ON A RELATION BETWEEN POTENTIALS FOR PLURIHARMONIC MAPS AND PARA-PLURIHARMONIC MAPS

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Abstract. In this paper, we show that one can interrelate pluriharmonic maps with para-pluriharmonic maps by means of the loop group method. As an appendix, we give examples for the interrelation between pluriharmonic maps and para-pluriharmonic maps. Moreover, we investigate the relation among CMC-surfaces by use of such maps.

1. Introduction

Let $f_1 : (M_1, J) \to G_1/H_1$ be a pluriharmonic map from a complex manifold (M_1, J), and let $f_2 : (M_2, I) \to G_2/H_2$ be a para-pluriharmonic map from a para-complex manifold (M_2, I), where G_i/H_i are affine symmetric spaces. Then, the loop group method enables us to obtain a pluriharmonic potential $(\eta_\lambda, \tau_\lambda)$ and a para-pluriharmonic potential $(\eta_\theta, \tau_\theta)$ from f_1 and f_2, respectively; and furthermore, the method enables us to construct pluriharmonic maps and para-pluriharmonic maps from their potentials, respectively (see Section 3).

Plurharmonic maps $f_1 : (M_1, J) \to G_1/H_1$	Para-plurharmonic maps $f_2 : (M_2, I) \to G_2/H_2$
$\Updownarrow$	$\Updownarrow$
Pluriharmonic potentials $(\eta_\lambda, \tau_\lambda)$	Para-pluriharmonic potentials $(\eta_\theta, \tau_\theta)$

The goal of this paper is to interrelate $f_1 : (M_1, J) \to G_1/H_1$ with $f_2 : (M_2, I) \to G_2/H_2$ by interrelating $(\eta_\lambda, \tau_\lambda)$ with $(\eta_\theta, \tau_\theta)$. In this paper, we demonstrate that one can indeed locally interrelate a pluriharmonic map with a para-pluriharmonic map in the case where its potential satisfies the *morphing condition* (M) (see Theorem 4.3.1).

The notions of a pluriharmonic map and a para-pluriharmonic map are generalized notions of a harmonic map from a Riemann surface Σ^2 and a Lorentz harmonic map from a Lorentz surface Σ^2_1, respectively. Consequently, Theorem 4.3.1 enables us to interrelate harmonic maps from Σ^2 with Lorentz harmonic maps from Σ^2_1. Harmonic maps f_1 from Σ^2 or Lorentz harmonic maps f_2 from Σ^2_1 into S^2, H^2 or S^2_1 give rise to constant mean curvature surfaces (CMC-surfaces, for short) in $\mathbb{R}^3$, spacelike CMC-surfaces in $\mathbb{R}^3_1$ or

2000 *Mathematics Subject Classification.* Primary 53C43; Secondary 58E20.

Key words and phrases. potential, pluriharmonic map, para-pluriharmonic map, the loop group method.

timelike CMC-surfaces in $\mathbb{R}^3_1$; and vice versa. For this reason, one can interrelate CMC-surfaces in $\mathbb{R}^3$ or $\mathbb{R}^3_1$ with other CMC-surfaces in $\mathbb{R}^3$ or $\mathbb{R}^3_1$ by means of Theorem 4.3.1. In the appendix, we present concrete examples of the method developed in this paper; and moreover, we investigate the relation among CMC-surfaces by use of such maps.

This paper is organized as follows: In Section 2 we recall the basic definitions and results concerning para-complex manifolds, para-pluriharmonic maps and pluriharmonic maps. In Section 3 we review elementary facts and results about the loop group method; and we study the relation between para-pluriharmonic or pluriharmonic maps and loop groups. In Section 4 we prove the main Theorem 4.3.1. Finally, in Section 5 we actually interrelate some pluriharmonic maps with para-pluriharmonic maps by means of Theorem 4.3.1.

