$\mathbf{SO}(n)\backslash\mathbf{SO}_0(n,1)$ HAS POSITIVE CURVATURES

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Abstract. The Lie group $\mathrm{SO}_0(n,1)$ has the left-invariant metric coming from the Killing-Cartan form. The maximal compact subgroup $\mathrm{SO}(n)$ of the isometry group acts from the left. The geometry of the quotient space of the homogeneous submersion $\mathrm{SO}_0(n,1) \to \mathrm{SO}(n)\backslash\mathrm{SO}_0(n,1)$ is investigated. The space is expressed as a warped product. Its group of isometries and sectional curvatures are calculated.

0. Introduction

On the Lie group $G = \mathrm{SO}_0(n,1)$, we give a left-invariant metric which comes from the Killing-Cartan form. The maximal compact subgroup $\mathrm{SO}(n)$ $= \mathrm{SO}(n) \times \{1\}$ is denoted by K. Then the group of isometries is

$$\mathrm{Isom}_0(G) = G \times K,$$

the left translation by G and the right translation by K. Thus, there are two actions of K, $\ell(K) \subset G$ and $r(K) = K$.

The homogeneous Riemannian submersion by the isometric $r(K)$-action (which is free and proper)

$$\mathrm{SO}(n) \to \mathrm{SO}_0(n,1) \to \mathrm{SO}_0(n,1)/\mathrm{SO}(n)$$

is very well understood; $\mathrm{SO}_0(n,1)/\mathrm{SO}(n)$ is the n-dimensional hyperbolic space $\mathbb{H}^n$.

It is the purpose of this paper to study the homogeneous Riemannian submersion by the $\ell(K)$-action

$$\mathrm{SO}(n) \to \mathrm{SO}_0(n,1) \to \mathrm{SO}(n)\backslash\mathrm{SO}_0(n,1).$$

It can be seen that this space $\mathcal{H}^n = \mathrm{SO}(n)\backslash\mathrm{SO}_0(n,1)$ is diffeomorphic to $\mathbb{H}^n$, but metrically it is not as nice as the case of right actions. More specifically, it will be shown that the metric is not conformal to $\mathbb{H}^n$, and the space has fewer symmetries. The following facts will be proven:

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1. $\text{Isom}_0(\text{SO}(n)\backslash\text{SO}_0(n,1)) = r(\text{SO}(n))$, and it has one fixed point $\{\mathbf{i}\}$, (Theorem 4.4).
2. $\mathcal{H}^n - \{\mathbf{i}\}$ is a warped product $(1,\infty) \times_{e^{2\phi}} S^{n-1}$, (Theorem 4.11).
3. The sectional curvature κ satisfies: $0 < \kappa \leq 5$, and $\kappa = 5$ is achieved only at $\mathbf{i}$, (Theorem 4.15).

1. Iwasawa Decomposition

1.1. We shall establish some notation first. Let

$$J = \begin{bmatrix} -I_p & 0 \\ 0 & I_q \end{bmatrix},$$

where I_p and I_q are the identity matrices of size p and q. The group $O(p,q)$ is the subgroup of $\text{GL}(p+q,\mathbb{R})$ satisfying $AJA^t = J$. It has 4 connected components (for $p, q > 0$) and we denote the connected component of the identity by $\text{SO}_0(p,q)$. It is a semi-simple Lie group. The Iwasawa decomposition is best described on its Lie algebra. We specialize to $\text{SO}_0(n,1)$.

1.2. Let e_{ij} denote the matrix whose (i,j)-entry is 1 and 0 elsewhere. The standard metric on $\text{SO}_0(n,1)$ is given by the orthonormal basis for the Lie algebra

$$E_{ij} = \epsilon_{ij} e_{ij} + e_{ji}, \quad 1 \leq i < j \leq n+1,$$

where $\epsilon_{ij} = -1$ if $j < n+1$ and $\epsilon_{ij} = 1$ if $j = n+1$.

An Iwasawa decomposition KAN is defined as follows. Let

$$N_i = E_{i,n} + E_{i,n+1}, \text{ for } i = 1, 2, \ldots, n-1,$$

be a basis for the nilpotent Lie algebra $\mathfrak{n}$; $A_1 = E_{n,n+1}$ be a basis for the abelian $\mathfrak{a}$. The compact subalgebra $\mathfrak{k} = \mathfrak{so}(n)$ is sitting in $\mathfrak{so}(n+1)$ as blocked diagonal matrices $\mathfrak{so}(n) \oplus (0)$. For an explicit discussion of such a decomposition using positive roots, see, for example, [2].

1.3. It is well known that $NA(= AN)$ forms a (solvable) subgroup. As a Riemannian *subspace*, NA is an Einstein space; i.e., has a Ricci tensor which is proportional to the metric. However, our concern here is NA, not as a subspace, but rather as a *quotient space* of G because it provides a smooth cross-section for both $G \longrightarrow G/K$ and $G \longrightarrow K\backslash G$.

1.4. From now on, in a slight abuse of notation, '$r(K)$-action' means the right action of $K = \text{SO}(n)$ on either $\text{SO}_0(n,1)$ or $\mathcal{H}^n$ under appropriate situations. Also '$\ell(K)$-action' means the left action of $K = \text{SO}(n)$ on either $\text{SO}_0(n,1)$ or $\mathbb{H}^n$. Note that $\mathcal{H}^n$ (respectively, $\mathbb{H}^n$) does not have an $\ell(K)$-action (respectively, $r(K)$-action).

2. $\mathcal{H}^2 = \mathrm{SO}(2)\backslash\mathrm{SO}_0(2,1)$

2.1 (Metric on $\mathrm{SO}_0(2,1)$)**.** We shall study the case when $n = 2$ first, because this is the building block for the general case. The orthonormal basis for the Lie algebra $\mathfrak{so}(2,1)$ is

$$E_{13} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad E_{23} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad E_{12} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The Lie algebras for the Iwasawa decomposition are

$$\mathfrak{k} = \langle E_{12} \rangle, \quad \mathfrak{a} = \langle A_1 \rangle \quad \text{and} \quad \mathfrak{n} = \langle N_1 \rangle,$$

where

$$A_1 = E_{23} \quad \text{and} \quad N_1 = E_{13} + E_{12}.$$

The corresponding Lie subgroups are denoted by K, A and N, respectively.

2.2 (Global trivialization of $\mathcal{H}^2$)**.** In order to study $\mathrm{SO}(2)\backslash\mathrm{SO}_0(2,1)$, it is advantageous to use the notation $\mathrm{SO}_0(2,1) = NAK$ rather than KAN. That is, every element p of $\mathrm{SO}_0(2,1)$ is uniquely written as a product

$$p = nak, \quad n \in N,\ a \in A,\ k \in K.$$

The nilpotent subgroup N is normalized by A, and NA forms a subgroup. We give a global coordinate to NA by

$$\begin{aligned} \varphi :\ \mathbb{R} \times \mathbb{R}^+ &\longrightarrow NA \\ (x, y) &\mapsto e^{xN_1} e^{\ln(y) A_1}. \end{aligned} \tag{2.1}$$

Note that this is different from the restriction of the exponential map $\exp : \mathfrak{so}(2,1) \to \mathrm{SO}_0(2,1)$. Sometimes we shall suppress φ and write (x, y) for $\varphi(x, y)$.

2.3 (Comparison with $\mathrm{SL}(2,\mathbb{R})$)**.** We use the standard isomorphism of Lie algebras $\mathfrak{sl}(2,\mathbb{R})$ and $\mathfrak{so}(2,1)$, sending the basis

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \tfrac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \tfrac{1}{2}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

to the basis

$$N_1, \quad A_1, \quad -E_{12}.$$

With the above identification φ in diagram (2.1), we see the following correspondence:

$$\mathbb{R} \times \mathbb{R}^+ \ni (x, y) \longleftrightarrow \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{y} & 0 \\ 0 & \frac{1}{\sqrt{y}} \end{bmatrix} \longleftrightarrow e^{xN_1} e^{\ln(y) A_1} \in NA.$$

For the compact subgroup SO(2), this isomorphism yields a 2-to-1 covering transformation

$$\begin{bmatrix} \cos\frac{z}{2} & -\sin\frac{z}{2} \\ \sin\frac{z}{2} & \cos\frac{z}{2} \end{bmatrix} \longleftrightarrow \begin{bmatrix} \cos z & \sin z & 0 \\ -\sin z & \cos z & 0 \\ 0 & 0 & 1 \end{bmatrix} = e^{z(-E_{12})}.$$

Therefore, in order to conform with the ordinary Möbius transformations of SO(2) on the upper half-plane model, the group $\mathrm{SO}(2) \subset \mathrm{SO}_0(2,1)$ will be parametrized by $e^{z(-E_{12})}$ rather than by $e^{zE_{12}}$.

