REMARKS ON THE HAMILTONIAN FOR THE FERMIONIC UNITARY GAS MODEL

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Abstract. We consider a quantum system in dimension three composed by a group of N identical fermions, with mass $1/2$, interacting via zero-range interaction with a group of M identical fermions of a different type, with mass $m/2$. Exploiting a renormalization procedure, we construct the corresponding quadratic (or energy) form and define the so-called Ter-Martirosyan-Skornyakov extension H_α, which is the natural candidate as a possible Hamiltonian of the system. In the particular case $M = 1$, under a suitable condition on the parameters m, N, we show that the quadratic form is unbounded from below. In the same setting we prove that H_α is not a self-adjoint and bounded from below operator and this in particular suggests that the so-called Thomas effect could occur.

1. Introduction

In many models in condensed matter physics and statistical mechanics a gas of n quantum particles in $\mathbb{R}^3$ is described through the formal Hamiltonian

$$H = -\sum_{i=1}^{n} \frac{1}{2m_i}\Delta_{x_i} + \mu \sum_{\substack{i,j=1 \\ i<j}}^{n} \delta(x_i - x_j) \tag{1.1}$$

where $x_i \in \mathbb{R}^3$, m_i is the mass of the i-th particle, Δ_{x_i} is the free laplacian relative to the coordinate x_i and μ is the strength of the δ, or zero-range, interaction acting between each pair of particles of the gas. To simplify the notation we fix $\hbar = 1$.

One reason of interest for the Hamiltonian (1.1) is that it is a simple but non trivial modification of the free Hamiltonian and then it can be used for concrete computations of relevant physical properties of the quantum gas. It is worth to mention that in recent years these models have been widely used in the physical literature, for systems of bosons or fermions, possibly with harmonic confining potential. In particular the limiting case of infinite two-body scattering length, or unitary limit, is also considered (see the reviews [2], [8] and also [19], [20], [16], [4]). From the mathematical point of view an Hamiltonian of the type (1.1) in the appropriate Hilbert space is defined as a self-adjoint extension of the free Hamiltonian restricted to a domain of smooth functions vanishing on each hyperplane $x_i = x_j$. The most used techniques for the concrete construction of such extensions are Krein's theory of self-adjoint extensions

and limiting procedure of smooth approximating Hamiltonians (in the sense of the resolvent or the quadratic form).
The problem is completely understood in the case $n = 2$, where one is reduced to study a fixed δ interaction in the relative coordinate and all the self-adjoint extensions can be explicitly constructed (we refer to [1] for a complete mathematical analysis of this case).
A direct generalization of the same construction to the case $n > 2$ naturally leads to the definition of the so-called Ter-Martirosyan-Skornyakov extension H_α. Roughly speaking, such extension is a symmetric operator acting on a set of functions ψ which are smooth outside the hyperplanes $x_i = x_j$, $i, j = 1, \dots, n$, while on each hyperplane they exhibit the following singular behavior

$$\psi \simeq \frac{\mathfrak{f}_{ij}}{|x_i - x_j|} + \alpha\, \mathfrak{f}_{ij} + o(1) \quad \text{for } |x_i - x_j| \to 0 \tag{1.2}$$

where $\mathfrak{f}_{ij}$ is a function defined on the hyperplane $x_i = x_j$ and α is a real parameter. One can see that α^{-1} is proportional to the two-body scattering length and therefore the unitary limit is obtained for $\alpha = 0$.
As a matter of fact the operator H_α is not self-adjoint and all its self-adjoint extensions are unbounded from below due to the presence of an infinite sequence of energy levels E_k going to $-\infty$ for $k \to \infty$. This result was first rigorously proved for a system of (at least) three identical bosons in [14] (see also [11] for the case of three different particles) using the theory of self-adjoint extensions. This effect, known as Thomas effect, prevents H_α from being a good physical Hamiltonian. Notice that the effect is absent in dimension two ([5]). We also mention that the Thomas effect can be considered as the counterpart in zero-range interaction models of the well known Efimov effect, i.e. the appearance of an infinite sequence of negative energy levels accumulating at zero for a three-bosons system with two-body resonant interaction potentials.
It is expected that the Thomas effect could be absent if the Hilbert space of states is appropriately restricted, e.g. introducing suitable symmetry constraints on the wave function. Indeed this was first rigorously proved in [15],[12] for a system of two identical fermions plus a different particle, with all equal masses. In this case, due to the antisymmetry of the wave function, the two fermions can only interact with the different particle and this makes the Hamiltonian less singular. Strangely enough, this result cannot be generalized to the case of a system composed by N identical fermions plus a different particle. When all masses are equal, it was shown in [5] that the quadratic form associated to H_α is unbounded from below for N sufficiently large. As we explain in section 5, this implies that in such a case the operator H_α cannot be self adjoint and bounded from below. The result was proved by evaluating the quadratic form on an explicit sequence of trial functions. It is remarkable that the trial functions must be chosen in the p-wave, contrary to the case of bosonic case where the s-wave is required.
In this paper we shall approach the following problem. Let us consider a system in dimension three made of two subsystems $\mathcal{A}$ and $\mathcal{B}$, where $\mathcal{A}$ consists of N identical fermions of one kind and $\mathcal{B}$ of M identical fermions of another kind. We assume that no interaction is present between particles of the same species while each particle of $\mathcal{A}$ interacts with each particle of $\mathcal{B}$

through a zero-range potential. Without loss of generality, we fix the mass of a particle in $\mathcal{A}$ equal to 1/2 and the mass of a particle in $\mathcal{B}$ equal to $m/2$.
The first mathematical problem is to construct the corresponding Ter-Martirosyan-Skornyakov extension and to show that it is self-adjoint and bounded from below or, on the contrary, it is only symmetric and, possibly, there is Thomas effect. In this generality the problem is open and one can only stress that the answer seems to be strongly dependent on the physical parameters m, N, M. Here, as a first step, we shall construct the renormalized quadratic form and the Ter-Martirosyan-Skornyakov extension for arbitrary values of the parameters.
In the simpler case $M = 1$, it has been conjectured by R.A. Minlos[1] that the Ter-Martirosyan-Skornyakov extension is

i) self-adjoint and bounded from below if $\Lambda(m, N) < 1$,
ii) only symmetric and the Thomas effect occurs if $\Lambda(m, N) > 1$,

where

$$\Lambda(m, N) \equiv (N-1)\frac{2(m+1)^3}{\pi\sqrt{m(m+2)}}\int_0^{\arcsin\frac{1}{m+1}} dx\, x \sin x \tag{1.3}$$

Notice that, for N fixed, $\Lambda(m, N)$ is a positive, decreasing function of m, with $\lim_{m\to 0}\Lambda(m, N) = \infty$ and $\lim_{m\to\infty}\Lambda(m, N) = 0$. Therefore there is a unique critical value of the mass $m_c(N)$ such that $\Lambda(m, N) < 1$ for $m > m_c(N)$ and $\Lambda(m, N) > 1$ for $m < m_c(N)$.
The conjecture is known to be true for $N = 2$ with a critical mass $m_c(2) \sim (13.607)^{-1}$ (see e.g. [2] and references therein for the physical literature and [17] for a rigorous result). Recently, the case $N = 3$ has been approached in [4] where, exploiting analytical and numerical arguments, it is shown that there is Thomas effect if $m < (13.384)^{-1}$. A further result has been obtained in [13] where it is proved that for $N \le 4$ and m sufficiently large the Ter-Martirosyan-Skornyakov extension is self-adjoint and bounded from below.
Our main result in this paper is the proof that for $\Lambda(m, N) > 1$ the Ter-Martirosyan-Skornyakov extension is not a self-adjoint and bounded from below operator. This does not prove the existence of the Thomas effect for $\Lambda(m, N) > 1$ but, in our opinion, it strongly suggests that this is in fact the case. We stress that a more detailed analysis is required in order to give a complete proof of part ii) of the conjecture. We also underline that the part i) of the conjecture is still an open problem.
The paper is organized as follows.
In section 2, following the arguments of [5], we describe the limiting procedure to obtain the quadratic form F_α which is naturally associated to the system in the general case of N fermions of one type and M fermions of another type.
In section 3 we introduce the corresponding Ter-Martirosyan-Skornyakov extension H_α and we show that its mean value coincides with F_α restricted to the operator domain.
In section 4 we restrict to the case $M = 1$ and we explicitly show that for $\Lambda(m, N) > 1$ the quadratic form F_α is unbounded from below.

[1]Private communication

In section 5 we recall some results of the theory of positive self-adjoint extensions of positive symmetric operators and then we show that H_α is not self-adjoint and bounded from below if $M = 1$ and $\Lambda(m, N) > 1$.
In the appendix we collect some technical results which are used in previous sections.
With an abuse of notation, the scalar product and the norm of various L^2-spaces introduced throughout the paper will be all denoted by the same symbols $(\cdot, \cdot)$, $\|\cdot\|$. Moreover the Fourier transform of f will be denoted by $\hat{f}$.

2. Limiting procedure for the quadratic form

In this section we describe a limiting procedure for the construction of the quadratic form naturally associated to the Hamiltonian of a system composed by two species $\mathcal{A}$ and $\mathcal{B}$ of identical fermions described in the introduction. Following the argument of [5], we first regularize the formal Hamiltonian and the corresponding quadratic form introducing an ultra-violet cut-off and then we remove the cut-off with a suitable renormalization of the coupling constant. We stress that our aim here is only to identify the limit. The rigorous control of the limiting procedure is outside the scope of the paper.
The Hilbert space of the system is denoted by $L^2_a(\mathbb{R}^{3(N+M)})$ and the formal Hamiltonian describing the dynamics is

$$(Hu)(\boldsymbol{x}_N, \boldsymbol{y}_M) = (H_0 u)\,(\boldsymbol{x}_N, \boldsymbol{y}_M) - \mu \sum_{(i,j)} \delta(y_j - x_i) u(\boldsymbol{x}_N, \boldsymbol{y}_M) \tag{2.1}$$

where

$$H_0 = -\Delta_{\boldsymbol{x}_N} - \frac{1}{m}\Delta_{\boldsymbol{y}_M} \tag{2.2}$$

$\boldsymbol{x}_N = (x_1, \ldots, x_N) \in \mathbb{R}^{3N}$, $x_i \in \mathbb{R}^3$, $\boldsymbol{y}_M = (y_1, \ldots, y_M) \in \mathbb{R}^{3M}$, $y_j \in \mathbb{R}^3$; moreover we introduce the following short-hand notation for three types of sums used in the sequel

$$\sum_{(i,j)} = \sum_{i=1}^{N} \sum_{j=1}^{M} \qquad \sum_{(i,j)\neq(l,h)} = \sum_{\substack{i=1\\ i\neq l}}^{N} \sum_{\substack{j=1\\ j\neq h}}^{M} \qquad \sum_{(i,j)\neq(l,h)}{}' = \sum_{\substack{i,l=1\\ (i,j)\neq(l,h)}}^{N} \sum_{j,h=1}^{M} \tag{2.3}$$

As we already remarked, the expression (2.1) does not define an operator in $L^2_a(\mathbb{R}^{3(N+M)})$. In order to obtain a well defined operator the first step is to regularize the expression (2.1) and this is more conveniently done in the Fourier space. Using the representation

$$\delta(y_j - x_i) = \frac{1}{(2\pi)^3} \int dw \, e^{iw(y_j - x_i)} \tag{2.4}$$

a direct computation yields

$$
\begin{aligned}
(\hat{H}\hat{u})(\boldsymbol{p}_N,\boldsymbol{k}_M) &= h_0(\boldsymbol{p}_N,\boldsymbol{k}_M)\hat{u}(\boldsymbol{p}_N,\boldsymbol{k}_M) - \frac{\mu}{(2\pi)^3}\sum_{(i,j)}\int dz\,\hat{u}(\hat{\boldsymbol{p}}_i, p_i+z, \hat{\boldsymbol{k}}_j, k_j - z) \\
&= h_0(\boldsymbol{p}_N,\boldsymbol{k}_M)\hat{u}(\boldsymbol{p}_N,\boldsymbol{k}_M) - \frac{2^{3/2}\mu}{(2\pi)^3}\sum_{(i,j)}\int dz\,\hat{u}\Big(\hat{\boldsymbol{p}}_i, \frac{p_i+k_j}{2}+\frac{z}{\sqrt{2}}, \hat{\boldsymbol{k}}_j, \frac{k_j-z}{2}-\frac{z}{\sqrt{2}}\Big) \qquad (2.5)
\end{aligned}
$$

where

$$
h_0(\boldsymbol{p}_N,\boldsymbol{k}_M) = \boldsymbol{p}_N^2 + \frac{\boldsymbol{k}_M^2}{m} \qquad (2.6)
$$

$$
\hat{\boldsymbol{p}}_i = (p_1,\dots,p_{i-1},p_{i+1},\dots,p_N) \in \mathbb{R}^{3(N-1)} \qquad (2.7)
$$

$$
\hat{u}\Big(\hat{\boldsymbol{p}}_i, r, \hat{\boldsymbol{k}}_j, s\Big) = \hat{u}\Big(p_1,\dots,p_{i-1},r,p_{i+1},\dots,p_N,k_1,\dots,k_{j-1},s,k_{j+1},\dots,k_M\Big) \qquad (2.8)
$$