Acknowledgments. Many thanks are due to the members of GeometrieWerkstatt at the Universität Tübingen. The first named author would like to express his sincere gratitude to Wayne Rossman, Hui Ma, Yoshihiro Ohnita, and David Brander for their encouragement; and he is grateful to Lars Schäfer for his valuable advice.

2. Pluriharmonic maps and para-pluriharmonic maps

2.1. **Para-complex manifolds.** We first recall the notion of a para-complex manifold, in order to introduce the notion of a para-pluriharmonic map.

Definition 2.1.1 (cf. Libermann [20], [21, p. 82, p. 83]).

(i) Let M be a $2n$-dimensional real smooth manifold, and let $\mathfrak{X}M$ denote the Lie algebra of smooth vector fields on M. Then M is called a *para-complex manifold*, if there exists a smooth $(1,1)$-tensor field I on M such that

(1) $I^2 = \mathrm{id}$;
(2) $\dim_{\mathbb{R}} T^+_p M = n = \dim_{\mathbb{R}} T^-_p M$ for each $p \in M$;
(3) $[IX, IY] - I[IX, Y] - I[X, IY] + [X, Y] = 0$ for any $X, Y \in \mathfrak{X}M$,

where $T^{\pm}_p M$ denotes the $\pm$-eigenspace of I_p (= the value of I at p) in T_pM.

(ii) Let (M, I) and (M', I') be two para-complex manifolds. Then a smooth map $f : (M, I) \to (M', I')$ is called *para-holomorphic* (resp. *para-antiholomorphic*), if it satisfies $df \circ I = I' \circ df$ (resp. $df \circ I = -I' \circ df$).

Every para-complex manifold can be endowed with a set of special, local coordinates $(x^1_\alpha, \cdots, x^n_\alpha, y^1_\alpha, \cdots, y^n_\alpha)$ which are called *para-holomorphic coordinates*:

Proposition 2.1.2 (cf. Kaneyuki-Kozai [16, p. 83]). *Let (M, I) be a para-complex manifold with $\dim_{\mathbb{R}} M = 2n$. Then, M has an atlas $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ with U_α open and $\varphi_\alpha = (x^1_\alpha, \cdots, x^n_\alpha, y^1_\alpha, \cdots, y^n_\alpha)$ a coordinate map satisfying*

(1) *$I(\partial/\partial x^a_\alpha) = \partial/\partial x^a_\alpha$ and $I(\partial/\partial y^a_\alpha) = -\partial/\partial y^a_\alpha$ for all $1 \le a \le n$;*
(2) *$\partial y^b_\beta/\partial x^a_\alpha = 0 = \partial x^b_\beta/\partial y^a_\alpha$ on $U_\alpha \cap U_\beta \neq \emptyset$ for all $1 \le a, b \le n$.*

A Lorentz surface and a one sheeted hyperboloid are one of the examples of para-complex manifold.

2.2. **Para-pluriharmonic maps.**

2.2.1. Now, let us recall the notion of a para-pluriharmonic map:

DEFINITION 2.2.1 (cf. Schäfer [25, p. 72]). Let (M, I) be a para-complex manifold with $\dim_{\mathbb{R}} M = 2n$, and let N be a smooth manifold with a torsion-free affine connection ∇^N. Then a smooth map $f : (M, I) \to (N, \nabla^N)$ is called *para-pluriharmonic*, if it satisfies

$$(\nabla df)\big(\frac{\partial}{\partial y^a}, \frac{\partial}{\partial x^b}\big) = 0 \quad \text{for all } 1 \leq a, b \leq n, \tag{P}$$

for any local para-holomorphic coordinate $(x^1, \cdots, x^n, y^1, \cdots, y^n)$ on (M, I). Here ∇ denotes the connection on $\mathrm{End}(TM, f^{-1}TN)$ which is induced from D and ∇^N, where D is any para-complex (i.e., $DI = 0$) torsion-free affine connection on (M, I).

REMARK 2.2.2. Every para-complex manifold admits a para-complex torsion-free affine connection (cf. [25, p. 64]).