2.4 (Riemannian metric on $\mathcal{H}^2$)**.** With the Riemannian metric on $\mathrm{SO}_0(2,1)$ induced by the orthonormal basis $\{E_{13}, E_{23}, E_{12}\}$, the group of isometries is

$$\mathrm{Isom}_0(\mathrm{SO}_0(2,1)) = \mathrm{SO}_0(2,1) \times \mathrm{SO}(2).$$

The subgroup $\mathrm{SO}(2) \subset \mathrm{SO}_0(2,1)$ acts on $\mathrm{SO}_0(2,1)$ as left translations, $\ell(K)$, freely and properly, yielding a submersion. The quotient space $\mathrm{SO}(2)\backslash\mathrm{SO}_0(2,1)$ acquires a unique Riemannian metric that makes the projection, $\mathrm{proj} : \mathrm{SO}_0(2,1) \longrightarrow \mathrm{SO}(2)\backslash\mathrm{SO}_0(2,1)$, a Riemannian submersion. It has a natural smooth (non-metric) cross section NA in $KNA = NAK$.

At any $p \in \mathrm{SO}_0(2,1)$, the vector $\ell(p)_*(E_{ij})$ is just matrix multiplication pE_{ij}, and

$$\{pE_{13}, pE_{23}, pE_{12}\}$$

is an orthonormal basis at p. The isometric $\ell(K)$-action induces a homogeneous foliation on $\mathrm{SO}_0(2,1)$. The leaf passing through p is Kp, the orbit containing p. Therefore, the vertical vector is $E_{12}\,p$. We can find a new orthonormal basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, where the last vector $\mathbf{v}_3$ is the normalized $E_{12}\,p$. More explicitly, we write $E_{12}p$ as a combination of the above orthonormal basis:

$$\mathbf{u}_3 = E_{12}p = g_1(p)\ pE_{13} + g_2(p)\ pE_{23} + g_3(p)\ pE_{12},$$

and set

$$\mathbf{u}_1 = -g_3(p)\ pE_{13} + g_1(p)\ pE_{12}.$$

Then take the cross product $\mathbf{u}_3 \times \mathbf{u}_1$ as $\mathbf{u}_2$. Now normalize $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ to get $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

Thus, $\{\mathbf{v}_1, \mathbf{v}_2\}$ is an orthonormal basis for the horizontal distribution to the homogeneous foliation generated by the $\ell(K)$-action. We want the projection, $\mathrm{proj} : \mathrm{SO}_0(2,1) \to \mathcal{H}^2 = \mathrm{SO}(2)\backslash\mathrm{SO}_0(2,1)$, to be an isometry on the horizontal spaces. Since we are using the global coordinate system

$$\mathbb{R} \times \mathbb{R}^+ \xrightarrow[\cong]{\varphi} NA \xrightarrow[\cong]{\mathrm{proj}|_{NA}} \mathcal{H}^2 = \mathrm{SO}(2)\backslash\mathrm{SO}_0(2,1) \tag{2.2}$$

on $\mathcal{H}^2$, we take the projection $T_p(\mathrm{SO}_0(2,1)) = T_p(NAK) \to T_p(NA)$ for $p \in NA$. Expressing the images of $\mathbf{v}_1, \mathbf{v}_2$ by this projection in terms of

$\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\}$, we get

$$\mathbf{w}_1 = -\frac{\sqrt{\left(x^2+1\right)^2+y^4}}{\sqrt{2}y}\frac{\partial}{\partial x}\Big|_{(x,y)} - \frac{\sqrt{2}x\left(x^2+1\right)}{\sqrt{\left(x^2+1\right)^2+y^4}}\ \frac{\partial}{\partial y}\Big|_{(x,y)}$$

$$\mathbf{w}_2 = y\sqrt{\frac{2x^2y^2}{\left(x^2+1\right)^2+y^4}+1}\ \frac{\partial}{\partial y}\Big|_{(x,y)}.$$

Proposition 2.5. *The Riemannian metric on the quotient of the Riemannian submersion* $\mathrm{SO}_0(2,1) \to \mathcal{H}^2 = \mathrm{SO}(2)\backslash\mathrm{SO}_0(2,1)$ *is given by the orthonormal basis* $\{\mathbf{w}_1, \mathbf{w}_2\}$.

The space $\mathcal{H}^2$ is always assumed to have this metric.

2.6 (Subgroup NA with the left-invariant metric)**.** We mention that the left-invariant metric restricted on the subgroup NA yields a space isometric to the quotient $\mathbb{H}^2 = \mathrm{SO}_0(2,1)/\mathrm{SO}(2)$: The subgroup NA with the Riemannian metric induced from that of $\mathrm{SO}_0(2,1)$ has an orthonormal basis $\{\frac{1}{\sqrt{2}}N_1,\ A_1\}$ at the identity, while the quotient $\mathrm{SO}_0(2,1)/\mathrm{SO}(2)$ is isometric to the Lie group NA with a new left-invariant metric coming from the orthonormal basis $\{N_1,\ A_1\}$. These two are isometric by $(x,y) \mapsto (\sqrt{2}x, y)$, and have the same constant sectional curvatures -1. Similar statements are true for general n.

2.7 (Global trivialization of $\mathbb{H}^2$)**.** With the same global coordinate system

$$\mathbb{R}\times\mathbb{R}^+ \xrightarrow[\cong]{\ \varphi\ } NA \xrightarrow[\cong]{\ \mathrm{proj}|_{NA}\ } \mathbb{H}^2 = \mathrm{SO}_0(2,1)/\mathrm{SO}(2),$$

$\mathbb{H}^2$ has the orthonormal basis $\{y\frac{\partial}{\partial x}, y\frac{\partial}{\partial y}\}$ (on the plane $\mathbb{R}\times\mathbb{R}^+$). Then the projection $T_p(\mathrm{SO}_0(2,1)) = T_p(NAK) \to T_p(NA)$ for $p \in NA$ is a Riemannian submersion, that is, an isometry on the horizontal spaces.

2.8 (Special point $(0,1)$)**.** Note also that the vector fields $\{\mathbf{w}_1, \mathbf{w}_2\}$ are globally defined and smooth (including the point $(0,1)$). This fact is significant because we shall use the fact that our space with the point $(0,1)$ removed is a warped product to calculate curvatures etc. Since the curvature is a smooth function of the orthonormal basis, the curvatures at the point $(0,1)$ will simply be the limit of the curvature, $\lim_{(x,y)\to(0,1)}\kappa(x,y)$.

2.9 ($r(K)$-action on $\mathcal{H}^2$ vs. $\ell(K)$-action on $\mathbb{H}^2$)**.** Observe that $r(K)$ normalizes (in fact, centralizes) the left action $\ell(K)$, and hence, it induces an isometric action on the quotient $K\backslash G$. We need to study this isometric $r(K)$-action in detail.

First we consider the isometric action $\ell(K)$ on the hyperbolic space $\mathbb{H}^2 = G/K$. For $p \in NA$ and $k \in K$, suppose $k \cdot p = p_1 k_1$. Then $k \cdot (pK) = p_1 K$. That is

$$\ell(k) \cdot \bar{p} = \bar{p}_1 \text{ in } G/K \quad (\text{if } k \cdot p = p_1 k_1 \text{ for some } k_1).$$

Now for our $r(K)$-action on $\mathcal{H}^2 = K\backslash G$, let $p \in NA$ and $k \in K$. Suppose $p \cdot k = k_2 p_2$. Then $(Kp) \cdot k = Kp_2$. That is

$$r(k) \cdot \bar{p} = \bar{p}_2 \text{ in } K\backslash G \quad (\text{if } p \cdot k = k_2 p_2 \text{ for some } k_2).$$

Proposition 2.10. *In xy-coordinate for $\mathcal{H}^2 = \mathrm{SO}(2)\backslash \mathrm{SO}_0(2,1)$ (upper half-plane), the isometric $r(K)$-action on $\mathcal{H}^2$ is given by:*

For $\hat{z} = e^{z(-E_{12})} = \begin{bmatrix} \cos z & \sin z & 0 \\ -\sin z & \cos z & 0 \\ 0 & 0 & 1 \end{bmatrix} \in K$ *and* $(x, y) \in \mathcal{H}^2$,

$$\begin{aligned} r(\hat{z}) \cdot (x,y) = \frac{1}{2y}\Big(&-(-x^2+y^2-1)\sin z + 2xy\cos z, \\ &(-x^2+y^2-1)\cos z + 2xy\sin z + x^2+y^2+1 \Big). \end{aligned}$$

In vector notation,

$$r(\hat{z}) \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos z & -\sin z \\ \sin z & \cos z \end{bmatrix} \left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 0 \\ \frac{1+x^2+y^2}{2y} \end{bmatrix} \right) + \begin{bmatrix} 0 \\ \frac{1+x^2+y^2}{2y} \end{bmatrix}.$$

2.11. Note that $r(\hat{z})$ is a "Euclidean rotation" with an appropriate center. More precisely, each (x, y) is on the Euclidean circle centered at $\left(0, \frac{1+x^2+y^2}{2y}\right)$ with radius $\sqrt{x^2 + \left(y - \frac{1+x^2+y^2}{2y}\right)^2}$, and $r(\hat{z})$ rotates the point (x, y) along this circle. This can be seen by calculations.

The $\ell(K)$-action on $\mathbb{H}^2$ is the genuine Möbius transformation, and is given by

$$\ell(\hat{z}) \cdot (x,y) = \frac{1}{L}\Big(\big(x^2+y^2-1\big)\sin z + 2x\cos z, 2y \Big)$$

with

$$L = -\big(x^2+y^2-1\big)\cos z + 2x\sin z + x^2 + y^2 + 1.$$

The relation between $r(K)$-action on $\mathcal{H}^2$ and $\ell(K)$-action on $\mathbb{H}^2$ will be stated in Proposition 3.1 more clearly.

