RESULTS ON COUPLED RICCI AND HARMONIC MAP FLOWS

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Abstract. We explore Ricci flow coupled with harmonic map flow, both as it arises naturally in certain bundle constructions related to Ricci flow and as a geometric flow in its own right. In the first case, we generalize a theorem of Knopf that demonstrates convergence and stability of certain locally $\mathbb{R}^N$-invariant Ricci flow solutions. In the second case, we prove a version of Hamilton's compactness theorem for the coupled flow, and then generalize it to the category of étale Riemannian groupoids. We also provide a detailed example of solutions to the flow on the Lie group Nil^3.

Contents

1. Introduction

The harmonic map flow for maps between Riemannian manifolds, introduced by Eells and Sampson [5], was one of the first geometric flows and pioneered the use of parabolic PDE in solving geometric problems. Given $(\mathcal{M}, g)$ and $(\mathcal{N}, h)$ and a map ϕ_0 between them, it is

$$\frac{\partial}{\partial t}\phi = \tau_{g,h}\phi, \qquad \phi(0) = \phi_0, \tag{1.1}$$

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where $\tau_{g,h}\phi$ is the tension field of ϕ. The Ricci flow, though weakly parabolic, is similar in spirit: both flows attempt to improve certain objects via a heat-type flow. These flows are also related beyond formal similarities. For example, the DeTurck trick, which modifies Ricci flow by diffeomorphisms to make it strictly parabolic, was used to prove short-time existence and uniqueness of Ricci flow on closed manifolds in [4]. Hamilton subsequently observed that these diffeomorphisms actually solve a harmonic map flow [8].

In [14], Lott studied Ricci flow on manifolds that are twisted principal bundles and where the solutions are locally invariant under the action of the structure group. This construction is used to obtain information on collapse under Ricci flow of three-dimensional manifolds. A special case of this construction appears in [13], where the manifolds have the structure of a flat $\mathbb{R}^N$-vector bundle and their study is motivated by structure that appears in examples of expanding Ricci solitons. It turns out that solitons in this context naturally involve harmonic maps, and the Ricci flow can be interpreted as a coupling of Ricci flow and harmonic map flow (see Proposition 2.5 below). This observation is a new phenomenon, distinct from the DeTurck trick.

Of course, one can also study the coupling of these two flows as a subject of independent interest. The case where the target manifold for the harmonic map flow is a subset of the real line was considered by List in [12]. This coupling has relevance to general relativity and solutions of the Einstein equations. The case of arbitrary target manifolds was subsequently studied by Müller in [17] under the name $(RH)_\alpha$ flow (although we will replace α with c). This flow has a number of nice properties and can be better behaved than the Ricci or harmonic map flow taken separately.

1.1. **Summary of results.** In [10], Knopf proved convergence and stability of certain Ricci flow solutions in Lott's twisted principal bundle context, where the dimension of the total space is three. In Section 2, we review the necessary constructions and extend the result to arbitrary dimension in some cases. See Equation (2.2) for the locally $\mathbb{R}^N$-invariant Ricci flow system.

Theorem 1.2. *Let $\mathbf{g} = (g, A, G)$ be a locally $\mathbb{R}^N$-invariant metric on a product $\mathbb{R}^N \times \mathcal{B}$, where $\mathcal{B}$ is compact and orientable. Suppose that A vanishes and G is constant, and that either*

(i) *g has constant sectional curvature $-1/2(n-1)$, or*
(ii) *$\mathcal{B} = \mathcal{S}^2$ and g has constant positive sectional curvature.*

Then for any $\rho \in (0,1)$, there exists $\theta \in (\rho, 1)$ such that the following holds.

There exists a $(1+\theta)$ little-Hölder neighborhood $\mathcal{U}$ of $\mathbf{g}$ such that for all initial data $\widetilde{\mathbf{g}}(0) \in \mathcal{U}$, the unique solution $\widetilde{\mathbf{g}}(t)$ of rescaled locally $\mathbb{R}^N$-invariant Ricci flow exists for all $t \geq 0$ and converges exponentially fast in the $(2+\rho)$-Hölder norm to a limit metric $\mathbf{g}_\infty = (g_\infty, A_\infty, G_\infty)$ such that A_∞ vanishes, G_∞ is constant, and

in case (i), g_∞ is hyperbolic, and
in case (ii), g_∞ has constant positive sectional curvature.

In Section 3, we consider the general coupling of Ricci and harmonic map flows, and prove a version of Hamilton's compactness theorem in this context. We also prove a version of the theorem for étale Riemannian groupoids, a setting which is well-suited for discussion of convergence of solutions under the flow, especially when collapse is involved. See Equation (3.2) for the $(RH)_c$ flow.

Theorem 1.3. *Let* $\{(\mathcal{M}_k^n, g_k(t), \phi_k(t), O_k)\}$ *be a sequence of complete, pointed* $(RH)_c$ *flow solutions, with* $0, t \in (\alpha, \omega)$*,* $c(t)$ *non-increasing, and* $\phi_k(t)$ *mapping* $\mathcal{M}_k$ *into a closed Riemannian manifold* $(\mathcal{N}, h)$*, such that*

(a) *the geometry is uniformly bounded: for all* k*,*

$$\sup_{(x,t)\in\mathcal{M}_k\times(\alpha,\omega)} |\operatorname{Rm}_k|_k \leq C_1$$

for some C_1 *independent of* k*;*

(b) *the initial injectivity radii are uniformly bounded below: for all* k*,*

$$\operatorname{inj}_{g_k(0)}(O_k) \geq \iota_0 > 0,$$

for some ι_0 *independent of* k*.*

Then there is a subsequence such that

$$\big(\mathcal{M}_k, g_k(t), \phi_k(t), O_k\big) \longrightarrow \big(\mathcal{M}_\infty, g_\infty(t), \phi_\infty(t), O_\infty\big),$$

where the limit is also a pointed, complete, $(RH)_c$ *flow solution.*

If we do not assume any injectivity radius bound, then we have convergence to

$$\big(\mathcal{G}_\infty, g_\infty(t), \phi_\infty(t), O_\infty\big),$$

a complete, pointed, n*-dimensional, étale Riemannian groupoid with map* ϕ_∞ *on the base.*

We conclude with a detailed example of $(RH)_c$ solutions on the Lie group Nil^3, where the metrics are left-invariant and the map is a harmonic real-valued function. The behavior of these solutions depends strongly on the coupling function, although it is similar to that of Ricci flow solutions if the function decays fast enough as $t \to \infty$.

2. Locally $\mathbb{R}^N$-invariant Ricci flow

2.1. **Setup.** The manifolds that we will consider in this section have a special bundle structure. Let $\mathcal{B}$ be a connected, oriented, compact manifold, and let $\mathcal{E} \xrightarrow{p} \mathcal{B}$ be a flat $\mathbb{R}^N$-vector bundle. We consider $\mathcal{M}$ to be a principal $\mathbb{R}^N$-bundle over $\mathcal{B}$, twisted by $\mathcal{E}$. That is, there exists a smooth map

$$\mathcal{E} \times_{\mathcal{B}} \mathcal{M} = \bigcup_{b\in\mathcal{B}} \mathcal{E}_b \times \mathcal{M}_b \longrightarrow \mathcal{M}$$

that, over each point $b \in \mathcal{B}$, gives a free and transitive action that is consistent with the flat connection on $\mathcal{E}$. This means that if $\mathcal{U} \subset \mathcal{B}$ is such that $\mathcal{E}_{\mathcal{U}} \to \mathcal{U}$ is trivializiable, then $\pi^{-1}(\mathcal{U})$ has a free $\mathbb{R}^N$ action. Let $\mathcal{M}$ have a connection A such that $A|_{\pi^{-1}(U)}$ is an $\mathbb{R}^N$-valued connection. If we assume that $\mathcal{M}$ also has a flat connection itself, then A is an $\mathbb{R}^N$-valued 1-form.

We will use this bundle structure to describe local coordinates for $\mathcal{M}$. Let $\mathcal{U} \subseteq \mathcal{B}$ be an open set such that $\mathcal{E}_{\mathcal{U}} \to \mathcal{U}$ is trivializable and has a local section $\sigma\colon \mathcal{U} \to \pi^{-1}(\mathcal{U})$. Additonally, let $\rho\colon \mathbb{R}^n \to \mathcal{U}$ be a parametrization of $\mathcal{U}$, with coordinates x^α, and let e_i be a basis for $\mathbb{R}^N$. Then we obtain coordinates (x^α, x^i) on $\pi^{-1}(\mathcal{U})$ via

$$\begin{aligned} \mathbb{R}^n \times \mathbb{R}^N &\longrightarrow \pi^{-1}(\mathcal{U}) \\ (x^\alpha, x^i) &\longmapsto (x^i e_i) \cdot \sigma\big(\rho(x^\alpha)\big) \end{aligned}$$

where $\cdot$ denotes the free $\mathbb{R}^N$-action described above.

Let $\mathbf{g}$ be a Riemannian metric on $\mathcal{M}$ such that the $\mathbb{R}^N$-action is a local isometry. With respect to the coordinates above, one may write

$$\mathbf{g} = \sum_{\alpha,\beta=1}^{n} g_{\alpha\beta}\,dx^\alpha\,dx^\beta + \sum_{i,j=1}^{N} G_{ij}\big(dx^i + \textstyle\sum_{\alpha=1}^n A^i_\alpha\,dx^\alpha\big)\big(dx^j + \textstyle\sum_{\beta=1}^n A^j_\beta\ ,dx^\beta\big)$$

(2.1)

$$= g_{\alpha\beta}\,dx^\alpha\,dx^\beta + G_{ij}(dx^i + A^i_\alpha\,dx^\alpha)(dx^j + A^j_\beta\ ,dx^\beta).$$

We will write this informally as $\mathbf{g} = (g, A, G)$, where $g(b) = g_{\alpha\beta}(b)\,dx^\alpha\,dx^\beta$ is locally a Riemannian metric on $\mathcal{U} \subset \mathcal{B}$, $A(b) = A^i_\alpha(b)\,dx^\alpha$ is locally the pullback by σ of a connection on $\pi^{-1}(\mathcal{U}) \to \mathcal{U}$, and $G(b) = G_{ij}(b)\,dx^i\,dx^j$ is an inner product on the fiber $\mathcal{M}_b$.

2.2. **The rescaled flow.** In [14], Lott considered metrics of the form (2.1) that evolve under Ricci flow, which are called *locally* $\mathbb{R}^N$*-invariant* solutions. He showed that the Ricci flow equation for $(M, \mathbf{g})$ becomes three equations: one for each of g, A, and G (see [14, Equation (4.10)]). To study the asymptotic stabilty of this system, Knopf transformed it into an equivalent one that has legitimate fixed points (see [10, Equation (1.3)]). Let $s(t)$ be a function and c a constant. Then the transformed system is [1]

(2.2a)

$$\frac{\partial}{\partial t} g_{\alpha\beta} = -2R_{\alpha\beta} + \frac{1}{2} G^{ik}G^{j\ell}\nabla_\alpha G_{ij}\nabla_\beta G_{k\ell} + g^{\gamma\delta}G_{ij}(dA)^i_{\alpha\gamma}(dA)^j_{\beta\delta} - s g_{\alpha\beta},$$

(2.2b)

$$\frac{\partial}{\partial t} A^i_\alpha = -(\delta dA)^i_\alpha + g^{\beta\gamma}G^{ij}\nabla^\gamma G_{jk}(dA)^k_{\beta\alpha} - \frac{1+c}{2} s A^i_\alpha,$$

(2.2c)

$$\frac{\partial}{\partial t} G_{ij} = \Delta G_{ij} - g^{\alpha\beta}G^{k\ell}\nabla_\alpha G_{ik}\nabla^\beta G_{\ell j} - \frac{1}{2} g^{\alpha\gamma}g^{\beta\delta}G_{ik}G_{j\ell}(dA)^k_{\alpha\beta}(dA)^\ell_{\gamma\delta} + csG_{ij}.$$

We call this system a *rescaled locally* $\mathbb{R}^N$*-invariant Ricci flow.*

The case where the bundle connection is flat (i.e., A vanishes) was studied in [13], in the context of structures that arise from certain expanding Ricci solitons on low-dimensional manifolds. There and in the more general setting, certain Ricci flow solutions give rise to a (twisted) harmonic map $G\colon \mathcal{B} \to \mathrm{SL}(N,\mathbb{R})/\,\mathrm{SO}(N)$ (the target being the space of symmetric positive-definite bilinear forms of fixed determinant) together with a "soliton-like" equation relating the metrics g and G. These are the *harmonic-Einstein equations.*

We will need a byproduct of this fact. Write $\mathcal{S}_N = \mathrm{SL}(N,\mathbb{R})/\,\mathrm{SO}(N)$. The tangent space $T_G\mathcal{S}_N$ at $G \in \mathcal{S}_N$ consists of symmetric bilinear forms with no trace. There is a Riemannian metric on $T_G\mathcal{S}_N$ defined by

$$\overline{g}_G(X, Y) = \mathrm{tr}(G^{-1}XG^{-1}Y) = G^{ij}X_{jk}G^{k\ell}Y_{\ell i}. \tag{2.3}$$

[1]Here $\nabla_\alpha = \frac{\partial}{\partial x^\alpha}$, $(dA)^i_{\alpha\beta} = \nabla_\alpha A^i_\beta - \nabla_\beta A^i_\alpha$, $(\delta dA)^i_\alpha = -g^{\beta\gamma}\nabla^\gamma(dA)^i_{\beta\alpha}$, and $\Delta G_{ij} = g^{\alpha\beta}\nabla_\alpha\nabla_\beta G_{ij} = g^{\alpha\beta}(\frac{\partial^2}{\partial x^\alpha \partial x^\beta}G_{ij} - \Gamma^\gamma_{\alpha\beta}\frac{\partial}{\partial x^\gamma}G_{ij})$, where Γ represents the Christoffel symbols of g.