The following lemma implies that the equation (P) in Definition 2.2.1 is independent of the choice of para-complex torsion-free affine connections on (M, I):

LEMMA 2.2.3. *Let (M, I) be a para-complex manifold with $\dim_{\mathbb{R}} M = 2n$, and let D be any para-complex torsion-free affine connection on (M, I). Then, every local para-holomorphic coordinate $(x^1, \cdots, x^n, y^1, \cdots, y^n)$ on (M, I) satisfies $D_{\partial^a_+} \partial^b_- = 0 = D_{\partial^a_-} \partial^b_+$ for all $1 \leq a, b \leq n$. Here, $\partial^a_+ := \partial/\partial x^a$ and $\partial^a_- := \partial/\partial y^a$.*

PROOF. It follows from $DI = 0$ that for any $1 \leq a, b \leq n$,

$$I(D_{\partial^a_+} \partial^b_-) = D_{\partial^a_+} I(\partial^b_-) - (D_{\partial^a_+} I)\partial^b_- = D_{\partial^a_+} I(\partial^b_-) = -D_{\partial^a_+} \partial^b_-.$$

This yields $D_{\partial^a_+} \partial^b_- \in T^- M$. Similarly one has $D_{\partial^b_-} \partial^a_+ \in T^+ M$. Therefore we conclude

$$T^- M \ni D_{\partial^a_+} \partial^b_- = D_{\partial^b_-} \partial^a_+ + [\partial^a_+, \partial^b_-] = D_{\partial^b_-} \partial^a_+ \in T^+ M$$

because the torsion of D is free. Thus $D_{\partial^a_+} \partial^b_- = 0 = D_{\partial^b_-} \partial^a_+$. □

2.2.2. Our goal in this subsection is to show Proposition 2.2.4 (below) which will play an important role in Section 3. First, let us fix the setting and the notation of the proposition.

Let G be a connected matrix group, and let σ be an involution of G. We denote by H the fixed point set of σ in G, and get an affine symmetric space $(G/H, \sigma)$. Let (M, I) be a para-complex manifold of dimension $2n$, and let F be a smooth map from (M, I) into G. Then we consider:

(2.2.1) π: the projection from G onto G/H,

(2.2.2) ∇^1: the canonical affine connection on $(G/H, \sigma)$ (see [22, p. 54] for the definition of the canonical affine connection), (2.2.3) $\alpha := F^{-1} \cdot dF$: the pullback of the left-invariant Maurer-Cartan form on G along F,

(2.2.4) $\mathfrak{g} := \operatorname{Lie} G, \quad \mathfrak{h} := \operatorname{Fix}(\mathfrak{g}, d\sigma), \quad \mathfrak{m} := \operatorname{Fix}(\mathfrak{g}, -d\sigma)$,

(2.2.5) $\alpha_{\mathfrak{h}}$ (resp. $\alpha_{\mathfrak{m}}$): the $\mathfrak{h}$-component (resp. the $\mathfrak{m}$-component) of α with respect to $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$,

(2.2.6) $\alpha_{\mathfrak{h}}^{\pm} := (1/2) \cdot (\alpha_{\mathfrak{h}} \pm {}^{t}I(\alpha_{\mathfrak{h}})), \quad \alpha_{\mathfrak{m}}^{\pm} := (1/2) \cdot (\alpha_{\mathfrak{m}} \pm {}^{t}I(\alpha_{\mathfrak{m}}))$,

(2.2.7) $\partial_{-}\alpha_{\mathfrak{m}}^{+} + [\alpha_{\mathfrak{h}}^{-} \wedge \alpha_{\mathfrak{m}}^{+}] = 0$ as an abbreviation for $\partial_{-}^{a}(\alpha_{\mathfrak{m}}(\partial_{+}^{b})) + [\alpha_{\mathfrak{h}}(\partial_{-}^{a}), \alpha_{\mathfrak{m}}(\partial_{+}^{b})] = 0$ for all $1 \leq a, b \leq n$, where $(x^1, \cdots, x^n, y^1, \cdots, y^n)$ is any local para-holomorphic coordinate system on (M, I),

