the action of φ on F and its fixed points are as in one of the following pictures. Here thick line represents a fixed curve, $\circ$ an isolated fixed point and $\bullet$ an intermediate fixed point on F.

Type I_0^*

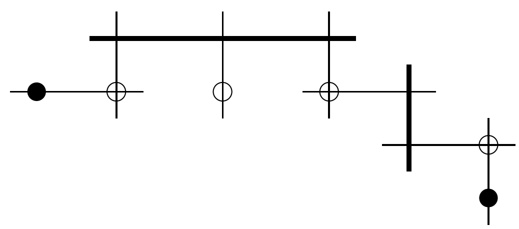


(ii), (iii): φ permutes E_i.

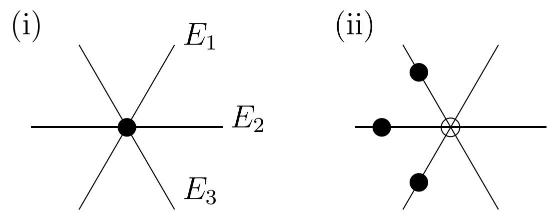
Type II



Type II^*

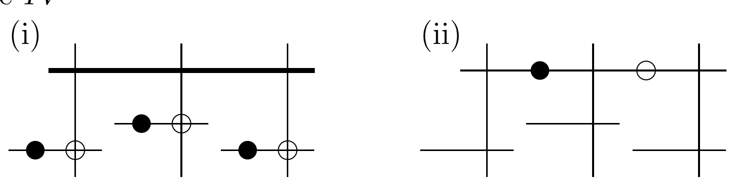


Type IV



φ permutes E_i.

Type IV^*



φ permutes three branches.

Proof. We begin with the following.

Lemma 3.7. If $C \simeq \mathbb{P}^1$ is a (-2)-curve in F which is preserved and not fixed by φ, then C has one isolated fixed point and the other is intermediate or on a fixed curve.

Proof. By the topological Lefschetz formula C has two fixed points. Let us choose an inhomogeneous coordinate z on C such that two fixed points are $z = 0$ and $z = \infty$. If φ acts on z as a scalar $a \neq 1$, then φ

acts on the local coordinate z^{-1} near ∞ by a^{-1}. By (1), we see that $\{a, a^{-1}\} = \{\zeta_3, \zeta_3^{-1}\}$ and the lemma follows. $\square$

The result for cases I_0^*, II^* and IV^* follows easily from this lemma: we first classify the symmetry of the configuration and then can determine the location of fixed points. The lemma helps us to determine the types of fixed points.

For the case IV, we easily see that either (i) three curves are permuted or (ii) three curves are preserved. In each case the center Q is a fixed point, and we can blow up the center to obtain the exceptional curve E. The automorphism φ lifts up to the blow up. In case (i) φ acts on $E \simeq \mathbb{P}(T_Q X)$ nontrivially. Hence the center is intermediate by (1). In case (ii), the three intersection points of E and strict transforms are fixed by φ. Thus E is fixed by φ and by (1) the center is isolated.

In case of type II we have to exclude the possibilities of isolated fixed points. First we note that the whole cusp curve cannot be fixed. This is because X^φ contains smooth curves only. Thus there are two fixed points P, Q on F, where P is a smooth point and Q is the cusp. Assume that either P or Q or both are isolated fixed points. We consider the minimal resolution $\sigma\colon Y \to X/\varphi$ of the quotient. Let $G := F/\varphi \subset X/\varphi$. Then it is easy to see that G is a smooth Weil divisor on X/φ with $(G^2) = 0$. (In fact, in an appropriate coordinate F is isomorphic to the cuspidal cubic $\{zy^2 = x^3\} \subset \mathbb{P}^2$ equipped with the order three automorphism $(x, y, z) \mapsto (\zeta_3 x, y, z)$. Here P is $(0, 1, 0)$ and Q is $(0, 0, 1)$.)

Since in either case the isolated fixed point is of type $\frac{1}{3}(1, 1)$, the exceptional curves E_i $(i \leq 2)$ of σ are (-3)-curves and we obtain the relation

$$\sigma^* G - \sum_i \frac{1}{3} E_i = \overline{G},$$

where $\overline{G}$ is the strict transform. But here the self-intersection number of LHS is $(G^2) - \sum(3/9) = -(1/3)$ or $-(2/3) \notin \mathbb{Z}$, which is impossible since Y is smooth near $\overline{G}$. Hence we see that there are no isolated fixed points inside cusp fiber F. This concludes Proposition 3.6.

$\square$

Remark 3.8. (1) If φ is of elliptic type, then $C^{(g)}$ intersects every fiber and F has at least one intermediate fixed point. Thus I_0^*-(i) and I_0^*-(ii) do not occur.

(2) If moreover f admits a fixed (-2)-curve which is a section, then obviously IV-(i) and IV^*-(ii) cannot occur.

(3) On the other hand, if φ is not of elliptic type, these actions can arise. See Examples 3.3, 3.4.

3.2. **Automorphisms of elliptic type.** We keep the elliptic fibration f of Proposition 3.5. In this section we describe singular fibers of f and the action of φ on fibers combinatorically, relying on the results of previous subsection. First we recall the following

Proposition 3.9 ([AS], [Ta])**.** There exist exactly eleven fixed lattices for non-symplectic automorphisms of elliptic type. The correspondence between S_X^φ and the fixed locus X^φ is as follows.

No.	S_X^φ	X^φ
1	U	$C^{(5)} \amalg \mathbb{P}^1$
2	$U(3)$	$C^{(4)}$
3	$U \oplus A_2$	$C^{(4)} \amalg \mathbb{P}^1 \amalg \{pt\}$
4	$U(3) \oplus A_2$	$C^{(3)} \amalg \{pt\}$
5	$U \oplus A_2^{\oplus 2}$	$C^{(3)} \amalg \mathbb{P}^1 \amalg \{pt\} \times 2$
6	$U(3) \oplus A_2^{\oplus 2}$	$C^{(2)} \amalg \{pt\} \times 2$
7	$U \oplus E_6$	$C^{(3)} \amalg \mathbb{P}^1 \times 2 \amalg \{pt\} \times 3$
8	$U \oplus A_2^{\oplus 3}$	$C^{(2)} \amalg \mathbb{P}^1 \amalg \{pt\} \times 3$
9	$U \oplus E_8$	$C^{(3)} \amalg \mathbb{P}^1 \times 3 \amalg \{pt\} \times 4$
10	$U \oplus E_6 \oplus A_2$	$C^{(2)} \amalg \mathbb{P}^1 \times 2 \amalg \{pt\} \times 4$
11	$U \oplus E_8 \oplus A_2$	$C^{(2)} \amalg \mathbb{P}^1 \times 3 \amalg \{pt\} \times 5$

We divide these eleven fixed lattices into *Jacobian types* and *non-Jacobian types*. Non-Jacobian types consists of No. 2,4,6 and Jacobian types include others. Equivalently S_X^φ in the table is of Jacobian type if there exists an embedding of lattices $U \subset S_X^\varphi$.