The tension field of $G\colon \mathcal{B} \to \mathcal{S}_N$, with respect to the metrics g and $\overline{g}$, has components

$$\tau_{g,\overline{g}}(G)_{ij} = \Delta G_{ij} + g^{\alpha\beta} \sum_{\substack{p<q\\ r<s}}^{N} ({}^{\mathcal{S}_N}\Gamma \circ G)^{ij}_{pq,rs} \nabla_\alpha G_{pq} \nabla_\beta G_{rs}. \tag{2.4}$$

The reader is invited to compare this definition to the general formulation in (3.1) below.

Proposition 2.5. *The evolution equation for G from* (2.2) *is a modified harmonic map flow for $G\colon \mathcal{B} \to \mathcal{S}_N$. More precisely,*

$$\frac{\partial}{\partial t} G_{ij} = \tau_{g(t),\overline{g}}(G)_{ij} - \frac{1}{2} g^{\alpha\gamma} g^{\beta\delta} G_{ik} G_{j\ell} (dA)^k_{\alpha\beta} (dA)^\ell_{\gamma\delta} + csG_{ij}.$$

Proof. What we are really claiming is that

$$\Delta G_{ij} - g^{\alpha\beta} G^{k\ell} \nabla_\alpha G_{ik} \nabla_\beta G_{\ell j} = \tau_{g,\overline{g}}(G)_{ij}. \tag{2.6}$$

The map G has energy

$$E(G) = \frac{1}{2} \int_B g^{\alpha\beta} \operatorname{tr}(G^{-1} \nabla_\alpha G^{-1} \nabla_\beta G) \, dV.$$

In [13, Proposition 4.17] it is shown that the variational equation of this energy is precisely

$$\Delta G_{ij} - g^{\alpha\beta} G^{k\ell} \nabla_\alpha G_{ik} \nabla_\beta G_{\ell j} = 0.$$

It follows from general harmonic map theory that, as it came from the variation of the energy of a map, the quantity on the left must be the tension field $\tau_{g(t),\overline{g}}(G)$ from (2.4). □

Remark 2.7. It is possible to verify equation (2.6) directly with a (lengthy) computation.

Now, we want to describe the fixed points of (2.2), with the proper choices of s and c. There are two cases that we will consider.

Lemma 2.8. *Let* $(\mathbb{R}^N \times \mathcal{B}, \mathbf{g}(t))$ *be a Riemannian product that solves* (2.2)*, such that* (B, g) *is nonflat and Einstein. Choose coordinates so that G is constant,* $A = 0$*, and* $g(t) = -Kt\, g(-K)$*, where* $K = \pm 1$ *is the Einstein constant such that* $2\operatorname{Rc}[g(-K)] = Kg(-K)$*. (Note that $g(t)$ exists for $t < 0$ if $K = 1$ and for $t > 0$ if $K = -1$.) The choices $s = -K$ and $c = 0$ make* $\mathbf{g}(0)$ *a stationary solution for (2.2).*

With these choices, we call (2.2) the *K-rescaled locally $\mathbb{R}^N$-invariant Ricci flow* system.

Next, given any smooth function $f\colon \mathcal{B} \to \mathbb{R}$, we define

$$\fint_{\mathcal{B}} f \, \mathrm{d}\mu = \frac{\int_{\mathcal{B}} f \, \mathrm{d}\mu}{\int_{\mathcal{B}} \mathrm{d}\mu}.$$

Let $V(t)$ denote the volume of $(\mathcal{B}, g(t))$ and define

$$r = R - \frac{1}{4} |\nabla G|^2 - \frac{1}{2} |dA|^2,$$

where everything is computed with respect to g. Because

$$\frac{dV}{dt} = - \int_{\mathcal{B}} r \, \mathrm{d}\mu - \frac{n}{2} sV(t),$$

it follows that V is fixed if and only if

$$s = -\frac{2}{n} \oint_{\mathcal{B}} r \, \mathrm{d}\mu. \tag{2.9}$$

Lemma 2.10. *Let* $(\mathbb{R}^N \times \mathcal{B}, \mathbf{g}(t))$ *be a Riemannian product that solves* (2.2)*, such that* (B, g) *is Einstein. Choose coordinates such that* G *is constant and* $A = 0$*. For any* t_0 *in its time domain of existence, taking* $c = 0$ *and* s *as in* (2.9) *makes* $\mathbf{g}(0) = (\sigma^{-1}(t_0) g(t_0), 0, G)$ *into a stationary solution of (2.2) for any choice of positive antiderivative* $\sigma(t)$ *of* s*.*

With these choices, we call (2.2) the *volume-rescaled locally* $\mathbb{R}^N$*-invariant Ricci flow* system.

In this section, we prove the following results that imply convergence in the little-Hölder spaces as defined in [10]. It is a generalization of [10, Theorems 1 & 2].

Theorem 2.11. *Let* $\mathbf{g} = (g, A, G)$ *be a locally* $\mathbb{R}^N$*-invariant metric of the form (2.1) on a product* $\mathbb{R}^N \times \mathcal{B}$*, where* $\mathcal{B}$ *is compact and orientable. Suppose that* A *vanishes and* G *is constant, and that either*

(i) *g has constant sectional curvature* $-1/2(n-1)$*, or*
(ii) $\mathcal{B} = \mathcal{S}^2$ *and* g *has constant positive sectional curvature.*

Then for any $\rho \in (0, 1)$*, there exists* $\theta \in (\rho, 1)$ *such that the following holds.*

There exists a $(1 + \theta)$ *little-Hölder neighborhood* $\mathcal{U}$ *of* $\mathbf{g}$ *such that for all initial data* $\widetilde{\mathbf{g}}(0) \in \mathcal{U}$*, the unique solution* $\widetilde{\mathbf{g}}(\tau)$ *of (2.2) exists for all* $t \geq 0$ *and converges exponentially fast in the* $(2+\rho)$*-Hölder norm to a limit metric* $\mathbf{g}_\infty = (g_\infty, A_\infty, G_\infty)$ *such that* A_∞ *vanishes,* G_∞ *is constant, and*

> *in case (i), with choices of c and s as in Lemma 2.8, g_∞ is hyperbolic, and*
> *in case (ii), with choices of c and s as in Lemma 2.10, g_∞ has constant positive sectional curvature.*

2.3. **Linearization at a stationary solution of rescaled flow.** Consider a fixed point of the flow (2.2) on a Riemannian product $(\mathbb{R}^N \times \mathcal{B}, \mathbf{g})$. From Lemmas (2.8) and (2.10), we can assume that g is Einstein with $2\,\mathrm{Rc}(g) = Kg$, A is identically zero, and G is constant. Also, $c = 0$ and s is a (known) function.

To analyze the stability near a fixed point, we must compute the linearization of the flow. Write $\mathbf{g}_0 = (g_0, 0, G_0)$ for such a fixed point. Let

$$\tilde{\mathbf{g}}(\epsilon) = \big(\tilde{g}(\epsilon), \tilde{A}(\epsilon), \tilde{G}(\epsilon)\big) \tag{2.12}$$

be a variation of $\mathbf{g}$ such that

$$\tilde{\mathbf{g}}(0) = \mathbf{g}_0, \quad \frac{\partial}{\partial \epsilon}\Big|_{\epsilon=0} \tilde{\mathbf{g}} = \mathbf{h} = (h, B, F). \tag{2.13}$$

Let Δ_ℓ denote the Lichnerowicz Laplacian acting on symmetric $(2,0)$-tensor fields. In coordinates,

$$\Delta_\ell h_{ij} = \Delta h_{ij} + 2R_{ipqj} h^{pq} - R_i^k h_{kj} - R_j^k h_{ik}. \tag{2.14}$$

Also, let $H = \mathrm{tr}_g\, h$ and $H_0 = \mathrm{tr}_{g_0} h$.

Lemma 2.15. *The linearization of (2.2) at a fixed point* $\mathbf{g}_0 = (g_0, 0, G_0)$ *with* $2\,\mathrm{Rc} = K g_0$ *and* G_0 *constant acts on* $\mathbf{h} = (h, B, F)$ *by*

$$\frac{\partial}{\partial t} h_{\alpha\beta} = \Delta_\ell h_{\alpha\beta} + \nabla_\alpha (\delta h)_\beta + \nabla_\beta (\delta h)_\alpha + \nabla_\alpha \nabla_\beta H_0 + X, \tag{2.16a}$$

$$\frac{\partial}{\partial t} B^i_\alpha = -(\delta d B)^i_\alpha + k B^i_\alpha, \tag{2.16b}$$

$$\frac{\partial}{\partial t} F_{ij} = \Delta F_{ij}, \tag{2.16c}$$

where

$$X = \begin{cases} K h_{\alpha\beta} & \text{in case (i)} \\ 2K(h_{\alpha\beta} - \frac{1}{n}(g_0)_{\alpha\beta} \oint_B H_0 \, \mathrm{d}\mu) & \text{in case (ii)} \end{cases}$$

and

$$k = \begin{cases} \frac{K}{2} & \text{in case (i)} \\ K & \text{in case (ii)} \end{cases} .$$

Proof. With a variation of $\mathbf{g}$ as in (2.12) and (2.13), we must compute

$$\frac{\partial}{\partial \epsilon}\Big|_{\epsilon=0} \left(\frac{\partial}{\partial t} \tilde{\mathbf{g}}(\epsilon) \right).$$

Here and in the next lemma, we will use standard variational formulas for geometric objects like g^{-1}, Γ, Rc, R, $\mathrm{d}\mu$, and $\oint R \, \mathrm{d}\mu$. See [3, Section 3.1], for example.

Considering the first equation, we have

$$\frac{\partial}{\partial t} h_{\alpha\beta} = \Delta_\ell h_{\alpha\beta} + \nabla_\alpha (\delta h)_\beta + \nabla_\beta (\delta h)_\alpha + \nabla_\alpha \nabla_\beta H_0 - \frac{\partial}{\partial \epsilon}\Big|_{\epsilon=0} (s \tilde{g}_{\alpha\beta}).$$

In case (i),

$$\frac{\partial}{\partial \epsilon}\Big|_{\epsilon=0} (s \tilde{g}_{\alpha\beta}) = K h_{\alpha\beta},$$

and in case (ii),

$$\begin{aligned} \frac{\partial}{\partial \epsilon}\Big|_{\epsilon=0} (s \tilde{g}_{\alpha\beta}) &= \frac{\partial}{\partial \epsilon}\Big|_{\epsilon=0} \left(\frac{2}{n} \tilde{g}_{\alpha\beta} \oint_B \tilde{r} \, \mathrm{d}\tilde{\mu} \right) \\ &= 2K \left(h_{\alpha\beta} - \frac{1}{n} (g_0)_{\alpha\beta} \oint_B H_0 \, \mathrm{d}\mu_0 \right). \end{aligned}$$

For the second equation, we have

$$\frac{\partial}{\partial t} B^i_\alpha = -(\delta d B)^i_\alpha + \frac{\partial}{\partial \epsilon}\Big|_{\epsilon=0} \left(\frac{1}{2} s \tilde{A}^i_\alpha \right).$$

In case (i),

$$\frac{\partial}{\partial \epsilon}\Big|_{\epsilon=0} \left(\frac{1}{2} s \tilde{A}^i_\alpha \right) = \frac{K}{2} B^i_\alpha,$$

and in case (ii),

$$\begin{aligned} \frac{\partial}{\partial \epsilon}\Big|_{\epsilon=0} \left(\frac{1}{2} s \tilde{A}^i_\alpha \right) &= \frac{\partial}{\partial \epsilon}\Big|_{\epsilon=0} \left(\frac{1}{n} \tilde{A}^i_\alpha \oint_B \tilde{r} \, \mathrm{d}\tilde{\mu} \right) \\ &= K B^i_\alpha. \end{aligned}$$

Here, we use that $r = nK$ when g_0 is Einstein.

