SCATTERING OF WAVE MAPS FROM $\mathbb{R}^{2+1}$ TO GENERAL TARGETS

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ABSTRACT. We show that smooth, compactly supported radially symmetric Wave Maps U from $\mathbb{R}^{2+1}$ to a compact target manifold N scatter. The result will follow from the work of Christdoulou and Tahvildar-Zadeh, and Struwe, upon proving that for $\lambda' \in (0,1)$, energy does not concentrate in the set $K^{\lambda'}_{\frac{5}{8}T,\frac{7}{8}T} = \{(x,t) \in \mathbb{R}^{2+1} |\ |x| \le \lambda' t, t \in [\frac{5}{8}T, \frac{7}{8}T]\}$.

1. Introduction

In this work we consider the initial value problem for Wave Maps from $\mathbb{R}^{2+1}$ to a compact target manifold $(N, \langle\cdot,\cdot\rangle)$,

$$\begin{cases} \partial_\alpha \partial^\alpha U = B(U)(\partial_\alpha U, \partial^\alpha U) \\ U(x,0) = U_0(x), \partial_t U(x,0) = U_1(x),\ x \in \mathbb{R}^2, \end{cases}$$

where B is the second fundamental form of $(N, \langle\cdot,\cdot\rangle)$. Much is known about this system; we refer readers to [6], [3], and references therein.

Concerning radially symmetric Wave Maps, Christdoulou and Tahvildar-Zadeh in [2] proved global well posedness for smooth Wave Maps to targets that satisfied certain bounds on the second fundamental form, in addition to being either compact or having bounded structure functions. These results were obtained by showing that energy does not concentrate at the origin, along with pointwise estimates on the fundamental solution to the linear problem.

Struwe in [4] extended this result to radially symmetric Wave Maps from $\mathbb{R}^{2+1}$ to spheres S^k, and later in [5] to general targets, by showing with energy estimates and rescalling, that energy cannot concentrate at the origin. Concerning asymptotic behavior for radially symmetric Wave Maps, Christdoulou and Tahvildar-Zadeh in [1] proved pointwise estimates which imply scattering for smooth, compactly supported Wave Maps to targets satisfying the same conditions as in [2].

We will use similar methods as in [4], [5], and [1] to prove our main result.

Theorem 1.1. *For a smooth, compactly supported radially symmetric Wave Map U to a compact target manifold N, there exists a function*

U_+ such that

$$\lim_{t\to\infty} \|U(x,t) - \cos(t\sqrt{-\Delta})U_0 - \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}}(U_1 + U_+)\|_{\dot{H}^1} = 0.$$

In Section 2, we review the work done on radially symmetric Wave Maps, with emphasis on results which we will use to prove Theorem 1.1, in Section 3. We use the following notation. Let

$$K^{\lambda}_{S,T} = \{(x,t) \in \mathbb{R}^{2+1} |\ |x| \leq \lambda t, t \in [S,T]\}.$$

Energy will be denoted by either

$$E(\lambda, \lambda'; t) = \int_{B_{\lambda' t}(0)\backslash B_{\lambda t}(0)} \langle \partial_\alpha U, \partial_\alpha U\rangle \, dx,$$

or

$$E(S,t) = \int_S \langle \partial_\alpha U, \partial_\alpha U\rangle \, dx.$$

With $r = |x|$, we will denote light cone coordinates as $u = t - r$, $v = t + r$. The statement '$a \lesssim b$' will mean the quantity a is less than b multiplied by a fixed constant.

2. A brief review of radially symmetric wave maps

We will prove our main result by showing that energy does not concentrate in the set $K^{\lambda'}_{\frac{5}{8}T,\frac{7}{8}T}$. Scattering will then follow by the work of Struwe in [5], and Christdoulou and Tahvildar-Zadeh in [1]. We briefly describe these results here.