A natural regularization of (2.5) is the following Hamiltonian depending on the cut-off $R>0$

$$
\begin{aligned}
&(\hat{H}_R\hat{u})(\boldsymbol{p}_N,\boldsymbol{k}_M) = h_0(\boldsymbol{p}_N,\boldsymbol{k}_M)\hat{u}(\boldsymbol{p}_N,\boldsymbol{k}_M) \\
&-\mu_R\sum_{(i,j)} 1_R\Big(\frac{p_i-k_j}{\sqrt{2}}\Big)\int dz\, 1_R(z)\hat{u}\Big(\hat{\boldsymbol{p}}_i, \frac{p_i+k_j}{2}+\frac{z}{\sqrt{2}}, \hat{\boldsymbol{k}}_j, \frac{k_j-z}{2}-\frac{z}{\sqrt{2}}\Big) \qquad (2.9)
\end{aligned}
$$

where μ_R is a new coupling constant explicitly dependent on the cut-off and 1_R is the characteristic function of the ball in $\mathbb{R}^3$ of radius R and center in the origin. It is obviously true that (2.9) defines a lower bounded self-adjoint operator for any $R>0$ with the same domain of the free Hamiltonian. The next step is to compute the quadratic form associated to (2.9) and then to take the limit $R\to\infty$ for a suitably chosen μ_R. The identification of the limit is easier if one introduces the following "volume charges" for $i=1,\dots,N$, $j=1,\dots,M$

$$
\hat{\rho}^R_{ij}(\boldsymbol{p}_N,\boldsymbol{k}_M) = \mu_R 1_R\Big(\frac{p_i-k_j}{\sqrt{2}}\Big)\int dz\, 1_R(z)\hat{u}\Big(\hat{\boldsymbol{p}}_i, \frac{p_i+k_j}{2}+\frac{z}{\sqrt{2}}, \hat{\boldsymbol{k}}_j, \frac{k_j-z}{2}-\frac{z}{\sqrt{2}}\Big) \qquad (2.10)
$$

and the corresponding "potentials" produced by $\hat{\rho}^R_{ij}$

$$
\widehat{G^\lambda\rho^R}(\boldsymbol{p}_N,\boldsymbol{k}_M) = \sum_{(i,j)}\widehat{G^\lambda\rho^R_{ij}}(\boldsymbol{p}_N,\boldsymbol{k}_M) = \sum_{(i,j)}\frac{\hat{\rho}^R_{ij}(\boldsymbol{p}_N,\boldsymbol{k}_M)}{h_0(\boldsymbol{p}_N,\boldsymbol{k}_M)+\lambda} \qquad (2.11)
$$

where $\lambda > 0$. Hence a direct computation yields

$$
\begin{aligned}
(\hat{u}, \hat{H}_R \hat{u}) &= \int d\boldsymbol{p}_N d\boldsymbol{k}_M \, h_0 |\hat{u}|^2 - \sum_{(i,j)} \int d\boldsymbol{p}_N d\boldsymbol{k}_M \, \overline{\hat{u}} \, \hat{\rho}^R_{ij} \\
&= \int d\boldsymbol{p}_N d\boldsymbol{k}_M \Big[(h_0 + \lambda) |\hat{u} - \widehat{G^\lambda \rho^R}|^2 - \lambda |\hat{u}|^2 \Big] \\
&- \int d\boldsymbol{p}_N d\boldsymbol{k}_M (h_0 + \lambda) |\widehat{G^\lambda \rho^R}|^2 + 2 \,\mathrm{Re} \int d\boldsymbol{p}_N d\boldsymbol{k}_M \, \overline{\hat{u}} \, (h_0 + \lambda) \widehat{G^\lambda \rho^R} - \sum_{(i,j)} \int d\boldsymbol{p}_N d\boldsymbol{k}_M \, \overline{\hat{u}} \, \hat{\rho}^R_{ij} \\
&= \int d\boldsymbol{p}_N d\boldsymbol{k}_M \Big[(h_0 + \lambda) |\hat{u} - \widehat{G^\lambda \rho^R}|^2 - \lambda |\hat{u}|^2 \Big] \\
&- \sum_{(i,j) \neq (l,h)}{}' \int d\boldsymbol{p}_N \boldsymbol{k}_M \frac{\overline{\hat{\rho}^R_{ij}} \, \hat{\rho}^R_{lh}}{h_0 + \lambda} - \sum_{(i,j)} \int d\boldsymbol{p}_N \boldsymbol{k}_M \frac{|\hat{\rho}^R_{ij}|^2}{h_0 + \lambda} + \sum_{(i,j)} \int d\boldsymbol{p}_N d\boldsymbol{k}_M \, \overline{\hat{u}} \, \hat{\rho}^R_{ij}
\end{aligned} \tag{2.12}
$$

where we have used (2.9), (2.10), (2.11) and the fact that

$$
\mathrm{Im} \int d\boldsymbol{p}_N d\boldsymbol{k}_M \, \overline{\hat{u}} \, \hat{\rho}^R_{ij} = 0 \tag{2.13}
$$

Let us define the following "surface charges" for $i = 1, \dots, N$, $j = 1, \dots, M$

$$
\hat{\xi}^R_{ij}(q, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j) = \mu_R \int dz \, 1_R(z) \hat{u}\Big(\hat{\boldsymbol{p}}_i, \frac{q+z}{\sqrt{2}}, \hat{\boldsymbol{k}}_j, \frac{q-z}{\sqrt{2}} \Big) \tag{2.14}
$$

We notice that

$$
\hat{\xi}^R_{ij}\Big(\frac{p_i + k_j}{\sqrt{2}}, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j \Big) = \mu_R \int dz \, 1_R(z) \hat{u}\Big(\hat{\boldsymbol{p}}_i, \frac{p_i + k_j}{2} + \frac{z}{\sqrt{2}}, \hat{\boldsymbol{k}}_j, \frac{p_i + k_j}{2} - \frac{z}{\sqrt{2}} \Big) \tag{2.15}
$$

and therefore

$$
\hat{\rho}^R_{ij}(\boldsymbol{p}_N, \boldsymbol{k}_M) = 1_R\Big(\frac{p_i - k_j}{\sqrt{2}} \Big) \hat{\xi}^R_{ij}\Big(\frac{p_i + k_j}{\sqrt{2}}, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j \Big) \tag{2.16}
$$

Let us rewrite the last two integrals in (2.12). We have

$$
\int d\boldsymbol{p}_N d\boldsymbol{k}_M \, \overline{\hat{u}} \, \hat{\rho}^R_{ij} = \mu_R^{-1} \int dq d\hat{\boldsymbol{p}}_i d\hat{\boldsymbol{k}}_j \, |\hat{\xi}^R_{ij}(q, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j)|^2 \tag{2.17}
$$

and

$$
\begin{aligned}
\int d\boldsymbol{p}_N d\boldsymbol{k}_M \, \frac{|\hat{\rho}^R_{ij}|^2}{h_0 + \lambda} &= \int d\boldsymbol{p}_N d\boldsymbol{k}_M \, 1_R\Big(\frac{p_i - k_j}{\sqrt{2}} \Big) \frac{|\hat{\xi}^R_{ij}\Big(\frac{p_i + k_j}{\sqrt{2}}, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j \Big)|^2}{h_0 + \lambda} \\
&= \int dq d\hat{\boldsymbol{p}}_i d\hat{\boldsymbol{k}}_j \, |\hat{\xi}^R_{ij}(q, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j)|^2 \, \mathcal{I}_R(q, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j)
\end{aligned} \tag{2.18}
$$

where we have introduced the integration variables $q=\frac{p_i+k_j}{\sqrt{2}}$, $z=\frac{p_i-k_j}{\sqrt{2}}$ and we have defined

$$\mathcal{I}_R(q,\hat{\boldsymbol{p}}_i,\hat{\boldsymbol{k}}_j)=\int dz\,\frac{1_R(z)}{\frac{m+1}{2m}z^2+\frac{m-1}{m}\,q\cdot z+\gamma},\qquad \gamma=\frac{m+1}{2m}\,q^2+\hat{\boldsymbol{p}}_i^2+\frac{1}{m}\hat{\boldsymbol{k}}_j^2+\lambda \tag{2.19}$$

Using (2.17), (2.18) in (2.12) we have

$$\begin{aligned}(\hat{u},\hat{H}_R\hat{u}) &= \int d\boldsymbol{p}_N d\boldsymbol{k}_M\Big[(h_0+\lambda)|\hat{u}-\widehat{G^\lambda\rho^R}|^2-\lambda|\hat{u}|^2\Big]\\ &+\sum_{(i,j)}\int dq d\hat{\boldsymbol{p}}_i d\hat{\boldsymbol{k}}_j\,|\hat{\xi}^R_{ij}(q,\hat{\boldsymbol{p}}_i,\hat{\boldsymbol{k}}_j)|^2\left(\mu_R^{-1}-\mathcal{I}_R(q,\hat{\boldsymbol{p}}_i,\hat{\boldsymbol{k}}_j)\right)-\sum_{(i,j)\neq(l,h)}{}'\int d\boldsymbol{p}_N\boldsymbol{k}_M\frac{\overline{\hat{\rho}^R_{ij}}\,\hat{\rho}^R_{lh}}{h_0+\lambda}\end{aligned} \tag{2.20}$$

For $R\to\infty$ one has

$$\begin{aligned}\mathcal{I}_R(q,\hat{\boldsymbol{p}}_i,\hat{\boldsymbol{k}}_j) &= \frac{2m}{m+1}\int dz\,\frac{1_R(z)}{z^2}-\frac{2m}{m+1}\int dz\,\frac{\frac{m-1}{m}\,q\cdot z+\gamma}{z^2\left(\frac{m+1}{2m}z^2+\frac{m-1}{m}\,q\cdot z+\gamma\right)}+o(1)\\ &= \frac{8\pi\,m}{m+1}R-2\pi^2\left(\frac{2m}{m+1}\right)^{3/2}\sqrt{\frac{2}{m+1}q^2+\hat{\boldsymbol{p}}_i^2+\frac{1}{m}\hat{\boldsymbol{k}}_j^2+\lambda}+o(1)\end{aligned} \tag{2.21}$$

where in the last line we have used the explicit integration

$$\int dz\,\frac{\delta\cdot z+\gamma}{z^2(z^2+\delta\cdot z+\gamma)}=\pi^2\sqrt{4\gamma-\delta^2}\,,\qquad \delta^2<4\gamma \tag{2.22}$$

Therefore, in order to obtain a non trivial limit for $R\to\infty$, we fix

$$\mu_R^{-1}=\frac{8\pi\,m}{m+1}R+\alpha \tag{2.23}$$

where $\alpha\in\mathbb{R}$ is a new coupling constant. At least formally, with this choice we can remove the cut-off and define the renormalized quadratic form as the limit of (2.20) for $R\to\infty$. More precisely, we are lead to the following definition of quadratic form

$$\begin{aligned}G_\alpha(u) &= \int d\boldsymbol{p}_N d\boldsymbol{k}_M\Big[(h_0(\boldsymbol{p}_N,\boldsymbol{k}_M)+\lambda)|(\hat{u}-\sum_{(i,j)}\widehat{G^\lambda\xi_{ij}})(\boldsymbol{p}_N,\boldsymbol{k}_M)|^2-\lambda|\hat{u}(\boldsymbol{p}_N,\boldsymbol{k}_M)|^2\Big]\\ &+\sum_{(i,j)}\int dq d\hat{\boldsymbol{p}}_i d\hat{\boldsymbol{k}}_j\left(\alpha+b\sqrt{h_1(q,\hat{\boldsymbol{p}}_i,\hat{\boldsymbol{k}}_j)+\lambda}\right)|\hat{\xi}_{ij}(q,\hat{\boldsymbol{p}}_i,\hat{\boldsymbol{k}}_j)|^2\\ &-\sum_{(i,j)\neq(l,h)}{}'\int d\boldsymbol{p}_N d\boldsymbol{k}_M\frac{\overline{\hat{\xi}_{ij}}\left(\frac{p_i+k_j}{\sqrt{2}},\hat{\boldsymbol{p}}_i,\hat{\boldsymbol{k}}_j\right)\hat{\xi}_{lh}\left(\frac{p_l+k_h}{\sqrt{2}},\hat{\boldsymbol{p}}_l,\hat{\boldsymbol{k}}_h\right)}{h_0(\boldsymbol{p}_N,\boldsymbol{k}_M)+\lambda}\end{aligned} \tag{2.24}$$

where

$$b = 2\pi^2 \left(\frac{2m}{m+1}\right)^{3/2} \tag{2.25}$$

$$h_1(q, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j) = \frac{2}{m+1} q^2 + \hat{\boldsymbol{p}}_i^2 + \frac{\hat{\boldsymbol{k}}_j^2}{m} \tag{2.26}$$

$$\hat{\xi}_{ij}(q, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j) = \lim_{R\to\infty} \hat{\xi}^R_{ij}(q, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j) \tag{2.27}$$

and the potential produced by the surface charges $\hat{\xi}_{ij}$ is given by

$$\sum_{(i,j)} \widehat{G^\lambda \xi_{ij}}(\boldsymbol{p}_N, \boldsymbol{k}_M) = \sum_{(i,j)} \frac{\hat{\xi}_{ij}\left(\frac{p_i+k_j}{\sqrt{2}}, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j\right)}{h_0(\boldsymbol{p}_N, \boldsymbol{k}_M) + \lambda} \tag{2.28}$$