(2.2.8) $T^{\pm}M$: the subbundle of the tangent bundle TM determined by the ± 1-eigenspace of I in TM,

(2.2.9) $[\alpha_{\mathfrak{m}}^{\pm} \wedge \alpha_{\mathfrak{m}}^{\pm}] = 0$ as an abbreviation of: $[\alpha_{\mathfrak{m}} \wedge \alpha_{\mathfrak{m}}] \equiv 0$ on $T^{+}M \times T^{+}M$ and on $T^{-}M \times T^{-}M$,

(2.2.10) $\mathbb{C}^{*} := \mathbb{C} \setminus \{0\}$.

Now, we are in a position to state

PROPOSITION 2.2.4. *With the above setting and notation, the following statements* (a) *and* (b) *are equivalent*:

(a) *A map* $f := \pi \circ F : (M, I) \to (G/H, \nabla^{1})$ *is para-pluriharmonic and satisfies* $[\alpha_{\mathfrak{m}}^{+} \wedge \alpha_{\mathfrak{m}}^{+}] = 0 = [\alpha_{\mathfrak{m}}^{-} \wedge \alpha_{\mathfrak{m}}^{-}]$;

(b) $d\alpha^{\mu} + (1/2) \cdot [\alpha^{\mu} \wedge \alpha^{\mu}] = 0$ *for any* $\mu \in \mathbb{C}^{*}$, *where* $\alpha^{\mu} := \alpha_{\mathfrak{h}} + \mu^{-1} \cdot \alpha_{\mathfrak{m}}^{+} + \mu \cdot \alpha_{\mathfrak{m}}^{-}$.

In order to prove the above proposition, we first show

LEMMA 2.2.5. $f = \pi \circ F : (M, I) \to (G/H, \nabla^{1})$ *is a para-pluriharmonic map if and only if* $\partial_{-}\alpha_{\mathfrak{m}}^{+} + [\alpha_{\mathfrak{h}}^{-} \wedge \alpha_{\mathfrak{m}}^{+}] = 0$ (*cf.* (2.2.7)).

PROOF. Lemma 2.2.3 allows us to reduce the equation (P) in Definition 2.2.1 as follows: $(\nabla df)(\partial_{-}^{a}, \partial_{+}^{b}) = \nabla^{1}_{\partial_{-}^{a}}\big(df(\partial_{+}^{b})\big)$. This implies that

$$f = \pi \circ F \text{ is para-pluriharmonic if and only if } \beta\big(\nabla^{1}_{\partial_{-}^{a}}\big(df(\partial_{+}^{b})\big)\big) = 0$$

because $\beta : T(G/H) \to G/H \times \mathfrak{g}$ is injective (see [5] or [15, p. 403] for β). Accordingly, it suffices to show that

$$\beta\big(\nabla^{1}_{\partial_{-}^{a}}\big(df(\partial_{+}^{b})\big)\big) = 0 \text{ if and only if } \partial_{-}\alpha_{\mathfrak{m}}^{+} + [\alpha_{\mathfrak{h}}^{-} \wedge \alpha_{\mathfrak{m}}^{+}] = 0. \tag{2.2.11}$$

To prove this we note first that it is known that ∇^{1} coincides with the canonical affine connection of the second kind (cf. [22, p. 53]). Therefore, Proposition 1.4 and Lemma 1.1 in [15, p. 404, p. 403] assure that

$$\beta\big(\nabla^{1}_{\partial_{-}^{a}}\big(df(\partial_{+}^{b})\big)\big) = \partial_{-}^{a}\big(\beta\big(df(\partial_{+}^{b})\big)\big) - \big[\beta\big(df(\partial_{-}^{a})\big), \beta\big(df(\partial_{+}^{b})\big)\big].$$