Proposition 3.10. If φ is of elliptic type and of Jacobian type, then after a suitable rechoice of the zero-element in Corollary 3.2, the fibration f of Proposition 3.5 has a section which is fixed by φ.

Proof. If there exists a fixed (-2)-curve that intersects fibers, then together with the curve $C^{(g)}$ they define a generically at least 3 to 1 map onto the base. By Proposition 3.5 (2),(3) this implies that the (-2)-curve is a fixed section.

Let us assume that every fixed (-2)-curve is inside fibers. We denote by m (resp. n) the number of type II^* (resp. type IV^*-(i)) fibers of f in the notation of Proposition 3.6. We remark that they are the only fibers that may have a fixed (-2)-curve, see also Remark 3.8. Then we obtain

$$2m + n = \#(\text{fixed } (-2)\text{-curve}),$$
$$4m + 3n \leq \#(\text{isolated fixed points of } \varphi).$$

In No.s 1,3,5,7,9,11 there exist no solutions to this restriction, hence at least one fixed (-2)-curve is outside the fibers.

In No.s 8 and 10, we have solutions $(m, n) = (0, 1)$ and $= (1, 0)$ respectively, and we have to make a rechoice of suitable f. This is related to the isomorphisms of lattices

$$U \oplus A_2^{\oplus 3} \simeq U(3) \oplus E_6 \quad (\text{resp. } U \oplus E_6 \oplus A_2 \simeq U(3) \oplus E_8).$$

We see from this isomorphism that in each case S_X^φ has two inequivalent zero-elements, D_1 and D_2, such that $(D_1, S_X^\varphi) = \mathbb{Z}$ and $(D_2, S_X^\varphi) = 3\mathbb{Z}$. By the proof of Lemma 3.1 each D_i can be sent to a nef element with the same property for $i = 1, 2$.

Suppose in No. 8 that we have one IV^*-(i) fiber F. Since the components of this fiber are all preserved by φ, we can consider the sublattice $L \subset S_X^\varphi$ generated by $C = C^{(2)}$ and the seven components of F. We will give an explicit isomorphism $L \simeq U(3) \oplus E_6$.

The intersection numbers can be read off from Proposition 3.6. C intersects each multiplicity one component e_1, e_2, e_3 transversally. The seven components other than e_3 constitutes a lattice M isomorphic to E_6. Let e_i^* denote the element in M^* which is dual to e_i, $i = 1, 2$. We can check that $e_1^* + e_2^* \in M$. Then we obtain

$$\begin{aligned} L &= \langle M, e_3, C \rangle \\ &= \langle M, F, C - e_1^* - e_2^* \rangle \\ &\simeq U(3) \oplus E_6. \end{aligned} \tag{2}$$

From this isomorphism, $(m, n) = (0, 1)$ occurs only if $(F, S_X^\varphi) = 3\mathbb{Z}$. Hence if we choose D_1 with $(D_1, S_X^\varphi) = \mathbb{Z}$ as the beginning element in Lemma 3.1 the solution $(m, n) = (0, 1)$ does not occur. In this fibration at least one fixed (-2)-curve is outside fibers.

The proof for No. 10 is the same. $\square$

We obtain the models for Jacobian type.

Theorem 3.11. Let φ be a non-symplectic automorphism of order three of elliptic type and of Jacobian type. Then X has an elliptic pencil $f : X \to \mathbb{P}^1$ which is stable under φ and has a fixed section. The set of singular fibers $\mathrm{Sing}(f)$ is one of the following.

No.	$\mathrm{Sing}(f)$	$\mathrm{Sing}(Z)$	$\rho(Z)$

1	$12 \cdot II$	$A_1(2)$	1
3a	IV-(ii)$+10 \cdot II$	$A_2(12)$	2
3b	I_0^*-(iii)$+9 \cdot II$	$A_2(12) + A_1$	1
5a	$2 \cdot IV$-(ii)$+8 \cdot II$	$A_3(11)$	3
5b	IV-(ii)$+I_0^*$-(iii)$+7 \cdot II$	$A_3(11) + A_1$	2
5c	$2 \cdot I_0^*$-(iii)$+6 \cdot II$	$A_3(11) + 2A_1$	1
7	IV^*-(i)$+8 \cdot II$	$D_4(2)$	2
8a	$3 \cdot IV$-(ii)$+6 \cdot II$	$D_4(1)$	4
8b	$2 \cdot IV$-(ii)$+I_0^*$-(iii)$+5 \cdot II$	$D_4(1) + A_1$	3
8c	IV-(ii)$+2 \cdot I_0^*$-(iii)$+4 \cdot II$	$D_4(1) + 2A_1$	2
8d	$3 \cdot I_0^*$-(iii)$+3 \cdot II$	$D_4(1) + 3A_1$	1
9	$II^* + 7 \cdot II$	$D_5(2)$	1
10a	IV^*-(i)$+IV$-(ii)$+6 \cdot II$	$D_5(1)$	3
10b	IV^*-(i)$+I_0^*$-(iii)$+5 \cdot II$	$D_5(1) + A_1$	2
11a	$II^* + IV$-(ii)$+5 \cdot II$	$D_6(1)$	2
11b	$II^* + I_0^*$-(iii)$+4 \cdot II$	$D_6(1) + A_1$	1

We treat two columns on the right in the next section, see Theorem 4.6.

Proof. f is the one obtained in Proposition 3.10. Let m and n be the numbers of singular fibers of type II^* and IV^*-(i) respectively. As in Proposition 3.10 we obtain

$$2m + n = \#(\text{fixed } (-2)\text{-curves}) - 1,$$
$$4m + 3n \leq \#(\text{isolated fixed points of } \varphi).$$

In every case this has a unique solution. Thus we know the number of singular fibers of type II^* and IV^*-(i). From Proposition 3.6 and Remark 3.8 we see that other fibers are of type either I_0^*-(iii), II or IV-(ii). The restriction on the number of isolated fixed points and the topological Euler number of X leads us to the table. □

Perhaps we may say that "type IV-(ii) fibers deform into I_0^*-(iii) fibers under the φ-equivariant deformations preserving Jacobian elliptic fibrations".

In non-Jacobian types, there are no fixed (-2)-curves, hence $C^{(g)}$ is a tri-section by Proposition 3.5.

Theorem 3.12. Let φ be a non-symplectic automorphism of order three of elliptic type and of non-Jacobian type. Let f be the elliptic fibration of Proposition 3.5. Then the singular fibers of f are as in the following table.