In [1], the authors prove a series of energy estimates, which are then used in a bootstrap argument. We mention two in particular that will be used later. For $0 < \lambda' < \lambda'' < 1$,

$$\lim_{t\to\infty} E(\lambda', \lambda''; t) = 0, \tag{2.1}$$

and

$$\lim_{T\to\infty} \frac{1}{T} \int\int_{K^{\lambda'}_{T/2.T}} \|U_t\|^2 = 0. \tag{2.2}$$

Their bootstrap argument hinges on the Bondi energy decaying for large u,

$$\mathcal{E}(u) \equiv \int_u^\infty r\|\partial_v U\|^2 \, dv \to 0 \text{ as } u \to \infty. \tag{2.3}$$

In order to control $\mathcal{E}(u)$, define (see [1], page 34)

$$\mathcal{E}_{\lambda'}(u) \equiv \int_{((1+\lambda')/(1-\lambda'))u}^\infty 2r\|\partial_v U\|^2 \, dv, \tag{2.4}$$

which will approach 0 as $u \to \infty$, and observe that for $u = (1 - \lambda')t$ (ibid, page 43),

$$\frac{1}{T}\int_{\frac{5}{8}T}^{\frac{7}{8}T} \mathcal{E}(u)\, dt = \frac{1}{T}\int_{\frac{5}{8}T}^{\frac{7}{8}T} [\mathcal{E}_{\lambda'}(u) + E(0, \lambda'; t)]\, dt.$$

By using assumptions on the second fundamental form of N, along with energy estimates, the authors show (ibid, page 42)

$$\lim_{T\to\infty} \frac{1}{T}\int_{\frac{5}{8}T}^{\frac{7}{8}T} E(0, \lambda'; t)\, dt = 0, \tag{2.5}$$

which implies the necessary decay on $\mathcal{E}(u)$.

This is the only place where the bounds on the second fundamental form come into play. The rest of the paper is a bootstrap argument, and the fact that in appropriate coordinates the nonlinearlity in (1.1) can be controlled by (ibid, page 43)

$$|B(\partial_\alpha U, \partial^\alpha U)| \lesssim |\partial_u U||\partial_v U|, \tag{2.6}$$

to prove the following result (ibid, page 31, and page 45),

Theorem 2.1. *Let $\mathcal{C}_u^+$ (resp. $\mathcal{C}_u^-$) be the interior of the future (resp. past) light cone with vertex at $(t = u, r = 0)$ in $M = \mathbb{R}^{2,1}$. For a smooth, compactly supported radially symmetric Wave Map U satisfying* (2.5),

$$diam(U(\mathcal{C}_u^+)) \le \frac{c}{\sqrt{u}}.$$

Futhermore, U obeys the estimates

$$\|\partial_v U\| \le \frac{c}{v^{\frac{3}{2}}}, \|\partial_u U\| \le \frac{c}{v^{\frac{1}{2}}(|u| + 1)}. \tag{2.7}$$

In [5], it is shown that energy does not concentrate at the origin at some time T, since this is the only only obstacle to global well posedness by [2]. Arguing by contradiction, one finds a radius $R(t)$ such that

$$\varepsilon_1 \le E(B_{R(t)}(0), t) \le 2\varepsilon_1 \le \liminf_{t\to T} E(B_{T-t}(0), t), \tag{2.8}$$

in addition to

$$\lim_{t\to T} R(t)/(T - t) = 0. \tag{2.9}$$

Using estimates on the kinetic energy, one can find a sequence of intervals $\{(t_l - R(t_l), t_l + R(t_l))\}$ with $t_l \to T$ so that

$$\lim_{l\to\infty} \frac{1}{R(t_l)} \int_{(t_l - R(t_l), t_l + R(t_l))} \left(\int_{B_{T-t}(0)} \|U_t\|^2\, dx \right) dt = 0.$$

By rescaling with $U_l(t,x) \equiv U(t_l + R_l t, R_l x)$, one obtains a sequence of Wave Maps $\{U_l\}$ with

$$\lim_{l\to\infty} \int_{(-1,1)} \left(\int_{D_l(t)} \|\partial_t U_l\|^2 \, dx \right) dt = 0. \tag{2.10}$$

where $D_l(t) = \{x|\ R_l|x| \le t_l + R_l t\}$.