Let us compute each term on the right-hand side of the above equation. We note that $f^*\beta = \mathrm{Ad}\, F \cdot \alpha_{\mathfrak{m}}$ (cf. [15, p. 409]) implies

$$\begin{aligned}\partial_-^a\big(\beta\big(df(\partial_+^b)\big)\big) &= \partial_-^a\big((f^*\beta)(\partial_+^b)\big) = \partial_-^a\big(F\cdot\alpha_{\mathfrak{m}}(\partial_+^b)\cdot F^{-1}\big)\\ &= (\partial_-^a F)\cdot\alpha_{\mathfrak{m}}(\partial_+^b)\cdot F^{-1} + F\cdot\partial_-^a\big(\alpha_{\mathfrak{m}}(\partial_+^b)\big)\cdot F^{-1} - F\cdot\alpha_{\mathfrak{m}}(\partial_+^b)\cdot F^{-1}\cdot(\partial_-^a F)\cdot F^{-1}\\ &= \mathrm{Ad}\, F\cdot\big\{\partial_-^a\big(\alpha_{\mathfrak{m}}(\partial_+^b)\big) + \big[F^{-1}\cdot(\partial_-^a F), \alpha_{\mathfrak{m}}(\partial_+^b)\big]\big\}\\ &= \mathrm{Ad}\, F\cdot\big\{\partial_-^a\big(\alpha_{\mathfrak{m}}(\partial_+^b)\big) + \big[\alpha(\partial_-^a), \alpha_{\mathfrak{m}}(\partial_+^b)\big]\big\}\,.\end{aligned}$$

Moreover, $f^*\beta = \mathrm{Ad}\, F \cdot \alpha_{\mathfrak{m}}$ yields

$$\big[\beta\big(df(\partial_-^a)\big), \beta\big(df(\partial_+^b)\big)\big] = \big[(f^*\beta)(\partial_-^a), (f^*\beta)(\partial_+^b)\big] = \mathrm{Ad}\, F\cdot\big[\alpha_{\mathfrak{m}}(\partial_-^a), \alpha_{\mathfrak{m}}(\partial_+^b)\big].$$

Therefore we obtain

$$\beta\big(\nabla^1_{\partial_-^a}\big(df(\partial_+^b)\big)\big) = \mathrm{Ad}\, F\cdot\big\{\partial_-^a\big(\alpha_{\mathfrak{m}}(\partial_+^b)\big) + \big[\alpha_{\mathfrak{h}}(\partial_-^a), \alpha_{\mathfrak{m}}(\partial_+^b)\big]\big\}\,.$$

Hence we have shown (2.2.11). □

Proof of Proposition 2.2.4. First we rewrite the expression $d\alpha^\mu + (1/2)\cdot[\alpha^\mu\wedge\alpha^\mu]$. Since $\alpha = F^{-1}\cdot dF$ we have $d\alpha + (1/2)\cdot[\alpha\wedge\alpha] = 0$. From $[\mathfrak{h},\mathfrak{h}]\subset\mathfrak{h}$, $[\mathfrak{h},\mathfrak{m}]\subset\mathfrak{m}$ and $[\mathfrak{m},\mathfrak{m}]\subset\mathfrak{h}$ we obtain $d\alpha_{\mathfrak{h}} + (1/2)\cdot\big([\alpha_{\mathfrak{h}}\wedge\alpha_{\mathfrak{h}}] + [\alpha_{\mathfrak{m}}\wedge\alpha_{\mathfrak{m}}]\big) = 0 = d\alpha_{\mathfrak{m}} + [\alpha_{\mathfrak{h}}\wedge\alpha_{\mathfrak{m}}]$. Thus we can assert that

$$\begin{aligned}&d\alpha_{\mathfrak{h}} + \frac{1}{2}\cdot[\alpha_{\mathfrak{h}}\wedge\alpha_{\mathfrak{h}}] + [\alpha_{\mathfrak{m}}^+\wedge\alpha_{\mathfrak{m}}^-] = -\frac{1}{2}\cdot\big([\alpha_{\mathfrak{m}}^+\wedge\alpha_{\mathfrak{m}}^+] + [\alpha_{\mathfrak{m}}^-\wedge\alpha_{\mathfrak{m}}^-]\big)\,,\\ &d\alpha_{\mathfrak{m}} + [\alpha_{\mathfrak{h}}\wedge\alpha_{\mathfrak{m}}] = 0.\end{aligned}\tag{2.2.12}$$