In the quadratic form (2.24) the particles in the two groups $\mathcal{A}$ and $\mathcal{B}$ are still considered distinguishable. Since we want to describe fermions, the final step of the construction is to take into account the requirement of antisymmetry. From (2.27), (2.14) it is easily seen that

$$\hat{\xi}_{ij}(q, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j) = (-1)^{i+j}\, \hat{\xi}_{11}(q, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j) \equiv (-1)^{i+j}\, \hat{\xi}(q, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j) \tag{2.29}$$

which in particular means that the interaction is completely described by the unique surface charge $\hat{\xi}_{11} \equiv \hat{\xi}$. We also denote

$$\widehat{G^\lambda \xi}(\boldsymbol{p}_N, \boldsymbol{k}_M) = \sum_{(i,j)} \frac{(-1)^{i+j} \hat{\xi}\left(\frac{p_i+k_j}{\sqrt{2}}, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j\right)}{h_0(\boldsymbol{p}_N, \boldsymbol{k}_M) + \lambda} \tag{2.30}$$

Moreover for the products of surface charges in (2.24) we have

$$\begin{aligned}
&\overline{\hat{\xi}_{ij}}\Big(\frac{p_i+k_j}{\sqrt{2}}, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j\Big)\hat{\xi}_{lh}\Big(\frac{p_l+k_h}{\sqrt{2}}, \hat{\boldsymbol{p}}_l, \hat{\boldsymbol{k}}_h\Big) = \overline{\hat{\xi}}\Big(\frac{p_1+k_1}{\sqrt{2}}, \hat{\boldsymbol{p}}_1, \hat{\boldsymbol{k}}_1\Big)\hat{\xi}\Big(\frac{p_2+k_2}{\sqrt{2}}, \hat{\boldsymbol{p}}_2, \hat{\boldsymbol{k}}_2\Big) \quad \text{if} \quad i \neq l \;\; j \neq h \\
&\overline{\hat{\xi}_{ij}}\Big(\frac{p_i+k_j}{\sqrt{2}}, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j\Big)\hat{\xi}_{ih}\Big(\frac{p_i+k_h}{\sqrt{2}}, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_h\Big) = -\overline{\hat{\xi}}\Big(\frac{p_1+k_1}{\sqrt{2}}, \hat{\boldsymbol{p}}_1, \hat{\boldsymbol{k}}_1\Big)\hat{\xi}\Big(\frac{p_1+k_2}{\sqrt{2}}, \hat{\boldsymbol{p}}_1, \hat{\boldsymbol{k}}_2\Big) \quad \text{if} \quad i = l \;\; j \neq h \\
&\overline{\hat{\xi}_{ij}}\Big(\frac{p_i+k_j}{\sqrt{2}}, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j\Big)\hat{\xi}_{lj}\Big(\frac{p_l+k_j}{\sqrt{2}}, \hat{\boldsymbol{p}}_l, \hat{\boldsymbol{k}}_j\Big) = -\overline{\hat{\xi}}\Big(\frac{p_1+k_1}{\sqrt{2}}, \hat{\boldsymbol{p}}_1, \hat{\boldsymbol{k}}_1\Big)\hat{\xi}\Big(\frac{p_2+k_1}{\sqrt{2}}, \hat{\boldsymbol{p}}_2, \hat{\boldsymbol{k}}_1\Big) \quad \text{if} \quad i \neq l \;\; j = h
\end{aligned} \tag{2.31}$$

Taking into account the above symmetry constraints in (2.24), we finally arrive at the following quadratic form

$$F_\alpha(u) = \mathcal{F}^\lambda(u) + \Phi^\lambda_\alpha(\xi) \tag{2.32}$$

where

$$\mathcal{F}^\lambda(u) = \int d\boldsymbol{p}_N d\boldsymbol{k}_M \Big[(h_0(\boldsymbol{p}_N, \boldsymbol{k}_M) + \lambda)|(\hat{u} - \widehat{G^\lambda \xi})(\boldsymbol{p}_N, \boldsymbol{k}_M)|^2 - \lambda|\hat{u}(\boldsymbol{p}_N, \boldsymbol{k}_M)|^2\Big] \tag{2.33}$$

$$\Phi^\lambda_\alpha(\xi) = NM\Big[\alpha\|\xi\|^2 + \Phi^\lambda_0(\xi) + \Phi^\lambda_1(\xi) + \Phi^\lambda_2(\xi) + \Phi^\lambda_3(\xi)\Big] \tag{2.34}$$

and

$$\Phi_0^\lambda(\xi) = b\int dq d\hat{\boldsymbol{p}}_1 d\hat{\boldsymbol{k}}_1 \sqrt{h_1(q,\hat{\boldsymbol{p}}_1,\hat{\boldsymbol{k}}_1)+\lambda}\,|\hat{\xi}(q,\hat{\boldsymbol{p}}_1,\hat{\boldsymbol{k}}_1)|^2 \tag{2.35}$$

$$\Phi_1^\lambda(\xi) = (N-1)\int d\boldsymbol{p}_N d\boldsymbol{k}_M \frac{\overline{\hat{\xi}}\left(\frac{p_1+k_1}{\sqrt{2}},\hat{\boldsymbol{p}}_1,\hat{\boldsymbol{k}}_1\right)\hat{\xi}\left(\frac{p_2+k_1}{\sqrt{2}},\hat{\boldsymbol{p}}_2,\hat{\boldsymbol{k}}_1\right)}{h_0(\boldsymbol{p}_N,\boldsymbol{k}_M)+\lambda} \tag{2.36}$$

$$\Phi_2^\lambda(\xi) = (M-1)\int d\boldsymbol{p}_N d\boldsymbol{k}_M \frac{\overline{\hat{\xi}}\left(\frac{p_1+k_1}{\sqrt{2}},\hat{\boldsymbol{p}}_1,\hat{\boldsymbol{k}}_1\right)\hat{\xi}\left(\frac{p_1+k_2}{\sqrt{2}},\hat{\boldsymbol{p}}_1,\hat{\boldsymbol{k}}_2\right)}{h_0(\boldsymbol{p}_N,\boldsymbol{k}_M)+\lambda} \tag{2.37}$$

$$\Phi_3^\lambda(\xi) = -(N-1)(M-1)\int d\boldsymbol{p}_N d\boldsymbol{k}_M \frac{\overline{\hat{\xi}}\left(\frac{p_1+k_1}{\sqrt{2}},\hat{\boldsymbol{p}}_1,\hat{\boldsymbol{k}}_1\right)\hat{\xi}\left(\frac{p_2+k_2}{\sqrt{2}},\hat{\boldsymbol{p}}_2,\hat{\boldsymbol{k}}_2\right)}{h_0(\boldsymbol{p}_N,\boldsymbol{k}_M)+\lambda} \tag{2.38}$$

We also fix the following domain of definition of F_α

$$D(F_\alpha)=\left\{u \in L^2_a(\mathbb{R}^{3(N+M)}) \,|\, u = w^\lambda + G^\lambda\xi,\; w^\lambda \in H^1(\mathbb{R}^{3(N+M)}),\; \xi \in H^{1/2}(\mathbb{R}^{3(N+M-1)})\right\} \tag{2.39}$$

where $H^s(\mathbb{R}^d)$, $s \in \mathbb{R}$, $d \in \mathbb{N}$, denotes the standard Sobolev space

$$H^s(\mathbb{R}^d) = \left\{u \in L^2(\mathbb{R}^d) \mid \int_{\mathbb{R}^d} d\boldsymbol{p}\,(\boldsymbol{p}^2+1)^s|\hat{u}(\boldsymbol{p})|^2 < \infty\right\} \tag{2.40}$$

In the Appendix we shall show that the definition (2.34) is well-posed, i.e. $|\Phi_i^\lambda(\xi)| < \infty$, $i = 1,2,3$, for any $\xi \in H^{1/2}(\mathbb{R}^{3(N+M-1)})$.

3. The Ter-Martirosyan-Skornyakov extension

In this section we introduce the Ter-Martirosyan-Skornyakov extension H_α, i.e. the symmetric operator which is usually considered as a possible candidate for the description of the dynamics of our system. Then we show that the energy form naturally associated with it coincides with the quadratic form defined in the previous sections. We define the operator H_α as follows. Let us introduce the $3(N+M-1)$-dimensional hyperplanes in $\mathbb{R}^{3(N+M)}$

$$\Gamma_{ij} = \left\{(\boldsymbol{x}_N,\boldsymbol{y}_M) \in \mathbb{R}^{3(N+M)} \mid x_i = y_j\right\} \tag{3.1}$$

and the open domain

$$\Omega = \mathbb{R}^{3(N+M)} \setminus \bigcup_{(i,j)} \Gamma_{ij} \tag{3.2}$$

Then

$$D(H_\alpha)=\Big\{u\in L^2_a(\mathbb{R}^{3(N+M)})\mid u=w^\lambda+G^\lambda\xi,\; w^\lambda\in H^2(\mathbb{R}^{3(N+M)}),\;\xi\in H^{3/2}(\mathbb{R}^{3(N+M-1)}),$$

$$8\pi^{3/2}\widehat{w^\lambda|_{\Gamma_{ij}}}(\sqrt{2}q,\hat{\boldsymbol{p}}_i,\hat{\boldsymbol{k}}_j)=\Big(\alpha\,\hat{\xi}_{ij}+\sum_{(l,h)}\mathcal{T}^\lambda_{ij,lh}\hat{\xi}_{lh}\Big)(q,\hat{\boldsymbol{p}}_i,\hat{\boldsymbol{k}}_j)\Big\} \tag{3.3}$$

$$(H_\alpha+\lambda)u=(H_0+\lambda)w^\lambda \tag{3.4}$$

where the operator $\mathcal{T}^\lambda_{ij,lh}$ acting on the surface charges $\hat{\xi}_{lh}$ is defined in the following way

$$\mathcal{T}^\lambda_{ij,lh}\hat{\xi}_{lh}(q,\hat{\boldsymbol{p}}_i,\hat{\boldsymbol{k}}_j)=\begin{cases} b\sqrt{h_1(q,\hat{\boldsymbol{p}}_i,\hat{\boldsymbol{k}}_j)+\lambda}\;\hat{\xi}_{ij}(q,\hat{\boldsymbol{p}}_i,\hat{\boldsymbol{k}}_j) & (i,j)=(l,h)\\ -8\pi^{3/2}\;\widehat{G^\lambda\xi_{lh}|_{\Gamma_{ij}}}(\sqrt{2}q,\hat{\boldsymbol{p}}_i,\hat{\boldsymbol{k}}_j) & (i,j)\neq(l,h)\end{cases} \tag{3.5}$$

It is useful to give explicit expressions for the non diagonal terms in (3.5). We distinguish the three possible cases: $l\neq i$ and $h\neq j$, $l=i$ and $h\neq j$, $l\neq i$ and $h=j$.