With these estimates, it can be shown that U_l converges to a Harmonic Map from $\mathbb{R}^2$ to N. Specifically, (2.10) shows that the function that U_l converges to satisfies a Harmonic Map equation, (2.8) shows this function has bounded energy, and (2.9) shows that the limit is a map from all of $\mathbb{R}^2$ to N. Since N is compact, such a map must be constant. With some geometric estimates, one can then show that the total energy of U_l tends to 0, contradicting the lower bound in (2.8). In particular, Struwe proved the following result in [5].

Theorem 2.2. *If $\{U_l\}$ is a sequence of radially symmetric Wave Maps from $\mathbb{R}^{2+1}$ to a compact manifold N satisfying*

$$E(D_l(t), t) \le E_0 < \infty, \tag{2.11}$$

(2.10) *with some $D_l(t)$ obeying* $\limsup_{l\to\infty} D_l(t) = \mathbb{R}^2$*, then $E(D_l(t), t) \to 0$ as $l \to \infty$.*

3. Proof of main result

With the results from the previous section, we prove Theorem 1.1. Using Theorem 2.2, we will show (2.5), then use this fact to apply Theorem 2.1.

We argue the decay of energy by contradiction. Suppose it is not true that

$$\lim_{T\to\infty} \frac{1}{T} \int_{\frac{5}{8}T}^{\frac{7}{8}T} E(0, \lambda'; t)\, dt = 0.$$

Since energy is positive and bounded,

$$\limsup_{T\to\infty} \frac{1}{T} \int_{\frac{5}{8}T}^{\frac{7}{8}T} E(0, \lambda'; t)\, dt = \eta,$$

where $0 < \eta < \infty$. Pick $\{T_n\}_{n\in\mathbb{N}}$ such that $\lim_{T\to\infty} \frac{1}{T_n} \int_{\frac{5}{8}T_n}^{\frac{7}{8}T_n} E(0, \lambda'; t)\, dt = \eta$.

In order to produce the sequence U_l in Theorem 2.2, we require the lower bound in (2.8), which we now prove. By energy conservation (for $t < t'$ and $R > 0$),

$$E(B_R(0), t) \le E(B_{R+(t'-t)}(0), t'), \tag{3.1}$$

any energy that enters or leaves $K^{\lambda'}_{\frac{5}{8}T, \frac{7}{8}T}$ must pass through the surrounding region. By (2.1), energy just outside $K^{\lambda'}_{\frac{5}{8}T, \frac{7}{8}T}$ must decay with time. This keeps energy from rapidly fluxuating in $K^{\lambda'}_{\frac{5}{8}T, \frac{7}{8}T}$, so after

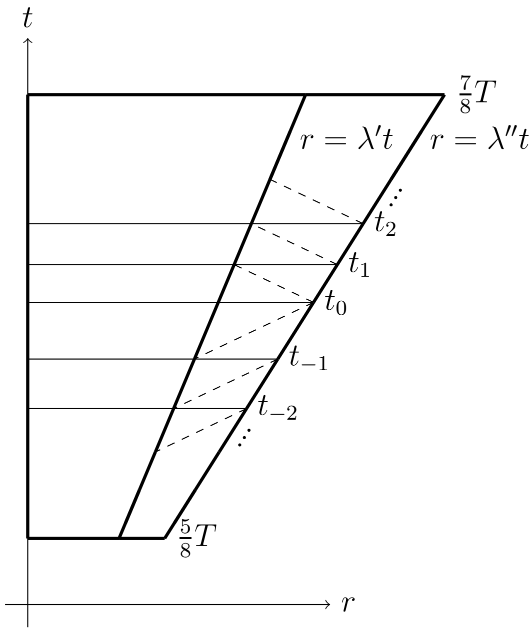

FIGURE 1. Construction for the sequence $\{t_l\}$. Dashed lines either have slope 1 or -1.

sufficient time, the energy at a fixed time in $K^{\lambda'}_{\frac{5}{8}T,\frac{7}{8}T}$ must stay away from 0. This argument is formalized in the following lemma.

