Sharp bounds on the rate of convergence of the empirical covariance matrix*

Radosław ADAMCZAK† Alexander E. LITVAK Alain PAJOR
Nicole TOMCZAK-JAEGERMANN‡

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Abstract

Let $X_1, \ldots, X_N \in \mathbb{R}^n$ be independent centered random vectors with log-concave distribution and with the identity as covariance matrix. We show that with overwhelming probability one has

$$\sup_{x \in S^{n-1}} \Big| \frac{1}{N} \sum_{i=1}^{N} \left(|\langle X_i, x\rangle|^2 - \mathbb{E}|\langle X_i, x\rangle|^2 \right) \Big| \leq C\sqrt{\frac{n}{N}},$$

where C is an absolute positive constant. This result is valid in a more general framework when the linear forms $(\langle X_i, x\rangle)_{i\leq N, x\in S^{n-1}}$ and the Euclidean norms $(|X_i|/\sqrt{n})_{i\leq N}$ exhibit uniformly a sub-exponential decay. As a consequence, if A denotes the random matrix with columns (X_i), then with overwhelming probability, the extremal singular values $\lambda_{\min}$ and $\lambda_{\max}$ of $AA^\top$ satisfy the inequalities $1 - C\sqrt{\frac{n}{N}} \leq \frac{\lambda_{\min}}{N} \leq \frac{\lambda_{\max}}{N} \leq 1 + C\sqrt{\frac{n}{N}}$ which is a quantitative version of Bai-Yin theorem [4] known for random matrices with i.i.d. entries.

Let $X \in \mathbb{R}^n$ be a centered random vector whose covariance matrix is the identity and X_1, ..., X_N be independent copies of X. Let A be a random $n \times N$ matrix whose columns are (X_i). By $\lambda_{\min}$ (resp. $\lambda_{\max}$) we denote the smallest (resp. the largest) singular number of the matrix of empirical covariance $AA^\top$. In the study of the local regime in the random matrix theory of particular interest is the limit behaviour of extremal values of the spectrum of $AA^\top$. In the case of Wishart Ensemble when the coordinates of X are independent, the Bai-Yin theorem [4] establishes the convergence of $\lambda_{\min}/N$ and $\lambda_{\max}/N$ when $n, N \to \infty$ and $n/N \to \beta \in (0,1)$, under the assumption of a finite fourth moment. In this note we study the asymptotic non-limit behaviour (also called "non-asymptotic" in Statistics) i.e. we look for sharp upper and lower bounds for singular values in terms of n and N, when n and N are sufficiently large. For example, for Gaussian matrices it is known that singular values satisfy inequalitites

$$1 - C\sqrt{\frac{n}{N}} \leq \frac{\lambda_{\min}}{N} \leq \frac{\lambda_{\max}}{N} \leq 1 + C\sqrt{\frac{n}{N}} \tag{1}$$

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with probability close to 1. We obtain the same estimates for large class of random matrices, which in particular do not require that entries of the matrix are independent or that X_i's are identically distributed. Note that the natural question about convergence of singular values in such a case is still open (see [2] for the case of X_i having uniform distribution on a rescaled ℓ_p^n ball).

The natural scalar product and Euclidean norm on $\mathbb{R}^n$ are denoted by $\langle \cdot, \cdot \rangle$ and $|\cdot|$. We also denote by the same notation $|\cdot|$ the cardinality of a set. By C, C_1, c etc. we will denote absolute positive constants.

Let $X_1, \ldots, X_N$ be a sequence of random vectors in $\mathbb{R}^n$ (not necessarily identically distributed). We say that it is uniformly ψ_1 if for some $\psi > 0$,

$$\sup_{i \le N} \sup_{y \in S^{n-1}} \| \, |\langle X_i, y \rangle| \, \|_{\psi_1} \le \psi, \tag{2}$$

where for a random variable $Y \in \mathbb{R}$, $\|Y\|_{\psi_1} = \inf \{C > 0 \, ; \, \mathbb{E} \exp(|Y|/C) \le 2\}$. We say that it satisfies the boundedness condition with constant K (for some $K \ge 1$) if

$$\mathbb{P}\left(\max_{i \le N} |X_i|/\sqrt{n} > K \max\{1, (N/n)^{1/4}\} \right) \le \exp\left(-\sqrt{n}\right). \tag{3}$$

The main result of this note is the following theorem.