(1) $l\neq i$ and $h\neq j$

$$G^\lambda\xi_{lh}|_{\Gamma_{ij}}(x_i,\hat{\boldsymbol{x}}_i,\hat{\boldsymbol{y}}_j)=\frac{1}{(2\pi)^{\frac{3}{2}(N+M)}}\int d\hat{\boldsymbol{p}}_i d\hat{\boldsymbol{k}}_j\,e^{i\left(\hat{\boldsymbol{p}}_i\cdot\hat{\boldsymbol{x}}_i+\hat{\boldsymbol{k}}_j\cdot\hat{\boldsymbol{y}}_j\right)}\int dp_i dk_j\,e^{i(p_i+k_j)x_i}\frac{\hat{\xi}_{lh}\left(\frac{p_l+k_h}{\sqrt{2}},\hat{\boldsymbol{p}}_l,\hat{\boldsymbol{k}}_h\right)}{h_0(\boldsymbol{p}_N,\boldsymbol{k}_M)+\lambda}$$

$$=\frac{(-1)^{l+h}}{(2\pi)^{\frac{3}{2}(N+M)}}\int dq d\hat{\boldsymbol{p}}_i d\hat{\boldsymbol{k}}_j\,e^{i\left(\sqrt{2}q\cdot x_i+\hat{\boldsymbol{p}}_i\cdot\hat{\boldsymbol{x}}_i+\hat{\boldsymbol{k}}_j\cdot\hat{\boldsymbol{y}}_j\right)}\int ds\,\frac{\hat{\xi}\left(\frac{p_l+k_h}{\sqrt{2}},\hat{\boldsymbol{p}}_l|_{p_i=\frac{q+s}{\sqrt{2}}},\hat{\boldsymbol{k}}_h|_{k_j=\frac{q-s}{\sqrt{2}}}\right)}{\tilde{h}_0(q,s,\hat{\boldsymbol{p}}_i,\hat{\boldsymbol{k}}_j)+\lambda} \tag{3.6}$$

where

$$\tilde{h}_0(q,s,\hat{\boldsymbol{p}}_i,\hat{\boldsymbol{k}}_j)=\frac{m+1}{2m}q^2+\frac{m+1}{2m}s^2+\frac{m-1}{m}q\cdot s+\hat{\boldsymbol{p}}_i^2+\frac{1}{m}\hat{\boldsymbol{k}}_j^2 \tag{3.7}$$

Then the Fourier transform reads

$$\widehat{G^\lambda\xi_{lh}|_{\Gamma_{ij}}}(\sqrt{2}q,\hat{\boldsymbol{p}}_i,\hat{\boldsymbol{k}}_j)=\frac{(-1)^{l+h}}{8\pi^{3/2}}\int ds\,\frac{\hat{\xi}\left(\frac{p_l+k_h}{\sqrt{2}},\hat{\boldsymbol{p}}_l|_{p_i=\frac{q+s}{\sqrt{2}}},\hat{\boldsymbol{k}}_h|_{k_j=\frac{q-s}{\sqrt{2}}}\right)}{\tilde{h}_0(q,s,\hat{\boldsymbol{p}}_i,\hat{\boldsymbol{k}}_j)+\lambda} \tag{3.8}$$

A similar computation can be done for the other two cases.

(2) $l=i$ and $h\neq j$

$$\widehat{G^\lambda\xi_{ih}|_{\Gamma_{ij}}}(\sqrt{2}q,\hat{\boldsymbol{p}}_i,\hat{\boldsymbol{k}}_j)=\frac{(-1)^{i+h}}{8\pi^{3/2}}\int ds\,\frac{\hat{\xi}\left(\frac{q+s}{2}+\frac{k_h}{\sqrt{2}},\hat{\boldsymbol{p}}_i,\hat{\boldsymbol{k}}_h|_{k_j=\frac{q-s}{\sqrt{2}}}\right)}{\tilde{h}_0(q,s,\hat{\boldsymbol{p}}_i,\hat{\boldsymbol{k}}_j)+\lambda} \tag{3.9}$$

(3) $l\neq i$ and $h=j$

$$\widehat{G^\lambda\xi_{lj}|_{\Gamma_{ij}}}(\sqrt{2}q,\hat{\boldsymbol{p}}_i,\hat{\boldsymbol{k}}_j)=\frac{(-1)^{l+j}}{8\pi^{3/2}}\int ds\,\frac{\hat{\xi}\left(\frac{p_l}{\sqrt{2}}+\frac{q-s}{2},\hat{\boldsymbol{p}}_l|_{p_i=\frac{q+s}{\sqrt{2}}},\hat{\boldsymbol{k}}_j\right)}{\tilde{h}_0(q,s,\hat{\boldsymbol{p}}_i,\hat{\boldsymbol{k}}_j)+\lambda} \tag{3.10}$$

The last equality in (3.3) should be considered as the boundary condition satisfied by u on Γ_{ij} and it connects the regular and the singular part of an element of $D(H_\alpha)$. It is easy to verify that the operator H_α,$D(H_\alpha)$ is independent of the choice of $\lambda > 0$ and it is symmetric.
In the next proposition we show that our definition of H_α,$D(H_\alpha)$ coincides with the standard definition usually found in the literature, except for an irrelevant modification of the coupling constant α.

Proposition 3.1. *Let $u \in D(H_\alpha)$. Then*

$$H_\alpha u|_\Omega = H_0 u|_\Omega \tag{3.11}$$

$$\lim_{|x_i - y_j| \to 0} |x_i - y_j| u(\boldsymbol{x}_N, \boldsymbol{y}_M) = \mathfrak{f}_{ij}(x_i, \hat{\boldsymbol{x}}_i, \hat{\boldsymbol{y}}_j) \tag{3.12}$$

$$\lim_{|x_i - y_j| \to 0} \left(u(\boldsymbol{x}_N, \boldsymbol{y}_M) - \frac{\mathfrak{f}_{ij}(x_i, \hat{\boldsymbol{x}}_i, \hat{\boldsymbol{y}}_j)}{|x_i - y_j|} \right) = \alpha_0\, \mathfrak{f}_{ij}(x_i, \hat{\boldsymbol{x}}_i, \hat{\boldsymbol{y}}_j) \tag{3.13}$$

where

$$\mathfrak{f}_{ij}(x_i, \hat{\boldsymbol{x}}_i, \hat{\boldsymbol{y}}_j) = (-1)^{i+j}\, \frac{2\sqrt{\pi}\, m}{m+1}\, \xi(\sqrt{2} x_i, \hat{\boldsymbol{x}}_i, \hat{\boldsymbol{y}}_j) \tag{3.14}$$

$$\alpha_0 = \frac{\sqrt{2}(m+1)}{8\pi^2 m}\, \alpha \tag{3.15}$$

Proof
Taking into account (3.4) and the fact that $(H_0 + \lambda) G^\lambda \xi_{ij}|_\Omega = 0$ (see (6.28)), we have

$$\begin{aligned} H_\alpha u|_\Omega &= (H_\alpha + \lambda) u|_\Omega - \lambda u|_\Omega = (H_0 + \lambda) w^\lambda|_\Omega - \lambda u|_\Omega = (H_0 + \lambda)\Big(u - \sum_{(i,j)} G^\lambda \xi_{ij}\Big)|_\Omega - \lambda u|_\Omega \\ &= H_0 u|_\Omega \end{aligned} \tag{3.16}$$

Let us characterize the singularity of an element of (3.3) at the hyperplane Γ_{ij}. Exploiting (6.29), for $|x_i - y_j| \to 0$ we have

$$\begin{aligned} u(\boldsymbol{x}_N, \boldsymbol{y}_M) &= w^\lambda(\boldsymbol{x}_N, \boldsymbol{y}_M) + \sum_{(l,h) \neq (i,j)} G^\lambda \xi_{lh}(\boldsymbol{x}_N, \boldsymbol{y}_M) + G^\lambda \xi_{ij}(\boldsymbol{x}_N, \boldsymbol{y}_M) \\ &= w^\lambda(\boldsymbol{x}_N, \boldsymbol{y}_M) + \sum_{(l,h) \neq (i,j)} G^\lambda \xi_{lh}(\boldsymbol{x}_N, \boldsymbol{y}_M) + \frac{1}{|x_i - y_j|} \frac{2\sqrt{\pi}\, m}{m+1}\, \xi_{ij}(\sqrt{2} x_i, \hat{\boldsymbol{x}}_i, \hat{\boldsymbol{y}}_j) \\ &\quad - \frac{b}{(2\pi)^{\frac{3}{2}(N+M)}} \int dq d\hat{\boldsymbol{p}}_i d\hat{\boldsymbol{k}}_j\, e^{i\left(\sqrt{2} x_i q + \hat{\boldsymbol{x}}_i \cdot \hat{\boldsymbol{p}}_i + \hat{\boldsymbol{y}}_j \cdot \hat{\boldsymbol{k}}_j\right)} \sqrt{h_1(q, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j) + \lambda}\; \hat{\xi}_{ij}(q, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j) + o(1) \end{aligned} \tag{3.17}$$

We notice that

$$\lim_{|x_i - y_j| \to 0} |x_i - y_j| \Big(w^\lambda(\boldsymbol{x}_N, \boldsymbol{y}_M) + \sum_{(l,h) \neq (i,j)} G^\lambda \xi_{lh}(\boldsymbol{x}_N, \boldsymbol{y}_M) \Big) = 0 \tag{3.18}$$

Therefore from (3.17) we obtain (3.12). Moreover

$$\lim_{|x_i-y_j|\to 0}\left(u(\boldsymbol{x}_N,\boldsymbol{y}_M)-\frac{\mathfrak{f}_{ij}(x_i,\hat{\boldsymbol{x}}_i,\hat{\boldsymbol{y}}_j)}{|x_i-y_j|}\right)=\lim_{|x_i-y_j|\to 0}\left(u(\boldsymbol{x}_N,\boldsymbol{y}_M)-\frac{2\sqrt{\pi}\,m\,\xi_{ij}(\sqrt{2}x_i,\hat{\boldsymbol{x}}_i,\hat{\boldsymbol{y}}_j)}{|x_i-y_j|(m+1)}\right)$$
$$=w^\lambda(\boldsymbol{x}_N,\boldsymbol{y}_M)|_{\Gamma_{ij}}+\sum_{(l,h)\neq(i,j)}G^\lambda\xi_{lh}(\boldsymbol{x}_N,\boldsymbol{y}_M)|_{\Gamma_{ij}}$$
$$-\frac{b}{(2\pi)^{\frac{3}{2}(N+M)}}\int dq d\hat{\boldsymbol{p}}_i d\hat{\boldsymbol{k}}_j\, e^{i\left(\sqrt{2}x_i q+\hat{\boldsymbol{x}}_i\cdot\hat{\boldsymbol{p}}_i+\hat{\boldsymbol{y}}_j\cdot\hat{\boldsymbol{k}}_j\right)}\sqrt{h_1(q,\hat{\boldsymbol{p}}_i,\hat{\boldsymbol{k}}_j)+\lambda}\;\hat{\xi}_{ij}(q,\hat{\boldsymbol{p}}_i,\hat{\boldsymbol{k}}_j)$$
$$\equiv f(x_i,\hat{\boldsymbol{x}}_i,\hat{\boldsymbol{y}}_j) \tag{3.19}$$

The computation of f is more easily done in the Fourier space. Exploiting (3.5) we have

$$\widehat{f}(\sqrt{2}q,\hat{\boldsymbol{p}}_i,\hat{\boldsymbol{k}}_j)$$
$$=\widehat{w|_{\Gamma_{ij}}}(\sqrt{2}q,\hat{\boldsymbol{p}}_i,\hat{\boldsymbol{k}}_j)+\sum_{(l,h)\neq(i,j)}\widehat{G^\lambda\xi_{lh}|_{\Gamma_{ij}}}(\sqrt{2}q,\hat{\boldsymbol{p}}_i,\hat{\boldsymbol{k}}_j)-\frac{1}{8\pi^{3/2}}(\mathcal{T}^\lambda_{ij,ij}\hat{\xi}_{ij})(q,\hat{\boldsymbol{p}}_i,\hat{\boldsymbol{k}}_j)$$
$$=\widehat{w|_{\Gamma_{ij}}}(\sqrt{2}q,\hat{\boldsymbol{p}}_i,\hat{\boldsymbol{k}}_j)-\frac{1}{8\pi^{3/2}}\left(\sum_{(l,h)}\mathcal{T}^\lambda_{ij,lh}\hat{\xi}_{lh}\right)(q,\hat{\boldsymbol{p}}_i,\hat{\boldsymbol{k}}_j)$$
$$=\frac{\alpha}{8\pi^{3/2}}(-1)^{i+j}\hat{\xi}(q,\hat{\boldsymbol{p}}_i,\hat{\boldsymbol{k}}_j) \tag{3.20}$$

where, in the last line, we have used the boundary condition in (3.3). Taking the inverse Fourier transform and using (3.14), (3.15) we find

$$f(x_i,\hat{\boldsymbol{x}}_i,\hat{\boldsymbol{y}}_j)=\alpha_0\,\mathfrak{f}_{ij}(x_i,\hat{\boldsymbol{x}}_i,\hat{\boldsymbol{y}}_j) \tag{3.21}$$

concluding the proof of the proposition.

□

The next step is to verify that the mean value of the operator H_α coincides with our quadratic form F_α restricted to $D(H_\alpha)$.