No.	Sing(f)	condition
2	$x \cdot II + y \cdot IV$-(i)	$x + 2y = 12$
4a	IV-(ii) $+x \cdot II + y \cdot IV$-(i)	$x + 2y = 10$
4b	I_0^*-(iii) $+x \cdot II + y \cdot IV$-(i)	$x + 2y = 9$
4c	IV^*-(ii) $+x \cdot II + y \cdot IV$-(i)	$x + 2y = 8$
6a	$2 \cdot IV$-(ii) $+x \cdot II + y \cdot IV$-(i)	$x + 2y = 8$
6b	I_0^*-(iii)$+IV$-(ii) $+x \cdot II + y \cdot IV$-(i)	$x + 2y = 7$
6c	$2 \cdot I_0^*$-(iii) $+x \cdot II + y \cdot IV$-(i)	$x + 2y = 6$
6d	IV-(ii) $+IV^*$-(ii) $+x \cdot II + y \cdot IV$-(i)	$x + 2y = 6$
6e	IV^*-(ii) $+I_0^*$-(iii) $+x \cdot II + y \cdot IV$-(i)	$x + 2y = 5$
6f	$2 \cdot IV^*$-(ii) $+x \cdot II + y \cdot IV$-(i)	$x + 2y = 4$

Proof. The proof is the same as Jacobian case, we use the restriction on fixed (-2)-curves, isolated fixed points and the topological Euler number. □

4. From $K3$ to log del Pezzo

We use the results of last section to obtain the list of singularities of log del Pezzo surface Z of index three with the multiple smooth divisor property. Our discussion depends on the following observation.

Proposition 4.1. Let X be a $K3$ surface and φ a non-symplectic automorphism of finite order n such that X^φ contains a curve C of genus $g \geq 2$. Then using the natural morphisms

$$\begin{aligned} X \xrightarrow{\nu} X_0 &:= \operatorname{Proj} \oplus_{m\geq 0} H^0(X, \mathcal{O}_X(mC)) \\ \xrightarrow{\pi} Z &:= \operatorname{Proj} \oplus_{m\geq 0} H^0(X, \mathcal{O}_X(mC))^\varphi = X_0/\varphi, \end{aligned}$$

we get a log del Pezzo surface Z whose index divides n. Moreover Z satisfies the following condition:
• $|-nK_Z| \ni (n-1)C_0$, where $C_0 = \pi\nu(C)$ is a smooth curve which does not meet singularities.

Proof. Since C is nef and big, ν is a birational morphism which contracts every (-2)-curve on X disjoint from C (a very special case of Basepoint-free theorem, [KM, Theorem 3.3]). We note that for any k, the fixed point set X^{φ^k} consists of C, (-2)-curves disjoint from C and some isolated fixed points. Therefore the induced action of φ^k on X_0

has only fixed curve $\nu(C)$ and some isolated fixed points. Also note that $\nu(C)$ is disjoint from singularities. The ramification formula of π is therefore

$$0 \sim K_{X_0} = \pi^* K_Z + (n-1)\nu(C). \tag{3}$$

Since $\nu(C)$ is ample and π is a finite morphism, $-K_Z$ is ample. Since X_0 has only quotient singularities, so does Z. Thus Z is a log del Pezzo surface. Moreover, if ω is the nowhere vanishing holomorphic two form on X, then φ acts trivially on $\omega^{\otimes n}$. It descends to a nowhere vanishing section of nK_Z over $Z - \mathrm{Sing}(Z) - \pi\nu(C)$ and this is the local generator of $\mathcal{O}_Z(nK_Z)$ around singularities. Hence nK_Z is Cartier. The last condition follows from (3) by applying π_*. $\square$

Let Z be a log del Pezzo surface of index 3. We say Z satisfies the *multiple smooth divisor property* if the linear system $|-3K_Z|$ contains a divisor $2C$ with C a smooth curve that does not meet $\mathrm{Sing}(Z)$. As the previous proposition implies, this is a natural necessary condition for Z when we consider correspondence with $K3$ surfaces. We will show in Section 5 that conversely for any log del Pezzo surface Z of index three with this multiple smooth divisor property there exists (X, φ) for which the construction of Proposition 4.1 leads to Z. We note that, the analogous condition for index 2 corresponds to the smooth divisor theorem of [AN].

In the following we examine singularities of Z which is obtained from (X, φ) via Proposition 4.1. We use the elliptic fibrations obtained in Theorems 3.11, 3.12.

Lemma 4.2. Let f be the elliptic fibration of Theorem 3.11 and 3.12. Suppose there exists a (-2)-curve E which is disjoint from $C = C^{(g)}$. Then one of the following holds.

(1) $E \subset X^{\varphi}$.
(2) E is a fiber component.
(3) E, $\varphi(E)$ and $\varphi^2(E)$ are mutually disjoint and they are sections of f.

Proof. Let us assume that $E \not\subset X^{\varphi}$ and $(E, F) \geq 1$, where F is the general fiber. We note that $(C, F) = 2$ or 3 respectively for Theorems 3.11 or 3.12, by Proposition 3.5.

We first consider the case $\varphi(E) = E$. Since F is general, $F \cap E$ is not the fixed point of E, hence $(F, E) \geq 3$. Let us derive a contradiction using the Hodge index theorem. The divisor $C + bE - \frac{(C^2)}{(C,F)}F$ is orthogonal to C for any $b \in \mathbb{R}$, hence we should have the self-intersection

$$\left(\left(C + bE - \frac{(C^2)}{(C,F)}F\right)^2\right) \leq 0.$$

This function on b takes the maximum at $b = -(C^2)(E,F)/2(C,F)$ and we deduce

$$\begin{aligned} 0 &\geq \left(\left(C - \frac{(C^2)(E,F)}{2(C,F)}E - \frac{(C^2)}{(C,F)}F\right)^2\right) \\ &= (C^2)\left(\frac{(C^2)(E,F)^2}{2(C,F)^2} - 1\right). \end{aligned}$$

This is possible only if $(C^2) = 2, (E,F) = 3$ and $(C,F) = 3$, namely in the case No. 6. In this case the two fixed points of $\varphi|_E$ are both isolated since $(C,E) = 0$. But this is a contradiction to Lemma 3.7.

Next we consider the case $\varphi(E) \neq E$. Again by the Hodge index theorem, any divisor $aE + b\varphi(E) + c\varphi^2(E)$ has a non-positive (and negative if it is effective) self-intersection number. From this we see that $E, \varphi(E)$ and $\varphi^2(E)$ are disjoint. We put $D = E + \varphi(E) + \varphi^2(E)$. Then $(D,F) \geq 3$ and this number is divisible by 3. As in the previous case, the function $\left(\left(C + bD - \frac{(C^2)}{(C,F)}F\right)^2\right)$ on $b \in \mathbb{R}$ takes non-positive value, and its maximum is

$$\begin{aligned} 0 &\geq \left(\left(C - \frac{(C^2)(D,F)}{6(C,F)}D - \frac{(C^2)}{(C,F)}F\right)^2\right) \\ &= (C^2)\left(\frac{(C^2)(D,F)^2}{6(C,F)^2} - 1\right). \end{aligned}$$

This inequality holds only if

- $(C,F) = 2, (C^2) = 2$ and $(D,F) = 3$ (No. 8, 10, 11), or
- $(C,F) = 3, (C^2) = 2, 4, 6$ and $(D,F) = 3$ (Non-Jacobian types).

In any case E is a section of f. $\square$

Lemma 4.3. We use the same notation and assumptions as in Lemma 4.2 (3). In cases of Jacobian type, the curve E does not exist.