Theorem 1 *Let N, n be positive integers and $\psi, K \ge 1$. Let $X_1, \ldots, X_N$ be independent random vectors in $\mathbb{R}^n$ satisfying (2) and (3). Then with probability at least $1 - 2\exp(-c\sqrt{n})$ one has*

$$\sup_{x \in S^{n-1}} \left| \frac{1}{N} \sum_{i=1}^{N} \left(|\langle X_i, x \rangle|^2 - \mathbb{E} |\langle X_i, x \rangle|^2 \right) \right| \le C \, (\psi + K)^2 \; \sqrt{\frac{n}{N}}.$$

Theorem 1 improves estimates obtained in [1] for log-concave isotropic vectors. There, we considered essentially the case of N proportional to n, which was sufficient to answer the question of Kannan, Lovász and Simonovits [6], however, for bigger N, the results were off by a logarithmic factor. The theorem above removes this factor completely leading to the best possible estimate for an arbitrary N, that is to an estimate of the same order as in the Gaussian case.

As a consequence, we obtain in our setting, the following quantitative version of Bai-Yin theorem [4] known for random matrices with i.i.d. entries.

Corollary 1 *Let A be a random $n \times N$ matrix, whose columns $X_1, \ldots, X_N$ are isotropic random vectors satisfying the assumptions of Theorem 1. Then with probability at least $1 - 2\exp(-c\sqrt{n})$,*

$$1 - C \, (\psi + K)^2 \sqrt{\frac{n}{N}} \le \frac{\lambda_{\min}}{N} \le \frac{\lambda_{\max}}{N} \le 1 + C \, (\psi + K)^2 \sqrt{\frac{n}{N}}.$$

To emphasize the strength of the above results we observe that conditions (2) and (3) are valid for many classes of distributions.

Example 1 Random vectors uniformly distributed on the Euclidean ball of radius $K\sqrt{n}$ clearly satisfy (3). They also satisfy (2) with $\psi = CK$.

Example 2 Log-concave isotropic random vectors in $\mathbb{R}^n$. Recall that a random vector is isotropic if its covariance matrix is the identity and it is log-concave if its distribution has a log-concave density. Such vectors satisfy (2) and (3) for appropriate absolute constants ψ and K. The boundedness condition follows from Paouris' theorem ([7]) and is explicitly written e.g., in [1], Lemma 3.1. We would like to remark that a version of Theorem 1 with a weaker probability estimate was proved by Aubrun in the case of isotropic log-concave random vectors under an additional assumption of unconditionality (see [3]).

Example 3 Any isotropic random vectors $(X_i)_{i\le N}$ in $\mathbb{R}^n$, satisfying the Poincaré inequality with constant L, i.e. such that $\mathrm{Var}(f(X_i)) \le L^2 \mathbb{E}|\nabla f(X_i))|^2$ for all compactly supported smooth functions, satisfy (2) with $\psi = CL$ and (3) with $K = CL$. The question from [5] whether all log-concave isotropic random vectors satisfy the Poincaré inequality with an absolute constant is one of the major open problems in the theory of log-concave measures.

The proof of Theorem 1 is close to arguments in Section 4.3 of [1], however it uses a choice of parameters more appropriate for the case considered here, and a new approximation argument. We need additional notations. Let $1 \le m \le N$. By U_m we denote the subset of all vectors in S^{N-1} having at most m non-zero coordinates. For an $n \times N$ matrix A we let

$$A_m = \sup_{z \in U_m} |Az|. \tag{4}$$

The main technical tool is the following result which is the "in particular" part of Theorem 3.13 from [1] in which one needs to adjust corresponding constants and to take a union bound.

Theorem 2 *Let $X_1, \ldots, X_N$ be as in Theorem 1, let A be a random $n \times N$ matrix whose columns are the X_i's. Then for every $t \ge 1$ one has*

$$\mathbb{P}\left(\exists m \quad A_m \ge C\psi t \max\{\sqrt{m}\ln\frac{2N}{m}, \sqrt{n}\} + 6\max_{i\le N}|X_i|\right) \le \exp\left(-t\sqrt{n}\right).$$

Proof of Theorem 1. We assume $N \ge n$, otherwise Theorem 2 implies Theorem 1. For $x \in S^{n-1}$ set

$$S(x) = \left|\frac{1}{N}\sum_{i=1}^{N}\left(|\langle X_i, x\rangle|^2 - \mathbb{E}|\langle X_i, x\rangle|^2\right)\right|.$$