2.12. Both $r(K)$- and $\ell(K)$-actions have a unique fixed point at $(0,1)$, and all the other orbits are Euclidean circles centered on the y-axis. This implies that the geometry is completely determined by the geometry at the points on the y-axis (more economically, on the subset $[1,\infty)$ of the y-axis). The orthonormal bases at the points of y-axis are important. From Proposition 2.5, we have

Corollary 2.13. *At $(0,y)\in\mathcal{H}^2$ with $y>1$, the orthonormal system is*

$$\begin{aligned}\mathbf{w}_1 &= -\sqrt{\cosh(2\ln y)}\frac{\partial}{\partial x}\Big|_{(0,y)}\\ \mathbf{w}_2 &= y\frac{\partial}{\partial y}\Big|_{(0,y)}.\end{aligned}$$

With the orthonormal basis on the upper half-plane model given in Proposition 2.5, we can calculate the sectional curvature.

Theorem 2.14. *On the space $\mathcal{H}^2=\mathrm{SO}(2)\backslash\mathrm{SO}_0(2,1)$, the sectional curvature at (x,y) is*

$$\kappa(x,y)=\frac{4y^2\left(x^4+2x^2\left(y^2+1\right)+y^4+3y^2+1\right)}{\left(x^4+2x^2\left(y^2+1\right)+y^4+1\right)^2}.$$

In particular, $0<\kappa\le 5$ and the maximum 5 is attained at the point $(0,1)$.

2.15. Note that, because of the isometric $r(K)$-action (see Proposition 2.10), it is enough to know the curvatures at the points on the y-axis,

$$\kappa(0,y)=\frac{4y^2(1+3y^2+y^4)}{(1+y^4)^2}.$$

As we shall see in Proposition 3.1, the $r(K)$-orbits will be the geometric concentric circles centered at $(0,1)$. These are Euclidean circles with different centers, see Proposition 2.10. Over these $r(K)$-orbits, $\kappa(x,y)$ is constant, of course. In fact, on the geometric circle of radius $|\ln y|$, the curvature is $\kappa(0,y)$.

Here are graphs of the sectional curvatures. Figure 1(a) shows that $\kappa=5$ is the maximum at $(0,1)$. The level curves are the geometric circles centered at $(0,1)$ of $\mathcal{H}^2$.

3. $\mathcal{H}^2=\mathrm{SO}(2)\backslash\mathrm{SO}_0(2,1)$ vs. $\mathbb{H}^2=\mathrm{SO}_0(2,1)/\mathrm{SO}(2)$

Recall that both spaces $\mathrm{SO}(2)\backslash\mathrm{SO}_0(2,1)$ and $\mathrm{SO}_0(2,1)/\mathrm{SO}(2)$ have isometric actions by circles, $r(K)$ and $\ell(K)$, respectively. The trivialization functions φ and $\mathrm{proj}|_{NA}\circ\varphi$ in diagram (2.2) will be suppressed sometimes.

From the weak G-equivariant diffeomorphism from $K\backslash G$ to G/K given by $Kg\mapsto g^{-1}K$, we can define $\tau:\mathbb{R}\times\mathbb{R}^+\longrightarrow\mathbb{R}\times\mathbb{R}^+$ as in the following

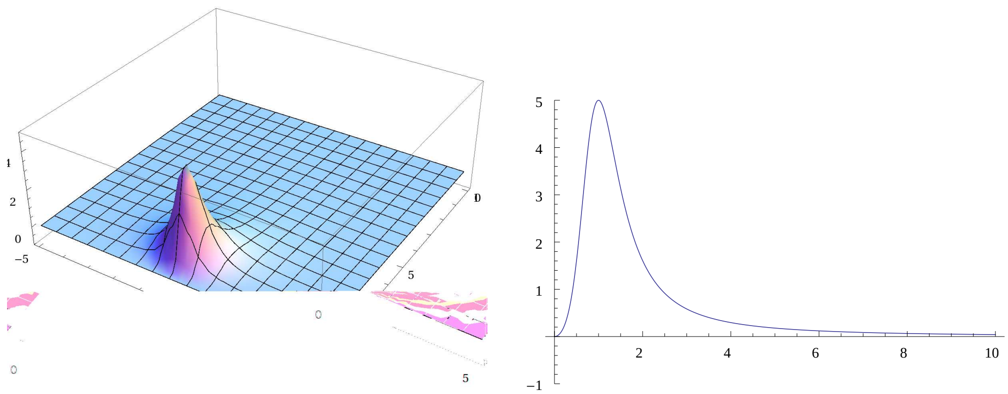

FIGURE 1. κ for $(-5 < x < 5,\ 0 < y < 10)$, and the cross section at $x = 0$

Proposition 3.1. *For $x \in \mathbb{R}^1$ and $y \in \mathbb{R}^+$, $(0,y)(x,1)(0,y)^{-1} = (yx,1)$ (with the notation in the diagram (2.2)) so that*

$$(x,y)^{-1} = (-\tfrac{x}{y}, \tfrac{1}{y}).$$

The map

$$\tau:\ \mathcal{H}^2 = \mathrm{SO}(2)\backslash\mathrm{SO}_0(2,1) \longrightarrow \mathbb{H}^2 = \mathrm{SO}_0(2,1)/\mathrm{SO}(2)$$

(as a map $\mathbb{R}\times\mathbb{R}^+ \longrightarrow \mathbb{R}\times\mathbb{R}^+$) defined by

$$\tau(x,y) = (-\tfrac{x}{y}, \tfrac{1}{y})$$

has the following properties:

(1) *τ is a weakly $\mathrm{SO}(2)$-equivariant diffeomorphism of period 2. More precisely,*

$$\tau(r(\hat{z})\cdot(x,y)) = \ell(\hat{z}^{-1})\cdot\tau(x,y)$$

for $\hat{z} \in \mathrm{SO}(2)$. In other words, the identification of $\mathcal{H}^2$, $\mathbb{H}^2$ and NA with $\mathbb{R}\times\mathbb{R}^+$ as sets permits some abuse of τ and gives the following relation between $r(K)$-action and $\ell(K)$-action: $r(\hat{z})\cdot(x,y) = \tau\left(\ell(\hat{z}^{-1})\cdot\tau(x,y)\right)$.

(2) *τ leaves the geometric circles centered at $(0,1)$ in each geometry invariant. That is, for $m > 0$, the Euclidean circle*

$$x^2 + (y - \cosh(\ln m))^2 = \sinh^2(\ln m)$$

is a geometric circle centered at $(0,1)$ with radius $|\ln m|$, in both geometries, and the map τ maps such a circle to itself. These circles are $r(K)$-orbits in $\mathcal{H}^2$ and $\ell(K)$-orbits in $\mathbb{H}^2$ (when $\mathcal{H}^2$ and $\mathbb{H}^2$ are identified with $\mathbb{R}\times\mathbb{R}^+$) at the same time.

(3) *τ gives a 1-1 correspondence between the two sets of all the geodesics passing through $(0,1)$ in the two geometries $\mathcal{H}^2$ and $\mathbb{H}^2$. In fact, τ maps the y-axis to itself and half-circles $\{(x-\alpha)^2 + y^2 = \alpha^2 + 1\}_{\alpha\in\mathbb{R}}$ to hyperbolas $\{x^2 + 2\alpha xy - y^2 + 1 = 0\}_{\alpha\in\mathbb{R}}$.*

Proof. (1) Observe that $\tau(x, y)$ corresponds to the inverse of $\varphi(x, y)$ in the group NA. In fact, we have

$$\varphi(\tau(x, y)) = (\varphi(x, y))^{-1}.$$

For $\varphi(x, y) \in NA$ and $\hat{z} \in K$, one can find $k \in K$ for which

$$k \cdot \varphi(x, y) \cdot \hat{z} \in NA.$$

Thus,

$$\begin{aligned} \varphi\big(\tau(r(\hat{z}) \cdot (x, y))\big) &= \big(\varphi(r(\hat{z}) \cdot (x, y))\big)^{-1} \\ &= (k \cdot \varphi(x, y) \cdot \hat{z}\big)^{-1} \\ &= \hat{z}^{-1} \cdot (\varphi(x, y))^{-1} \cdot k^{-1} \\ &= \hat{z}^{-1} \cdot \big(\varphi(\tau(x, y))\big) \cdot k^{-1} \\ &= \varphi\big(\ell(\hat{z})^{-1} \cdot \tau(x, y)\big). \end{aligned}$$

(2) Any (x, y) lies on the Euclidean circle centered at $(0, c)$, where $c = \frac{1+x^2+y^2}{2y}$ and radius $r = \sqrt{x^2 + \left(y - \frac{1+x^2+y^2}{2y}\right)^2}$. In particular, $(0, m)$ lies on the Euclidean circle centered at $(0, c)$, where $c = \cosh(\ln m)$ and radius $r = |\sinh(\ln m)|$. Note $m = \cosh(\ln m) + \sinh(\ln m)$ and $\frac{1}{m} = \cosh(\ln m) - \sinh(\ln m)$, which show that both $(0, m)$ and $(0, \frac{1}{m})$ lie on the same circle. Then, in $\mathbb{R} \times \mathbb{R}^+$,

$$r(K) \cdot (0, m) = \tau(\ell(K) \cdot (0, \tfrac{1}{m})) = \ell(K) \cdot (0, \tfrac{1}{m}) = \ell(K) \cdot (0, m)$$

shows this circle is both $r(K)$-orbit of the point $(0, m)$ (in $\mathcal{H}^2$) and its $\ell(K)$-orbit (in $\mathbb{H}^2$) at the same time. Since both $r(K)$-action on $\mathcal{H}^2$ and $\ell(K)$-action on $\mathbb{H}^2$ are isometric, every point on the circle has the same distance from $(0, 1)$ in each geometry.