Proof. Our tool is an explicit basis of S_X^{φ} obtained from fiber components, c.f. (2) in the proof of Proposition 3.10. Let S be the fixed (-2)-curve which is a section of f. Then $U_0 = \langle F, S \rangle$ gives a sublattice isomorphic to U.

In case of IV-(ii) fiber, the two components l_1, l_2 which are disjoint from S constitute a sublattice A_{IV} of S_X^{φ} isomorphic to A_2 and orthogonal to U_0. In case of I_0^*-(iii) fiber, we put three simple components disjoint from S as $l, \varphi(l), \varphi^2(l)$ and the multiple component as m. Then m and $l^+ := l + \varphi(l) + \varphi^2(l)$ constitute a sublattice $A_{I_0^*}$ of S_X^{φ} isomorphic to A_2 and orthogonal to U_0. Similarly in cases of IV^*-(i) and II^*, the choice of basis is clear.

Let us give a proof in detail for No. 8. The other cases are similar. Note that in No. 8 there are three singular fibers of type IV-(ii) or I_0^*-(iii) which corresponds to three components A_2 in S_X^{φ}. Using the basis above for each A_2, we obtain the explicit basis for S_X^{φ} as

$$S_X^{\varphi} = U_0 \oplus A_{IV} \oplus A_{I_0^*} \oplus \cdots,$$

where the right-hand-side should be replaced suitably.

Suppose that there exists E as in Lemma 4.2 (3). Then $D = E + \varphi(E) + \varphi^2(E) \in S_X^{\varphi}$. D satisfies the following intersection relations.

$$(D, F) = 3, \ (D, S) = 0. \tag{4}$$

(5)
$$\begin{cases} (D, l_1) = (D, l_2) = 0 & \text{if } E \text{ meets zero component of } IV\text{-(ii)} \\ (D, l_1) = 3, \ (D, l_2) = 0 & \text{if } E \text{ meets } l_1 \\ (D, l_1) = 0, \ (D, l_2) = 3 & \text{if } E \text{ meets } l_2 \end{cases}$$

(6)
$$\begin{cases} (D, m) = (D, l^+) = 0 & \text{if } E \text{ meets zero component of } I_0^*\text{-(iii)} \\ (D, m) = 0, \ (D, l^+) = 3 & \text{if } E \text{ meets } l^+ \end{cases}$$

Since $U_0 \oplus A_{IV} \oplus A_{I_0^*} \oplus \cdots$ is an orthogonal direct sum, D has an expression $D = D_{U_0} + D_{A_{IV}} + D_{A_{I_0^*}} + \cdots$ and each D_{U_0} etc. can be computed separately from the relations above. In fact the first relation (4) shows that the $D_{U_0} = 3S + 6F$. The second relation (5) shows that $D_{A_{IV}} = 0$ or $-2l_1 - l_2$ or $-2l_2 - l_1$. The third relation (6) gives that $D_{A_{I_0^*}} = 0$ or $-3m - 2l^+$. Now we compute $(-2l_1 - l_2)^2 = (-2l_2 - l_1)^2 = -6$, $(-3m - 2l^+)^2 = -6$. Then we see

$$\begin{aligned} (D^2) &= (D_{U_0}^2) + (D_{A_{IV}}^2) + (D_{A_{I_0^*}}^2) + \cdots \\ &= 18 + (0 \text{ or } -6) + (0 \text{ or } -6) + \cdots \\ &\geq 18 - 6 - 6 - 6 \\ &= 0 \end{aligned}$$

since there are only three A_2 components. But actually D consists of three disjoint (-2)-curves, hence $(D^2) = -6$. Thus we obtain a contradiction. (In the same way, in case No. 10 we obtain $(D^2) \geq 18-12-6=0$ and in case No. 11 we obtain $(D^2) \geq 18-0-6=12$.) □

In non-Jacobian cases disjoint sections can exist. To proceed, it suffices to classify the linear equivalence class of $D = E+\varphi(E)+\varphi^2(E)$ by the following obvious lemma.

Lemma 4.4. Assume that the linear system of a divisor $D_1 \in S_X$ contains an effective divisor E_1 which is a disjoint union of (negative definite) ADE configurations. Then E_1 is the only divisor in $|D_1|$, namely $H^0(\mathcal{O}_X(D_1)) = 1$.

Proof. This is because ADE configurations can be contracted to normal singularities. □

Lemma 4.5. We use the same notation and assumptions as in Lemma 4.2 (3). In non-Jacobian cases, we have the following possibilities for $D = E + \varphi(E) + \varphi^2(E)$. The notation of divisors will be explained in the proof. For each class of $D \in S_X^{\varphi}$, E is unique (up to φ) if exists.

No.	$D \in S_X^{\varphi}$	intersection relations
2	$C - 2F$	
4a	$C' - 2F$	E meets l_1
	$C' - F - 2l_2 - l_3$	E meets l_2
	$C' - F - l_2 - 2l_3$	E meets l_3
4b	$C' - 2F$	E meets n
	$C' - F - 3m - 2L$	E meets L
4c	$C' - 2F$	E meets N
6a	$C' - 2F$	E meets l_1^1 and l_1^2
	$C' - F - 2l_2^2 - l_3^2$	E meets l_1^1 and l_2^2
	$C' - F - l_2^2 - 2l_3^2$	E meets l_1^1 and l_3^2
	$C' - F - 2l_2^1 - l_3^1$	E meets l_2^1 and l_1^2
	$C' - 2l_2^1 - l_3^1 - 2l_2^2 - l_3^2$	E meets l_2^1 and l_2^2
	$C' - 2l_2^1 - l_3^1 - l_2^2 - 2l_3^2$	E meets l_2^1 and l_3^2
	$C' - F - l_2^1 - 2l_3^1$	E meets l_3^1 and l_1^2
	$C' - l_2^1 - 2l_3^1 - 2l_2^2 - l_3^2$	E meets l_3^1 and l_2^2
	$C' - l_2^1 - 2l_3^1 - l_2^2 - 2l_3^2$	E meets l_3^1 and l_3^2
6b	$C' - 2F$	E meets n^1 and l_1^2
	$C' - F - 2l_2^2 - l_3^2$	E meets n^1 and l_2^2
	$C' - F - l_2^2 - 2l_3^2$	E meets n^1 and l_3^2
	$C' - F - 3m^1 - 2L^1$	E meets L^1 and l_1^2
	$C' - 3m^1 - 2L^1 - 2l_2^2 - l_3^2$	E meets L^1 and l_2^2

	$C' - 3m^1 - 2L^1 - l_2^2 - 2l_3^2$	E meets L^1 and l_3^2
6c	$C' - 2F$	E meets n^1 and n^2
	$C' - F - 3m^2 - 2L^2$	E meets n^1 and L^2
	$C' - F - 3m^1 - 2L^1$	E meets L^1 and n^2
	$C' - 3m^1 - 2L^1 - 3m^2 - 2L^2$	E meets L^1 and L^2
6d	$C' - 2F$	E meets l_1^1 and N^2
	$C' - F - 2l_2^1 - l_3^1$	E meets l_2^1 and N^2
	$C' - F - l_2^1 - 2l_3^1$	E meets l_3^1 and N^2
6e	$C' - 2F$	E meets N^1 and n^2
	$C' - F - 3m^2 - 2L^2$	E meets N^1 and L^2
6f	$C' - 2F$	E meets N^1 and N^2

Proof. The idea is the same as Lemma 4.3. The point is the construction of explicit basis for S_X^φ.