By a direct computation we obtain

$$\begin{aligned}d\alpha^\mu + \frac{1}{2}\cdot[\alpha^\mu\wedge\alpha^\mu] =& d\alpha_{\mathfrak{h}} + \frac{1}{2}\cdot[\alpha_{\mathfrak{h}}\wedge\alpha_{\mathfrak{h}}] + [\alpha_{\mathfrak{m}}^+\wedge\alpha_{\mathfrak{m}}^-]\\ &+ \mu^{-1}\cdot\big(d\alpha_{\mathfrak{m}}^+ + [\alpha_{\mathfrak{h}}\wedge\alpha_{\mathfrak{m}}^+]\big) + \mu\cdot\big(d\alpha_{\mathfrak{m}}^- + [\alpha_{\mathfrak{h}}\wedge\alpha_{\mathfrak{m}}^-]\big)\\ &+ \frac{1}{2}\cdot\mu^{-2}\cdot[\alpha_{\mathfrak{m}}^+\wedge\alpha_{\mathfrak{m}}^+] + \frac{1}{2}\cdot\mu^2\cdot[\alpha_{\mathfrak{m}}^-\wedge\alpha_{\mathfrak{m}}^-].\end{aligned}$$

Consequently, by virtue of (2.2.12) one can rewrite $d\alpha^\mu + (1/2)\cdot[\alpha^\mu\wedge\alpha^\mu]$ as follows:

$$\begin{aligned}d\alpha^\mu + \frac{1}{2}\cdot[\alpha^\mu\wedge\alpha^\mu] =& \mu^{-1}\cdot\big(d\alpha_{\mathfrak{m}}^+ + [\alpha_{\mathfrak{h}}\wedge\alpha_{\mathfrak{m}}^+]\big) + \mu\cdot\big(d\alpha_{\mathfrak{m}}^- + [\alpha_{\mathfrak{h}}\wedge\alpha_{\mathfrak{m}}^-]\big)\\ &+ \frac{1}{2}\cdot(\mu^{-2}-1)\cdot[\alpha_{\mathfrak{m}}^+\wedge\alpha_{\mathfrak{m}}^+] + \frac{1}{2}\cdot(\mu^2-1)\cdot[\alpha_{\mathfrak{m}}^-\wedge\alpha_{\mathfrak{m}}^-].\end{aligned}\tag{2.2.13}$$

(a)→(b): Suppose that $f = \pi\circ F$ is para-pluriharmonic and satisfies $[\alpha_{\mathfrak{m}}^+\wedge\alpha_{\mathfrak{m}}^+] = 0 = [\alpha_{\mathfrak{m}}^-\wedge\alpha_{\mathfrak{m}}^-]$. Then, (2.2.13) yields

$$d\alpha^\mu + \frac{1}{2}\cdot[\alpha^\mu\wedge\alpha^\mu] = \mu^{-1}\cdot\big(d\alpha_{\mathfrak{m}}^+ + [\alpha_{\mathfrak{h}}\wedge\alpha_{\mathfrak{m}}^+]\big) + \mu\cdot\big(d\alpha_{\mathfrak{m}}^- + [\alpha_{\mathfrak{h}}\wedge\alpha_{\mathfrak{m}}^-]\big)\,.$$

So it suffices to show

$$d\alpha_{\mathfrak{m}}^+ + [\alpha_{\mathfrak{h}}\wedge\alpha_{\mathfrak{m}}^+] = 0 = d\alpha_{\mathfrak{m}}^- + [\alpha_{\mathfrak{h}}\wedge\alpha_{\mathfrak{m}}^-].$$