In xy-coordinates, the equations for geodesics in $\mathcal{H}^2$ are a system of 2 equations

$$\begin{aligned} 0 =& x''(t)\Big(2x(t)^2 y(t)^3 + \left(x(t)^2 + 1\right)^2 y(t) + y(t)^5\Big)^2 \\ &- 2y(t)x'(t)y'(t)\Big(x(t)^6\left(4y(t)^2 + 2\right) + x(t)^4\left(6y(t)^4 + 8y(t)^2\right)\Big) \\ &- 2y(t)x'(t)y'(t)\Big(2x(t)^2\left(2y(t)^6 + y(t)^4 + 2y(t)^2 - 1\right) + x(t)^8 + y(t)^8 - 1\Big) \\ &- 4x(t)y(t)^2 x'(t)^2\left(x(t)^2 + 1\right)^2 \\ &+ x(t)y'(t)^2\left(4\left(x(t)^2 + 1\right)y(t)^6 + 2\left(3x(t)^4 + 4x(t)^2 + 1\right)y(t)^4\right) \\ &+ x(t)y'(t)^2\Big(4\left(x(t)^2 + 1\right)^3 y(t)^2 + \left(x(t)^2 + 1\right)^4 + y(t)^8\Big) \end{aligned}$$

and

$$\begin{aligned}0 =& y(t)\left(2x(t)^2\left(y(t)^2+1\right)+x(t)^4+y(t)^4+1\right)^2 y''(t)\\ &-4x(t)y(t)x'(t)y'(t)\left(x(t)^4\left(3y(t)^2+1\right)+x(t)^2\left(3y(t)^4+4y(t)^2-1\right)\right)\\ &-4x(t)y(t)x'(t)y'(t)\left(x(t)^6+y(t)^6+y(t)^4+y(t)^2-1\right)\\ &+2y(t)^2x'(t)^2\left(3x(t)^4\left(y(t)^2-1\right)+x(t)^2\left(y(t)^2+1\right)\left(3y(t)^2-5\right)\right)\\ &+2y(t)^2x'(t)^2\left(x(t)^6+y(t)^6-y(t)^4-y(t)^2-1\right)\\ &+y'(t)^2\left(2x(t)^6\left(y(t)^2+1\right)+4x(t)^4y(t)^2-2x(t)^2\left(y(t)^6+y(t)^4-y(t)^2+1\right)\right)\\ &+y'(t)^2\left(x(t)^8-\left(y(t)^4+1\right)^2\right).\end{aligned}$$

One can readily check that

$$\gamma(t)=(0,e^t)\in\mathcal{H}^2,\ 0\le t\le|\ln(m)|=|\ln(\tfrac{1}{m})|$$

is a unit-speed geodesic, and therefore, $\mathrm{Length}(\gamma)=|\ln(m)|$. This is the geometric radius of the circle centered at $\mathbf{i}=(0,1)\in\mathcal{H}^2$.

(3) Let $\mathcal{G}_{\mathcal{H}^2}$ and $\mathcal{G}_{\mathbb{H}^2}$ be the sets of all the unit-speed geodesics starting from $\mathbf{i}$ in $\mathcal{H}^2$ and $\mathbb{H}^2$, respectively. Then

$$\mathcal{G}_{\mathcal{H}^2}=\{r(k)\cdot\gamma(\bullet):\mathbb{R}\longrightarrow\mathcal{H}^2\}_{k\in K}$$

and

$$\mathcal{G}_{\mathbb{H}^2}=\{l(k)\cdot\gamma(\bullet):\mathbb{R}\longrightarrow\mathbb{H}^2\}_{k\in K},$$

since $\gamma\in\mathcal{G}_{\mathcal{H}^2}\cap\mathcal{G}_{\mathbb{H}^2}$.

The 1-1 correspondence between $\mathcal{G}_{\mathcal{H}^2}$ and $\mathcal{G}_{\mathbb{H}^2}$ by τ comes from the weak equivariance of τ and the fact $\tau(\gamma(t))=\gamma(-t)$. In fact, for $k\in K$ and $t\in\mathbb{R}$,

$$r(k)\cdot\gamma(t)=\tau(\ell(k^{-1})\cdot\tau(\gamma(t)))=\tau(\ell(k^{-1})\cdot\gamma(-t))=\tau(\ell(k^{-1})\cdot\ell(\hat{\pi})\cdot\gamma(t)).$$

Finally, we can check easily that for each $\alpha\in\mathbb{R}$, the hyperbola $x^2+2\alpha xy-y^2+1=0$, a $\mathcal{H}^2$-geodesic, corresponds to the half-circle $(x-\alpha)^2+y^2=\alpha^2+1$, a $\mathbb{H}^2$-geodesic. □

Theorem 3.2. *The space $\mathcal{H}^2-\{\mathbf{i}\}$ is isometric to the warped product $B\times_{e^{2\phi}}S^1$, where $B=(1,\infty)=\{(0,y):\ 1<y<\infty\}\subset\mathcal{H}^2$ has the induced metric; that is, $|\frac{\partial}{\partial t}(t_0)|=\frac{1}{t_0}$ for $t_0\in(1,\infty)$, S^1 has the standard metric; and $e^{2\phi(t)}=\frac{\sinh^2(\ln t)}{\cosh(2\ln t)}$.*

Proof. The crucial points are that $r(K)\subset\mathrm{Isom}(\mathcal{H}^2)$ and that all the other orbits are circles, except for the one fixed point $\mathbf{i}=(0,1)$. This will make our space a warped product of S^1 by the base space B, and we need to find a map ϕ in $B\times_{e^{2\phi}}S^1$. The $r(K)$-orbit through $(0,y)\in\mathcal{H}^2$ is, by Proposition 2.10,

$$r(\hat{z})\cdot(0,y)=\big(-\sinh(\ln y)\sin z,\ \sinh(\ln y)\cos z+\cosh(\ln y)\big).$$

Define a map

$$f:B\times_{e^{2\phi}}S^1\longrightarrow\mathcal{H}^2=\mathrm{SO}(2)\backslash\mathrm{SO}_0(2,1)$$

by

$$\begin{aligned} f(t,\hat{z}) &= f(t,\hat{z}\cdot\hat{0}) \\ &= r(\hat{z}^{-1})\cdot(0,t) \\ &= r(\widehat{-z})\cdot(0,t) \\ &= \big(\sinh(\ln t)\sin z,\ \sinh(\ln t)\cos z+\cosh(\ln t)\big). \end{aligned}$$

Note the definition of f does not depend on $e^{2\phi}$ and it is weakly equivariant with the $r(\mathrm{SO}(2))$-action without the concept of isometry yet. Since f maps the base $B\times\hat{0}$ of the warped product to the y-axis of $\mathcal{H}^2$, it is enough to find $e^{2\phi}$ which makes f isometric on $B\times\hat{0}$.

Recall that $\mathcal{H}^2$ has an orthonormal basis

$$\left\{-\sqrt{\cosh(2\ln t)}\frac{\partial}{\partial x}\Big|_{(0,t)},\ t\frac{\partial}{\partial y}\Big|_{(0,t)}\right\}$$

at $f(t,\hat{0})=(0,t)$, $t>1$, see Corollary 2.13. Also note that the metric on $B\times_{e^{2\varphi}}S^1$ is given by the orthonormal basis

$$\left\{t\frac{\partial}{\partial t}\Big|_{(t,\hat{z})},\ -e^{-\phi(t)}\frac{\partial}{\partial\hat{z}}\Big|_{(t,\hat{z})}\right\}$$

at $(t,\hat{z})$. Observe

$$\begin{aligned} f_*\left(\frac{\partial}{\partial t}\Big|_{(t,\hat{0})}\right) &= \frac{d(f\circ t)}{dt}\Big|_{(t,\hat{0})} \\ &= \frac{\partial}{\partial t}\big(f(t,\hat{z})\big)\Big|_{z=0} \\ &= \frac{1}{t}\Big(\cosh(\ln t)\sin z\frac{\partial}{\partial x}\Big|_{f(t,\hat{z})} \\ &\qquad + \big(\cosh(\ln t)\cos z+\sinh(\ln t)\big)\frac{\partial}{\partial y}\Big|_{f(t,\hat{z})}\Big)\Big|_{z=0} \\ &= \frac{\partial}{\partial y}\Big|_{f(t,\hat{0})} \\ &= \frac{\partial}{\partial y}\Big|_{(0,t)} \end{aligned}$$

and, we have

$$f_*\left(t\frac{\partial}{\partial t}\Big|_{(t,\hat{0})}\right) = t\frac{\partial}{\partial y}\Big|_{(0,t)}.$$