No. 2: In this case obviously $S_X^\varphi = \langle C, F\rangle$. Assume that E exists. Then $D = E + \varphi(E) + \varphi^2(E) \in S_X^\varphi$ satisfies $(D, C) = 0, (D, F) = 3$. Hence $D \sim C - 2F$. The uniqueness of E follows from the previous lemma.

No. 4a: Let us put the three components of IV-(ii) fiber as l_1, l_2, l_3. Then

$$\langle C' := C + l_2 + l_3, F\rangle \oplus \langle l_2, l_3\rangle \simeq U(3) \oplus A_2$$

is a basis of S_X^φ. When E meets l_1, we have relations

$$(D, C) = 0, (D, F) = 3, (D, l_2) = (D, l_3) = 0.$$

Hence $D \sim C' - 2F$. Next when E meets l_2, the relation becomes

$$(D, C) = 0, (D, F) = 3, (D, l_2) = 3, (D, l_3) = 0.$$

Then $D \sim C' - F - 2l_2 - l_3$. Similarly when E meets l_3, $D \sim C' - F - l_2 - 2l_3$.

No. 4b: This has a I_0^*-(iii) fiber. We denote by n the simple component preserved by φ, by $l, \varphi(l), \varphi^2(l)$ the other simple components and by m the double component. Then

$$\langle C' = C + 2m + L, F\rangle \oplus \langle m, L\rangle \simeq U(3) \oplus A_2$$

is a basis, where $L = l + \varphi(l) + \varphi^2(l)$. When E meets n, the relation is

$$(D, C) = 0, (D, F) = 3, (D, m) = 0, (D, L) = 0.$$

Thus $D \sim C' - 2F$. When E meets L, the relation becomes

$$(D, C) = 0, (D, F) = 3, (D, m) = 0, (D, L) = 3.$$

Then $D \sim C' - F - 3m - 2L$.

No. 4c: We denote by m the triple component of IV^*-(ii) and by $l, \varphi(l), \varphi^2(l)$ three double components and by $n, \varphi(n), \varphi^2(n)$ three simple components. We put $L = l + \varphi(l) + \varphi^2(l), N = n + \varphi(n) + \varphi^2(n)$. Then

$$\langle C' = C + 2m + L, F\rangle \oplus \langle m, L\rangle \simeq U(3) \oplus A_2$$

is the basis of S_X^φ. E can possibly meet only N and we have then

$$(D, C) = 0, (D, F) = 3, (D, m) = 0, (D, L) = 0.$$

Thus $D \sim C' - 2F$.

For cases of No. 6, we avoid describing the computations in detail. The notation is the same as No. 4 for fiber components and we use upper indices $l_1^1, l_1^2, \cdots$ to distinguish two reducible fibers corresponding to $A_2^{\oplus 2}$. We can choose the basis as

No.	the basis of S_X^φ
6a	$\langle C' = C + l_2^1 + l_3^1 + l_2^2 + l_3^2, F\rangle \oplus \langle l_2^1, l_3^1\rangle \oplus \langle l_2^2, l_3^2\rangle$
6b	$\langle C' = C + 2m^1 + L^1 + l_2^2 + l_3^2, F\rangle \oplus \langle m^1, L^1\rangle \oplus \langle l_2^2, l_3^2\rangle$
6c	$\langle C' = C + 2m^1 + L^1 + 2m^2 + L^2, F\rangle \oplus \langle m^1, L^1\rangle \oplus \langle m^2, L^2\rangle$
6d	$\langle C' = C + l_2^1 + l_3^1 + 2m^2 + L^2, F\rangle \oplus \langle l_2^1, l_3^1\rangle \oplus \langle m^2, L^2\rangle$
6e	$\langle C' = C + 2m^1 + L^1 + 2m^2 + L^2, F\rangle \oplus \langle m^1, L^1\rangle \oplus \langle m^2, L^2\rangle$
6f	$\langle C' = C + 2m^1 + L^1 + 2m^2 + L^2, F\rangle \oplus \langle m^1, L^1\rangle \oplus \langle m^2, L^2\rangle$

and the result is as in the table. □

Theorem 4.6. Let (X, φ) be a non-symplectic automorphism of order three on a $K3$ surface. Let Z be the log del Pezzo surface constructed in Proposition 4.1. If φ is of Jacobian type, then $\mathrm{Sing}(Z)$ and the Picard number $\rho(Z)$ are as in the table of Theorem 3.11. If φ is of non-Jacobian type, then $\mathrm{Sing}(Z)$ is one of the following possibilities. $\rho(Z)$ can be seen as

$$\rho(Z) = a - \mathrm{rk}(\text{rational double points}),$$

where $a = 2, 4, 6$ according to No. 2, 4, 6.

No.	$\mathrm{Sing}(Z)$		
2	(nonsing.)	A_1	
4a	$A_1(1)$	$A_1(1) + A_1$	
4b	$A_1(1) + A_1$	$A_1(1) + 2A_1$	$A_1(1) + A_2$
4c	$A_1(1) + A_2$	$A_1(1) + A_3$	
6a	$2A_1(1) + kA_1$	$(0 \leq k \leq 3)$	
6b	$2A_1(1) + A_1$	$2A_1(1) + A_1$	

	$2A_1(1) + A_2$	$2A_1(1) + A_1 + A_2$	
6c	$2A_1(1) + 2A_1$	$2A_1(1) + 3A_1$	$2A_1(1) + A_1 + A_2$
	$2A_1(1) + A_1 + A_3$	$2A_1(1) + 2A_2$	$2A_1(1) + A_3$
6d	$2A_1(1) + A_2$	$2A_1(1) + A_3$	
6e	$2A_1(1) + A_1 + A_2$	$2A_1(1) + A_3 + A_1$	$2A_1(1) + A_4$
6f	$2A_1(1) + 2A_2$	$2A_1(1) + A_5$	

Proof. The construction of Z from X is given by

- the contraction of all (-2)-curves disjoint from $C^{(g)}$, and
- the quotient by φ.

Equivalently we can construct Z as in the following diagram:

$$\begin{tikzcd}
X \arrow[dr, "\pi"] & & Z_r \arrow[dl, "\nu"'] \arrow[dr, "f"] & \\
& Y = X/\varphi & & Z
\end{tikzcd}$$

where ν is the minimal resolution and f is the contraction away from the transform of $C^{(g)}$ (Z_r is the one called right resolution of Z, see the next section). In this diagram, a singularity $A_1(1)$ on Y appears from an isolated fixed point on X. A (-2)-curve E on X which is preserved and not fixed by φ is mapped to a (-1)-curve on Z_r. We can compute the diagram explicitly for each case of Theorems 3.11, 3.12. Since we already know all the curves disjoint from $C^{(g)}$, Lemmas 4.2, 4.3 and 4.5, the computation of $\mathrm{Sing}(Z)$ can be done.