Proposition 3.2. *If $u\in D(H_\alpha)$ then $(u,H_\alpha u)=F_\alpha(u)$.*

Proof

Let us introduce the tubular neighborhood Γ^ε_{ij}, for $\varepsilon>0$, of the hyperplane Γ_{ij}

$$\Gamma^\varepsilon_{ij}=\left\{(\boldsymbol{x}_N,\boldsymbol{y}_M)\in\mathbb{R}^{3(N+M)}\,|\,|x_i-y_j|\leq\varepsilon\right\} \tag{3.22}$$

and the open domain

$$\Omega^\varepsilon=\mathbb{R}^{3(N+M)}\setminus\bigcup_{(i,j)}\Gamma^\varepsilon_{ij} \tag{3.23}$$

Taking into account (3.11), for any $u \in D(H_\alpha)$ we can write

$$(u, H_\alpha u) = \lim_{\varepsilon \to 0} \int_{\Omega^\varepsilon} d\boldsymbol{x}_N d\boldsymbol{y}_M \, \bar{u} \, H_0 u \tag{3.24}$$

The r.h.s. of (3.24) can be computed using the definition (3.3) and the equation (6.28) proved in proposition 6.3. In fact we have

$$\begin{aligned}
(u, H_\alpha u) &= \lim_{\varepsilon \to 0} \int_{z^\varepsilon} d\boldsymbol{x}_N d\boldsymbol{y}_M \left(\overline{w^\lambda} + \sum_{(i,j)} G^\lambda \bar{\xi}_{ij} \right) (H_0 + \lambda) \left(w^\lambda + \sum_{(i,j)} G^\lambda \xi_{ij} \right) - \lambda \int d\boldsymbol{x}_N d\boldsymbol{y}_M \, |u|^2 \\
&= \int d\boldsymbol{x}_N d\boldsymbol{y}_M \, \overline{w^\lambda} (H_0 + \lambda) w^\lambda - \lambda \int d\boldsymbol{x}_N d\boldsymbol{y}_M \, |u|^2 + \sum_{(i,j)} \int d\boldsymbol{x}_N d\boldsymbol{y}_M \, G^\lambda \bar{\xi}_{ij} \, (H_0 + \lambda) w^\lambda \\
&= \mathcal{F}^\lambda(u) + 8\pi^{3/2} \sum_{(i,j)} \int dq d\hat{\boldsymbol{p}}_i d\hat{\boldsymbol{k}}_j \, \bar{\hat{\xi}}_{ij}(q, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j) \widehat{w^\lambda|_{\Gamma_{ij}}}(\sqrt{2} q, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j) \\
&= \mathcal{F}^\lambda(u) + \sum_{(i,j)} \int dq d\hat{\boldsymbol{p}}_i d\hat{\boldsymbol{k}}_j \, \bar{\hat{\xi}}_{ij}(q, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j) \left(\alpha \, \hat{\xi}_{ij} + \sum_{(l,h)} \mathcal{T}^\lambda_{ij,lh} \hat{\xi}_{lh} \right) (q, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j)
\end{aligned} \tag{3.25}$$

where in the last line we have used the boundary condition satisfied by u on Γ_{ij} (see (3.3)). Now we closely look at the last term appearing in r.h.s. of (3.25) and show that they reconstruct $\Phi^\lambda_\alpha(\xi)$. First we have

$$\alpha \sum_{(i,j)} \int dq d\hat{\boldsymbol{p}}_i d\hat{\boldsymbol{k}}_j \, \bar{\hat{\xi}}_{ij}(q, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j) \widehat{\xi_{ij}}(q, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j) = \alpha N M \|\xi\|^2 \tag{3.26}$$

Using (3.5) the diagonal terms can be written as

$$\begin{aligned}
&\sum_{(i,j)} \int dq d\hat{\boldsymbol{p}}_i d\hat{\boldsymbol{k}}_j \, \bar{\hat{\xi}}_{ij}(q, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j) \mathcal{T}^\lambda_{ij,ij} \hat{\xi}_{ij}(q, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j) \\
&= \sum_{(i,j)} \int dq d\hat{\boldsymbol{p}}_i d\hat{\boldsymbol{k}}_j \, b \sqrt{h_1(q, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j) + \lambda} \, |\xi(q, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j)|^2 = N M \Phi^\lambda_0(\xi)
\end{aligned} \tag{3.27}$$

Concerning the non diagonal terms, we use the explicit expression of $\widehat{G^\lambda \xi_{lh}|_{\Gamma_{ij}}}$ and we find

$$\sum_{(i,j) \neq (l,h)}{}' \int dq d\hat{\boldsymbol{p}}_i d\hat{\boldsymbol{k}}_j \, \bar{\hat{\xi}}_{ij}(q, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j) \mathcal{T}^\lambda_{ij,lh} \hat{\xi}_{lh}(q, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j) = N M (\Phi^\lambda_1(\xi) + \Phi^\lambda_2(\xi) + \Phi^\lambda_3(\xi)) \tag{3.28}$$

The proof of the proposition is concluded.

□

For any physical application the crucial point is to show that the symmetric operator H_α is a good Hamiltonian for our system, i.e. to prove that it is self-adjoint and bounded from below (stability condition). As we already remarked, in general the problem is open and we expect that the answer can be positive only under appropriate conditions on the physical parameters of the system N, M, m. Exploiting the representation theorem of self-adjoint operators (see e.g. [9]), the result could be obtained proving that the associated quadratic form F_α is closed and bounded from below. We leave untouched such question in this paper. In the following we shall concentrate on the "negative" result, i.e. we shall prove that in the case $M = 1$, $\Lambda(m, N) > 1$ the quadratic form F_α is unbounded from below.

4. Unboundedness from below of the quadratic form for $M = 1$

From now on we shall limit ourselves to the case $M = 1$. The quadratic form Φ^λ_α (see (2.34)) reads

$$\Phi^\lambda_\alpha(\xi) = N\alpha\|\xi\|^2 + N\Phi^\lambda_0(\xi) + N\Phi^\lambda_1(\xi) \tag{4.1}$$

$$\Phi^\lambda_0(\xi) = b\int dq d\hat{\boldsymbol{p}}_1 \sqrt{h_1(q,\hat{\boldsymbol{p}}_1)+\lambda}\,|\hat{\xi}(q,\hat{\boldsymbol{p}}_1)|^2, \qquad h_1(q,\hat{\boldsymbol{p}}_1) = \frac{2}{m+1}q^2 + \hat{\boldsymbol{p}}_1^2 \tag{4.2}$$

$$\Phi^\lambda_1(\xi) = (N-1)\int d\boldsymbol{p}_N dk \frac{\overline{\hat{\xi}}\left(\frac{p_1+k}{\sqrt{2}},\hat{\boldsymbol{p}}_1\right)\hat{\xi}\left(\frac{p_2+k}{\sqrt{2}},\hat{\boldsymbol{p}}_2\right)}{h_0(\boldsymbol{p}_N,k)+\lambda}, \qquad h_0(\boldsymbol{p}_N,k) = \boldsymbol{p}_N^2 + \frac{1}{m}k^2 \tag{4.3}$$

The regular part $\mathcal{F}$ of the quadratic form is written as in (2.33) where the potential $G^\lambda\xi$ is now given by

$$\widehat{G^\lambda\xi}(\boldsymbol{p}_N,k) = \sum_i \frac{(-1)^{i+1}\xi\left(\frac{p_i+k}{\sqrt{2}},\hat{\boldsymbol{p}}_i\right)}{h_0(\boldsymbol{p}_N,k)+\lambda} \tag{4.4}$$

With the above notation we have

$$F_\alpha(u) = \mathcal{F}^\lambda(u) + \Phi^\lambda_\alpha(\xi) \tag{4.5}$$

In the next proposition we show that the form (4.5) is unbounded from below under a suitable condition on the parameters m, N.

Proposition 4.1. *If $\Lambda(m,N) > 1$ then there exists a sequence $u_n \in D(F_\alpha)$, $\inf_n \|u_n\| > 0$, such that $F_\alpha(u_n) \to -\infty$ for $n \to \infty$.*

Proof

We fix $\lambda > 0$ and consider a sequence of trial functions of the form

$$u_n = G^\lambda \xi_n \tag{4.6}$$

and therefore $F_\alpha(u_n) = \Phi^\lambda_\alpha(\xi_n) - \lambda\|u_n\|^2$. The chosen sequence takes a much simpler expression if we use the function η, defined by (6.12), instead of ξ. In fact we choose

$$\eta(x,\hat{\boldsymbol{p}}_1) = \frac{1}{n} f\left(\frac{x}{n}\right) g(\hat{\boldsymbol{p}}_1) \tag{4.7}$$

where $g \in \mathcal{S}(\mathbb{R}^{3N-3})$, $\|g\| = 1$ and f is a smooth function which will be specified later. Exploiting the estimate (6.30), one can easily check that the sequence u_n satisfies the condition $\inf_n \|u_n\| > 0$. Using the same change of variables of proposition 6.2 we obtain

$$\Phi_0^\lambda(\xi_n) = n^2 \frac{2\pi^2 m^{9/2}(m+2)^{3/2}}{(m+1)^3} \int dx d\hat{\boldsymbol{p}}_1 \sqrt{\frac{m(m+2)}{(m+1)^2}x^2 + \frac{(m-1)p_2^2 + \hat{\boldsymbol{p}}_1^2 + \lambda}{n^2}}\, |f(x)|^2 |g(\hat{\boldsymbol{p}}_1)|^2 \tag{4.8}$$

$$\Phi_1^\lambda(\xi_n) = n^2(N-1)\frac{m^{9/2}(m+2)^{3/2}}{(m+1)^3} \int d\hat{\boldsymbol{p}}_1 dx dy \frac{\overline{f}(x) f(y) |g(\hat{\boldsymbol{p}}_1)|^2}{x^2 + y^2 + \frac{2}{m+1} x \cdot y + \frac{1}{n^2}(\hat{\boldsymbol{p}}_1^2 + \lambda)} \tag{4.9}$$

Let us compute the leading terms of $\Phi_0^\lambda(\xi_n)$ and $\Phi_1^\lambda(\xi_n)$ for $n \to \infty$. Using the inequality

$$\sqrt{\frac{m(m+2)}{(m+1)^2}x^2 + \frac{(m-1)p_2^2 + \hat{\boldsymbol{p}}_1^2 + \lambda}{n^2}} - \frac{\sqrt{m(m+2)}}{m+1}|x| \leq \frac{1}{n}\sqrt{(m-1)p_2^2 + \hat{\boldsymbol{p}}_1^2 + \lambda}$$

it follows

$$\Phi_0^\lambda(\xi_n) = n^2 \frac{2\pi^2 m^5 (m+2)^2}{(m+1)^4} \int dx\, |x|\, |f(x)|^2 + \mathcal{O}(n) \tag{4.10}$$

Now we prove that

$$\Phi_1^\lambda(\xi_n) = n^2(N-1)\frac{m^{9/2}(m+2)^{3/2}}{(m+1)^3} \int dx dy\, \frac{\overline{f}(x) f(y)}{x^2 + y^2 + \frac{2}{m+1} x \cdot y} + \mathcal{O}(n) \tag{4.11}$$

We have

$$\begin{aligned} &\Phi_1^\lambda(\xi_n) - n^2(N-1)\frac{m^{9/2}(m+2)^{3/2}}{(m+1)^3} \int dx dy\, \frac{\overline{f}(x) f(y)}{x^2 + y^2 + \frac{2}{m+1} x \cdot y} \\ &= n^2(N-1)\frac{m^{9/2}(m+2)^{3/2}}{(m+1)^3} \int d\hat{\boldsymbol{p}}_1 |g(\hat{\boldsymbol{p}}_1)|^2 \int dx dy\, \overline{f}(x) f(y)\, T_n(x, y; \hat{\boldsymbol{p}}_1^2 + \lambda) \end{aligned} \tag{4.12}$$

where we have defined

$$T_n(x, y; \hat{\boldsymbol{p}}_1^2 + \lambda) = -\frac{\hat{\boldsymbol{p}}_1^2 + \lambda}{n^2} \frac{1}{\left(x^2 + y^2 + \frac{2}{m+1} x \cdot y\right)\left(x^2 + y^2 + \frac{2}{m+1} x \cdot y + \frac{1}{n^2}(\hat{\boldsymbol{p}}_1^2 + \lambda)\right)} \tag{4.13}$$

For any $n, \hat{\boldsymbol{p}}_1, \lambda$, the integral kernel (4.13) defines a Hilbert-Schmidt operator and therefore its norm can be estimated as follows

$$\begin{aligned} \|T_n(\hat{\boldsymbol{p}}_1^2 + \lambda)\|^2 &\leqslant \int dx dy\, |T_n(x, y; \hat{\boldsymbol{p}}_1^2 + \lambda)|^2 \\ &\leqslant \frac{(\hat{\boldsymbol{p}}_1^2 + \lambda)^2 (m+1)^4}{n^4 m^4} \int dx dy\, \frac{1}{(x^2 + y^2)^2 \left(x^2 + y^2 + \frac{m+1}{n^2 m}(\hat{\boldsymbol{p}}_1^2 + \lambda)\right)^2} \\ &= \frac{c}{n^2} \frac{(\hat{\boldsymbol{p}}_1^2 + \lambda)(m+1)^3}{m^3} \end{aligned} \tag{4.14}$$

where c is a numerical constant. Using this estimate in (4.12), we obtain (4.11). Moreover we notice that

$$N\alpha\|\xi_n\|^2 = \mathcal{O}(n) \tag{4.15}$$

Therefore, from (4.10), (4.11), (4.15), we have

$$F_\alpha(u_n) = \Phi_\alpha^\lambda(\xi_n) - \lambda\|u_n\|^2 = n^2 N \frac{2\pi^2 m^5 (m+2)^2}{(m+1)^4}\,\widetilde{\Phi}(f) + \mathcal{O}(n) \tag{4.16}$$

where

$$\widetilde{\Phi}(f) = \int dx|x||f(x)|^2 + \frac{N-1}{2\pi^2}\frac{m+1}{\sqrt{m(m+2)}}\int dxdy \frac{\overline{f}(x)f(y)}{x^2+y^2+\frac{2}{m+1}x\cdot y} \tag{4.17}$$