Thus, if

$$f_*\left(e^{-\phi(t)}\frac{\partial}{\partial\hat{z}}\Big|_{(t,\hat{0})}\right) = -\sqrt{\cosh\left(2\ln t\right)}\frac{\partial}{\partial x}\Big|_{(0,t)}, \tag{3.1}$$

then f will be an isometry. Now,

$$
\begin{aligned}
(3.2)\qquad f_*\left(e^{-\phi(t)}\frac{\partial}{\partial\hat z}\Big|_{(t,\hat 0)}\right) &= e^{-\phi(t)}\frac{d(f\circ\hat z)}{d\hat z}\Big|_{(t,\hat 0)}\\
&= e^{-\phi(t)}\Big(\sinh(\ln t)\cos z\frac{\partial}{\partial x}\Big|_{f(t,\hat z)}\\
&\qquad -\sinh(\ln t)\sin z\frac{\partial}{\partial y}\Big|_{f(t,\hat z)}\Big)\Big|_{z=0}\\
&= e^{-\phi(t)}\sinh(\ln t)\frac{\partial}{\partial x}\Big|_{(0,t)}.
\end{aligned}
$$

From the equalities (3.1) and (3.2), the condition is then

$$-\sqrt{\cosh(2\ln t)} = e^{-\phi(t)}\sinh(\ln t),$$

which implies $e^{2\phi(t)} = \frac{\sinh^2(\ln t)}{\cosh(2\ln t)}$. □

We calculate $\kappa(0,y)$ again using the warped product. The result conforms with Theorem 2.14.

Corollary 3.3. *For* $(t,0)\in B\times_{e^{2\phi}} S^1$,

$$\kappa(t,0) = \frac{4t^2(1+3t^2+t^4)}{(1+t^4)^2}.$$

Proof. From $e^{2\phi(t)} = \frac{\sinh^2(\ln t)}{\cosh(2\ln t)}$, we get

$$\{t\tfrac{\partial}{\partial t}|_{(t,0)}, -\tfrac{\sqrt{\cosh(2\ln t)}}{\sinh(\ln t)}\tfrac{\partial}{\partial\hat z}|_{(t,0)}\}$$

is an orthonormal basis at $(t,0)\in B\times_{e^{2\phi}} S^1$ and

$$\phi(t) = \ln(\sinh(\ln t)) - \tfrac12\ln(\cosh(2\ln t)).$$

Since ϕ is constant along each circle,

$$
\begin{aligned}
\nabla\phi|_{(t,0)} &= \langle\nabla\phi, t\tfrac{\partial}{\partial t}\rangle\, t\tfrac{\partial}{\partial t}\,|_{(t,0)}\\
&= (t\tfrac{\partial\phi}{\partial t})\, t\tfrac{\partial}{\partial t}\,|_{(t,0)}\\
&= (\coth(\ln t) - \tanh(2\ln t))\, t\tfrac{\partial}{\partial t}\,|_{(t,0)}\ .
\end{aligned}
$$

For tangent vectors $T_1, T_2\in T(S^1)$ and $X\in T(S^1)^\perp$ in the warped product, we have

$$R(X,T)Y = \big(h_\phi(X,Y) + \langle\nabla\phi, X\rangle\langle\nabla\phi, Y\rangle\big)T$$

and so

$$\langle R(X,T)T, Y\rangle_\phi = -e^{2\phi}|T|^2_{S^1}\big(h_\phi(X,X) + \langle\nabla\phi, X\rangle^2\big),$$

where h_ϕ is a hessian form, see [1, p.60, Proposition2.2.2, Corollary 2.2.1]. Since

$$\begin{aligned} h_\phi(t\tfrac{\partial}{\partial t}, t\tfrac{\partial}{\partial t}) &= \langle \nabla_{t\frac{\partial}{\partial t}} \nabla\phi,\, t\tfrac{\partial}{\partial t}\rangle \\ &= -\operatorname{csch}^2(\ln t) - 2\operatorname{sech}^2(2\ln t) \\ &= -\operatorname{csch}^2(\ln t) - 2 + 2\tanh^2(2\ln t), \end{aligned}$$

$$\begin{aligned} \kappa(t\tfrac{\partial}{\partial t}, -\tfrac{\sqrt{\cosh(2\ln t)}}{\sinh(\ln t)}\tfrac{\partial}{\partial \hat{z}}) &= -\big(\langle \nabla\phi, t\tfrac{\partial}{\partial t}\rangle^2 + h_\phi(t\tfrac{\partial}{\partial t}, t\tfrac{\partial}{\partial t})\big) \\ &= 1 - 3\tanh^2(2\ln t) - 2\coth(\ln t)\cdot\tanh(2\ln t) \\ &= \tfrac{4t^2(1+3t^2+t^4)}{(1+t^4)^2}. \end{aligned}$$ □

Remark 3.4. The following are well known: the space $\mathbb{H}^2 - \{\mathbf{i}\}$ is isometric to the warped product $(0,1)\times_{e^{2\psi}} S^1$, where $(0,1)\subset\mathbb{H}^2$ has the induced metric from $\mathbb{H}^2$, that is, $|\frac{\partial}{\partial t}(t_0)| = \frac{1}{t_0}$ for $t_0\in(0,1)$; S^1 has the standard metric; and $e^{2\psi(t)} = \sinh^2(\ln t)$.

The isometry can be given by

$$\tilde{f} : (0,1)\times_{e^{2\psi}} S^1 \longrightarrow \mathbb{H}^2 - \{\mathbf{i}\}$$

defined by

$$\tilde{f}(s,\hat{u}) = \ell(\hat{u})\cdot(0,s).$$

See, for example, [1, p.58, Theorem 2.2.1].

Corollary 3.5. *The map τ induces a map on the warped products*

$$\tau' : (1,\infty)\times_{e^{2\phi}} S^1 \longrightarrow (0,1)\times_{e^{2\psi}} S^1$$

given by

$$\tau'(t,\hat{z}) = (\tfrac{1}{t},\hat{z}),$$

which is SO(2)*-equivariant and satisfies $\tilde{f}\circ\tau' = \tau\circ f$.*

The following commutative diagram shows more detail:

$$\begin{array}{ccc} (1,\infty)\times_{e^{2\phi}} S^1 & \xrightarrow{\tau'} & (0,1)\times_{e^{2\psi}} S^1 \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle \tilde{f}} \\ \mathcal{H}^2 = \mathrm{SO}(2)\backslash\mathrm{SO}_0(2,1) & \xrightarrow{\tau} & \mathbb{H}^2 = \mathrm{SO}_0(2,1)/\mathrm{SO}(2) \end{array}$$

$$\begin{array}{ccc} (t,\hat{z}\cdot\hat{0}) = (t,\hat{z}) & \xrightarrow{\tau'} & (\tfrac{1}{t},\hat{z}) = (\tfrac{1}{t},\hat{z}\cdot\hat{0}) \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle \tilde{f}} \\ r(\widehat{-z})\cdot(0,t) & \xrightarrow{\tau} & \ell(\hat{z})\cdot(-\tfrac{0}{t},\tfrac{1}{t}) \end{array}$$

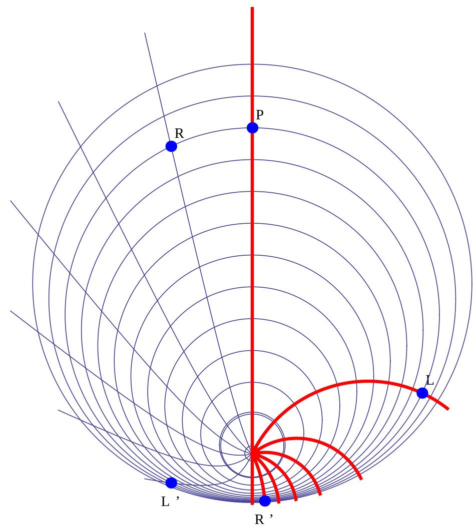

FIGURE 2. Geometric circles and orthogonal geodesics in two geometries. $R = r(\frac{\hat{\pi}}{7}) \cdot P$, $L = \ell(\frac{\hat{\pi}}{7}) \cdot P$ and $R' = \tau(R)$, $L' = \tau(L)$.

4. THE GENERAL CASE: $\mathrm{SO}(n)\backslash\mathrm{SO}_0(n,1)$

4.1 (Subgroup NA with the left-invariant metric)**.** As is well known, the subgroup NA has the structure of a solvable Lie group $N \rtimes A$, where

$$N \cong \mathbb{R}^{n-1}, \quad A \cong \mathbb{R}^{+}.$$

The subgroup NA with the Riemannian metric induced from that of $\mathrm{SO}_0(n,1)$ has an orthonormal basis

$$\{\tfrac{1}{\sqrt{2}}N_1,\ \tfrac{1}{\sqrt{2}}N_2,\ \ldots,\ \tfrac{1}{\sqrt{2}}N_{n-1},\ A_1\}.$$

at the identity while the quotient $\mathrm{SO}_0(n,1)/\mathrm{SO}(n)$ is isometric to the Lie group NA with a new left-invariant metric coming from the orthonormal basis

$$\{N_1,\ N_2,\ \ldots,\ N_{n-1},\ A_1\}.$$

These two are isometric by $(\mathbf{x}, y) \mapsto (\sqrt{2}\mathbf{x}, y)$, and have the same constant sectional curvatures -1.