The computations for Jacobian cases are straightforward. We can draw a detailed picture using Theorem 3.11, Lemmas 4.2, 4.3. We omit the explanation.

For non-Jacobian cases, we take up No. 4b case for example. The other cases are done in a similar way. In No. 4b we have a I_0^*-(iii) fiber. By Lemma 4.5 we have the following possibilities.

(1) There are no sections.
(2) Only $D \sim C' - 2F$ is realized by sections.
(3) Only $D \sim C' - F - 3m - 2L$ is realized by sections.
(4) Both $D \sim C' - 2F$ and $D \sim C' - F - 3m - 2L$ are realized by sections.

But (4) does not occur, because $(C' - 2F, C' - F - 3m - 2L) = -3 < 0$ hence it cannot happen that both are effective divisors without common components. In (1) we have one $A_1(1)$ corresponding to the isolated fixed point and one A_1 corresponding to the three permuted simple components of I_0^*-(iii). In (2) the three sections produce one more

A_1. On the other hand in (3) the three sections intersect the three components of I_0^*-(iii) and produce A_2 instead of A_1. $\square$

Corollary 4.7. Except for the case $S_X^{\varphi} \simeq U(3)$, we obtain a log del Pezzo surface of index three by Proposition 4.1.

Remark 4.8. In cases other than No. 7 and 10, the log del Pezzo surface Z with the maximal rank of rational double points has the Picard number $\rho(Z) = 1$. In particular in No.s 8d, 9 and 11b we get Z with non-cyclic quotient singularities, hence different from (quasi-smooth) toric examples.

5. From log del Pezzo to $K3$

In this section we prove the following theorem.

Theorem 5.1. Let Z be a log del Pezzo surface of index three. Assume that it satisfies the multiple smooth divisor property, namely the linear system $|-3K_Z|$ contains a divisor $2C$ with C a non-singular curve which does not meet singularities. Then there exist a $K3$ surface X and a non-symplectic automorphism φ of order three of elliptic type on X such that Z can be obtained from (X, φ) by the construction of Proposition 4.1.

The proof uses standard constructions and classification theory of surfaces. We begin with the following remark. By [H, III, Corollary 7.9.] C is connected. Let g be the genus of C. Since C is located in the smooth locus, and by the conditions, the genus formula shows

$$2g - 2 = (C^2) + (C, K_Z) = -(C, K_Z)/2 > 0. \tag{7}$$

Hence $g \geq 2$.

5.1. **Right resolution.** Let $\tilde{f} : \widetilde{Z} \to Z$ be the minimal resolution of singularities. We denote by E_i an exceptional curve over a singularity of index three and a_i its discrepancy. We consider the following blowings up of $\widetilde{Z}$.

- If exceptional curves satisfy $(E_i, E_j) = 1$ and $a_i = -1/3$, $a_j = -2/3$ then we blow up at $E_i \cap E_j$.
- If exceptional curves satisfy $(E_i, E_j) = 1$ and $a_i = a_j = -2/3$, then after the blow up at $E_i \cap E_j$, we again blow up the two intersection points of the three exceptional curves.
- We remark that there exist no exceptional curves E_i, E_j such that $(E_i, E_i) = 1$ and both have discrepancies $-1/3$, see Table 2.

We do this process for all pairs (E_i, E_j). Then we obtain the surface $Z_r \xrightarrow{f} Z$, which we call the *right resolution* of Z, whose exceptional divisor over a singularity of index three is a successive union of the unit chain

$$\begin{array}{ccccc} (-3) & - & (-1) & - & (-6) \\ -\frac{1}{3} & & 0 & & -\frac{2}{3} \end{array},$$

(or one (-3)-curve for $A_1(1)$ or one (-6)-curve for $A_1(2)$). For example, the minimal resolution of $A_3(12)$ with components E_1, E_2, E_3 in this order as in Table 2 will be blown up to the chain

$$\begin{array}{ccccccccccccc} E_1 & & & & E_2 & & & & & & & & E_3 \\ (-3) & - & (-1) & - & (-6) & - & (-1) & - & (-3) & - & (-1) & - & (-6) \\ -\frac{1}{3} & & 0 & & -\frac{2}{3} & & 0 & & -\frac{1}{3} & & 0 & & -\frac{2}{3} \end{array}.$$

The point is that curves with nonzero discrepancies are disjoint to each other on Z_r. Let p (resp. q) be the number of (-3)-curves (resp. (-6)-curves) in the exceptional locus $\mathrm{Exc}(f)$ of Z_r. Now we relabel (-3)-curves as $E_i, 1 \le i \le p$ and (-6)-curves as $F_i, 1 \le i \le q$. Then the comparison of canonical bundles for f shows

$$K_{Z_r} \equiv f^* K_Z - \frac{1}{3}\sum_p E_i - \frac{2}{3}\sum_q F_i.$$

5.2. **Branched covering.** Since $3K_Z$ is Cartier, we have the relation

$$\begin{aligned} -3K_{Z_r} &= f^*(-3K_Z) + \sum_p E_i + 2\sum_q F_i \\ &\sim 2C + \sum_p E_i + 2\sum_q F_i, \end{aligned} \tag{8}$$

where we denoted the strict transform of C on Z_r by the same C. By taking the branched cover with branch $2C + \sum E_i + 2\sum F_i$ together with normalization, we get a triple cover $\pi\colon \widetilde{X} \to Z_r$, simply branched over the disjoint union $C \sqcup (\sqcup E_i) \sqcup (\sqcup F_i)$.

We put $\widetilde{E}_i = \pi^*(E_i)_{\mathrm{red}}, \widetilde{F}_i = \pi^*(F_i)_{\mathrm{red}}$ and $\widetilde{C} = \pi^*(C)_{\mathrm{red}}$. They are (-1)-curves, (-2)-curves and a curve of genus g on $\widetilde{X}$ respectively. We have the ramification formula

$$3K_{\widetilde{X}} = 3\pi^* K_{Z_r} + 6\widetilde{C} + 6\sum_p \widetilde{E}_i + 6\sum_q \widetilde{F}_i$$

and by substituting (8), we get $3K_{\widetilde{X}} \sim 3\sum_p \widetilde{E}_i$. Since $\widetilde{E}_i$ are disjoint (-1)-curves, we can contract them and we get a surface X with $3K_X \sim 0$.

5.3. X **is a** $K3$ **surface.** Let us show that X is a $K3$ surface. By $3K_X \sim 0$, it is a minimal surface with Kodaira dimension $\kappa = 0$. Recall that the class of minimal algebraic surfaces with $\kappa = 0$ consists of $K3$ surfaces, Enriques surfaces, abelian surfaces and bi-elliptic (or hyperelliptic) surfaces [BHPV]. An Enriques surface has $K_Y \not\sim 0$ and $2K_Y \sim 0$, so X is not an Enriques surface. Among other three surfaces, we can distinguish $K3$ surfaces by showing $H^1(X, \mathcal{O}_X) = 0$. Clearly this is equivalent to saying $H^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) = 0$. We will prove this vanishing.