Thus the problem is reduced to find f such that $\widetilde{\Phi}(f) < 0$. We introduce polar coordinates (ρ,θ,φ) in $\mathbb{R}^3$ and denote the standard measure on S^2 by dz. We further specialize our choice of the trial function by

$$f(\rho,\theta,\varphi) = \; a(\rho)\,\cos\theta \tag{4.18}$$

where the radial part a will be specified later. Then we have

$$\widetilde{\Phi}(f) = \frac{4\pi}{3}\int_0^{+\infty} d\rho\,\rho^3|a(\rho)|^2 + \frac{N-1}{2\pi^2}\frac{m+1}{\sqrt{m(m+2)}}\int_0^{+\infty} d\rho_1 d\rho_2\,\rho_1^2\rho_2^2\,\overline{a}(\rho_1)a(\rho_2)$$
$$\int_{S^2} dz_1 dz_2 \frac{\cos\theta_1\cos\theta_2}{\rho_1^2+\rho_2^2+\frac{2}{m+1}\rho_1\rho_2\cos\theta_{12}} \tag{4.19}$$

where θ_{12} is the angle between x and y. With the change of variable $e^x = \rho$ we arrive at

$$\widetilde{\Phi}(f) = \frac{4\pi}{3}\int_{\mathbb{R}} dx\,|e^{2x}a(e^x)|^2 + \frac{N-1}{4\pi^2}\frac{m+1}{\sqrt{m(m+2)}}\int_{\mathbb{R}} dx_1 dx_2\; e^{2x_1}\overline{a}(e^{x_1})e^{2x_2}a(e^{x_2})$$
$$\int_{S^2} dz_1 dz_2 \frac{\cos\theta_1\cos\theta_2}{\cosh(x_1-x_2)+\frac{1}{m+1}\cos\theta_{12}} \tag{4.20}$$

Both terms appearing in (4.20) can be diagonalized by Fourier transform, see e.g. [6], and we get

$$\widetilde{\Phi}(f) = \int dk\;|d(k)|^2\,S(k) \qquad d(k) = \frac{1}{\sqrt{2\pi}}\int dx\,e^{-ixk}e^{2x}a(e^x) \tag{4.21}$$

where $S(k)$ is the continuous and even function given by

$$S(k) = \frac{4\pi}{3} + \frac{N-1}{2\pi}\frac{m+1}{\sqrt{m(m+2)}}\int_{S^2} dz_1 dz_2 \cos\theta_1\cos\theta_2 \frac{\sinh\gamma k}{\sin\gamma\,\sinh\pi k} \tag{4.22}$$

and $\gamma = \arccos\left(\frac{\cos\theta_{12}}{m+1}\right)$. Now we study the sign of $S(0)$ and in particular we show that if $\Lambda(m,N) > 1$, see (1.3), then $S(0) < 0$. We have

$$S(0) = \frac{4\pi}{3} + \frac{N-1}{2\pi^2}\frac{m+1}{\sqrt{m(m+2)}}\int_{S^2} dz_1 dz_2 \cos\theta_1\cos\theta_2 \frac{\gamma}{\sin\gamma} \tag{4.23}$$

The angular integral can be computed introducing $\varphi_{12} = \varphi_1 - \varphi_2$ and performing the change of variables $(\varphi_1, \theta_1, \varphi_2, \theta_2) \to (\varphi_1, \theta_1, \varphi_{12}, \theta_{12})$. Then we have

$$\begin{aligned}
&\int_{S^2} dz_1 dz_2 \cos\theta_1 \cos\theta_2 \,\frac{\gamma}{\sin\gamma} \\
&= \int_{S^2} dz_1 dz_{12} \cos\theta_1 (\cos\theta_1 \cos\theta_{12} + \sin\theta_1 \sin\theta_{12} \cos(\varphi_1 - \varphi_{12}))\,\frac{\gamma}{\sin\gamma} \\
&= \frac{8\pi^2}{3} \int_{-1}^{1} dy\, \frac{y\ \arccos\frac{y}{m+1}}{\sin\arccos\frac{y}{m+1}} \\
&= \frac{8\pi^2}{3} \int_{-1}^{1} dy\, \frac{y\ \left(\frac{\pi}{2} - \arcsin\frac{y}{m+1}\right)}{\cos\arcsin\frac{y}{m+1}} \\
&= -\frac{16\pi^2}{3} \int_{0}^{1} dy\, \frac{y\ \arcsin\frac{y}{m+1}}{\cos\arcsin\frac{y}{m+1}} \\
&= -\frac{16\pi^2}{3} (m+1)^2 \int_0^{\arcsin\frac{1}{m+1}} dx\ x\ \sin x
\end{aligned} \tag{4.24}$$

where we have used the trigonometric identity $\arccos z = \pi/2 - \arcsin z$ and used the change of variables $y = (m+1)\sin x$. From (4.23) and (4.24) we have

$$S(0) = \frac{4\pi}{3}\left(1 - (N-1)\frac{2(m+1)^3}{\pi\sqrt{m(m+2)}} \int_0^{\arcsin\frac{1}{m+1}} dx\ x\ \sin x\right) \equiv \frac{4\pi}{3}\left(1 - \Lambda(m,N)\right) \tag{4.25}$$

which is negative for $\Lambda(m,N) > 1$. The last step is to fix the radial function a such that $d(k)$ is, roughly speaking, supported around $k = 0$. We choose

$$a(\rho) = c\ \frac{\sqrt{\beta}}{\rho^2 \cosh\left(\frac{\rho^\beta + \rho^{-\beta}}{2}\right)} \qquad\qquad \beta > 0 \tag{4.26}$$

A straightforward calculation gives

$$d(k) = \frac{c}{\sqrt{\beta}}\ \widehat{h}\left(\frac{k}{\beta}\right), \qquad\qquad h(x) = \frac{1}{\cosh(\cosh x)} \tag{4.27}$$

We fix c such that $\|d\| = \|h\| = 1$. For β sufficientely small $\widetilde{\Phi}(f) < 0$ and the proof is complete.

□

5. On the Ter-Martirosyan-Skornyakov extension for $M = 1$

In this section we prove that for $M = 1$ and $\Lambda(m, N) > 1$ the Ter-Martirosyan-Skornyakov extension defined by (3.3) and (3.4) does not define a self-adjoint and bounded below operator in $L^2_a(\mathbb{R}^{3(N+1)})$.

First we recall some facts from Birman-Krein theory of positive extensions of a given symmetric and positive operator on a Hilbert space ([10], [3], [7]). Proofs can also be found in [18] where a detailed discussion of the original Russian literature is given.

Let S_0 be a symmetric and positive operator on an Hilbert space $\mathcal{H}$ and let $\mathcal{N}$ be the kernel of S_0^*. We shall denote the Friedrichs extension of S_0 by S_F. Notice that since S_0 is positive then S_F is positive and has a bounded inverse.

The main result of the Birman-Krein theory is that the positive self-adjoint extensions of S_0 are in a one-to-one correspondence with positive operators on $\mathcal{N}$. More precisely, if S is a positive self-adjoint extension of S_0 then there exists a positive operator $B : D(B) \subseteq \mathcal{N} \to \mathcal{N}$ such that

$$D(S) = \{u \in \mathcal{H} \,|\, u = \phi + S_F^{-1}(Bf + g) + f,\ \phi \in D(S_0),\ f \in D(B),\ g \in \mathcal{N} \cap D(B)^{\perp}\} \tag{5.1}$$

and

$$Su = S_0^*|_{D(S)} u = S_0 \phi + Bf + g \tag{5.2}$$

Notice that the closure of $D(B)$ may be a proper subspace of $\mathcal{N}$.

Let us specialize this general result to our concrete case. We have $\mathcal{H} = L^2_a(\mathbb{R}^{3(N+1)})$ and $S_0 = \tilde{H}_0 + \lambda$, $\lambda > 0$, where $\tilde{H}_0$ is the free Hamiltonian restricted to

$$D(\tilde{H}_0) = \left\{u \in L^2_a(\mathbb{R}^{3(N+1)}) \,|\, u \in H^2(\mathbb{R}^{3(N+1)}), \quad u|_{\Gamma_i} = 0, \quad i = 1, \ldots, N\right\} \tag{5.3}$$

where $\Gamma_i = \{(\boldsymbol{x}_N, y) \in \mathbb{R}^{3(N+1)} \,|\, x_i = y\}$. Moreover (see e.g. [7], [13])

$$\mathcal{N} = \left\{u \in L^2_a(\mathbb{R}^{3(N+1)}) \,|\, u = G^{\lambda}\mu,\ \mu \in H^{-1/2}(\mathbb{R}^{3N})\right\} \tag{5.4}$$

The Friedrichs extension of $\tilde{H}_0 + \lambda$ is $H_F + \lambda$, where H_F is the free Laplacian with domain

$$D(H_F) = \left\{u \in L^2_a(\mathbb{R}^{3(N+1)}) \,|\, u \in H^2(\mathbb{R}^{3(N+1)})\right\} \tag{5.5}$$

We shall denote $\mathcal{G}^{\lambda} = (H_F + \lambda)^{-1}$. Notice that $\mathcal{G}^{\lambda} : L^2(\mathbb{R}^{3(N+1)}) \to H^2(\mathbb{R}^{3(N+1)})$ while $G^{\lambda} : H^{-1/2}(\mathbb{R}^{3N}) \to L^2(\mathbb{R}^{3(N+1)})$ even if they act in the same way as multiplication operators in Fourier space. By the Birman-Krein theory we have that any self-adjoint positive extension of $\tilde{H}_0 + \lambda$ is given by $H_B + \lambda$, where

$$\begin{aligned} D(H_B) = \big\{&u \in L^2_a(\mathbb{R}^{3(N+1)}) \,|\, u = \varphi^{\lambda} + \mathcal{G}^{\lambda}(BG^{\lambda}\mu + G^{\lambda}\nu) + G^{\lambda}\mu,\ \varphi^{\lambda} \in D(\tilde{H}_0), \\ &\mu, \nu \in H^{-1/2}(\mathbb{R}^{3N}),\ G^{\lambda}\mu \in D(B),\ G^{\lambda}\nu \in \mathcal{N} \cap D(B)^{\perp}\big\} \end{aligned} \tag{5.6}$$

$$(H_B + \lambda)u = (H_0 + \lambda)\varphi^{\lambda} + BG^{\lambda}\mu + G^{\lambda}\nu \tag{5.7}$$

where $B : D(B) \subseteq \mathcal{N} \to \mathcal{N}$ is a positive operator. Exploiting this fact we can prove the following result.