4.2 (Global trivialization of $\mathbb{H}^n$)**.** With the Riemannian metric on $\mathrm{SO}_0(n,1)$ induced by the orthonormal basis $\{E_{ij} :\ 1 \leq i < j \leq n+1\}$, the group of isometries is

$$\mathrm{Isom}_0(\mathrm{SO}_0(n,1)) = \mathrm{SO}_0(n,1) \times \mathrm{SO}(n).$$

The subgroup $\mathrm{SO}(n) \subset \mathrm{SO}_0(n,1)$ acts on $\mathrm{SO}_0(n,1)$ as left translations, $\ell(K)$, freely and properly, yielding a submersion. The quotient space $\mathrm{SO}(n)\backslash\mathrm{SO}_0(n,1)$ acquires a unique Riemannian metric that makes the projection, proj :

$\mathrm{SO}_0(n,1) \longrightarrow \mathrm{SO}(n)\backslash\mathrm{SO}_0(n,1)$, a Riemannian submersion. It has a natural smooth (non-metric) cross section NA in $KNA = NAK$.

A map

$$\begin{aligned} \varphi :\ \mathbb{R}^{n-1} \times \mathbb{R}^{+} &\longrightarrow NA \\ (\mathbf{x}, y) &\mapsto e^{\sum_{i=1}^{n-1} x_i N_i} e^{\ln(y) A_1}, \end{aligned}$$

where $\mathbf{x} = (x_1, \ldots, x_{n-1})$, gives rise to a global trivialization for the subgroup NA and our space $\mathcal{H}^n$. Thus, we shall use $(\mathbf{x}, y)$ to denote a point in $\mathcal{H}^n \cong NA$.

4.3. Note, for $\mathbf{x} \in \mathbb{R}^{n-1}$ and $y \in \mathbb{R}^{+}$,

$$(\mathbf{0}, y)(\mathbf{x}, 1)(\mathbf{0}, y)^{-1} = (y\mathbf{x}, 1).$$

Even though we use the local trivialization $\mathcal{H}^n = \mathrm{SO}(n)\backslash\mathrm{SO}_0(n,1) \to NA$, the metric on $\mathcal{H}^n$ is not related to the group structure of NA. That is, the metric is neither left-invariant nor right-invariant.

Theorem 4.4. $\mathrm{Isom}_0(\mathrm{SO}(n)\backslash\mathrm{SO}_0(n,1)) = r(\mathrm{SO}(n))$.

Proof. The normalizer of $\ell(\mathrm{SO}(n))$ in $\mathrm{Isom}_0(\mathrm{SO}(n,1)) = \ell(\mathrm{SO}_0(n,1)) \times r(\mathrm{SO}(n))$ is $\ell(\mathrm{SO}(n)) \times r(\mathrm{SO}(n))$. Since $\ell(\mathrm{SO}(n))$ acts ineffectively on the quotient, only $r(\mathrm{SO}(n))$ acts effectively on the quotient as isometries. Thus, $\mathrm{Isom}_0(\mathrm{SO}(n)\backslash\mathrm{SO}_0(n,1)) \supset r(\mathrm{SO}(n))$.

Suppose these are not equal. Then there exists a point whose orbit contains an open subset, since the $r(\mathrm{SO}(n))$-orbits are already codimension 1. This implies the sectional curvature is constant on such an open subset. But this is impossible by Theorem 4.15. Notice that, for the calculation of the sectional curvature, we only need the inequality above. □

For $a \in A$ and $k \in \mathrm{SO}(n-1) \times \mathrm{SO}(1) \subset K = \mathrm{SO}(n)$,

$$ak = ka$$

and

$$(Ka) \cdot k = Kka = Ka$$

so that the stabilizer of $r(\mathrm{SO}(n))$ at $a = \varphi(\mathbf{0}, y),\ y \neq 1,\ y \in \mathbb{R}^{+}$, contains $\mathrm{SO}(n-1) \times \mathrm{SO}(1)$. Let S be the only subgroup of $K = \mathrm{SO}(n)$ properly containing $\mathrm{SO}(n-1) \times \mathrm{SO}(1)$. Then $\mathrm{SO}(n-1) \times \mathrm{SO}(1)$ has index 2 in S, and no element of $S - \mathrm{SO}(n-1) \times \mathrm{SO}(1)$ can fix a. Thus, we have

Corollary 4.5. *For the $r(\mathrm{SO}(n))$-action on $\varphi^{-1}(NA) = \mathbb{R}^{n-1} \times \mathbb{R}^{+}$, the stabilizer at $(\mathbf{0}, y),\ y \neq 1$, is $\mathrm{SO}(n-1) \times \mathrm{SO}(1)$.*

This can also be proved from the similar fact on $\mathbb{H}^n$ using the weak $\mathrm{SO}(n)$-equivariant map.

4.6 (Embedding of $\mathrm{SO}_0(2,1)$ into $\mathrm{SO}_0(n,1)$)**.** Consider the subgroup $\mathrm{SO}_0(2,1)$ of $\mathrm{SO}_0(n,1)$, as

$$I_{n-2} \times \mathrm{SO}_0(2,1) \subset \mathrm{SO}_0(n,1),$$

where I_{n-2} is the identity matrix of size $n-2$. For $k \in \ell(K)$ and $p \in \mathrm{SO}_0(2,1)$, $k \cdot p \in \mathrm{SO}_0(2,1)$ if and only if $k \in \mathrm{SO}_0(2,1)$. Therefore, the space $\ell(\mathrm{SO}(2))\backslash\mathrm{SO}_0(2,1)$ is isometrically embedded into $\ell(K)\backslash\mathrm{SO}_0(n,1)$. With this embedding, there is an orthonormal basis for this 2-dimensional subspace:

$$\begin{aligned}
\mathbf{w}_{n-1} &= c\,\frac{\partial}{\partial x_{n-1}}\Big|_{(\mathbf{0},y)}\\
\mathbf{w}_n &= y\frac{\partial}{\partial y}\Big|_{(\mathbf{0},y)}
\end{aligned}$$

where $c = -\sqrt{\cosh(2\ln y)}$.

4.7 (Orthonormal basis of $\mathcal{H}^n$)**.** The right action of a matrix $k = \exp(\frac{\pi}{2} \cdot E_{j,n-1}) \in K$ $(j < n)$ maps $(\mathbf{x}, y) = (x_1, \dots, x_j, \dots,\ x_{n-1},\ y) \in \mathcal{H}^n$ to $(\mathbf{x}', y) = (x_1, \dots, x_{n-1}, \dots, -x_j, y) \in \mathcal{H}^n$. (i.e., exchanges the $(n-1)$st and jth slot). More precisely, $\varphi(\mathbf{x}, y) \cdot k = k' \cdot \varphi(\mathbf{x}', y)$ in $\mathrm{SO}_0(n,1)$ for some $k' \in \mathrm{SO}(n)$. By applying such a right action on NA for $j = 1, 2, \dots, n-2$, we get the orthonormal system at $(0, \dots, 0, y) \in \mathcal{H}^n$ with $y > 1$:

$$\begin{aligned}
\mathbf{w}_1 &= c\,\frac{\partial}{\partial x_1}\Big|_{(\mathbf{0},y)}\\
\mathbf{w}_2 &= c\,\frac{\partial}{\partial x_2}\Big|_{(\mathbf{0},y)}\\
\mathbf{w}_3 &= c\,\frac{\partial}{\partial x_3}\Big|_{(\mathbf{0},y)}\\
&\cdots\\
\mathbf{w}_{n-1} &= c\,\frac{\partial}{\partial x_{n-1}}\Big|_{(\mathbf{0},y)}\\
\mathbf{w}_n &= y\,\frac{\partial}{\partial y}\Big|_{(\mathbf{0},y)}
\end{aligned}$$

where $c = -\sqrt{\cosh(2\ln y)}$. As before, we denote the upper half-space $\mathbb{R}^{n-1} \times \mathbb{R}^+$ with this metric by $\mathcal{H}^n$. The above shows that the metric is very close to being conformal to the standard $\mathbb{R}^n$.

4.8. Recall that both spaces $\mathcal{H}^n = \mathrm{SO}(n)\backslash\mathrm{SO}_0(n,1)$ and $\mathbb{H}^n = \mathrm{SO}_0(n,1)/\mathrm{SO}(n)$ have isometric actions by the maximal compact subgroup, $r(K)$ and $\ell(K)$, respectively. The latter has more isometries, $\ell(\mathrm{SO}_0(n,1))$.