For the third equation, we use Proposition 2.5 to write

$$\frac{\partial}{\partial\tau}G_{ij} = \tau_{g(t),\overline{g}}(G)_{ij} - \frac{1}{2}g^{\alpha\gamma}g^{\beta\delta}G_{ik}G_{j\ell}(dA)^k_{\alpha\beta}(dA)^\ell_{\gamma\delta},$$

where the first term is the tension field from (2.4). Then it is easy to see that

$$\frac{\partial}{\partial\epsilon}\Big|_{\epsilon=0}F_{ij} = \Delta F_{ij},$$

as desired. □

As in [10], we use the DeTurck trick to make the linear (2.2) system strictly parabolic. That is, to each equation in (2.2) we add a term consisting of the Lie derivative of the metric with respect to a carefully chosen family of vector fields $W(t)$. To this end, fix a background connection $\underline{\Gamma}$ and define

$$W^\gamma = g^{\alpha\beta}(\Gamma^\gamma_{\alpha\beta} - \underline{\Gamma}^\gamma_{\alpha\beta}), \qquad \gamma = 1,\dots,n \tag{2.17a}$$

$$(W_\flat)_k = (\delta A)_k, \qquad k = 1,\dots,N. \tag{2.17b}$$

Let ψ_t be diffeomorphisms generated by $W(t)$, with initial condition $\psi_0 = \mathrm{id}$. The one-parameter family of metrics $\psi_t^*\mathbf{g}(t)$ is the solution of the *rescaled* $\mathbb{R}^N$-*invariant Ricci–DeTurck flow.* We now take $\underline{\Gamma}$ to be the Levi-Civita connection of the stationary solution around which we linearize. Observe that a stationary solution $\mathbf{g}_0 = (g_0, 0, G_0)$ of (2.2) with $2\,\mathrm{Rc} = Kg_0$ and G_0 constant is then also a stationary solution of the rescaled Ricci–DeTurck flow.

Lemma 2.18. *The linearization of (2.2) at a fixed point* $\mathbf{g}_0 = (g_0, 0, G_0)$ *with* $2\,\mathrm{Rc} = Kg_0$ *is the autonomous, self-adjoint, strictly parabolic system*

$$\frac{\partial}{\partial t}\begin{pmatrix} h \\ B \\ F\end{pmatrix} = \mathbf{L}\begin{pmatrix} h \\ B \\ F\end{pmatrix} = \begin{pmatrix} \mathbf{L}_2 h \\ \mathbf{L}_1 B \\ \mathbf{L}_0 H\end{pmatrix},$$

where

$$\mathbf{L}_0 = \Delta, \tag{2.19}$$

$$\mathbf{L}_1 = \begin{cases} \Delta_1 + \frac{K}{2}\,\mathrm{id} & \textit{in case (i)} \\ \Delta_1 + K\,\mathrm{id} & \textit{in case (ii)} \end{cases}. \tag{2.20}$$

$$\mathbf{L}_2 = \begin{cases} \Delta_\ell + K\,\mathrm{id} & \textit{in case (i)} \\ \Delta_\ell + \Phi & \textit{in case (ii)} \end{cases} \tag{2.21}$$

Here $-\Delta_1 = d\delta + \delta d$ *denotes the Hodge–de Rham Laplacian acting on* 1*-forms, and*

$$\Phi(h) = 2K\left(h - \frac{1}{n}g_0 \fint_B H_0\,\mathrm{d}\mu\right).$$

Proof. The normalized Ricci-DeTurck flow is obtained by subtracting a Lie derivative from the right side of (2.2):

$$\frac{\partial}{\partial t}\mathbf{g} = -2\,\mathrm{Rc}[\mathbf{g}] - \mathcal{L}_W\mathbf{g},$$

so we must compute the linearization of this Lie derivative, as in Lemma 2.15.

First, since W^γ is defined the same way as it was in [10, Equations (3.6), (4.4)], the components of the form $(\mathcal{L}_W\mathbf{g})_{\alpha\beta}$ are unchanged from that paper. This means that when computing its ϵ-derivative, we get

$$\frac{\partial}{\partial\epsilon}\Big|_{\epsilon=0}(\mathcal{L}_W\mathbf{g})_{\alpha\beta} = \Delta_\ell h_{\alpha\beta} + \nabla_\alpha(\delta h)_\beta + \nabla_\beta(\delta h)_\alpha + \nabla_\alpha\nabla_\beta H_0.$$

Subtracting this from (2.16a) gives (2.21).

Next, we have

$$(\mathcal{L}_W\mathbf{g})_{\alpha i} = (d\delta A)^i_\alpha,$$

and so

$$\frac{\partial}{\partial\epsilon}\Big|_{\epsilon=0}(\mathcal{L}_W\tilde{\mathbf{g}})_{\alpha i} = (d\delta B)^i_\alpha.$$

Subtracting this from (2.16b) gives (2.20).

Finally, we have

$$(\mathcal{L}_W\mathbf{g})_{ij} = 0,$$

and so

$$\frac{\partial}{\partial\epsilon}\Big|_{\epsilon=0}(\mathcal{L}_W\tilde{\mathbf{g}})_{ij} = 0.$$

Subtracting this from (2.16c) gives (2.19). □

Now assume that $\mathbf{g}_0 = (g_0, 0, G_0)$ is a fixed point of the rescaled $\mathbb{R}^N$-invariant Ricci–DeTurck flow with G_0 constant and g_0 a metric of constant sectional curvature. In case (i), $\mathrm{sect}[g_0] = -1/2(n-1) < 0$, and in case (ii), $\mathrm{sect}[g_0] = (n-1)k > 0$. In case (ii), by passing to a covering space if necessary, we may assume that $\mathcal{B}^n$ is the round n-sphere of radius $\sqrt{1/k}$.

Recall that a linear operator L is *weakly (strictly) stable* if its spectrum is confined to the half plane $\mathrm{Re}\, z \leq 0$ (and is uniformly bounded away from the imaginary axis).

Because $\mathbf{L}$ is diagonal, we can determine its stability by examining its component operators. The conclusions we obtain here will hold below when we extend $\mathbf{L}$ to a complex-valued operator on a larger domain in which smooth representatives are dense.

Lemma 2.22. *Let $\mathbf{g}_0 = (g_0, 0, G_0)$ be a metric of the form (2.1) such that G_0 is constant and g_0 has constant sectional curvature. Then the linear system* (2.19)-(2.21) *has the following stability properties:*

The operator $\mathbf{L}_0$ is weakly stable.

The operator $\mathbf{L}_1$ is strictly stable.

If $n = 2$, then the operator $\mathbf{L}_2$ is weakly stable.

Let $n \geq 3$. In case (i), the operator $\mathbf{L}_2$ is strictly stable. In case (ii), the operator $\mathbf{L}_2$ is unstable.

Proof. It is well-known that $\mathbf{L}_0 = \Delta$, the Laplacian acting on $(2,0)$-forms, is weakly stable. The statements about $\mathbf{L}_2$ and $\mathbf{L}_3$ carry over directly from Lemmas 5 and 7 in [10]. □

We now turn to the proof of the the main theorem. See [10, Section 2] for summary of the machinery that is used in the proof.

Proof of Theorem 2.11. Following [10], the proof consists of four step. First, one must show that the complexified operator is sectorial. This depends only on Lemma 2.22, which has the same conclusion as [10, Lemmas 5 & 7]. Therefore, there is no modification to this step.

Second, one applies Simonett's theorem from [19]. This is valid by Step 1, and [10, Lemma 2]. Since that Lemma was stated in the full generality of our context, there is no modification to this step.

Third, one proves the uniqueness of a smooth center manifold consisting of fixed points of the flow (2.2). Since fixed points of this flow still coincide with those of the rescaled Ricci-DeTurck flow, there is no modification.

Fourth, one proves convergence of the metric. In both cases (i) and (ii), the arguments involved do not depend on the dimension N, so there is no modification. □

3. A compactness theorem

In this section, we consider the Ricci flow coupled with the harmonic map flow, or the $(RH)_c$ flow. We prove a version of Hamilton's Compactness Theorem for a class of such flows. This is first done in the category of smooth manifolds, where we assume uniform bounds on the curvatures and injectivity radii. We also prove a version in the category of étale Riemannian groupoids, where no information about the injectivity radii is needed. The compactness theorems presented here (like Hamilton's) provide subsequential convergence in general; in cases where stability theorems like those above apply, this can be improved to genuine asymptotic convergence. Let us recall the setup for the coupled flow in question.

3.1. **Definitions.** Let $(\mathcal{M}, g)$ be a closed Riemannian manifold, with $(\mathcal{N}, h)$ a closed target manifold. Let $\phi\colon \mathcal{M} \to \mathcal{N}$ be a smooth map. The Levi-Civita covariant derivative $\nabla^{T\mathcal{M}}$ of the metric g on $\mathcal{M}$ induces a covariant derivative $\nabla^{T^*\mathcal{M}}$ on the cotangent bundle, which satisfies

$$\nabla_X^{T^*\mathcal{M}}\omega(Y) = X\big(\omega(Y)\big) - \omega\left(\nabla_X^{T\mathcal{M}}Y\right).$$

By requiring a product rule and compatibility with the metric, we also have convariant derivatives on all tensor bundles

$$T_q^p(\mathcal{M}) = (T\mathcal{M})^{\otimes p} \otimes (T^*\mathcal{M})^{\otimes q}.$$

The Levi-Civita covariant derivative $\nabla^{T\mathcal{N}}$ of the metric h on N induces a covariant derivative $\nabla^{\phi^*T\mathcal{N}}$ on the pull-back bundle $\phi^*T\mathcal{N} \to \mathcal{M}$, given by

$$\nabla_X^{\phi^*T\mathcal{N}}\phi^*Y = \phi^*\left(\nabla_{\phi_*X}^{T\mathcal{N}}Y\right),$$

for $X \in \mathcal{T}(\mathcal{M})$ and $Y \in \mathcal{T}(\mathcal{N})$. As before, we get a covariant derivative on all tensor bundles over $\mathcal{M}$ of the form

$$T_q^p(\mathcal{M}) \otimes T_s^r(\phi^*\mathcal{N}) = (T\mathcal{M})^{\otimes p} \otimes (T^*\mathcal{M})^{\otimes q} \otimes (\phi^*T\mathcal{N})^{\otimes r} \otimes (\phi^*T^*\mathcal{N})^{\otimes s}.$$

We refer to them simply as ∇. In local coordinates,

$$\nabla\phi = \phi_* = \partial_i\phi^\lambda\, dx^i \otimes \partial_\lambda|_\phi \in \Gamma(T^*\mathcal{M} \otimes \phi^*T\mathcal{N}).$$

Similarly, if we write ${}^{\mathcal{N}}\nabla$ for $\nabla^{T\mathcal{N}}$, we have

$$\begin{aligned}\nabla^2\phi &= \left(\partial_i\partial_j\phi^\lambda - \Gamma_{ij}^k\partial_k\phi^\lambda + ({}^{\mathcal{N}}\Gamma \circ \phi)_{\mu\nu}^\lambda\partial_i\phi^\mu\phi_j^\nu\right) dx^i \otimes dx^j \otimes \partial_\lambda|_\phi \\ &\in \Gamma(T^*\mathcal{M} \otimes T^*\mathcal{M} \otimes \phi^*T\mathcal{N}).\end{aligned}$$

Additionally

$$\nabla\phi \otimes \nabla\phi = h_{\lambda\mu}\partial_i\phi^\lambda\partial_j\phi^\mu\, dx^i \otimes dx^j$$

and is a symmetric $(2,0)$-tensor on $\mathcal{M}$, and we define

$$\mathrm{S} = \mathrm{Rc} - c\nabla\phi \otimes \nabla\phi$$

where $c = c(t) \geq 0$ is a coupling function. Finally, the *tension field* of ϕ with respect to g and h was defined in (2.4), but the general description is

$$\tau_{g,h}\phi = \mathrm{tr}_g \nabla^2 \phi. \tag{3.1}$$

Now, the flow for initial data $(\mathcal{M}, g_0, \phi_0)$ is the system

$$\begin{aligned} \frac{\partial}{\partial t} g &= -2\mathrm{S} = -2\,\mathrm{Rc} + 2c\nabla\phi \otimes \nabla\phi \\ \frac{\partial}{\partial t}\phi &= \tau_{g,h}\phi \\ \big(g(0), \phi(0)\big) &= (g_0, \phi_0) \end{aligned} \tag{3.2}$$

For short, call this the $(RH)_c$ flow. We will assume that $c(t)$ is non-increasing.

As written here, this flow was introduced in [17] and is a generalization of one studied in [12]. Indeed, the latter considers the case when ϕ is a real-valued function.

Definition 3.3. A family $\{(\mathcal{M}^n, g(t), \phi(t), O)\}$ of complete, pointed Riemannian manifolds with maps

$$\phi(t) \colon \mathcal{M} \longrightarrow \mathcal{N}$$

that solves the system (3.2) with coupling function $c(t)$, for $t \in (\alpha, \omega)$, is a *complete, pointed* $(RH)_c$ *flow solution*.