Proposition 5.1. *If $M = 1$ and $\Lambda(m,N) > 1$ then the Ter-Martirosyan-Skornyakov extension H_α, defined by (3.3) and (3.4), is not a self-adjoint and bounded from below operator in $L^2_a(\mathbb{R}^{3(N+1)})$.*

Proof
We shall prove the proposition by contradiction, i.e. we assume that $H_\alpha + \lambda$ is a positive, self-adjoint operator for a sufficiently large $\lambda > 0$ and then we show that this leads to a contradiction with the result of the previous section. By definition we have

$$D(H_\alpha) = \Big\{ u \in L^2_a(\mathbb{R}^{3(N+1)}) \,|\, u = w^\lambda + G^\lambda \xi,\; w^\lambda \in H^2(\mathbb{R}^{3(N+1)}),\; \xi \in H^{3/2}(\mathbb{R}^{3N}),$$
$$8\pi^{3/2} \widehat{w^\lambda|_{\Gamma_i}}(\sqrt{2}q, \hat{\boldsymbol{p}}_i) = \big(\alpha\, \hat{\xi}_i + \sum_l \mathcal{T}^\lambda_{i,l}\, \hat{\xi}_l\big)(q, \hat{\boldsymbol{p}}_i) \Big\} \tag{5.8}$$

where

$$\mathcal{T}^\lambda_{i,l}\, \hat{\xi}_l(q, \hat{\boldsymbol{p}}_i) = \begin{cases} b\sqrt{h_1(q,\hat{\boldsymbol{p}}_i) + \lambda}\;\, \hat{\xi}_i(q,\hat{\boldsymbol{p}}_i) & i = l \\ -8\pi^{3/2}\, \widehat{G^\lambda \xi_l|_{\Gamma_i}}(\sqrt{2}q, \hat{\boldsymbol{p}}_i) & i \neq l \end{cases} \tag{5.9}$$

Let us assume that $H_\alpha + \lambda$ is a positive, self-adjoint extension of $\tilde{H}_0 + \lambda$. Then there exists a positive operator B in $\mathcal{N}$ such that $D(H_\alpha) = D(H_B)$. In particular for any $u = w^\lambda + G^\lambda \xi \in D(H_\alpha)$ there exist $\varphi^\lambda \in D(\tilde{H}_0)$ and $\mu, \nu \in H^{-1/2}(\mathbb{R}^{3N})$, with $G^\lambda \mu \in D(B)$ and $G^\lambda \nu \in \mathcal{N} \cap D(B)^\perp$, such that the following identity holds

$$u \equiv w^\lambda + G^\lambda \xi = \varphi^\lambda + \mathcal{G}^\lambda(B G^\lambda \mu + G^\lambda \nu) + G^\lambda \mu \tag{5.10}$$

From (5.10) we have $G^\lambda \xi = G^\lambda \mu$ and therefore, by (6.30), it follows $\xi = \mu$ and $\mu \in H^{3/2}(\mathbb{R}^{3N})$. Moreover, from propositions 3.1 and 6.3 we also obtain

$$\lim_{|x_i - y| \to 0} \left(u(\boldsymbol{x}_N, y) - \frac{2\sqrt{\pi}\, m}{(m+1)|x_i - y|} \xi_i(\sqrt{2}x_i, \hat{\boldsymbol{x}}_i) \right)$$
$$= \frac{\alpha}{(2\pi)^{3/2}} \xi_i(\sqrt{2}x_i, \hat{\boldsymbol{x}}_i) = \big(\mathcal{G}^\lambda(B G^\lambda \xi + G^\lambda \nu)\big)(\boldsymbol{x}_N, y)|_{\Gamma_i} - \sum_l \big(\mathscr{F}^{-1} \mathcal{T}^\lambda_{i,l} \hat{\xi}_l\big)(x_i, \hat{\boldsymbol{x}}_i) \tag{5.11}$$

where $\mathscr{F}^{-1}$ denotes the inverse Fourier transform. Formula (5.11) holds in particular for $\xi = 0$ and this means that $\nu = 0$. Then in the Fourier space formula (5.11) reads

$$\big(\alpha\, \hat{\xi}_i + \sum_l \mathcal{T}^\lambda_{i,l}\, \hat{\xi}_l\big)(q, \hat{\boldsymbol{p}}_i) = \big(\mathcal{G}^\lambda \widehat{B G^\lambda \xi}|_{\Gamma_i}\big)(q, \hat{\boldsymbol{p}}_i) \tag{5.12}$$

From (5.12) we obtain

$$\Phi^\lambda_\alpha(\xi) \equiv \sum_i \big(\hat{\xi}_i,\, \alpha\, \hat{\xi}_i + \sum_l \mathcal{T}^\lambda_{i,l}\, \hat{\xi}_l\big) = \sum_i \big(\hat{\xi}_i,\, \mathcal{G}^\lambda \widehat{B G^\lambda \xi}|_{\Gamma_i}\big) \tag{5.13}$$

By a direct computation one sees that the r.h.s. of (5.13) equals $(8\pi^{3/2})^{-1}(G^\lambda \xi, B\, G^\lambda \xi)$ and then we conclude

$$\Phi^\lambda_\alpha(\xi) = \frac{1}{8\pi^{3/2}} (G^\lambda \xi, B\, G^\lambda \xi) \geq 0 \tag{5.14}$$

On the other hand we know from the previous section that the form $\Phi^\lambda_\alpha(\xi)$ can be made negative for a suitably chosen ξ. Therefore we obtain a contradiction and the proposition is proved.

□

6. Appendix

In this appendix we collect some technical results used in the paper.

Lemma 6.1. *Let us consider following integral operator in* $L^2(\mathbb{R}^3)$

$$(Qu)(x) = \int dx' \, \frac{u(x')}{\sqrt{|x|}(x^2 + x'^2)\sqrt{|x'|}} \tag{6.1}$$

Then Q *is bounded and*

$$\|Q\| \leq 2\pi^2 \tag{6.2}$$

Proof

Introducing spherical coordinates $x = (r, z)$ and using the Schwartz inequality we have

$$|(u, Qv)| \leq 4\pi \int_0^\infty dr \, \mathfrak{u}(r) \int_0^\infty dr' \, \mathfrak{v}(r') \, \tilde{Q}(r, r') \tag{6.3}$$

where we have denoted

$$\mathfrak{u}(r) = r\Big(\int dz |u(r, z)|^2 \Big)^{1/2}, \qquad \mathfrak{v}(r) = r\Big(\int dz |v(r, z)|^2 \Big)^{1/2} \tag{6.4}$$

and $\tilde{Q}$ is the integral operator in $L^2(\mathbb{R}^+)$ with integral kernel

$$\tilde{Q}(r, r') = \frac{\sqrt{r\, r'}}{r^2 + r'^2} \tag{6.5}$$

The operator $\tilde{Q}$ can be explicitly diagonalized. It is sufficient to introduce the unitary operator

$$\mathcal{D} \;:\; L^2(\mathbb{R}^+) \to L^2(\mathbb{R}), \qquad (\mathcal{D}f)(y) = e^{y/2} f(e^y) \tag{6.6}$$

and to observe that

$$(\mathcal{D}\tilde{Q}\mathcal{D}^{-1}g)(y) = \frac{1}{2} \int dy' \, \frac{g(y')}{\cosh(y - y')} \tag{6.7}$$

Taking the Fourier transform, the above operator is reduced to the multiplication operator (see e.g.[6])

$$(\widehat{\mathcal{D}\tilde{Q}\mathcal{D}^{-1}}g)(k) = \frac{\pi}{2\cosh\frac{\pi}{2}k} \, \hat{g}(k) \tag{6.8}$$

and therefore the norm of $\tilde{Q}$ is $\frac{\pi}{2}$. Using this fact in (6.3) we conclude the proof.

□

Exploiting (6.2) we can now estimate Φ_i^λ, for $i = 1, 2, 3$.

Proposition 6.2. *There exist positive constants $C_i = C_i(N, M, m, \lambda)$ such that*

$$|\Phi_i^\lambda(\xi)| \le C_i\, \Phi_0^\lambda(\xi) \qquad i = 1, 2, 3 \tag{6.9}$$

Proof

Let us first consider Φ_1^λ defined in (2.36). We introduce the change of the integration variables $(p_1, p_2, k_1) \to (x, y, z)$ given by

$$\begin{cases} x = \left(\frac{m+1}{m(m+2)^2}\right)^{1/2} \big(k_1 + p_1 - (m+1)p_2\big) \\ y = \left(\frac{m+1}{m(m+2)^2}\right)^{1/2} \big(k_1 + p_2 - (m+1)p_1\big) \\ z = \left(\frac{1}{m(m+2)}\right)^{1/2} (k_1 + p_1 + p_2) \end{cases} \tag{6.10}$$

with inverse given by

$$\begin{cases} p_1 = \left(\frac{m}{m+2}\right)^{1/2} z - \left(\frac{m}{m+1}\right)^{1/2} y \\ p_2 = \left(\frac{m}{m+2}\right)^{1/2} z - \left(\frac{m}{m+1}\right)^{1/2} x \\ k_1 = \left(\frac{m}{m+1}\right)^{1/2} (x+y) + \left(\frac{m^3}{m+2}\right)^{1/2} z \end{cases} \tag{6.11}$$

Moreover we define

$$\eta(x, \hat{\boldsymbol{p}}_1, \hat{\boldsymbol{k}}_1) = \hat{\xi}\left(\sqrt{\frac{m(m+1)^2}{2(m+2)}}p_2 + \sqrt{\frac{m}{2(m+1)}}x, \sqrt{\frac{m}{m+2}}p_2 - \sqrt{\frac{m}{m+1}}x, p_3, \dots, p_N, \hat{\boldsymbol{k}}_1\right) \tag{6.12}$$

Then we have

$$\Phi_1^\lambda(\xi) = (N-1)|J_1| \int d\hat{\boldsymbol{p}}_1 d\hat{\boldsymbol{k}}_1 \int dxdy\, \frac{\eta(x, \hat{\boldsymbol{p}}_1, \hat{\boldsymbol{k}}_1)\eta(y, \hat{\boldsymbol{p}}_1, \hat{\boldsymbol{k}}_1)}{x^2 + y^2 + \frac{2}{m+1} x \cdot y + \hat{\boldsymbol{p}}_1^2 + \frac{1}{m}\hat{\boldsymbol{k}}_1^2 + \lambda} \tag{6.13}$$

where

$$|J_1| = \left|\frac{\partial(p_1, p_2, k_1)}{\partial(x, y, z)}\right| = \left(\frac{m^{3/2}\sqrt{m+2}}{m+1}\right)^3 \tag{6.14}$$

is the jacobian of the transformation of coordinates (6.10). Taking into account

$$x^2 + y^2 + \frac{2}{m+1} x \cdot y + \hat{\boldsymbol{p}}_1^2 + \frac{\hat{\boldsymbol{k}}_1^2}{m} \ge \frac{m}{m+1}(x^2 + y^2) \tag{6.15}$$

we have the following estimate

$$|\Phi_1^\lambda(\xi)| \leq (N-1)|J_1|\frac{m+1}{m}\int d\hat{\boldsymbol{p}}_1 d\hat{\boldsymbol{k}}_1 \int dx dy \, \frac{\sqrt{|x|}|\eta(x,\hat{\boldsymbol{p}}_1,\hat{\boldsymbol{k}}_1)|\sqrt{|y|}|\eta(y,\hat{\boldsymbol{p}}_1,\hat{\boldsymbol{k}}_1)|}{\sqrt{|x|}(x^2+y^2)\sqrt{|y|}} \tag{6.16}$$

Exploiting the estimate (6.2) we obtain

$$|\Phi_1^\lambda(\xi)| \leq 2\pi^2(N-1)|J_1|\frac{m+1}{m}\int d\hat{\boldsymbol{p}}_1 d\hat{\boldsymbol{k}}_1 \int dx \, |x||\eta(x,\hat{\boldsymbol{p}}_1,\hat{\boldsymbol{k}}_1)|^2 \tag{6.17}$$

Let us rewrite also $\Phi_0^\lambda(\xi)$ in terms of η defined by (6.12). It is convenient to introduce a further change of coordinates $(q,p_2) \to (x,q_2)$ given by

$$\begin{cases} x = \sqrt{\frac{m+1}{m(m+2)^2}}\left(\sqrt{2}\,q-(m+1)p_2\right) \\ q_2 = \frac{1}{\sqrt{m(m+2)}}\left(\sqrt{2}\,q+p_2\right) \end{cases} \tag{6.18}$$

with inverse given by

$$\begin{cases} q = \sqrt{\frac{m}{2(m+1)}}x + \sqrt{\frac{m(m+1)^2}{2(m+2)}}q_2, \\ p_2 = -\sqrt{\frac{m}{m+1}}x + \sqrt{\frac{m}{m+2}}q_2 \end{cases} \tag{6.19}$$

The diagonal term now reads

$$\Phi_0^\lambda(\xi) = b|J_2|\int d\hat{\boldsymbol{p}}_1 d\hat{\boldsymbol{k}}_1 dx \sqrt{\frac{m(m+2)}{(m+1)^2}x^2 + mp_2^2 + p_3^2 + \ldots + p_N^2 + \hat{\boldsymbol{p}}_1^2 + \lambda}\, |\eta(x,\hat{\boldsymbol{p}}_1,\hat{\boldsymbol{k}}_1)|^2 \tag{6.20}$$

where

$$|J_2| = m^3\left(\frac{m+2}{2(m+1)}\right)^{3/2} \tag{6.21}$$

is the jacobian of the transformation of coordinates (6.18). By the trivial estimate

$$\sqrt{\frac{m(m+2)}{(m+1)^2}x^2 + mp_2^2 + p_3^2 + \ldots + p_N^2 + \hat{\boldsymbol{p}}_1^2 + \lambda} \geq \sqrt{\frac{m(m+2)}{(m+1)^2}}\,|x| \tag{6.22}$$

and (6.17), (6.20) we conclude

$$|\Phi_1^\lambda(\xi)| \leq C_1\,\Phi_0^\lambda(\xi) \qquad C_1 = \mathrm{Max}\left\{1, \frac{2\pi^2(N-1)|J_1|\frac{m+1}{m}}{b|J_2|\sqrt{\frac{m(m+2)}{(m+1)^2}}}\right\} \tag{6.23}$$

Proceeding exactly in the same way we also have

$$|\Phi_2^\lambda(\xi)| \leq C_2\,\Phi_0^\lambda(\xi) \qquad C_2 = \mathrm{Max}\left\{1, \frac{2\pi^2(M-1)|J_1|\frac{m+1}{m}}{b|J_2|\sqrt{\frac{m(m+2)}{(m+1)^2}}}\right\} \tag{6.24}$$