Lemma 3.1. *Fix $\lambda' < \lambda'' < 1$. There is an $\alpha = \alpha(\lambda', \lambda'') \in (0, 1)$ such that for large enough n, $t \in [\frac{5}{8}T_n, \frac{7}{8}T_n]$, it follows that $\alpha\eta < E(0, \lambda'; t)$.*

Proof. From (2.1), we can pick n big enough so that $E(\lambda', \lambda''; t) < \beta\eta$ for $\beta = \beta(\lambda', \lambda'')$ to be chosen later and $t \geq \frac{5}{8}T_n$. For perhaps even larger n, we can have that

$$\left| \frac{1}{T_n} \int_{\frac{5}{8}T_n}^{\frac{7}{8}T_n} E(0, \lambda'; t)\, dt - \eta \right| < \gamma\eta \tag{3.2}$$

for $\gamma = \gamma(\lambda', \lambda'') \in (0, 1)$ which we will specify below.

Suppose for some $\tau_n \in [\frac{5}{8}T_n, \frac{7}{8}T_n]$, $E(0, \lambda'; \tau_n) \leq \alpha\eta$. We will show that

$$\frac{1}{T_n} \int_{\frac{5}{8}T_n}^{\frac{7}{8}T_n} E(0, \lambda'; t)\, dt =$$

$$\frac{1}{T_n} \int_{\frac{5}{8}T_n}^{\tau_n} E(0, \lambda'; t)\, dt + \frac{1}{T_n} \int_{\tau_n}^{\frac{7}{8}T_n} E(0, \lambda'; t)\, dt < (1 - \gamma)\eta, \tag{3.3}$$

which would contradict our assumption (3.2). We seperately estimate the two integrals in the middle of (3.3).

Let $\{t_l\}_{l\in\mathbb{Z}}$ be defined by $t_0 = \tau_n$, $t_l = (\frac{1+\lambda''}{1+\lambda'})^l t_0$ for $l > 0$, and $t_l = (\frac{1-\lambda''}{1-\lambda'})^l t_0$ for $l < 0$ (see Figure 1). Let $N_+ = N_+(\lambda', \lambda'')$ be the smallest number with $t_{N_+} \geq \frac{7}{8}T_n$, and $N_- = N_-(\lambda', \lambda'')$ the largest number with $t_{N_-} \leq \frac{5}{8}T_n$. Let $q_+ = \frac{1+\lambda''}{1+\lambda'}$ and $q_- = \frac{1-\lambda''}{1-\lambda'}$.

We first estimate the integral over $[\tau_n, \frac{7}{8}T_n]$. By (3.1),

$$E(0,\lambda';t_{l+1}) \leq E(0,\lambda';t_l) + E(\lambda',\lambda'';t_l).$$

From this, our bounds on $E(\lambda',\lambda'';t_l)$, and our assumption on $t_0 = \tau_n$,

$$\begin{aligned} E(0,\lambda';t_{l+1}) &\leq E(0,\lambda';t_0) + lE(\lambda',\lambda'';t_0) \\ &\leq \alpha\eta + l\beta\eta. \end{aligned}$$

Integrating over $[t_l, t_{l+1}]$,

$$\int_{t_l}^{t_l+1} E(0,\lambda';s)\,ds \leq (q_+ - 1)q_+^l \tau_n \alpha\eta + l(q_+ - 1)q_+^l \tau_n \beta\eta. \tag{3.4}$$