Example 3.4. Consider the special case of the twisted bundle construction, seen in [13]. Let $\mathcal{M}$ be an $\mathbb{R}^N$-vector bundle with flat connection, flat metric G on the fibers, and Riemannian base $(\mathcal{B}, g)$. In the notion of Section 2, write the metric on $\mathcal{M}$ as $\mathrm{g} = (g, 0, G)$. Then the fiber metrics constitute a map

$$G \colon \mathcal{B} \longrightarrow \mathrm{SL}(N,\mathbb{R})/\mathrm{SO}(N).$$

From [13, Equation (4.10)], Ricci flow on $\mathcal{M}$ becomes the pair of equations

$$\begin{aligned} \frac{\partial}{\partial t} g_{\alpha\beta} &= -2R_{\alpha\beta} + \frac{1}{2} G^{ij}\nabla_\alpha G_{jk} G^{k\ell}\nabla_\beta G_{\ell i} \\ \frac{\partial}{\partial t} G_{ij} &= g^{\alpha\beta}\nabla_\alpha\nabla_\beta G_{ij} - g^{\alpha\beta}\nabla_\alpha G_{ik} G^{kl}\nabla_\beta G_{\ell i}. \end{aligned}$$

But with the metric $\overline{g}$ on $\mathrm{SL}(N,\mathbb{R})/\mathrm{SO}(N)$ as in (2.3), we see that

$$\frac{1}{2} G^{ij}\nabla_\alpha G_{jk} G^{k\ell}\nabla_\beta G_{\ell i} = \frac{1}{4}(\nabla G \otimes \nabla G)_{\alpha\beta},$$

and Proposition 2.5 says that

$$g^{\alpha\beta}\nabla_\alpha\nabla_\beta G_{ij} - g^{\alpha\beta}\nabla_\alpha G_{ik} G^{kl}\nabla_\beta G_{\ell i} = \tau_{g,\overline{g}} G.$$

This means Ricci flow on $\mathcal{M}$ is precisely $(RH)_c$ flow on $\mathcal{B}$, with target manifold $(\mathrm{SL}(N,\mathbb{R})/\mathrm{SO}(N), \overline{g})$, maps G, and $c = 1/4$.

This gives many examples of $(RH)_c$ flow solutions. For instance, the homogeneous spaces in [13] that admit expanding Ricci solitons all have the bundle structure just described, so those Ricci flow solutions are $(RH)_c$ solutions.

Definition 3.5. A sequence $\{\mathcal{M}_k^n, g_k(t), \phi_k(t), O_k)\}$ of complete, pointed $(RH)_c$ flow solutions *converges* to $(\mathcal{M}_\infty^n, g_\infty(t), \phi_\infty(t), O_\infty)$ for $t \in (\alpha, \omega)$ if there exists

- an exhaustion $\{\mathcal{U}_k\}$ of $\mathcal{M}_\infty$ by open sets with $O_\infty \in \mathcal{U}_k$ for all k, and

- a family of diffeomorphisms $\{\Psi_k \colon \mathcal{U}_k \to \mathcal{V}_k \subset \mathcal{M}_k\}$ with $\Psi_k(O_\infty) = O_k$

such that

$$\left(\mathcal{U}_k \times (\alpha,\omega), \Psi_k^*\left(g_k(t)|_{\mathcal{V}_k} + dt^2\right), \Psi_k^*\phi_k|_{\mathcal{V}_k}\right)$$

converges uniformly in C^∞ on compact sets to

$$\left(\mathcal{M}_\infty^n \times (\alpha,\omega), g_\infty(t) + dt^2, \phi_\infty(t)\right).$$

Here, dt^2 is the standard metric on $(\alpha,\omega) \subset \mathbb{R}$.

We mention that we will use abbreviated notation for geometric objects associated with metric $g_k(t)$. For example, $\mathrm{Rc}_k(t)$ means $\mathrm{Rc}[g_k(t)]$, and ∇_k refers to the Levi-Civita covariant derivative corresponding to the metric $g_k(t)$. Also, an undecorated ∇ will refer to the Levi-Civita covariant derivative corresponding to a background metric.

3.2. **Statement of the theorem.** The original version of this theorem appears in [7] in the context Ricci flow, and was generalized to the groupoid setting in [13]. Such theorems are crucial in the study of geometric flows, especially regarding singularity models. For example, one often wishes to construct sequences of rescaled solutions to investigate the behavior at a singular time (possibly $T = +\infty$), and it is helpful to be able to extract convergent subsequences.

Theorem 3.6. *Let $\{(\mathcal{M}_k^n, g_k(t), \phi_k(t), O_k)\}$ be a sequence of complete, pointed $(RH)_c$ flow solutions, with $0, t \in (\alpha,\omega)$ and $c(t)$ non-increasing, such that*

(a) *the geometry is uniformly bounded: for all k,*

$$\sup_{(x,t)\in\mathcal{M}_k\times(\alpha,\omega)} |\operatorname{Rm}_k|_k \leq C_1$$

for some C_1 independent of k;

(b) *the initial injectivity radii are uniformly bounded below: for all k,*

$$\mathrm{inj}_{g_k(0)}(O_k) \geq \iota_0 > 0,$$

for some ι_0 independent of k.

Then there is a subsequence such that

$$\left(\mathcal{M}_k, g_k(t), \phi_k(t), O_k\right) \longrightarrow \left(\mathcal{M}_\infty, g_\infty(t), \phi_\infty(t), O_\infty\right),$$

where the limit is also a pointed, complete, $(RH)_c$ flow solution.

If we do not assume a bound on he injectivity radius bound, then we have convergence to

$$\left(\mathcal{G}_\infty, g_\infty(t), \phi_\infty(t), O_\infty\right),$$

a complete, pointed, n-dimensional, étale Riemannian groupoid with map ϕ_∞ on the base.

The idea of the proof is the same as in [7], and subsequently [12], although we follow the exposition found in [2, Chapter 3]. Briefly, the main ingredients are derivative estimates to bound the curvature and the derivatives of the map ϕ, a general compactness theorem of Hamilton, a technical lemma, and corollary of the Arzela-Ascoli theorem. Of course, many facts about the $(RH)_c$ flow, found in [17], are used along the way.

Example 3.7. Here is a way to obtain sequences of $(RH)_c$ flow solutions like those considered in the compactness theorem. To be clear about the dependence on the coupling constant, let us write a solution of the $(RH)_c$ flow as a triple $\big(g(t), \phi(t), c(t)\big)$. We can obtain a family of $(RH)_c$ flow solutions by performing a *blowdown*, a technique used extensively in [13] and [14]. For $s \in (0, \infty)$, define

$$\big(g_s(t), \phi_s(t), c_s(t)\big) = \left(\frac{1}{s} g(st), \frac{1}{s}\phi(st), s^2 c(st)\right).$$

Now we see that

$$\frac{\partial}{\partial t} g_s(t) = \left(\frac{\partial}{\partial t} g\right)(st) \quad \text{and} \quad \frac{\partial}{\partial t}\phi_s(t) = \left(\frac{\partial}{\partial t}\phi\right)(st),$$

and

$$\begin{aligned} \mathrm{S}[g_s(t), \phi_s(t), c_s(t)] &= -2\,\mathrm{Rc}[g_s(t)] + 2c_s(t)\nabla\phi_s(t) \otimes \nabla\phi_s(t) \\ &= -2\,\mathrm{Rc}[g(st)] + 2c(st)\nabla\phi(st) \otimes \nabla\phi(st) \\ &= \mathrm{S}[g(st), \phi(st), c(st)], \end{aligned}$$

$$\begin{aligned} \tau_{g_s(t),h}\phi_s(t) &= \mathrm{tr}_{g_s(t)}\, \nabla^2\phi_s(t) \\ &= \mathrm{tr}_{g(st)}\, \nabla^2\phi(st) \\ &= \tau_{g(st),h}\phi(st). \end{aligned}$$

Therefore, for each s, the blowdown gives another $(RH)_c$ solution. It is common to replace the continuous parameter s with a sequence $\{s_j\}$ converging to infinity.

3.3. **Two lemmas.** In this section we prove two lemmas that will be used in the proof of Theorem 3.6. The first is an analogue of [7, Lemma 2.4], [2, Lemma 3.11], and [12, Lemma 7.6].

Lemma 3.8. *Let $(\mathcal{M}^n, g)$ be a Riemannian manifold, with $\mathcal{K} \subset \mathcal{M}$ compact. Let $\{(g_k(t), \phi_k(t))\}$ be a sequence of solutions to the $(RH)_c$ flow, defined on $\mathcal{K} \times [\beta, \psi]$, where $t_0 \in [\beta, \psi]$. Suppose the following hold.*

The metrics $g_k(t_0)$ are uniformly equivalent to g on $\mathcal{K}$. That is, for all $x \in \mathcal{K}$, $V \in T_x\mathcal{M}$, and k, there is $C < \infty$ such that

$$C^{-1} g(V,V) \le g_k(t_0)(V,V) \le C g(V,V). \tag{3.9}$$

The covariant derivatives of $g_k(t_0)$ and ϕ_k with respect to g are uniformly bounded on K. That is, for all $p \ge 0$, there exist C_p, C'_p such that

$$\max_{x\in\mathcal{K}} |\nabla^{p+1} g_k(t_0)| \le C_p < \infty, \tag{3.10}$$

$$\max_{x\in\mathcal{K}} |\nabla^{p} \phi_k(t_0)| \le C'_p < \infty. \tag{3.11}$$

The covariant derivatives of Rm_k and ϕ_k with respect to $g_k(t)$ are uniformly bounded on $K \times [\beta, \psi]$. That is, for all $p \ge 0$, there exist C''_p, C'''_p such that

$$\max_{x\in\mathcal{K}} |\nabla^p_k \mathrm{Rm}_k|_k \le C''_p < \infty, \tag{3.12}$$

$$\max_{x\in\mathcal{K}} |\nabla^p_k \phi_k|_k \le C'''_p < \infty. \tag{3.13}$$

Then the following hold.

The metrics $g_k(t)$ are uniformly equivalent to g on $\mathcal{K} \times [\beta, \psi]$. That is, for all $x \in \mathcal{K}$, $V \in T_x\mathcal{M}$, k, there exists $B > 0$ such that

$$B^{-1} g(V,V) \leq g_k(t)(V,V) \leq Bg(V,V). \tag{3.14}$$

The time and covariant derivatives with respect to g of $g_k(t)$ and $\phi_k(t)$ are uniformly bounded on $\mathcal{K} \times [\beta, \psi]$. That is, for all p and q, there exist $\tilde{C}_{p,q}, \tilde{D}_{p,q}$ such that

$$\max_{x \in \mathcal{K}} \left| \frac{\partial^q}{\partial t^q} \nabla^p g_k(t) \right| \leq \tilde{C}_{p,q} < \infty, \tag{3.15}$$

$$\max_{x \in \mathcal{K}} \left| \frac{\partial^q}{\partial t^q} \nabla^p \phi_k(t) \right| \leq \tilde{D}_{p,q} < \infty. \tag{3.16}$$

Proof. First, note that throughout the proof we will follow standard practice in not indexing constants, and will often use the same symbol (e.g., C) for different constants within a sequence of inequalities.

To prove (a), we have

$$\frac{\partial}{\partial t} g_k(t) = -2\mathrm{S}_k(t) = -2\operatorname{Rc}_k(t) + 2c(t)\nabla_k \phi_k(t) \otimes \nabla_k \phi_k(t),$$

so that for $V \in T\mathcal{M}$,

$$\begin{aligned} \left| \frac{\partial}{\partial t} g_k(t)(V,V) \right| &= |-2\operatorname{Rc}_k(t)(V,V) + 2c(t)\nabla_k \phi_k(t) \otimes \nabla_k \phi_k(t)(V,V)| \\ &\leq 2|\operatorname{Rc}_k(t)||V|_k^2 + 2|c(t)||\nabla_k \phi_k(t)|^2 |V|_k^2 \\ &\leq C'|V|_k^2 \\ &= C' g_k(t)(V,V). \end{aligned}$$

This implies

$$|\partial_t \log g_k(t)(V,V)| = \left| \frac{\partial_t g_k(t)(V,V)}{g_k(t)(V,V)} \right| \leq C',$$

and thus for any $t_1 \in [\beta, \psi]$, we have

$$\int_{t_0}^{t_1} |\partial_t \log g_k(t)(V,V)|\, dt \leq C'|t_1 - t_0|.$$

This gives

$$\begin{aligned} C'|t_1 - t_0| &\geq \int_{t_0}^{t_1} |\partial_t \log g_k(t)(V,V)|\, dt \\ &\geq \left| \int_{t_0}^{t_1} \partial_t \log g_k(t)(V,V)\, dt \right| \\ &= \left| \log \frac{g_k(t_1)(V,V)}{g_k(t_0)(V,V)} \right|. \end{aligned}$$

Expanding this gives

$$-C'|t_1 - t_0| \leq \log \frac{g_k(t_1)(V,V)}{g_k(t_0)(V,V)} \leq C'|t_1 - t_0|,$$

and exponentiating gives

$$\exp(-C'|t_1 - t_0|) g_k(t_0)(V,V) \leq g_k(t_1) \leq \exp(C'|t_1 - t_0|) g_k(t_0)(V,V).$$

Combining this with the original hypotheses, we get

$$C^{-1}\exp(-C'|t_1-t_0|)g(V,V)\le g_k(t_1)\le C\exp(C'|t_1-t_0|)g(V,V).$$

Since t_1 was arbitrary, and since $C\exp(C'|t_1-t_0|)\le C\exp(C'|\psi-\beta|)=B$, this completes the proof of (a).