Let us consider Φ_3^λ defined in (2.38). Introducing the coordinates

$$v = \frac{p_1 + k_1}{\sqrt{2}}, \quad x = \frac{p_1 - k_1}{\sqrt{2}}, \quad z = \frac{p_2 + k_2}{\sqrt{2}}, \quad y = \frac{p_2 - k_2}{\sqrt{2}} \tag{6.25}$$

and exploiting the fact that $h_0(\boldsymbol{p}_N, \boldsymbol{k}_M) \geq m^{-1}(x^2 + y^2)$, we have

$$\begin{aligned}|\Phi_3^\lambda(\xi)| &\leq (N-1)(M-1)m \int dp_3..dp_N dk_3..dk_M dv dz \\ &\cdot \int dx dy \frac{\left|\hat{\xi}\Big(v, \frac{z+y}{\sqrt{2}}, p_3, .., p_N, \frac{z-y}{\sqrt{2}}, k_3, .., k_M\Big)\right| \left|\hat{\xi}\Big(z, \frac{v+x}{\sqrt{2}}, p_3, .., p_N, \frac{v-x}{\sqrt{2}}, k_3, .., k_M\Big)\right|}{x^2 + y^2}\end{aligned} \tag{6.26}$$

Using the estimate (6.2) we also obtain

$$\begin{aligned}|\Phi_3^\lambda(\xi)| &\leq (N-1)(M-1)\, 2\pi^2 m \int dp_3..dp_N dk_3..dk_M dv dz \\ &\cdot \int dx \sqrt{x^2 + v^2} \left|\hat{\xi}\Big(z, \frac{v+x}{\sqrt{2}}, p_3, .., p_N, \frac{v-x}{\sqrt{2}}, k_3, .., k_M\Big)\right|^2 \\ &= (N-1)(M-1)\, 2\pi^2 m \int dz d\hat{\boldsymbol{p}}_1 d\hat{\boldsymbol{k}}_1 \sqrt{p_2^2 + k_2^2}\, |\hat{\xi}(z, \hat{\boldsymbol{p}}_1, \hat{\boldsymbol{k}}_1)|^2 \\ &\leq (N-1)(M-1) \frac{2\pi^2 m}{b}\, \Phi_0^\lambda(\xi) \;\equiv\; C_3 \Phi_0^\lambda(\xi)\end{aligned} \tag{6.27}$$

and the proof of (6.9) is concluded.

□

In the next proposition we collect some useful properties of the potential produced by the surface charges ξ_{ij}.

Proposition 6.3. *For $\xi_{ij} \in L^2(\mathbb{R}^{3(N+M-1)})$ the corresponding potential $G^\lambda \xi_{ij}(\boldsymbol{x}_N, \boldsymbol{y}_M)$ satisfies*

$$\left[(H_0 + \lambda)\, G^\lambda \xi_{ij}\right] (\boldsymbol{x}_N, \boldsymbol{y}_M) = 8\pi^{3/2} \xi_{ij}(\sqrt{2}\, x_i, \hat{\boldsymbol{x}}_i, \hat{\boldsymbol{y}}_j)\, \delta(x_i - y_j) \tag{6.28}$$

in distributional sense. For $\xi_{ij} \in H^1(\mathbb{R}^{3(N+M-1)})$ the singularity for $|x_i - y_j| \to 0$ is characterized as follows

$$\begin{aligned}(G^\lambda \xi_{ij})(\boldsymbol{x}_N, \boldsymbol{y}_M) &= \frac{1}{|x_i - y_j|} \frac{2\sqrt{\pi}\, m}{m+1}\, \xi_{ij}(\sqrt{2} x_i, \hat{\boldsymbol{x}}_i, \hat{\boldsymbol{y}}_j) \\ -\frac{b}{(2\pi)^{\frac{3}{2}(N+M)}} &\int dq d\hat{\boldsymbol{p}}_i d\hat{\boldsymbol{k}}_j\, e^{i\left(\sqrt{2} x_i q + \hat{\boldsymbol{x}}_i \cdot \hat{\boldsymbol{p}}_i + \hat{\boldsymbol{y}}_j \cdot \hat{\boldsymbol{k}}_j\right)} \sqrt{h_1(q, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j) + \lambda}\; \hat{\xi}_{ij}(q, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j) + o(1)\end{aligned} \tag{6.29}$$

Moreover for $\xi_{ij} \in H^{-1/2}(\mathbb{R}^{3(N+M-1)})$ *one has*

$$c_1\!\int\! dq d\hat{\boldsymbol{p}}_i d\hat{\boldsymbol{k}}_j \, \frac{|\hat{\xi}\big(q,\hat{\boldsymbol{p}}_i,\hat{\boldsymbol{k}}_j\big)|^2}{\sqrt{q^2+\hat{\boldsymbol{p}}_i^2+\frac{\hat{\boldsymbol{k}}_j^2}{m}+\lambda}} \leq \|G^\lambda \xi_{ij}\|^2 \leq c_2\!\int\! dq d\hat{\boldsymbol{p}}_i d\hat{\boldsymbol{k}}_j \, \frac{|\hat{\xi}\big(q,\hat{\boldsymbol{p}}_i,\hat{\boldsymbol{k}}_j\big)|^2}{\sqrt{q^2+\hat{\boldsymbol{p}}_i^2+\frac{\hat{\boldsymbol{k}}_j^2}{m}+\lambda}} \tag{6.30}$$

where $c_1 = \pi^2 \min\{m,1\}$, $c_2 = \pi^2 \max\{m,1\}$.

Proof

Let us fix a test function ϕ and let us consider the definition (2.28) of $\widehat{G^\lambda \xi_{ij}}$. Then, exploiting Fourier transform, we have

$$\begin{aligned}
&\int\! d\boldsymbol{x}_N d\boldsymbol{y}_M \, \overline{\phi}(\boldsymbol{x}_N,\boldsymbol{y}_M) \left[(H_0+\lambda)\, G^\lambda \xi_{ij}\right](\boldsymbol{x}_N,\boldsymbol{y}_M) = \!\int\! d\boldsymbol{p}_N d\boldsymbol{k}_M \, \overline{\hat{\phi}}(\boldsymbol{p}_N,\boldsymbol{k}_M) \hat{\xi}_{ij}\!\left(\frac{p_1+k_j}{\sqrt{2}},\hat{\boldsymbol{p}}_i,\hat{\boldsymbol{k}}_j\right) \\
&= \frac{8\pi^{3/2}}{(2\pi)^{\frac{3}{2}(N+M)}} \int\! dx_i d\hat{\boldsymbol{x}}_i d\hat{\boldsymbol{y}}_j \, \xi_{ij}(\sqrt{2}x_i,\hat{\boldsymbol{x}}_i,\hat{\boldsymbol{y}}_j)\!\int\! d\boldsymbol{p}_N d\boldsymbol{k}_M \, \overline{\hat{\phi}}(\boldsymbol{p}_N,\boldsymbol{k}_M)\, e^{-i[(p_i+k_j)\cdot x_i+\hat{\boldsymbol{p}}_i\cdot\hat{\boldsymbol{x}}_i+\hat{\boldsymbol{k}}_j\cdot\hat{\boldsymbol{y}}_j)]} \\
&= 8\pi^{3/2} \int\! dx_i d\hat{\boldsymbol{x}}_i d\hat{\boldsymbol{y}}_j \, \xi_{ij}(\sqrt{2}x_i,\hat{\boldsymbol{x}}_i,\hat{\boldsymbol{y}}_j)\overline{\phi}(\hat{\boldsymbol{x}}_i,x_i,\hat{\boldsymbol{y}}_j,x_i)
\end{aligned} \tag{6.31}$$

and this proves (6.28). From (2.28) we also have

$$\begin{aligned}
G^\lambda \xi_{ij}(\boldsymbol{x}_N,\boldsymbol{y}_M) &= \frac{1}{(2\pi)^{\frac{3}{2}(N+M)}} \int\! d\boldsymbol{p}_N d\boldsymbol{k}_M \, e^{i(\boldsymbol{x}_N\cdot\boldsymbol{p}_N+\boldsymbol{y}_M\cdot\boldsymbol{k}_M)} \frac{\hat{\xi}_{ij}\left(\frac{p_1+k_j}{\sqrt{2}},\hat{\boldsymbol{p}}_i,\hat{\boldsymbol{k}}_j\right)}{h_0(\boldsymbol{p}_N,\boldsymbol{k}_M)+\lambda} \\
&= \frac{1}{(2\pi)^{\frac{3}{2}(N+M)}} \int\! dq d\hat{\boldsymbol{p}}_i d\hat{\boldsymbol{k}}_j \, e^{i\left[\frac{x_i+y_j}{\sqrt{2}}\cdot q+\hat{\boldsymbol{x}}_i\cdot\hat{\boldsymbol{p}}_i+\hat{\boldsymbol{y}}_j\cdot\hat{\boldsymbol{k}}_j\right]} \hat{\xi}_{ij}(q,\hat{\boldsymbol{p}}_i,\hat{\boldsymbol{k}}_j)\, \mathcal{L}(x_i-y_j,q,\hat{\boldsymbol{p}}_i,\hat{\boldsymbol{k}}_j)
\end{aligned} \tag{6.32}$$

where

$$\mathcal{L}(x_i-y_j,q,\hat{\boldsymbol{p}}_i,\hat{\boldsymbol{k}}_j) = \!\int\! dz \, \frac{e^{i\frac{x_i-y_j}{\sqrt{2}}\cdot z}}{\frac{m+1}{2m}z^2+\frac{m-1}{m}\,q\cdot z+\gamma}\,, \qquad \gamma = \frac{m+1}{2m}\,q^2+\hat{\boldsymbol{p}}_i^2+\frac{1}{m}\hat{\boldsymbol{k}}_j^2+\lambda \tag{6.33}$$

For $|x_i - y_j| \to 0$ the last integral is given by

$$\begin{aligned}
\mathcal{L}(x_i-y_j,q,\hat{\boldsymbol{p}}_i,\hat{\boldsymbol{k}}_j) &= \frac{2m}{m+1}\!\int\! dz \, \frac{e^{i\frac{x_i-y_j}{\sqrt{2}}\cdot z}}{z^2} - \frac{2m}{m+1}\!\int\! dz \, \frac{\frac{m-1}{m}q\cdot z+\gamma}{z^2\left(\frac{m+1}{2m}z^2+\frac{m-1}{m}\,q\cdot z+\gamma\right)} + o(1) \\
&= \frac{4\sqrt{2}\pi^2 m}{m+1}\frac{1}{|x_i-y_j|} - b\sqrt{h_1(q,\hat{\boldsymbol{p}}_i,\hat{\boldsymbol{k}}_j)+\lambda} + o(1)
\end{aligned} \tag{6.34}$$

where we have used the explicit integration (2.22). Using (6.34) in (6.32) we obtain (6.29). Finally for the proof of (6.30) we observe that

$$\|G^\lambda \xi_{ij}\|^2 = \int\! d\boldsymbol{p}_N d\boldsymbol{k}_M \, \frac{\left|\hat{\xi}\left(\frac{p_i+k_j}{\sqrt{2}},\hat{\boldsymbol{p}}_i,\hat{\boldsymbol{k}}_j\right)\right|^2}{\Big(h_0(\boldsymbol{p}_N,\boldsymbol{k}_M)+\lambda\Big)^2} \tag{6.35}$$

Introducing the coordinates $q = \frac{p_i + k_j}{\sqrt{2}}$, $v = \frac{p_i - k_j}{\sqrt{2}}$ and using the elementary inequality $-\frac{1}{2}(v^2 + q^2) \leq v \cdot q \leq \frac{1}{2}(v^2 + q^2)$ we have

$$c_1 \int dq d\hat{\boldsymbol{p}}_i d\hat{\boldsymbol{k}}_j \, |\hat{\xi}(q, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j)|^2 \mathcal{M} \leq \|G^\lambda \xi_{ij}\|^2 \leq c_2 \int dq d\hat{\boldsymbol{p}}_i d\hat{\boldsymbol{k}}_j \, |\hat{\xi}(q, \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{k}}_j)|^2 \mathcal{M} \tag{6.36}$$

where

$$\mathcal{M} = \frac{1}{\pi^2} \int dv \, \frac{1}{\left(v^2 + q^2 + \hat{\boldsymbol{p}}_i^2 + \frac{\hat{k}_j^2}{m} + \lambda\right)^2} \tag{6.37}$$

By an explicit computation of the above integral we obtain (6.30).

□

Remark 6.4. *We notice that, due to the singularity for $|x_i - y_j| \to 0$, the potential $G^\lambda \xi_{ij}$ does not belong to $H^1(\mathbb{R}^{3(N+M)})$ and therefore the decomposition $u = w + G^\lambda \xi$ for a generic element of the form domain (see (2.39)) is meaningful.*

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