By [EV, Claim 3.10], putting $B := 2C + \sum E_i + 2\sum F_i$, we have

$$\begin{aligned}\pi_* \mathcal{O}_{\widetilde{X}} &= \mathcal{O}_{Z_r} \oplus \left(\mathcal{O}_{Z_r}(K_{Z_r} + \lfloor \frac{1}{3}B \rfloor)\right) \oplus \left(\mathcal{O}_{Z_r}(2K_{Z_r} + \lfloor \frac{2}{3}B \rfloor)\right) \\ &= \mathcal{O}_{Z_r} \oplus \mathcal{O}_{Z_r}(K_{Z_r}) \oplus \mathcal{O}_{Z_r}(2K_{Z_r} + C + \sum F_i).\end{aligned}$$

Since Z_r is a rational surface [AN, Lemma 1.3], we know H^1 of the first two components vanish. Hence we have only to show

$$H^1\big(Z_r, \mathcal{O}_{Z_r}(2K_{Z_r} + C + \sum F_i)\big) = 0.$$

We use the exact sequence

$$\begin{aligned}0 &\to \mathcal{O}_{Z_r}(2K_{Z_r} + C + \sum F_i) \\ &\to \mathcal{O}_{Z_r}(2K_{Z_r} + C + \sum E_i + \sum F_i) \to \oplus \mathcal{O}_{E_i}(-1) \to 0.\end{aligned}$$

Taking the cohomology we obtain

$$\begin{aligned}0 = \oplus H^0(\mathcal{O}_{E_i}(-1)) &\to H^1(\mathcal{O}_{Z_r}(2K_{Z_r} + C + \sum F_i)) \\ &\to H^1(\mathcal{O}_{Z_r}(2K_{Z_r} + C + \sum E_i + \sum F_i)).\end{aligned} \tag{9}$$

In this sequence the last term can be written as $H^1(K_{Z_r} + \lceil K_{Z_r} + C + \frac{1}{3}\sum E_i + \frac{2}{3}\sum F_i \rceil)$. Since $\frac{1}{3}C$ is nef and big, numerical equivalence

$$K_{Z_r} + C + \frac{1}{3}\sum E_i + \frac{2}{3}\sum F_i \equiv \frac{1}{3}C$$

shows $H^1(\mathcal{O}_{Z_r}(2K_{Z_r} + C + \sum E_i + \sum F_i)) = 0$ by Kawamata-Viehweg vanishing theorem. Hence the middle term of (9) also vanishes and we have proved X is a $K3$ surface.

The rest is clear: the covering transformation of $\widetilde{X} \to Z_r$ produces a non-symplectic automorphism of order three on X. It is of elliptic type because C has genus ≥ 2. Thus Theorem 5.1 is proved.

Remark 5.2. It is easy to see from (7) that if Z has the multiple smooth divisor property then $K_Z^2 = 8(g-1)/3$. In particular, it is necessary for $3K_Z^2$ to be divisible by 8. This is a useful criterion. For example, the easiest log del Pezzo surface $Z = \mathbb{P}(1,1,3)$ has $3K_Z^2 = 25$ hence it does not satisfy multiple smooth divisor property.

6. Toric examples

In this section we collect examples of log del Pezzo surfaces of index three obtained as toric varieties. We thank Dr. T. Okada for providing us with the computer searching program. For the notation of singularities, we refer to Section 2.

Example 6.1. We put $Z := \mathbb{P}(1,1,6)$. It is easy to see that Z is a log del Pezzo surface of index three and has a singularity of type $A_1(2)$ at $(0,0,1)$. We remark $|-3/2K_Z| = |\mathcal{O}_Z(12)|$. Let C be an element of $\mathcal{O}_Z(12)$ defined by $\{x^{12} + y^{12} + z^2 + \cdots = 0\}$ where x, y and z are homogeneous coordinates of $\mathbb{P}(1,1,6)$. Then a smooth divisor C does not pass through $(0,0,1)$. Hence Z satisfies the multiple smooth divisor property.

Example 6.2. We consider the weighted hypersurface $Z := (4) \subset \mathbb{P}(1,1,1,3)$. Note Z is a log del Pezzo surface with a singular point of type $A_1(1)$. In particular the singular point is induced by $(0,0,0,1)$. We remark $\mathcal{O}_Z(K_Z) \simeq \mathcal{O}_Z(4-1-1-1-3) = \mathcal{O}_Z(-2)$. Let C be an element of $\mathcal{O}_Z(3)$ defined by $\{x^3 + y^3 + z^3 + w + \cdots = 0\}$ where x, y, z and w are homogeneous coordinates of $\mathbb{P}(1,1,1,3)$. Then a smooth divisor C does not pass through $(0,0,0,1)$. Hence Z satisfies the multiple smooth divisor property.

The following table contains some examples of log del Pezzo surfaces of index three. The first eighteen examples are in [Da, Theorem 1.3]. The notation ◯ implies that a log del Pezzo surface Z satisfies the multiple smooth divisor property. On the other hand $\times$ is not so.

Z	$\mathrm{Sing}(Z)$	K^2	$2C \in |-3K_Z|$
$\mathbb{P}(1,1,3)$	$A_1(1)$	$25/3$	$\times$
$\mathbb{P}(1,3,4)$	$A_1(1) + A_3$	$16/3$	◯
$\mathbb{P}(2,3,5)$	$A_1(1) + A_1 + A_4$	$10/3$	$\times$
$\mathbb{P}(1,1,2)/C_3$	$2A_1(1) + A_5$	$8/3$	◯
$\mathbb{P}(1,1,6)$	$A_1(2)$	$32/3$	◯
$\mathbb{P}(1,6,7)$	$A_1(2) + A_6$	$14/3$	$\times$
$\mathbb{P}(1,3,4)/C_2$	$A_1(2) + A_1 + A_7$	$8/3$	$\times$
$\mathbb{P}(1,2,3)/C_3$	$A_1(1) + A_1(2) + A_8$	2	$\times$
$\mathbb{P}^2/C_9$	$2A_2(12) + A_8$	1	$\times$