Next, we prove (3.15) and (3.16). Observe that

$$\left|\frac{\partial^q}{\partial t^q}\nabla^p g_k(t)\right| = \left|\nabla^p\frac{\partial^{q-1}}{\partial t^{q-1}}\frac{\partial}{\partial t}g_k(t)\right| = 2\left|\nabla^p\frac{\partial^{q-1}}{\partial t^{q-1}}\mathcal{S}_k(t)\right|, \tag{3.17}$$

$$\left|\frac{\partial^q}{\partial t^q}\nabla^p \phi_k(t)\right| = \left|\nabla^p\frac{\partial^{q-1}}{\partial t^{q-1}}\frac{\partial}{\partial t}\phi_k(t)\right| = \left|\nabla^p\frac{\partial^{q-1}}{\partial t^{q-1}}\tau_{g_k}\phi_k(t)\right|. \tag{3.18}$$

Recall that ∇ is the Levi-Civita covariant derivative corresponding to the background metric g. In general,

$$S_{ij}=R_{ij}-c\nabla_i\phi\nabla_j\phi,$$
$$(\tau_{g,h}\phi)^\lambda = g^{ij}\nabla_i\nabla_j\phi^\lambda,$$

and we have the following evolution equations for $\mathcal{S}$ and $\tau_{g,h}\phi$:

$$\frac{\partial}{\partial t}S_{ij}=\Delta_\ell S_{ij}+2c\,\tau_{g,h}\phi\,\nabla_i\nabla_j\phi-\dot{c}\nabla_i\phi\nabla_j\phi$$
$$\frac{\partial}{\partial t}\tau_{g,h}\phi = -g^{ik}g^{jl}S_{kl}(\nabla_i\nabla_j\phi)+g^{ij}\Big(\Delta(\nabla_i\phi\nabla_j\phi)-2\nabla_p\nabla_i\phi\nabla_p\nabla_j\phi$$
$$-R_{ip}\nabla_p\phi\nabla_j\phi-R_{jp}\nabla_p\phi\nabla_i\phi+2\left\langle{}^{\mathcal{N}}\mathrm{Rm}(\nabla_i\phi,\nabla_p\phi)\nabla_p\phi,\nabla_j\phi\right\rangle\Big).$$

To bound (3.17) and (3.18), we need to consider the evolution equations for all quantities involved, which appear in [17]:

- Christoffel symbols:

$$\frac{\partial}{\partial t}\Gamma^p_{ij}=-g^{pq}(\nabla_i R_{jq}+\nabla_j R_{iq}-\nabla_q R_{ij}-2c\nabla_i\nabla_j\phi\nabla_q\phi)$$

- Riemann:

$$\frac{\partial}{\partial t}R_{ijk\ell}=\nabla_i\nabla_k R_{j\ell}-\nabla_i\nabla_\ell R_{jk}-\nabla_j\nabla_k R_{i\ell}+\nabla_j\nabla_\ell R_{ik}-R_{ijq\ell}R_{kq}-R_{ijkq}R_{\ell q}$$
$$+2c\big(\nabla_i\nabla_k\phi\nabla_j\nabla_\ell\phi-\nabla_i\nabla_\ell\phi\nabla_j\nabla_k\phi-\left\langle{}^{\mathcal{N}}\mathrm{Rm}(\nabla_i\phi,\nabla_j\phi)\nabla_k\phi,\nabla_\ell\phi\right\rangle\big).$$

- Ricci:

$$\frac{\partial}{\partial t}R_{ij}=\Delta_\ell R_{ij}-2R_{iq}R_{jq}+2R_{ipjq}R_{pq}+2c\,\tau_{g,h}\phi\nabla_i\nabla_j\phi-2c\nabla_p\nabla_i\phi\nabla_p\nabla_j\phi$$
$$+2cR_{pijq}\nabla_p\phi\nabla_q\phi+2c\left\langle{}^{\mathcal{N}}\mathrm{Rm}(\nabla_i\phi,\nabla_p\phi)\nabla_p\phi,\nabla_j\phi\right\rangle.$$

In these equations, we used

$$\left\langle{}^{\mathcal{N}}\mathrm{Rm}(\nabla_i\phi,\nabla_j\phi)\nabla_j\phi,\nabla_i\phi\right\rangle := {}^{\mathcal{N}}R_{\kappa\mu\lambda\nu}\nabla_i\phi^\kappa\nabla_j\phi^\mu\nabla_i\phi^\lambda\nabla_j\phi^\nu,$$

and k was a coordinate index, not a sequence index.

The types of terms that will appear in the expansions of (3.17) and (3.18) therefore involve factors containing

$$\mathrm{S}_k,\mathrm{Rc}_k,\mathrm{Rm}_k,\nabla_k\phi_k,\tau_{g_k,h}\phi_k,{}^{\mathcal{N}}\mathrm{Rm}, \tag{3.19}$$

as well as time and covariant derivatives, whose norms we must show are bounded. Note that we can ingore the geometric factors coming from the manifold $\mathcal{N}$, since those quantities are bounded by compactness of $\mathcal{N}$ and by the chain rule.

Now, let us consider the case $p = 1, q = 0$ for (3.15) and (3.16). As in the proof of Lemma 3.11 in [2], we have

$$\frac{1}{2}|\nabla g_k(t)|_k \leq |\Gamma_k - \Gamma|_k \leq \frac{3}{2}|\nabla g_k(t)|_k. \tag{3.20}$$

That is, up to lowering/raising indices, the tensors $\nabla g_k(t)$ and $\Gamma_k - \Gamma$ are equivalent. Using the evolution of the Christoffel symbols, an estimation in normal coodinates gives

$$\left|\frac{\partial}{\partial t}(\Gamma_k - \Gamma)\right|_k^2 \leq 12|\nabla_k \operatorname{Rc}_k|_k^2 + 8\overline{c}|\nabla_k^2\phi_k|_k^2|\nabla_k\phi_k|_k^2 \leq C.$$

We can show that $|\Gamma_k - \Gamma|_k$ is bounded by integrating the above inequality:

$$\begin{aligned} C|t_1 - t_0| &\geq \int_{t_0}^{t_1} |\partial_t(\Gamma_k(t) - \Gamma)|_k \, dt \\ &\geq \left|\int_{t_0}^{t_1} \partial_t(\Gamma_k(t) - \Gamma)\, dt\right|_k \\ &\geq |\Gamma_k(t_1) - \Gamma|_k - |\Gamma_k(t_0) - \Gamma|_k. \end{aligned}$$

Since t_1 is arbitrary, we see that

$$\begin{aligned} |\Gamma_k(t) - \Gamma|_k &\leq C|t - t_0| + |\Gamma_k(t_0) - \Gamma|_k \\ &\leq C|t - t_0| + \frac{3}{2}|\nabla g_k(t_0)|_k \\ &\leq C|t - t_0| + \frac{3}{2}B|\nabla g_k(t_0)| \\ &\leq C. \end{aligned}$$

From this and (3.14) it follows that

$$\begin{aligned} |\nabla g_k(t)| &\leq C|\nabla g_k(t)|_k \\ &\leq C|\Gamma_k(t) - \Gamma|_k \\ &\leq C, \end{aligned}$$

and we also have

$$\begin{aligned} |\nabla\phi_k(t)| &\leq C|\nabla\phi_k(t)|_k \\ &\leq C\left(|(\nabla - \nabla_k)\phi_k(t)|_k + |\nabla_k\phi_k(t)|_k\right) \\ &\leq C\left(|\Gamma_k(t) - \Gamma|_k|\phi_k(t)| + |\nabla_k\phi_k(t)|_k\right) \\ &\leq C. \end{aligned}$$

This completes the case for $p = 1$, $q = 0$.

The general case will follow once we bound the norms of the quantities listed in (3.19) and their deriviatives. For this we need several preliminary bounds:

$$|\nabla^p \mathrm{S}_k(t)| \leq C|\nabla^p g_k(t)| + C', \tag{3.21}$$

$$|\nabla^p \phi_k(t)| \leq C'', \tag{3.22}$$

$$|\nabla^p g_k(t)| \leq C'''. \tag{3.23}$$

We prove these by induction. Consider (3.21). Since $\mathcal{S} = \operatorname{Rc} - c\nabla\phi \otimes \nabla\phi$, we have

$$\begin{aligned}|\mathrm{S}_k|_k &= |\operatorname{Rc}_k - c\nabla_k\phi_k \otimes \nabla_k\phi_k|_k \\ &\le |\operatorname{Rc}_k|_k + \overline{c}|\nabla_k\phi_k|_k^2 \\ &\le C,\end{aligned}$$

and

$$\begin{aligned}|\nabla_k \mathrm{S}_k|_k &= |\nabla_k \operatorname{Rc}_k - \nabla_k(c\nabla_k\phi_k \otimes \nabla_k\phi_k)|_k \\ &= |\nabla_k \operatorname{Rc}_k - \dot{c}\nabla_k\phi_k \otimes \nabla_k\phi_k - c\nabla_k(\nabla_k\phi_k \otimes \nabla_k\phi_k)|_k \\ &\le |\nabla_k \operatorname{Rc}_k|_k + |\dot{c}||\nabla_k\phi_k \otimes \nabla_k\phi_k|_k + 2\overline{c}|\nabla_k^2\phi_k \otimes \nabla_k\phi_k|_k \\ &\le C_1' + |\dot{c}||\nabla_k\phi_k|_k^2 + 2\overline{c}|\nabla_k^2\phi_k|_k|\nabla_k\phi_k|_k \\ &\le C.\end{aligned}$$

Now, we can use this to see that

$$\begin{aligned}|\nabla \mathrm{S}_k| &\le C|\nabla\mathcal{S}_k|_k \\ &\le C|(\nabla - \nabla_k)\mathcal{S}_k|_k + B^{3/2}|\nabla_k\mathcal{S}_k|_k \\ &\le C|\Gamma_k - \Gamma|_k|\mathcal{S}_k|_k + B^{3/2}|\nabla_k\mathcal{S}_k|_k \\ &\le C,\end{aligned}$$

so the base case is complete.

Assume that (3.21) holds for all $p < N$, and then consider $p = N$, for $N \ge 2$. Using the difference of powers formula, we have

$$\begin{aligned}|\nabla^N \mathrm{S}_k| &= \left|\sum_{i=1}^{N} \nabla^{N-i}(\nabla - \nabla_k)\nabla_k^{i-1}\mathrm{S}_k + \nabla_k^N \mathrm{S}_k\right| \\ &\le \sum_{i=1}^{N} |\nabla^{N-i}(\nabla - \nabla_k)\nabla_k^{i-1}\mathrm{S}_k| + |\nabla_k^N \mathrm{S}_k|.\end{aligned}$$

The goal now is to show that we can bound $|\nabla^{N-i}(\nabla - \nabla_k)\nabla_k^{i-1}\mathcal{S}_k|$. Recall that $\nabla - \nabla_k = \Gamma - \Gamma_k$ is a sum of terms of the form ∇g_k. In what follows, we will informally write this as $\sum \nabla g_k$.

Now, suppose $i = 1$. Then using the product rule repeatedly, we have

$$\begin{aligned}|\nabla^{N-1}(\nabla - \nabla_k)\mathrm{S}_k| &= |\nabla^{N-1}(\textstyle\sum \nabla g_k)\mathrm{S}_k| \\ &= \left|\sum_{j=0}^{N-1} \binom{N-1}{j} \nabla^{N-1-j}(\textstyle\sum \nabla g_k)\nabla^j \mathrm{S}_k\right| \\ &\le \sum_{j=0}^{N-1} \binom{N-1}{j} \sum |\nabla^{N-j} g_k||\nabla^j \mathrm{S}_k|.\end{aligned}$$

Each term here is bounded by inductive hypothsis.

Similarly, for $2 \le i \le N$, we have

$$|\nabla^{N-i}(\nabla - \nabla_k)\nabla_k^{i-1}\mathrm{S}_k| \le \sum_{j=0}^{N-i} \binom{N-i}{j} \sum |\nabla^{N-i-j+1} g_k||\nabla^j \nabla_k^{i-1}\mathrm{S}_k|.$$

We need to estimate the last factor. In general we have

$$\begin{aligned}
|\nabla^j \nabla_k^i \mathrm{S}_k| &= |[(\nabla - \nabla_k) + \nabla_k]^j \nabla_k^i \mathrm{S}_k| \\
&= \left| \sum_{l=0}^{j} \binom{j}{l} (\nabla - \nabla_k)^{j-l} \nabla_k^l \nabla_k^i \mathrm{S}_k \right| \\
&\leq \sum_{l=0}^{j} \binom{j}{l} |\nabla - \nabla_k|^{j-l} |\nabla_k^{l+i} \mathrm{S}_k| \\
&\leq \sum_{l=0}^{j} \binom{j}{l} \sum |\nabla g_k|^{j-l} |\nabla_k^{l+i} \mathrm{S}_k|.
\end{aligned}$$

This is also bounded by inductive hypothesis. Putting it all together (the assumptions of the lemma, the inductive hypotheses, equivalence of the norms) we have the desired bounds.