Proposition 4.9. *The map*

$$\tau:\ \mathcal{H}^n \longrightarrow \mathbb{H}^n$$

(as a map $\mathbb{R}^{n-1}\times\mathbb{R}^+ \longrightarrow \mathbb{R}\times\mathbb{R}^+$) defined by

$$\tau(\mathbf{x},y) = (-\tfrac{\mathbf{x}}{y},\tfrac{1}{y})$$

has the following properties:

(1) *τ is a weakly $\mathrm{SO}(n)$-equivariant diffeomorphism of period 2. More precisely,*

$$\tau(r(z)\cdot(\mathbf{x},y)) = \ell(z^{-1})\cdot\tau(\mathbf{x},y)$$

for $z\in\mathrm{SO}(n)$. In other words, the identification of $\mathcal{H}^n, \mathbb{H}^n, NA$, and $\mathbb{R}^{n-1}\times\mathbb{R}^+$ as sets permits the following abuse of τ and gives a following relation between $r(K)$-action and $\ell(K)$-action: $r(\hat{z})\cdot(x,y) = \tau\left(\ell(\hat{z}^{-1})\cdot\tau(x,y)\right)$.

(2) *τ leaves the geometric spheres centered at $\mathbf{i}=(\mathbf{0},1)$ in each geometry invariant. That is, in both geometries, for $m>0$, the Euclidean sphere*

$$|\mathbf{x}|^2 + (y-\cosh(\ln m))^2 = \sinh^2(\ln m)$$

is a geometric sphere centered at $\mathbf{i}=(\mathbf{0},1)$ with radius $|\ln m|$, in both geometries, and the map τ maps such a sphere to itself. These spheres are $r(K)$-orbits in $\mathcal{H}^n$ and $\ell(K)$-orbits in $\mathbb{H}^n$ (when $\mathcal{H}^n$ and $\mathbb{H}^n$ are identified with $\mathbb{R}^{n-1}\times\mathbb{R}^+$) at the same time.

(3) *τ gives a 1-1 correspondence between the two sets of all the geodesics passing through $\mathbf{i}$ in the two geometries $\mathcal{H}^n$ and $\mathbb{H}^n$.*

4.10. For the $\ell(K)$-action on the hyperbolic space $\mathbb{H}^n = G/K$, we can take the ray $\{\mathbf{0}\}\times(0,1]$ as a cross section to the $\ell(K)$-action. Clearly, $\{\mathbf{0}\}\times[1,\infty)$ is another cross section. The cross section to the $r(K)$-action on $\mathcal{H}^n = K\backslash G$ is the ray $\{\mathbf{0}\}\times[1,\infty)$. The action has a fixed point $\mathbf{i}=(\mathbf{0},1)$, and all the other orbits are $\mathrm{SO}(n)/\mathrm{SO}(n-1)\cong S^{n-1}\cong \mathrm{SO}(n-1)\backslash\mathrm{SO}(n)$. The geometry of the whole space $\mathcal{H}^n = K\backslash G$ is completely determined by the geometry on the line $\{\mathbf{0}\}\times[1,\infty)$ as shown below.

Theorem 4.11. *The space $\mathcal{H}^n - \{\mathbf{i}\}$ is isometric to the warped product $(1,\infty)\times_{e^{2\phi}} S^{n-1}$, where $(1,\infty)$ has the induced metric from $\{\mathbf{0}\}\times(1,\infty)\subset\mathcal{H}^n$, that is, $|\frac{\partial}{\partial t}(t_0)| = \frac{1}{t_0}$ for $t_0\in(1,\infty)$; S^{n-1} has the standard metric; and $e^{2\phi(t)} = \frac{\sinh^2(\ln t)}{\cosh(2\ln t)}$.*

Proof. The sphere $S^{n-1}\subset\mathbb{R}^n$ has a canonical $\mathrm{SO}(n)$-action by matrix multiplication. Choose the north pole $\mathbf{n}=(0,\dots,0,1)\in S^{n-1}$ as a base point. Then the $\mathrm{SO}(n)$-action induces an action on $(1,\infty)\times S^{n-1}$, acting trivially on the first factor. The space $\mathcal{H}^n - \{\mathbf{i}\}$ also has an (isometric) action by $r(\mathrm{SO}(n))$. Using these actions, we define

$$f:(1,\infty)\times S^{n-1}\longrightarrow \mathcal{H}^n - \{\mathbf{i}\}$$

by

$$f(t,a\cdot\mathbf{n}) = f(a\cdot(t,\mathbf{n})) = r(a^{-1})\cdot(\mathbf{0},t),$$

where $(\mathbf{0}, t) \in \mathbb{R}^{n-1} \times \mathbb{R}^+ \subset \mathcal{H}^n$. Since both actions have orbits S^{n-1}, and the stabilizers at $(t, \mathbf{n})$ and $(\mathbf{0}, t)$ are both $\mathrm{SO}(n-1) \times \mathrm{SO}(1) \subset \mathrm{SO}(n)$ (see Corollary 4.5), f is well-defined, bijective and smooth.

Consider the subgroup K_2,

$$K_2 = I_{n-2} \times \mathrm{SO}(2) \times I_1 \subset \mathrm{SO}(n) \times I_1 \subset \mathrm{SO}_0(n, 1).$$

By taking the intersection of $(1, \infty) \times S^{n-1}$ and $\mathcal{H}^n - \{\mathbf{i}\}$ with the last 2-dimensional plane, we get isometric embeddings

$$\begin{array}{ccc} (1,\infty) \times S^{n-1} & \longrightarrow & \mathcal{H}^n - \{\mathbf{i}\} \\ \cup\uparrow & & \cup\uparrow \\ (1,\infty) \times S^1 & \longrightarrow & \mathcal{H}^2 - \{\mathbf{i}\} \end{array}$$

Furthermore, when we give a warped product structure to $(1, \infty) \times S^1$ by the function $e^{2\phi(t)} = \frac{\sinh^2(\ln t)}{\cosh(2\ln t)}$, the restriction of the map f,

$$(1, \infty) \times_{e^{2\phi}} S^1 \longrightarrow \mathcal{H}^2 - \{\mathbf{i}\}$$

$$f(t, a \cdot \mathbf{n}) = f(a \cdot (t, \mathbf{n})) = r(a^{-1}) \cdot (\mathbf{0}, t),$$

where $(0, t) \in \mathcal{H}^2 \subset \mathcal{H}^n$, becomes an isometry by Theorem 3.2. Now it is clear that the $\mathrm{SO}(n)$-action on both spaces make the weakly equivariant map f a global isometry. Thus, the geometry of $\mathcal{H}^n$ is completely determined by the geometry on the cross section $\{\mathbf{0}\} \times (1, \infty) \subset \mathcal{H}^n$ to the $r(K)$-action. $\square$

4.12. The sectional curvature of a plane containing the $(1, \infty)$-direction in $(1, \infty) \times_{e^{2\phi}} S^{n-1}$ is easy to calculate, since such a plane is a rotation of corresponding plane for $\mathrm{SO}(2)\backslash\mathrm{SO}_0(2, 1)$ by $\mathrm{SO}(n)$. Thus, the curvature of such a plane is exactly the same as the 2-dimensional case.

4.13. For a general plane (not containing the $(1, \infty)$-direction), we need some work. Notice that $\{f_*^{-1}\mathbf{w}_1, \ldots, f_*^{-1}\mathbf{w}_{n-1}, f_*^{-1}\mathbf{w}_n\}$ is an orthonormal basis on $(1, \infty) \times_{e^{2\phi}} \{\mathbf{n}\}$ such that $f_*^{-1}\mathbf{w}_n$ is a normal vector to each sphere and the others are tangent to the sphere. By abusing notation, denote $f_*^{-1}\mathbf{w}_i$ as $\mathbf{w}_i$ again.

Lemma 4.14. *For $(\mathbf{0}, y) \in \mathcal{H}^n - \{(\mathbf{0}, 1)\} = (1, \infty) \times_{e^{2\phi}} S^{n-1}$, with $y > 1$, and $\mathbf{w}, \tilde{\mathbf{w}} \in \mathrm{Span}\{\mathbf{w}_1, \ldots \mathbf{w}_{n-1}\}$, with $|\mathbf{w}|_\phi = |\tilde{\mathbf{w}}|_\phi = 1$ and $\langle \mathbf{w}, \tilde{\mathbf{w}} \rangle_\phi = 0$, we have*

$$\kappa(a\mathbf{w}_n + b\mathbf{w},\, c\mathbf{w}_n + d\tilde{\mathbf{w}}) = (a^2d^2 + b^2c^2)\kappa(\mathbf{w}_n, \mathbf{w}) + b^2d^2\, \kappa(\mathbf{w}, \tilde{\mathbf{w}}).$$