$\mathbb{P}(1,5,9)$	$A_2(12)+A_4$	5	×
$\mathbb{P}(1,2,9)$	$A_2(12)+A_1$	8	◯
$\mathbb{P}(1,2,3)/C_3$	$A_2(12)+A_2+A_5$	2	×
$\mathbb{P}(1,1,2)/C_2\times C_3$	$2A_1(2)+A_{11}$	4/3	×
$\mathbb{P}(1,1,6)/C_2$	$A_3(11)+2A_1$	16/3	◯
$\mathbb{P}(1,4,15)$	$A_2(22)+A_3$	20/3	×
$\mathbb{P}(1,1,3)/C_5$	$A_2(22)+2A_4$	5/3	×
$\mathbb{P}(1,2,9)/C_2$	$A_3(12)+A_1+A_3$	4	×
$\mathbb{P}(1,1,6)/C_4$	$A_3(22)+2A_3$	8/3	×
$(4)\subset\mathbb{P}(1,1,1,3)$	$A_1(1)$	16/3	◯
$(7)\subset\mathbb{P}(1,1,1,6)$	$A_1(2)$	14/3	×
$(5)\subset\mathbb{P}(1,1,2,3)$	$A_1(1)+A_1$	10/3	×
$(8)\subset\mathbb{P}(1,1,2,6)$	$A_1(2)+A_1$	8/3	×
$(10)\subset\mathbb{P}(1,1,2,9)$	$A_2(12)$	5	×
$(6)\subset\mathbb{P}(1,1,3,3)$	$A_1(1)$	8/3	×
$(9)\subset\mathbb{P}(1,1,3,6)$	$A_1(1)+A_1(2)$	2	×
$(16)\subset\mathbb{P}(1,1,4,15)$	$A_2(22)$	20/3	×
$(10)\subset\mathbb{P}(1,1,5,9)$	$A_2(12)$	8	◯
$(12)\subset\mathbb{P}(1,1,6,6)$	$A_1(2)$	4/3	×
$(12)\subset\mathbb{P}(1,2,3,9)$	$A_2(12)+A_2$	2	×
$(18)\subset\mathbb{P}(1,2,9,9)$	$A_2(12)$	1	×
$(14)\subset\mathbb{P}(1,2,7,12)$	$A_3(11)+A_1$	16/3	◯
$(15)\subset\mathbb{P}(1,5,6,9)$	$A_2(12)+A_5$	2	×

Remark 6.3. For example, we consider $\mathbb{P}(1,2,9)/C_2$. The group $C_2 = \langle g\rangle$ acts on $\mathbb{P}(1,2,9)$ via $g:(x,y,z)\to(x,-y,-z)$.

We note $(x,-y,-z)=(e_4x,y,e_4^3z)=(-x,-y,z)$ where e_k is a primitive k-th root of unity. The fixed locus of the action consists of three points.

At $(0,0,1)$, we have the open set $\mathbb{A}^2_{x,y}/\langle g,h\rangle$ where $g:(x,y)\to(-x,-y)$ and $h:(x,y)\to(e_9x,e_9^2y)$. Then we have a singularity of type $1/18(1,5)$.

Similarly in other points we can calculate. Hence we have singularities of type A_1 at $(1,0,0)$, of type A_2 at $(0,1,0)$ and of type $A_3(12)$ at $(0,0,1)$.

Remark 6.4. There are two $\mathbb{P}(1,2,3)/C_3$ in Dias's list [Da, Theorem 1.3]. Indeed the group $C_3=\langle g_i\rangle$ has two actions on $\mathbb{P}(1,2,3)$, g_1 :

$(x, y, z) \to (x, e_3 y, e_3 z)$ and $g_2 : (x, y, z) \to (x, e_3 y, e_3^2 z)$ where e_3 is a primitive third root of unity.

In the case of $C_3 = \langle g_1 \rangle$, we note $(x, e_3 y, e_3 z) = (e_3 x, y, e_3 z) = (e_9^2 x, e_9^7 y, z)$. Then we have singularities of type $A_1(1)$ at $(1, 0, 0)$, of type $A_1(2)$ at $(0, 1, 0)$ and of type A_8 at $(0, 0, 1)$.

In the case of $C_3 = \langle g_2 \rangle$, we note $(x, e_3 y, e_3^2 z) = (e_3 x, y, e_3^2 z) = (e_9 x, e_9^5 y, z)$. Then we have singularities of type A_2 at $(1, 0, 0)$, of type A_5 at $(0, 1, 0)$ and of type $A_2(12)$ at $(0, 0, 1)$.

Remark 6.5. The group C_3 acts on $\mathbb{P}(1, 1, 2)$ via $g : (x, y, z) \to (x, e_3 y, e_3 z)$.

The group C_2 acts on $\mathbb{P}(1, 3, 4)$ via $g : (x, y, z) \to (x, -y, -z)$.

The group C_9 acts on $\mathbb{P}^2$ via $g : (x, y, z) \to (x, e_9 y, e_9^2 z)$.

The group $C_2 \times C_3$ acts on $\mathbb{P}(1, 1, 2)$ via $g : (x, y, z) \to (-x, e_3 y, -e_3 z)$.

The group C_2 acts on $\mathbb{P}(1, 1, 6)$ via $g : (x, y, z) \to (x, -y, -z)$.

The group C_5 acts on $\mathbb{P}(1, 1, 3)$ via $g : (x, y, z) \to (x, e_5 y, e_5^4 z)$.

The group C_4 acts on $\mathbb{P}(1, 1, 6)$ via $g : (x, y, z) \to (x, e_4 y, e_4^3 z)$.

References

[AN] V. Alexeev, V.V. Nikulin. *Del Pezzo and $K3$ surfaces*, MSJ Memoirs, Vol. **15**, Mathematical Society of Japan, Tokyo, 2006.

[AS] M. Artebani, A. Sarti, Non-symplectic automorphisms of order 3 on $K3$ surfaces, Math. Ann., **342** (2008), 903-921.

[BHPV] W. Barth, K. Hulek, C. Peters, A.Van de Ven, *Compact Complex Surfaces. Second Enlarged Edition*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3.Folge. A Series of Modern Surveys in Mathematics, **4**, Springer, 2004. xii+436 pp.

[Br] E. Brieskorn, Rationale Singularitäten komplexer Flächen, Inv. Math., **4** (1968), 336–358.

[Da] D.I. Dais, Classification of Toric log Del Pezzo Surfaces having Picard Number 1 and Index ≤ 3, Result.Math. **54** (2009), 219–252

[EV] H. Esnault and E. Viehweg, *Lectures on Vanishing Theorems*, Birkhäuser 1992.

[H] R. Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics, **52**, Springer-Verlag, New York-Heidelberg, 1977. xvi+496 pp.

[Kaw] Y. Kawamata, Crepant blowing-up of 3-dimensional canonical singularities and its application to degenerations of surfaces, Annals of Math., **127** (1988), 93-163.

[KM] J. Kollár, S. Mori, *Birational geometry of algebraic varieties*, Cambridge Tracts in Mathematics, **134**. Cambridge University Press, Cambridge, 1998. viii+254 pp.

[Na] N. Nakayama, Classification of log del Pezzo surfaces of index two, J. Math. Sci. Univ. Tokyo, **14** (2007), 293-498.

[Ni] V.V. Nikulin, Finite automorphism groups of Kählerian $K3$ surfaces, Trans. Moscow Math. Soc., **38** (1980), No 2, 71–135.

[O] K. Oguiso, On Jacobian fibrations on the Kummer surfaces of the product of non-isogenous elliptic curves, J. Math. Soc. Japan, **41** (1989), 651-680.

[Ta] S. Taki, Classification of non-symplectic automorphisms of order 3 on $K3$ surfaces, to appear in Math. Nachr.

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