Let $B > 0$ be a parameter which we specify later and observe that

$$\sup_{x\in S^{n-1}} S(x) \le \sup_{x\in S^{n-1}} \Bigg(\Big|\frac{1}{N}\sum_{i=1}^{N}\Big((|\langle X_i, x\rangle| \wedge B)^2 - \mathbb{E}\left(|\langle X_i, x\rangle| \wedge B\right)^2\Big)\Big|$$
$$+ \quad \frac{1}{N}\sum_{i=1}^{N}\left(|\langle X_i, x\rangle|^2 - B^2\right)\mathbf{1}_{\{|\langle X_i,x\rangle|\ge B\}} + \frac{1}{N}\mathbb{E}\sum_{i=1}^{N}\left(|\langle X_i, x\rangle|^2 - B^2\right)\mathbf{1}_{\{|\langle X_i,x\rangle|\ge B\}}\Bigg).$$

We denote the summands under the supremum by $S_1(x)$, $S_2(x)$, and $S_3(x)$, respectively.

Estimate for S_1: Given $x \in S^{n-1}$ and $i \le N$ let $Z_i = Z_i(x) = \left(|\langle X_i, x\rangle| \wedge B\right)^2 - \mathbb{E}\left(|\langle X_i, x\rangle| \wedge B\right)^2$. Then $|Z_i| \le B^2$. Moreover, since

$$\mathrm{Var}(Z_i) \le \mathbb{E}\left(|\langle X_i, x\rangle| \wedge B\right)^4 \le \mathbb{E}|\langle X_i, x\rangle|^4 \le C_1\psi^4,$$

we observe that $\sigma^2 = \frac{1}{N}\sum_{i=1}^N \mathrm{Var}(Z_i) \le C_1\psi^4$. Thus, by Bernstein's inequality

$$\mathbb{P}\left(S_1(x) \ge \theta\right) = \mathbb{P}\left(\frac{1}{N}\sum_{i=1}^N Z_i \ge \theta\right) \le \exp\left(-\frac{\theta^2 N}{2(C_1\psi^4 + B^2\theta/3)}\right).$$

It is well known that S^{n-1} admits a (1/3)-net $\mathcal{N}$ in the Euclidean metric such that $|\mathcal{N}| \le 7^n$. Then by the union bound we obtain that if

$$\theta^2 N > 8C_1\psi^4 n\ln 7 \quad \text{and} \quad \theta N > (8/3)B^2 n \ln 7 \tag{5}$$

then

$$\mathbb{P}\left(\sup_{x\in\mathcal{N}} S_1(x) \ge \theta\right) \le \exp\left(n\ln 7 - \frac{\theta^2 N}{2(C_1\psi^4 + B^2\theta/3)}\right) \le \exp\left(-\frac{\theta^2 N}{4(C_1\psi^4 + B^2\theta/3)}\right). \tag{6}$$

Estimates for S_2 and S_3: By Hölder's inequality and (2) we have, for some absolute constant $C_2 \ge 1$,

$$\sup_{x\in S^{n-1}} S_3(x) \le \frac{1}{N}\sum_{i=1}^N \sup_{x\in S^{n-1}} \|\langle X_i, x\rangle\|_4^2\, \mathbb{P}\left(|\langle X_i, x\rangle| \ge B\right)^{1/2} \le C_2\psi^2\exp(-B/\psi). \tag{7}$$

To estimate S_2, we will use the following notation

$$M = \max\{\psi^2 n, \max_{i\le N}|X_i|^2\}, \quad E_B = E_B(x) = \{i \le N\colon |\langle X_i, x\rangle| \ge B\}, \quad m = \sup_{x\in S^{n-1}} |E_B(x)|.$$

By the definition of A_m, we have for every $x \in S^{n-1}$

$$B^2|E_B| \le \sum_{i\in E_B} |\langle X_i, x\rangle|^2 \le \sup_{|E|\le m}\sum_{i\in E}|\langle X_i, x\rangle|^2 \le A_m^2,$$

which yields $B^2 m \le A_m^2$ and $NS_2(x) \le A_m^2$. Theorem 2 implies that for some absolute constant $C \ge C_2$, with probability at least $1 - \exp(-\sqrt{n})$ one has

$$B^2 m \le C\left(M + \psi^2 m\ln^2\frac{2N}{m}\right) \quad \text{and} \quad \sup_{x\in S^{n-1}} S_2(x) \le C\left(\frac{M}{N} + \psi^2\frac{m}{N}\ln^2\frac{2N}{m}\right). \tag{8}$$