The same method can be used to verify (3.22).

For (3.23), we have

$$\frac{\partial}{\partial t} \nabla^N g_k(t) = \nabla^N \frac{\partial}{\partial t} g_k(t) = -2 \nabla^N \mathcal{S}_k(t).$$

This implies

$$\begin{aligned}
\frac{\partial}{\partial t} |\nabla^N g_k|^2 &= 2 \left\langle \frac{\partial}{\partial t} \nabla^N g_k, \nabla^N g_k \right\rangle \\
&\leq \left| \frac{\partial}{\partial t} \nabla^N g_k(t) \right|^2 + |\nabla^N g_k(t)|^2 \\
&= 4 |\nabla^N \mathcal{S}_k|^2 + |\nabla^N g_k(t)|^2 \\
&\leq C |\nabla^N g_k|^2 + D
\end{aligned}$$

We can integrate this differential inequality to get

$$|\nabla^N g_k(t)|^2 \leq C,$$

as desired.

Using the arguments above, one can show that

$$|\nabla^p \nabla_k^q \operatorname{Rc}_k|, |\nabla^p \nabla_k^q \operatorname{Rm}_k|, |\nabla^p \nabla_k^q R_k|, |\nabla^p \nabla_k^q \mathrm{S}_k|, |\nabla^p \nabla_k^q \phi_k|$$

are bounded, independent of k.

Finally, we note that $\tau_{g_k,h} \phi_k$ and its derivatives have bounded norm. This follows from $\tau_{g,h} \phi = g^{ij} \nabla_i \nabla_j \phi$.

All terms are thus bounded, and we conclude that (3.17) and (3.18) are as well. □

The second lemma, which is a corollary of the Arzela-Ascoli theorem is a modification of [2, Corollary 3.15].

Lemma 3.24. *Let $(\mathcal{M}^n, g)$ be a Riemannian manifold, with $\mathcal{K} \subset \mathcal{M}$ compact and $p \in \mathbb{Z}^{\geq 0}$. Suppose $\{(g_k, \phi_k)\}$ is a sequence of Riemannian metrics on $\mathcal{K}$ and maps $\mathcal{K} \to \mathcal{N}$, where $\mathcal{N}$ is some fixed target manifold, such that*

$$\sup_{0 \leq \alpha \leq p+1} \max_{x \in \mathcal{K}} |\nabla^\alpha g_k| \leq C_1 < \infty,$$

$$\sup_{0\le\alpha\le p+1}\max_{x\in\mathcal{K}}|\nabla^\alpha\phi_k|\le C_2<\infty.$$

Addionally, suppose that there exists $\delta>0$ such that $|V|_k\ge\delta|V|$ for all $V\in T\mathcal{M}$. Then there exists a subsequence $\{(g_{k_j},\phi_{k_j})\}$, a Riemannian metric g_∞ on $\mathcal{K}$, and a smooth map $\phi_\infty\colon\mathcal{K}\to\mathcal{N}$ such that $(g_{k_j},\phi_{k_j})\to(g_\infty,\phi_\infty)$ in C^p as $k\to\infty$.

Proof. The existence of the subsequence will follow from the Arzela-Ascoli theorem, so we need to show that the collection of component functions $\{(g_k)_{ab}\}\cup\{(\phi_k)^\lambda\}$ is an equibounded and equicontinuous family. Equiboundedness follows from the hypotheses.

Now, in a fixed coordinate chart, by writing

$$\nabla_a(g_k)_{bc}=\partial_a(g_k)_{bc}-\Gamma^d_{ab}(g_k)_{dc}-\Gamma^d_{ac}(g_k)_{bd}$$

we see that bounds on $|\nabla g_k|$ give bounds on $|\partial_a(g_k)_{bc}|$. Similarly,

$$|\nabla_a(\phi_k)^\lambda|=|\partial_a(\phi_k)^\lambda|$$

is assumed to be bounded. Now, the mean value theorem for functions of several variables implies that

$$|(g_k)_{bc}(y)-(g_k)_{bc}(x)|\le C_1\,\mathrm{diam}(K),$$

for all $x,y\in K$ and all indices b,c, and similarly for components of ϕ_k. This means the family $\{(g_k)_{ab}\}\cup\{(\phi_k)^\lambda\}$ is equicontinuous in the chart. Since K is compact, we can take finitely many charts to see that there is a finite uniform bound. Now apply the Arzela-Ascoli theorem to obtain the limits g_∞ and ϕ_∞. The bounds on the metrics imply that g_∞ is also a metric, and clearly ϕ_∞ is smooth.

We have only demonstrated subsequential convergence in C^0. For C^p convergence, repeat the same arguments starting with covariant derivatives of g_k and ϕ_k, obtaining bounds on the higher partial derivatives. □

3.4. **The proof of the theorem.** We will need a result of Hamilton, Theorem 2.3 in [7], which he used to prove the original compactness theorem for Ricci flow.

Theorem 3.25. *Let $\{(\mathcal{M}^n_k,g_k,O_k)\}$ be a sequence of pointed, complete, Riemannian manifolds such that*

(a) *the geometry is uniformly bounded:*

$$|\nabla^p_k\,\mathrm{Rm}_k|_k\le C_p$$

on $\mathcal{M}_k$, for all $p\ge0$, all k, for C_p independent of k;

(b) *the injectivity radii are uniformly bounded below:*

$$\mathrm{inj}_k(O_k)\ge\iota_0\,0,$$

for some ι_0 independent of k.

Then there is a subsequence such that

$$(\mathcal{M}_k,g_k,O_k)\longrightarrow(\mathcal{M}_\infty,g_\infty,O_\infty),$$

where the limit is also a pointed, complete, Riemannian manifold.

We will also need the derivative estimate for the curvature and the map, Theorem 6.10 in [17]. This is a version of the Bernstein-Bando-Shi estimates for Ricci flow (see [3, Section 7.1] for exposition).

Theorem 3.26. *Let $(\mathcal{M}^n, g(t), \phi(t))$ solve the $(RH)_c$ flow for $t \in [0, \omega)$ and $c(t)$ non-increasing. Assume $0 < \underline{c} \leq c(t) \leq \overline{c} < \infty$ for all t, and that $\omega < \infty$. Suppose that the curvature is uniformly bounded:*

$$\sup_{\mathcal{M} \times [0,\omega)} |\operatorname{Rm}| \leq R_0.$$

Then there exists a constant $C = C(\underline{c}, \overline{c}, R_0, T, m, N) < \infty$ such that

$$\sup_{\mathcal{M} \times (0,\omega)} |\nabla \phi|^2 \leq \frac{C}{t},$$

$$\sup_{M \times (0,\omega)} \left(|\operatorname{Rm}|^2 + |\nabla^2 \phi|^2 \right) \leq \frac{C^2}{t^2}.$$

Moreover, there exist constants C_p depending on $p, \overline{c}$, m and N such that

$$\sup_{M \times (0,\omega)} \left(|\nabla^p \operatorname{Rm}|^2 + |\nabla^{p+2} \phi|^2 \right) \leq C_p \left(\frac{C}{t} \right)^{p+2}.$$

Now we prove the theorem, in the presense of a bound on the injectivity radius. The proof of the groupoid statement will appear in the next subsection.

Proof of Theorem 3.6. First, note that we may use a diagonalization argument, as in [7, Section 2], to show that we can assume that the interval of existence of the solutions is finite in length, that is,

$$-\infty < \alpha < \omega < \infty.$$

Since we are assuming that the curvatures are uniformly bounded, Theorem 3.26 applies to give uniform bounds on the derivatives of the curvatures and on the derivatives of the maps ϕ_k. With the former, and with the injectivity radius bound, we can use Theorem 3.25 to get pointed subsequential convergence of the metrics at a single time, say $0 \in (\alpha, \omega)$:

$$\big(\mathcal{M}_k, g_k(0), O_k\big) \to \big(\mathcal{M}_\infty, g_\infty, O_\infty\big).$$

The limit is a complete, pointed Riemannian manifold.

Unpacking this convergence, we have the existence of

- an exhaustion $\{\mathcal{U}_k\}$ of $\mathcal{M}_\infty$ by open sets with $O_\infty \in \mathcal{U}_k$ for all k, and
- a family of diffeomorphisms $\{\Psi_k \colon \mathcal{U}_k \to \mathcal{V}_k \subset \mathcal{M}_k\}$ with $\Psi_k(O_\infty) = O_k$

such that

$$\big(\mathcal{U}_k, \Psi_k^* g_k(0)|_{\mathcal{V}_k}\big) \longrightarrow (\mathcal{M}^n_\infty, g_\infty)$$

uniformly in C^∞ on compact sets.

The metrics and maps we are now interested in are $\bar{g}_k(t) = \Psi_k^* g_k(t)$ and $\bar{\phi}_k(t) = \Psi_k^* \phi_k(t)$.

Now we see that the hypotheses of the Lemma 3.8 are satisfied. For any compact $\mathcal{K} \subset \mathcal{M}_\infty$ and $[\beta, \psi] \subset [\alpha, \omega]$ containing 0, the collection $\{(\bar{g}_k(t), \bar{\phi}_k(t))\}$ is a sequence of $(RH)_c$ solutions on $K \times [\beta, \psi]$. Let g_∞ be the background metric and $t_0 = 0$.

The uniform convergence implies that the $\bar{g}_k(0)$ are uniformly equivalent to g_∞, and that the needed bounds hold. For example, using the equivalence of metrics and convergence at one time, we see that

$$\begin{aligned}|\nabla^p_\infty \bar{\phi}_k(0)|_\infty &\leq C|\nabla^p_\infty \bar{\phi}_k(0)|_{\bar{g}_k(0)}\\ &\leq C|\nabla^p_{\bar{g}_k(0)} \bar{\phi}_k(0)|_{\bar{g}_k(0)}\\ &\leq C|\nabla^p_{g_k(0)} \phi_k(0)|_{g_k(0)}\\ &\leq C,\end{aligned}$$

for large enough k.

By the lemma, we conclude that $\bar{g}_k(t)$ are uniformly equivalent to g_∞ on $\mathcal{K} \times [\beta, \psi]$, and that the time and space derivatives of $\bar{g}_k(t)$ and $\bar{\phi}_k(t)$ are uniformly bounded with respect to g_∞.

Now, the conditions of Lemma 3.24 are exactly satisfied by the implications of Lemma 3.8, so we have the desired subsequential convergence. Our limit solution is defined by

$$g_\infty(t) = \lim_{k\to\infty} \bar{g}_k(t), \quad \phi_\infty(t) = \lim_{k\to\infty} \bar{\phi}_k(t).$$

Finally, since all derivatives of the metric and the of map converge, the appropriate tensors converge, so that the limit is a metric/smooth map solving $(RH)_c$ flow. □

3.5. **The flow on groupoids.** In [13] and [14], Lott initiated the use of Riemannian groupoids in understanding the convergence of Ricci flow solutions, especially in the presense of collapsing. This idea has also been used in [6]. We will not review groupoid theory, as the souces above do this well. We will, however, mention two other general references. A comprehensive guide to the subject, with an emphasis on differential geometry, is a book by Mackenzie [15]. A more concise introduction, with an emphasis on foliation theory, is a book by Moerdijk and Mrčun [16].

A Lie groupoid $\mathcal{G} \rightrightarrows \mathcal{B}$ is *Riemannian* if the base $\mathcal{B}$ has a $\mathcal{G}$-invariant metric g. That is, if $\mathcal{U} \subset \mathcal{B}$ is open, $\sigma\colon \mathcal{U} \to \mathcal{G}$ is any local bisection, and $t\colon \mathcal{G} \to \mathcal{B}$ is the target map, then $(t\circ\sigma)^* g = g$. From this, we can construct the Ricci tensor $\mathrm{Rc}[g]$, which is a symmetric $(2,0)$-tensor on $\mathcal{B}$, and which is $\mathcal{G}$-invariant in the same sense as g. Therefore it makes sense to consider the Ricci flow on this groupoid:

$$\frac{\partial}{\partial t} g = -2\,\mathrm{Rc}\,.$$

Let $(\mathcal{N}, h)$ be another Riemannian manifold, thought of as a trivial groupoid, and consider $\phi\colon \mathcal{B} \to \mathcal{N}$ such that $\nabla\phi \otimes \nabla\phi$ is a $\mathcal{G}$-invariant $(2,0)$-tensor on $\mathcal{B}$. Additionally, the tension field $\tau_{g,h}\phi$ of ϕ is well-defined in the usual Riemannian manifold sense. Therefore, we have a well-defined coupling of Ricci flow and harmonic map flow:

$$\begin{aligned}\frac{\partial}{\partial t} g &= -2\,\mathrm{Rc} + 2c\nabla\phi\otimes\nabla\phi\\ \frac{\partial}{\partial t}\phi &= \tau_{g,h}\phi\end{aligned}$$

where $c(t)$ is a non-negative coupling function.