Adding these up,

$$\begin{aligned} \int_{\tau_n}^{\frac{7}{8}T_n} E(0,\lambda';s)\,ds &\leq \sum_{l=0}^{N_+-1} \int_{t_l}^{t_l+1} E(0,\lambda';s)\,ds \\ &\leq \sum_{l=0}^{N_+-1} ((q_+ - 1)q_+^l \tau_n \alpha\eta + l(q_+ - 1)q_+^l \tau_n \beta\eta) \\ &= (q_+ - 1)\frac{q_+^{N_+} - 1}{q_+ - 1}\tau_n\alpha\eta \\ &\quad + (q_+ - 1)[(\frac{N_+}{q_+ - 1} - \frac{q_+}{(1-q_+)^2})q_+^{N_+} + \frac{q_+}{(1-q_+)^2}]\tau_n\beta\eta \\ &\leq (q_+^{N_+} - 1)\frac{7}{8}T_n\alpha\eta \\ &\quad + [(N_+ - \frac{q_+}{1-q_+})q_+^{N_+} + \frac{q_+}{1-q_+}]\frac{7}{8}T_n\beta\eta. \end{aligned}$$

For the integral over $[\frac{5}{8}T_n, \tau_n]$ in (3.3), we use a similar argument,

$$\begin{aligned} \int_{\frac{5}{8}T_n}^{\tau_n} E(0,\lambda';s)\,ds &\leq \sum_{l=0}^{N_--1} ((q_- - 1)q_-^l \tau_n(\alpha\eta + l(q_- - 1)q_-^l \tau_n \beta\eta)) \\ &= (q_- - 1)\frac{q_+^{N_-} - 1}{q_- - 1}\tau_n\alpha\eta \\ &\quad + (q_- - 1)[(\frac{N_-}{q_- - 1} - \frac{q_-}{(1-q_-)^2})q_-^{N_-} + \frac{q_-}{(1-q_-)^2}]\tau_n\beta\eta \\ &\leq (q_-^{N_-} - 1)\frac{7}{8}T_n\alpha\eta \\ &\quad + [(N_- - \frac{q_-}{1-q_-})q_-^{N_-} + \frac{q_-}{1-q_-}]\frac{7}{8}T_n\beta\eta. \end{aligned}$$

Combining these,

$$\frac{1}{T_n}\int_{\frac{5}{8}T_n}^{\tau_n} E(0,\lambda';s)\,ds + \frac{1}{T_n}\int_{\tau_n}^{\frac{7}{8}T_n} E(0,\lambda';s)\,ds \le \frac{7}{8}(q_-^{N_-} + q_+^{N_+} - 2)\alpha\eta$$
$$+ [(N_+ - \frac{q_+}{1-q_+})q_+^{N_+} + \frac{q_+}{1-q_+}]\frac{7}{8}\beta\eta$$
$$+ [(N_- - \frac{q_-}{1-q_-})q_-^{N_-} + \frac{q_-}{1-q_-}]\frac{7}{8}\beta\eta.$$

Choosing α, β, and γ so that

$$\frac{7}{8}(q_-^{N_-} + q_+^{N_+} - 2)\alpha\eta + [(N_+ - \frac{q_+}{1-q_+})q_+^{N_+} + \frac{q_+}{1-q_+}]\frac{7}{8}\beta\eta$$
$$+ [(N_- - \frac{q_-}{1-q_-})q_-^{N_-} + \frac{q_-}{1-q_-}]\frac{7}{8}\beta\eta < (1-\gamma)\eta,$$

from which (3.3) follows, which contradicts (3.2). □

With this lemma, along with the results of Struwe and Tahvildar-Zadeh, we now prove Theorem 1.1.