Now we choose the parameters. Let $B = 2\sqrt{2C}\psi\ln(5N/n)$. Then (7) gives $S_3(x) \le C\psi^2\frac{n}{N} \le C\frac{M}{N}$ for all $x \in S^{n-1}$ and together with (8) it yields that with probability at least $1-\exp(-\sqrt{n})$ one has

$$\sup_{x\in S^{n-1}}(S_2(x) + S_3(x)) \le C\left((2M/N) + \psi^2(m/N)\ln^2(2N/m)\right).$$

It is easy to check that $M \ge \psi^2 m\ln^2(2N/m)$ on the set where (8) holds. Indeed, assume it is not so, thus $M < \psi^2 m\ln^2(2N/m)$. Then by (8) we observe that $B^2 \le 2C\psi^2\ln^2(2N/m)$, which implies

$$m \le 2N\exp(-B/\psi\sqrt{2C}) = 2n^2/25N.$$

By our hypothetical upper bound for M and since $f(m) = m\ln^2(2N/m)$ increases on $[1, 2N/e^2]$, we get

$$\psi^2 n \le M \le \psi^2(8n^2/25N)\ln^2(5N/n),$$

which is impossible.

It follows that

$$\mathbb{P}\left(\sup_{x\in S^{n-1}}(S_2(x)+S_3(x))\le 3C(M/N)\right)\ge 1-\exp(-\sqrt{n}).$$

Combining this estimate with (6), we get

$$\mathbb{P}\left(\sup_{x\in\mathcal{N}} S(x)\le \theta+3C\frac{M}{N}\right)\ge 1-\exp(-\sqrt{n})-\exp\left(-\frac{\theta^2 N}{4(C_1\psi^4+B^2\theta/3)}\right).$$

We now set $\theta = C_3\psi^2\,\sqrt{n/N}$, where C_3 is a sufficiently large absolute positive constant so that (5) is satisfied. Then using boundedness condition with constant K we obtain

$$\mathbb{P}\left(\sup_{x\in\mathcal{N}} S(x)\le \left(C_3\psi^2+3CK^2\right)\sqrt{n/N}\right)\ge 1-\exp(-\sqrt{n})-\exp\left(-cn\right)\ge 1-2\exp(-c\sqrt{n}),$$

where c is a sufficiently small positive constant. It proves the desired estimate on the (1/3)-net.

To pass from $\mathcal{N}$ to the whole sphere note that $S(x)$ can be written as $|\langle Tx,x\rangle|$, where T is a self-adjoint operator on $\mathbb{R}^n$. Thus, writing for each $x\in S^{n-1}$, $x=y+z$ with $y\in\mathcal{N}$ and $|z|\le 1/3$, we get

$$\|T\| = \sup_{x\in S^{n-1}}|\langle Tx,x\rangle|\le \sup_{y\in\mathcal{N}}|\langle Ty,y\rangle|+\frac{2}{3}\sup_{y\in\mathcal{N}}|Ty|+\sup_{|z|\le 1/3}|\langle Tz,z\rangle|\le \sup_{y\in\mathcal{N}} S(y)+\frac{7}{9}\|T\|,$$

which implies the desired estimate on the whole sphere S^{n-1}. □

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Radosław Adamczak
Institute of Mathematics,
University of Warsaw
Banacha 2, 02-097 Warszawa, Poland
R.Adamczak@mimuw.edu.pl

Alexander E. Litvak
Department of Mathematical and Statistical Sciences,
University of Alberta,
Edmonton, Alberta, Canada T6G 2G1
alexandr@math.ualberta.ca

Alain Pajor
Equipe d'Analyse et Mathématiques Appliquées,
Université Paris Est,
5 boulevard Descartes, Champs sur Marne, 77454 Marne-la-Vallee,
Cedex 2, France
alain.pajor@univ-mlv.fr

Nicole Tomczak-Jaegermann,
Department of Mathematical and Statistical Sciences,
University of Alberta,
Edmonton, Alberta, Canada T6G 2G1
nicole@ellpspace.math.ualberta.ca