Proof. For tangent vectors $T_1, T_2, T_3 \in T(S^{n-1})$ and $X \in T(S^{n-1})^\perp$ in the warped product, we have

$$R(T_1, T_2)T_3 = R_{S^{n-1}}(T_1, T_2)T_3 - e^{2\phi}\, |\nabla\phi|^2\, \big(\langle T_2, T_3\rangle_{S^{n-1}} T_1 - \langle T_1, T_3\rangle_{S^{n-1}} T_2\big),$$

$$R(X, T)Y = \big(h_\phi(X, Y) + \langle\nabla\phi, X\rangle\langle\nabla\phi, Y\rangle\big)T,$$

see [1, p.60, Proposition 2.2.2]. So,

$$\langle R(\tilde{\mathbf{w}}, \mathbf{w})\mathbf{w}, \mathbf{w}_n\rangle_\phi = 0 \quad \text{and} \quad \langle R(\mathbf{w}, \tilde{\mathbf{w}})\tilde{\mathbf{w}}, \mathbf{w}_n\rangle_\phi = 0,$$

also

$$\langle R(\mathbf{w}_n, \mathbf{w})\mathbf{w}_n, \tilde{\mathbf{w}}\rangle_\phi = e^{2\phi}\langle \mathbf{w}, \tilde{\mathbf{w}}\rangle_{S^1}\left(h_\phi(\mathbf{w}_n, \mathbf{w}_n) + \langle \nabla\phi, \mathbf{w}_n\rangle^2\right) = 0.$$

Using an isometric $r(K)$-action rotating the $\{\mathbf{w}_n, \mathbf{w}\}$-plane to $\{\mathbf{w}_n, \tilde{\mathbf{w}}\}$-plane, we have $\kappa(\mathbf{w}_n, \mathbf{w}) = \kappa(\mathbf{w}_n, \tilde{\mathbf{w}})$. Thus,

$$\begin{aligned}\kappa(a\mathbf{w}_n + b\mathbf{w},\, c\mathbf{w}_n + d\tilde{\mathbf{w}}) &= \langle R(a\mathbf{w}_n + b\mathbf{w},\, c\mathbf{w}_n + d\tilde{\mathbf{w}})(c\mathbf{w}_n + d\tilde{\mathbf{w}}),\, a\mathbf{w}_n + d\mathbf{w}\rangle_\phi \\ &= a^2d^2\kappa(\mathbf{w}_n, \tilde{\mathbf{w}}) + b^2c^2\kappa(\mathbf{w}_n, \mathbf{w}) + b^2d^2\kappa(\mathbf{w}, \tilde{\mathbf{w}}) \\ &= (a^2d^2 + b^2c^2)\kappa(\mathbf{w}_n, \mathbf{w}) + b^2d^2\,\kappa(\mathbf{w}, \tilde{\mathbf{w}}).\end{aligned}$$

□

Theorem 4.15 (The sectional curvature of the space $\mathcal{H}^n = \mathrm{SO}(n)\backslash\mathrm{SO}_0(n,1)$). *For $(\mathbf{0}, y) \in \mathcal{H}^n - \{(\mathbf{0}, 1)\} = (1, \infty) \times_{e^{2\phi}} S^{n-1}$, with $y > 1$, let σ be a 2-dimensional tangent plane at $(\mathbf{0}, y)$ whose angle with the y-axis is θ. Then its sectional curvature $\kappa(y, \theta) := \kappa(\sigma)$ is*

$$\kappa(y, \theta) = \cos^2\theta\,\frac{4y^2(1 + 3y^2 + y^4)}{(1 + y^4)^2} + \sin^2\theta\,\frac{2(1 + 2y^2 + 4y^4 + 2y^6 + y^8)}{(1 + y^4)^4}.$$

This curvature formula is valid for all $1 \le y < \infty$. Therefore $0 < \kappa_{(\mathbf{0},y)} \le 5$ for all $y \ge 1$, and at $y = 1$, $\kappa(1, \theta) = 5$ gives the maximum curvature for all $y \ge 1$.

Proof. It is obvious in the case of either $\theta = 0$ or $\theta = \frac{\pi}{2}$.

Assume $0 < \theta < \frac{\pi}{2}$. Let $\hat{\mathbf{w}}$ be the orthogonal projection of $\mathbf{w}_n$ to σ. There is a unique $\mathbf{w} \in T(S^{n-1})$, which lies in the plane $\{\hat{\mathbf{w}}, \mathbf{w}_n\}$, such that we can write $\hat{\mathbf{w}}$ as a linear combination of $\mathbf{w}_n$ and $\mathbf{w}$ with respect to θ: $\hat{\mathbf{w}} = r\cos\theta\ \mathbf{w}_n + r\sin\theta\ \mathbf{w}$ for some $r > 0$. Now let $\tilde{\mathbf{w}}$ be a unit vector in $\sigma \cap T(S^{n-1})$. Since $\tilde{\mathbf{w}}, \hat{\mathbf{w}} \in \sigma$,

$$0 = \langle \mathbf{w}_n, \tilde{\mathbf{w}}\rangle_\phi = \langle \hat{\mathbf{w}}, \tilde{\mathbf{w}}\rangle_\phi = \langle r\cos\theta\mathbf{w}_n + r\sin\theta\mathbf{w},\, \tilde{\mathbf{w}}\rangle_\phi = r\sin\theta\langle \mathbf{w}, \tilde{\mathbf{w}}\rangle_\phi,$$

which implies

$$\langle \mathbf{w}, \tilde{\mathbf{w}}\rangle_\phi = 0$$

and from the above lemma

$$\begin{aligned}\kappa(y, \theta) &= \kappa(\hat{\mathbf{w}}, \tilde{\mathbf{w}}) \\ &= \kappa(\cos\theta\ \mathbf{w}_n + \sin\theta\ \mathbf{w},\ \tilde{\mathbf{w}}) \\ &= \cos^2\theta\,\kappa(\mathbf{w}_n, \tilde{\mathbf{w}}) + \sin^2\theta\,\kappa(\mathbf{w}, \tilde{\mathbf{w}}) \\ &= \cos^2\theta\,\kappa(y) + \sin^2\theta\,\kappa(\mathbf{w}, \tilde{\mathbf{w}}),\end{aligned}$$

where $\kappa(y)$ is the curvature of any tangent 2-plane containing $\mathbf{w}_n$. Now, we get

$$|\,\mathbf{w}\,|_{S^{n-1}} = |\,\tilde{\mathbf{w}}\,|_{S^{n-1}} = e^{-\phi(y)}$$

with respect to the standard metric on S^{n-1} and, from the formula of $R(T_1, T_2)T_3$ in the proof of Lemma 4.14,

$$
\begin{aligned}
\kappa(\mathbf{w}, \tilde{\mathbf{w}}) &= \langle R(\mathbf{w}, \tilde{\mathbf{w}})\tilde{\mathbf{w}}, \mathbf{w}\rangle_\phi \\
&= e^{2\phi(y)} \langle R(\mathbf{w}, \tilde{\mathbf{w}})\tilde{\mathbf{w}}, \mathbf{w}\rangle_{S^{n-1}} \\
&= e^{2\phi(y)} \big(\kappa_{S^{n-1}}(\mathbf{w}, \tilde{\mathbf{w}}) - e^{2\phi(y)}\, |\nabla\phi|^2\, (|\,\mathbf{w}\,|^2_{S^{n-1}} |\,\tilde{\mathbf{w}}\,|^2_{S^{n-1}} - \langle \mathbf{w}, \tilde{\mathbf{w}}\rangle^2_{S^{n-1}})\big) \\
&= e^{2\phi(y)} \big(e^{-4\phi(y)} - e^{2\phi(y)}\, |\nabla\phi|^2\, e^{-4\phi(y)}\big) \\
&= e^{-2\phi(y)} - \langle \nabla\phi, \mathbf{w}_n\rangle^2 \\
&= \tfrac{\cosh(2\ln y)}{\sinh^2(\ln y)} - \big(\mathbf{w}_n(\phi)\big)^2 \\
&= \tfrac{2(y^4+1)}{(y^2-1)^2} - \big(y\tfrac{\partial\phi}{\partial y}\big)^2 \\
&= \frac{2(1+2y^2+4y^4+2y^6+y^8)}{(1+y^4)^2}.
\end{aligned}
$$

Thus,

$$\kappa(y,\theta) = \cos^2\theta\, \frac{4y^2(1+3y^2+y^4)}{(1+y^4)^2} + \sin^2\theta\, \frac{2(1+2y^2+4y^4+2y^6+y^8)}{(1+y^4)^2}.$$

By the remark after Proposition 2.5, by the continuity argument, this curvature formula is valid even at the removed point $(\mathbf{0}, 1)$ with $\kappa_{(\mathbf{0},1)} = 5$.

To estimate the values $\kappa(y,\theta)$, let

$$
\begin{aligned}
f(y) &= \frac{4y^2(1+3y^2+y^4)}{(1+y^4)^2} \\
g(y) &= \frac{2(1+2y^2+4y^4+2y^6+y^8)}{(1+y^4)^2}
\end{aligned}
$$

for $y > 1$. Then

$$0 < f(y) < 5 \quad \text{and} \quad 0 < g(y) < 5.$$

The relation,

$$\kappa(y,\theta) = \cos^2\theta\, f(y) + \sin^2\theta\, g(y) = \frac{f(y)+g(y)+\cos(2\theta)\big(f(y)-g(y)\big)}{2}$$

gives us the following inequality

$$\frac{f(y)+g(y) - |\,f(y)-g(y)\,|}{2} \le \kappa(y,\theta) \le \frac{f(y)+g(y) + |\,f(y)-g(y)\,|}{2},$$

so that

$$\min\{f(y), g(y)\} \le \kappa(y,\theta) \le \max\{f(y), g(y)\},$$

which shows $0 < \kappa(y,\theta) < 5$. □

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