Proof of Theorem 1.1. We closely follow the argument of Stuwe. Let E_0 denote the total energy of the system, $\mathcal{T} = \cup_n[\frac{5}{8}T_n, \frac{7}{8}T_n]$, and pick $R(t)$ so that for some η_0 with $2\eta_0$ less than $\liminf_{t\in\mathcal{T}} E(B_{\lambda t}(0), t)$, $\eta_0 < E(B_{R(t)}(0), t) < 2\eta_0$ for $t \in \mathcal{T}$. Cover $\mathcal{T}$ with countably many intervals $\Lambda_l \equiv (t_l - R(t_l), t_l + R(t_l))$, and with $R_l = R(t_l)$, note that from (2.2)

$$\frac{1}{R_l}\int_{\Lambda_l}\int_{B_{\lambda' t}(0)} \|\partial_t U\|^2\,dx\,dt \to 0, \tag{3.5}$$

as $l \to \infty$.

Since $\lim_{t\to\infty} E(\lambda', \lambda''; t) = 0$ for all $0 < \lambda' < \lambda'' < 1$, we have for any $\{t_n\} \subset \mathcal{T}$ with $\lim_{n\to\infty} t_n = \infty$,

$$\lim_{n\to\infty}\frac{R(t_n)}{t_n} = 0. \tag{3.6}$$

Rescale with $U_l(t,x) = U(t_l + R_l t, R_l x)$ so that

$$\int_{-1}^{1}\int_{D_l(t)} \|\partial_t U_l\|^2\,dx\,dt \to 0, \tag{3.7}$$

with

$$D_l(t) = \{x|\ R_l|x| \le \lambda'(t_l + R_l t)\}.$$

Then

$$\eta_0 \le E(D_l(t), t) \le E_0. \tag{3.8}$$

By Theorem 2.2 and (3.8), we have a contradiction. Therefore

$$\lim_{T\to\infty}\frac{1}{T}\int_{\frac{5}{8}T}^{\frac{7}{8}T} E(0,\lambda';t)\,dt = 0. \tag{3.9}$$

From (3.9), we can now apply Theorem 2.1 to show that U scatters. By translating in time, we can safely assume the initial condition is proscribed at $t = 1$, and write (1.1) as the following integral equation.

$$U(t,x) = \cos((t-1)\sqrt{-\Delta})U_0 + \frac{\sin((t-1)\sqrt{-\Delta})}{\sqrt{-\Delta}}U_1 + \int_1^t \frac{\sin((t-\tau)\sqrt{-\Delta})}{\sqrt{-\Delta}} B(U)(\partial_\alpha U, \partial^\alpha U)\, d\tau.$$

In order to prove scattering in the energy norm, it will suffice to show the quantity

$$\| \int_1^\infty \frac{\sin((-\tau)\sqrt{-\Delta})}{\sqrt{-\Delta}} B(U)(\partial_\alpha U, \partial^\alpha U)\, d\tau \|_{\dot{H}^1} \tag{3.10}$$

is finite.

Picking coordinates so that (2.6) applies, we can use energy estimates to bound (3.10) by

$$\|B(U)(\partial_\alpha U, \partial^\alpha U)\|_{L^1_t L^2_x} \lesssim \| |\partial_u U| |\partial_v U| \|_{L^1_t L^2_x}. \tag{3.11}$$

Applying the estimates in Theorem 2.1 to (3.11),

$$\begin{aligned} \| |\partial_u U| |\partial_v U| \|_{L^1_t L^2_x} &\lesssim \int_1^\infty \left(\int_0^\infty |\partial_u U|^2 |\partial_v U|^2 r\, dr \right)^{\frac{1}{2}} dt \\ &\lesssim \int_1^\infty \left(\int_0^\infty \frac{r\, dr}{(t+r)^4((t-r)^2+1)} \right)^{\frac{1}{2}} dt \\ &\lesssim \int_1^\infty \left(\frac{1}{t^4} \int_0^\infty \frac{r\, dr}{(t-r)^2+1} \right)^{\frac{1}{2}} dt \\ &\lesssim \int_1^\infty \frac{1}{t^{3/2}}\, dt < \infty. \end{aligned}$$

Therefore U scatters.

□

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