To use this approach to understand limits and convergence of $(RH)_c$ flow on Riemannian manifolds, we show how this groupoid setting can arise from the manifold setting. Let $(\mathcal{M}, g)$ and $(\mathcal{N}, h)$ be complete Riemannian manifolds, and $\phi\colon \mathcal{M} \to \mathcal{N}$

a smooth map. Select $\{p_i\}_{i\in I} \subset \mathcal{M}$ such that $\mathscr{U} = \{\mathcal{U}_i\}_{i\in I}$ is an open cover of $\mathcal{M}$, where the $\mathcal{U}_i$ are such that $\exp_{p_i}(0) = p_i \in \mathcal{U}_i$, and

$$\exp_{p_i}|_{B_{r_i}(0)} \colon B_{r_i}(0) \longrightarrow \mathcal{U}_i$$

is a diffeomorphism, for some sufficiently small $r_i > 0$. Put the metric $(\exp_{p_i})^* g$ on each $B_{r_i}(0)$. Call $\mathscr{U}$ an *open exponential cover* of $\mathcal{M}$.

As in [13, Example 5.7], from this we form a Riemannian groupoid $\mathcal{G}^{\mathscr{U}} \rightrightarrows \mathcal{B}^{\mathscr{U}}$, which is isometrically equivalent to the trivial groupoid $(\mathcal{M}, g)$. Set

$$\mathcal{B}^{\mathscr{U}} = \bigsqcup_{i\in I} B_{r_i}(0) = \{(i, v) \mid i \in I, v \in B_{r_i}(0)\},$$

$$\mathcal{G}^{\mathscr{U}} = \bigsqcup_{i,j\in I} \{(v_i, v_j) \in B_{r_i}(0) \times B_{r_j}(0) \mid \exp_{p_i}(v_i) = \exp_{p_j}(v_j)\}.$$

We will write elements of $\mathcal{B}^{\mathscr{U}}$ as $v_i = (i, v)$ and arrows as (v_i, v_j). Note that we always have $v_i = \exp_{p_i}^{-1}(x)$ for some $x \in U_i$.

The structure maps of this groupoid are defined as follows:

- source: $s(v_i, v_j) = v_i$
- target: $t(v_i, v_j) = v_j$
- unit: $u(v_i) = (v_i, v_i)$
- inverse: $(v_i, v_j)^{-1} = (v_j, v_i)$
- composition: $(v_j, v_k) \cdot (v_i, v_j) = (v_i, v_k)$

Call the étale Riemannian groupoid $\mathcal{G}^{\mathscr{U}} \rightrightarrows \mathcal{B}^{\mathscr{U}}$ the *Riemannian exponential groupoid* with respect to the open cover $\mathscr{U}$ of $\mathcal{M}$.

Proposition 3.27. *The $(RH)_c$ flow on a manifold $(\mathcal{M}, g, \phi)$ and target manifold $(\mathcal{N}, h)$ becomes $(RH)_c$ flow on the n-dimensional Riemannian exponential groupoid $(\mathcal{G}^{\mathscr{U}} \rightrightarrows \mathcal{B}^{\mathscr{U}}, g, \phi)$ associated to an open exponential cover $\mathscr{U}$ of $\mathcal{M}$.*

Proof. The map $\phi\colon \mathcal{M} \to \mathcal{N}$ induces a Lie groupoid morphism $\phi = (\phi_0, \phi_1)$ from $\mathcal{G}^{\mathscr{U}} \rightrightarrows \mathcal{B}^{\mathscr{U}}$ to the trivial groupoid $\mathcal{N} \rightrightarrows \mathcal{N}$. It is defined by

$$\phi_0(v_i) = \phi(\exp_{p_i}(v_i)),$$
$$\phi_1(v_i, v_j) = \phi(\exp_{p_i}(v_i)) = \phi(\exp_{p_j}(v_j)).$$

Thus we can write $\phi_0 = \phi_1 = \exp^* \phi$ for these induced maps. Note also that we could have defined them as

$$\phi_0(v_i) = \phi_0(\exp_{p_i}^{-1}(x)) = \phi(x),$$
$$\phi_1(v_i, v_j) = \phi_1(\exp_{p_i}^{-1}(x), \exp_{p_j}^{-1}(x)) = \phi(x).$$

It is easy to check that these maps are compatible with the structure maps of both groupoids. That is, the following diagram is commutative.

$$\begin{array}{ccc} \mathcal{G}^{\mathscr{U}} & \xrightarrow{\phi_1} & \mathcal{N} \\ u\uparrow\ \downarrow s,t & & u\uparrow\ \downarrow\downarrow s,t \\ \mathcal{B}^{\mathscr{U}} & \xrightarrow{\phi_0} & \mathcal{N} \end{array}$$

The main question is the $\mathcal{G}$-invariance of $\nabla\phi_0 \otimes \nabla\phi_0$. Let $\mathcal{U}_i \subset \mathcal{M}$ have coordinates (x^i), and let a neighborhood $\mathcal{V}_i$ of $\phi(p_i)$ have coordinates (y^α). Then $B_{r_i}(0) \subset \mathcal{B}^{\mathscr{U}}$ has coordinates (z^i), where

$$z^i = \exp_{p_i}^* x^i = x^i \circ \exp_{p_i},$$

and a coframe on $TB_{R_i}(0)$ is dz^i, where

$$dz^i = \exp_{p_i}^* dx^i = d(x^i \circ \exp_{p_i}).$$

To understand invariance, we must understand bisections of $\mathcal{G}^{\mathscr{U}} \rightrightarrows \mathcal{B}^{\mathscr{U}}$. Let σ be a bisection, say

$$\begin{aligned} \sigma\colon B_{r_i}(0) &\longrightarrow \mathcal{G}^{\mathscr{U}} \\ v_i &\longmapsto \big(\sigma_1(v_i), \sigma_2(v_i)\big) \end{aligned}$$

Since it is a bisection, we have $s \circ \sigma = \mathrm{id}_{\mathcal{B}^{\mathscr{U}}}$, and this implies $\sigma_1 = \mathrm{id}_{\mathcal{B}^{\mathscr{U}}}$. Therefore we write

$$\sigma(v_i) = \big(v_i, \tilde{\sigma}(v_i)\big),$$

where $\tilde{\sigma}(v_i)$ satisfies

$$\exp_{p_i} \tilde{\sigma}(v_i) = \exp_{p_i}(v_i).$$

Now we see that

$$(t \circ \sigma)(v_i) = t\big(v_i, \tilde{\sigma}(v_i)\big) = \tilde{\sigma}(v_i),$$

or $t \circ \sigma = \tilde{\sigma}$.

Now, the induced map $\phi_0\colon \mathcal{B}^{\mathscr{U}} \to \mathcal{N}$ has pushforward

$$(\phi_0)_* \in \Gamma(T^*\mathcal{B}^{\mathscr{U}} \otimes (\phi_0)^* T\mathcal{N}),$$

so

$$(\phi_0)_* = \frac{\partial \phi_0^\alpha}{\partial x^i}\, dx^i \otimes \left(\frac{\partial}{\partial y^\alpha}\right)_{\phi_0} = d\phi_0^\alpha \otimes \left(\frac{\partial}{\partial y^\alpha}\right)_{\phi_0}.$$

In any $B_{r_i}(0)$, we have

$$\begin{aligned} (t\circ\sigma)^* d\phi_0^\alpha &= d(\phi_0^\alpha \circ t \circ \sigma) \\ &= d(\phi^\alpha \circ \exp_{p_i} \circ \tilde{\sigma}) \\ &= d(\phi^\alpha \circ \exp_{p_i}) \\ &= d\phi_0^\alpha. \end{aligned}$$

If $f_0\colon B_{r_i}(0) \to \mathbb{R}$ is smooth, locally it is of the form $f_0 = f \circ \exp_{p_i}$ for some $f\colon \mathcal{U}_i \to \mathbb{R}$. Then

$$\begin{aligned} (t\circ\sigma)_* \left(\frac{\partial}{\partial y^\alpha}\right)_{\phi_0} f_0 &= \tilde{\sigma}_* \left(\frac{\partial}{\partial y^\alpha}\right)_{\phi_0} f_0 \\ &= \partial_\alpha (f_0 \circ \tilde{\sigma}) \\ &= \partial_\alpha (f \circ \exp_{p_i} \circ \tilde{\sigma}) \\ &= \partial_\alpha (f \circ \exp_{p_i}) \\ &= \left(\frac{\partial}{\partial y^\alpha}\right)_{\phi_0} f_0. \end{aligned}$$

From this, we conclude that $\nabla\phi_0(\phi_0)_*$ is a G_M-invariant tensor.

In general, a metric h on $T\mathcal{N}$ induces a metric h_ϕ on the pull-back bundle $\phi^*T\mathcal{N}$, given by

$$h_\phi(\xi, \eta) = h(\phi_*\xi, \phi_*\eta),$$

for all $\xi, \eta \in T\mathcal{M}$. In this way, we get a metric on $(\phi_0)^*T\mathcal{N}$, and it is $\mathcal{G}^{\mathscr{U}}$-invariant:

$$(t\circ\sigma)^* h_{\phi_0}(\xi,\eta) = h_{\phi_0}(\xi,\eta).$$

Thus $\nabla\phi_0 \otimes \nabla\phi_0$ is a $(2,0)$-tensor on $\mathcal{B}^{\mathscr{U}}$:

$$\nabla\phi_0 \otimes \nabla\phi_0 = (h_{\phi_0})_{\lambda\mu}\partial_i\phi_0^\lambda\partial_j\phi_0^\mu \, dz^i \otimes dz^j.$$

It is therefore $\mathcal{G}^{\mathscr{U}}$-invariant, and the $(RH)_c$ flow makes sense on $\mathcal{G}^{\mathscr{U}} \rightrightarrows \mathcal{B}^{\mathscr{U}}$. □

This proposition shows that this framework is at least non-vacuous. Before completing the proof of Theorem 3.6, we need a definition and a result of Lott.

Definition 3.28. Let $\{(\mathcal{G}_k \rightrightarrows \mathcal{B}_k, g_k, \phi_k, O_{x_k})\}$ be a sequence of pointed, n-dimensional Riemannian groupoids with maps into some fixed Riemannian manifold $(\mathcal{N}, h)$. Let $\{(\mathcal{G}_\infty \rightrightarrows \mathcal{B}_\infty, g_\infty, \phi_\infty, O_{x_\infty})\}$ be a pointed Riemannian groupoid with map $\phi_\infty\colon \mathcal{B}_\infty \to \mathcal{N}$. Let J_1 be the groupoid of 1-jets of local diffeomorphisms of $\mathcal{B}_\infty$. We say that

$$(\mathcal{G}_k \rightrightarrows \mathcal{B}_k, g_k, \phi_k, O_{x_k}) \longrightarrow (\mathcal{G}_\infty \rightrightarrows \mathcal{B}_\infty, g_\infty, \phi_\infty, O_{x_\infty})$$

in the *pointed smooth topology* if for all $R > 0$, the following hold.

- There are pointed diffeomorphisms $\Psi_{k,R}\colon B_R(O_{x_\infty}) \to B_R(O_{x_k})$, defined for large k, so that
$$\Psi^*_{k,R} g_k|_{B_R(O_{x_i})} \longrightarrow g_\infty|_{B_R(O_{x_\infty})}.$$
$$\Psi^*_{k,R} \phi_k|_{B_R(O_{x_i})} \longrightarrow \phi_\infty|_{B_R(O_{x_\infty})}.$$
- After conjugating by $\Psi_{k,R}$, the images of
$$s_k^{-1}\big(\overline{B_{R/2}(O_{x_k})}\big) \cap t_k^{-1}\big(\overline{B_{R/2}(O_{x_k})}\big)$$
converge in J_1 in the Hausdorff sense to the image of
$$s_\infty^{-1}\big(\overline{B_{R/2}(O_{x_\infty})}\big) \cap t_\infty^{-1}\big(\overline{B_{R/2}(O_{x_\infty})}\big)$$
in J_1.

The following is [13, Proposition 5.8].

Theorem 3.29. *Let $\{(\mathcal{M}_k, g_k, O_k)\}$ be a sequence of pointed complete n-dimensional Riemannian manifolds. Suppose that for each $p \geq 0$ and $r > 0$, there is some $C_{p,r} < \infty$ such that for all k,*

$$\max_{B_R(O_i)} |\nabla^p \operatorname{Rm}_k|_\infty \leq C_{p,r}.$$

Then there is a subsequence of $\{(M_k, O_k)\}$ that converges to some pointed n-dimensional Riemannian groupoid $(G_\infty \rightrightarrows B_\infty, g_\infty, O_{x_\infty})$ in the pointed smooth topology.

Now we can complete the proof.

Proof of Theorem 3.6. As Lott mentions, there is very little difference between the proofs of Hamilton's original theorem and [13, Theorem 5.12]. The same is true here. Namely, using Theorem 3.26, we obtain uniform bounds on the derivatives of the curvatures, which allow us to use Theorem 3.29. This is a version of Theorem 3.25 for groupoids, and gives subsequential convergence at one time to a pointed Riemannian groupoid.

To extend this to the whole time interval, we apply Lemma 3.8 and a version of 3.24, which gives another convergent subsequence. Hence we get a limiting metric and map, which together solve $(RH)_c$ flow on $\mathcal{M}$. □

Remark 3.30. As in [13, Section 5], Theorem 3.6 implies that the space of pointed n-dimensional $(RH)_c$ flow solutions with $\sup_t t|\operatorname{Rm}[g(t)]|_\infty < C$ is relatively compact among all $(RH)_c$ solutions on étale Riemannian groupoids. Let $\mathscr{S}_{n,C}$ be the closure of this space. It is easy to see that the blowdown procedure from Example 3.7 defines an $\mathbb{R}^+$-action on the compact space $\mathscr{S}_{n,C}$.

4. A detailed example of $(RH)_c$ flow

We conclude with an example of $(RH)_c$ flow on the three-dimensional nilpotent Lie group Nil^3, and compare the asympotics of the solutions with those for Ricci flow.

Consider

$$\mathrm{Nil}^3 \cong \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; x, y, z \in \mathbb{R} \right\} \subset \mathrm{SL}_3\,\mathbb{R}.$$

The obvious diffeomorphism with $\mathbb{R}^3$ provides global coordinates (x, y, z) in which the group multiplication is

$$(x, y, z) \cdot (z', y', z') = (x + x', y + y', z + z' + xy').$$

There is a frame of left-invariant vector fields,

$$F_1 = \frac{\partial}{\partial x}, \quad F_2 = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}, \quad F_3 = \frac{\partial}{\partial z},$$

and the only nontrivial Lie bracket relation is

$$[F_1, F_2] = F_3.$$

The dual coframe is

$$\theta^1 = dx, \quad \theta^2 = dy, \quad \theta^3 = dz - xdy.$$

A family of left-invariant metrics on Nil^3 is given by

$$g(t) = A(t)\,\theta^1 \otimes \theta^1 + B(t)\,\theta^2 \otimes \theta^2 + C(t)\,\theta^3 \otimes \theta^3, \tag{4.1}$$

and the corresponding Ricci $(2,0)$-tensors satisfy

$$-2\operatorname{Rc}(g(t)) = \frac{C}{B}\,\theta^1 \otimes \theta^1 + \frac{C}{A}\,\theta^2 \otimes \theta^2 - \frac{C^2}{AB}\,\theta^3 \otimes \theta^3.$$

Proposition 4.2. *Solutions of Ricci flow on* Nil^3 *of the form* (4.1) *have the following asympototics:*

$$\begin{aligned} A(t) &\sim A_0 K^{-1/3} t^{1/3}, \\ B(t) &\sim B_0 K^{-1/3} t^{1/3}, \\ C(t) &\sim C_0 K^{1/3} t^{-1/3}, \end{aligned} \tag{4.3}$$

for the constant $K = A_0 B_0 / 3C_0$.

Ricci flow on Nil^3 been studied extensively. See, for example, [9], [11], [1], [13], [6], [18], [20]. We want to study $(RH)_c$ flow on Nil^3. Consider a function

$$\phi\colon \big(\mathrm{Nil}^3, g(t)\big) \longrightarrow (\mathbb{R}, g_{\mathrm{can}}),$$

and let $c = c(t) \geq 0$ be a non-increasing function. For the resulting $(RH)_c$ flow system to remain a system of ordinary differential equations for the metric, we need

ϕ to be harmonic and $\nabla\phi \otimes \nabla\phi = d\phi \otimes d\phi$ to be a diagonal left-invariant tensor. It is not hard to see that the latter condition requires that

$$\phi(x, y, z) = ax + by,$$

for some $a, b \in \mathbb{R}$. Note that such a function is also a group homomorphism, and that

$$\tau_{g,g_{\rm can}}\phi = g^{ij}(\partial_i\partial_j\phi - \Gamma^k_{ij}\partial_k\phi) = 0,$$

so it is harmonic. Then

$$d\phi \otimes d\phi = a^2\, dx \otimes dx + ab(dx \otimes dy + dy \otimes dx) + b^2\, dy \otimes dy.$$

To keep the system diagonal, take $b = 0$, so that $d\phi \otimes d\phi = a^2\, \theta^1 \otimes \theta^1$. The $(RH)_c$ flow system is

$$\begin{aligned} \frac{d}{dt}A &= \frac{C}{B} + 2a^2 c, \\ \frac{d}{dt}B &= \frac{C}{A}, \\ \frac{d}{dt}C &= -\frac{C^2}{AB}. \end{aligned} \tag{4.4}$$

Let us first make a few general observations about the long-time behavior of A, B, and C. Set $f(t) = 2a^2c(t)$ for simplicity. Note that $\Phi = BC = B_0C_0$ is conserved, A and B are increasing, and C is decreasing. This implies

$$C' = -\frac{C^2}{AB} \geq -\frac{1}{A_0B_0}C^2,$$

and integrating tells us that

$$0 < \frac{A_0B_0C_0}{A_0B_0 + C_0t} \leq C(t) \leq C_0,$$

for $t \geq 0$. We conclude that $C(t) \to C_\infty \in [0, C_0)$ as $t \to \infty$. Similarly, we see that

$$C' = -\frac{C^3}{\Phi A} \geq -\frac{1}{\Phi A_0}C^3,$$

which implies

$$0 < \frac{A_0B_0C_0^2}{A_0B_0 + 2C_0^2t} \leq C(t)^2 \leq C_0^2.$$

This gives

$$\int_0^t C(s)^2\, ds \geq \int_0^t \frac{A_0B_0C_0^2}{A_0B_0 + 2C_0s}ds \longrightarrow \infty \tag{4.5}$$

as $t \to \infty$.

Next we use Φ to see that

$$A' = \frac{C^2}{\Phi} + f = \frac{\Phi}{B^2} + f, \tag{4.6}$$

which we integrate to obtain

$$A(t) = A_0 + \frac{1}{\Phi}\int_0^t C(s)^2\, ds + \int_0^t f(s)\, ds. \tag{4.7}$$

By (4.5), we have $A(t) \to \infty$ as $t \to \infty$, and we have a bound on the growth of A:

$$A(t) \leq A_0 + \frac{C_0^2}{\Phi}\int_0^t ds + f_0\int_0^t ds \leq A_0 + \left(\frac{C_0}{B_0} + f_0\right)t. \tag{4.8}$$

This implies

$$\int_0^t \frac{ds}{A(s)} \geq \int_0^t \frac{ds}{A_0 + \left(\frac{C_0}{B_0} + f_0\right)s} \longrightarrow \infty \tag{4.9}$$

as $t \to \infty$.

Finally,

$$(B^2)' = \frac{2\Phi}{A},$$

which implies

$$B(t)^2 = B_0^2 + 2\Phi \int_0^t \frac{ds}{A(s)}, \tag{4.10}$$

so, by (4.9), $B(t) \to \infty$ and $C(t) = \Phi/B(t) \to 0$ as $t \to \infty$.

4.1. **Constant coupling function.** Let us now consider the case when c (and therefore f, which we write as f_0) is a constant. From (4.6) we compute that

$$\lim_{t\to\infty} \frac{A(t)}{f_0 t} \stackrel{LH}{=} \lim_{t\to\infty} \frac{A'(t)}{f_0} = \lim_{t\to\infty} \left(\frac{C^2}{f_0\Phi} + 1\right) = 1,$$

so $A(t) \sim f_0 t$. Using (4.10) and $A \sim f_0 t \sim f_0(t+1)$, we have

$$B^2 \sim 2\Phi \int_0^t \frac{ds}{A(s)} \sim \frac{2\Phi}{f_0} \int_0^t \frac{ds}{s+1} \sim \frac{2\Phi}{f_0} \log t.$$

This gives the following.

Proposition 4.11. *Solutions of* $(RH)_c$ *flow on* Nil^3 *of the form* (4.1) *with map* $\phi(x,y,z) = ax$ *and* $c > 0$ *constant have the following asympototics:*

$$\begin{aligned} A(t) &\sim 2a^2ct, \\ B(t) &\sim \sqrt{\frac{B_0C_0}{a^2c}\log t}, \\ C(t) &\sim 2\sqrt{\frac{a^2cB_0C_0}{\log t}}. \end{aligned} \tag{4.12}$$

Note that if we attempt to take a limit of these solutions as $f_0 \to 0$, they *do not* converge in a naive sense to the solutions of Ricci flow from (4.3). To explain this, we examine certain coupling functions that decay as $t \to \infty$, and which yield behavior similar to that for Ricci flow.

4.2. **Nonconstant coupling function.** Now consider a coupling function such that

$$c(t) \sim \frac{1}{t^r},$$

where $r \geq 1$. We make the ansatz that $A(t) \sim \alpha t^p$, for some $\alpha, p > 0$ to be determined. From (4.8), it is consistent to assume that $0 < p \leq 1$. Then using (4.7),

$$\lim_{t\to\infty} \frac{A(t)}{\alpha t^p} \stackrel{LH}{=} \lim_{t\to\infty} \frac{A'(t)}{p\alpha t^{p-1}} = \lim_{t\to\infty} \frac{1}{p\alpha t^{p-1}} \left(\frac{\Phi}{B^2} + \frac{2a^2}{t^r}\right). \tag{4.13}$$

Finding this limit comes down to analyzing two limits:

$$\lim_{t\to\infty} \frac{1}{B^2 t^{p-1}}, \tag{4.14}$$

$$\lim_{t\to\infty} \frac{1}{t^{r+p-1}}. \tag{4.15}$$

Since $r \geq 1$ implies that (4.15) is zero for any $p > 0$, we need that

$$B \sim \beta t^{\frac{1-p}{2}}$$

for some $\beta > 0$. To find β, consider

$$B^2 \sim 2\Phi \int_0^t \frac{ds}{A(s)} \sim \frac{2\Phi}{\alpha} \int_0^t \frac{ds}{(s+1)^p} \sim \frac{2\Phi}{\alpha(1-p)} t^{1-p}.$$

This now implies

$$\lim_{t\to\infty} \frac{1}{B^2 t^{p-1}} = \frac{\alpha(1-p)}{2\Phi},$$

and so

$$1 \stackrel{?}{=} \lim_{t\to\infty} \frac{A(t)}{\alpha t^p} = \frac{\Phi}{p\alpha} \lim_{t\to\infty} \frac{1}{B^2 t^{p-1}} = \frac{\Phi}{p\alpha} \frac{\alpha(1-p)}{2\Phi} = \frac{1-p}{2p}.$$

For $A(t) \sim \alpha t^p$ we therefore need $p = 1/3$. From here, we obtain the asymptotic behavior. Modulo constants, it is that of the Ricci flow solutions (4.3).

Proposition 4.16. *Solutions of* $(RH)_c$ *flow on* Nil^3 *of the form* (4.1) *with map* $\phi(x, y, z) = ax$ *and* $c \sim 1/t^r$, $r \geq 1$, *have the following asympototics:*

$$\begin{aligned} A &\sim \alpha t^{1/3}, \\ B &\sim \sqrt{\frac{3\Phi}{\alpha}} t^{1/3}, \\ C &\sim \sqrt{\frac{\alpha\Phi}{3}} t^{-1/3}, \end{aligned} \tag{4.17}$$

for some constant α *depending on* r *and the initial data.*

The reason that the limit as $f_0 \to 0$ of the solutions (4.12) is not the Ricci flow solutions (4.3) lies in the integrability of the coupling function. Informally, the lack of such a limit results from

$$0 = \lim_{t\to\infty} \lim_{f_0\to 0} \int_0^t f_0\, ds \neq \lim_{f_0\to 0} \lim_{t\to\infty} \int_0^t f_0\, ds = \infty.$$

To be more precise, consider $A(t)$ as given in (4.7). When $r > 1$, $f(t)$ is integrable, allowing

$$\int_0^t C(s)^2\, ds$$

to dominate and produce growth like $t^{1/3}$. When $r < 1$, $f(t)$ is not integrable, and

$$\int_0^t f(s)\, ds$$

dominates to produce linear growth.

In numerical simulations, $0 < r < 1$ appears to be a transitionary region where solutions have properties of both (4.12) and (4.17). We were unable to obtain the precise asymptotics, but we expect that letting $r \to 0$ should recover (4.12) and letting $r \to \infty$ should recover (4.3).

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