Equation (P) in Definition 2.2.1 is symmetric with respect to the variables y^a and x^b. This and Lemma 2.2.5 imply that

$$f = \pi \circ F \text{ is para-pluriharmonic if and only if } \partial_- \alpha_{\mathfrak{m}}^+ + [\alpha_{\mathfrak{h}}^- \wedge \alpha_{\mathfrak{m}}^+] = 0$$
$$\text{if and only if } \partial_+ \alpha_{\mathfrak{m}}^- + [\alpha_{\mathfrak{h}}^+ \wedge \alpha_{\mathfrak{m}}^-] = 0.$$

Therefore it follows from $d\alpha_{\mathfrak{m}} + [\alpha_{\mathfrak{h}} \wedge \alpha_{\mathfrak{m}}] = 0$ (cf. (2.2.12)) that $d\alpha_{\mathfrak{m}}^+ + [\alpha_{\mathfrak{h}} \wedge \alpha_{\mathfrak{m}}^+] = 0 = d\alpha_{\mathfrak{m}}^- + [\alpha_{\mathfrak{h}} \wedge \alpha_{\mathfrak{m}}^-]$.

(b)→(a): Suppose that $d\alpha^\mu + (1/2) \cdot [\alpha^\mu \wedge \alpha^\mu] = 0$ for any $\mu \in \mathbb{C}^*$. We obtain $d\alpha_{\mathfrak{m}}^+ + [\alpha_{\mathfrak{h}} \wedge \alpha_{\mathfrak{m}}^+] = 0$ and $[\alpha_{\mathfrak{m}}^+ \wedge \alpha_{\mathfrak{m}}^+] = 0 = [\alpha_{\mathfrak{m}}^- \wedge \alpha_{\mathfrak{m}}^-]$ from (2.2.13). So Lemma 2.2.5 allows us to obtain the conclusion, if one has $\partial_- \alpha_{\mathfrak{m}}^+ + [\alpha_{\mathfrak{h}}^- \wedge \alpha_{\mathfrak{m}}^+] = 0$. But, this equation is immediate from $d\alpha_{\mathfrak{m}}^+ + [\alpha_{\mathfrak{h}} \wedge \alpha_{\mathfrak{m}}^+] = 0$. □

2.2.3. We recall the notion of the extended framing of a para-pluriharmonic map (cf. Definition 2.2.6). One will see that the framing is an element of the loop group $\widetilde{\Lambda} G_\sigma$ in Section 3.

Let $G^{\mathbb{C}}$ be a simply connected, simple, complex linear algebraic subgroup of $SL(m, \mathbb{C})$, let σ be a holomorphic involution of $G^{\mathbb{C}}$, and let ν be an antiholomorphic involution of $G^{\mathbb{C}}$ such that $[\sigma, \nu] = 0$ (i.e., $\sigma \circ \nu = \nu \circ \sigma$). Define $H^{\mathbb{C}}$, G and H by

$$H^{\mathbb{C}} := \mathrm{Fix}(G^{\mathbb{C}}, \sigma), \quad G := \mathrm{Fix}(G^{\mathbb{C}}, \nu), \quad H := \mathrm{Fix}(G, \sigma) = \mathrm{Fix}(H^{\mathbb{C}}, \nu). \tag{2.2.14}$$

Note that $(G/H, \sigma|_G)$ is an affine symmetric space. Now, let p_o be a base point in a simply connected para-complex manifold (M, I). Then, Proposition 2.2.4 assures that for any para-pluriharmonic map $f = \pi \circ F : (M, I) \to (G/H, \nabla^1)$ with $F(p_o) = \mathrm{id}$ and $[\alpha_{\mathfrak{m}}^\pm \wedge \alpha_{\mathfrak{m}}^\pm] = 0$, the $\mathfrak{g}^{\mathbb{C}}$-valued 1-form $\alpha^\mu = \alpha_{\mathfrak{h}} + \mu^{-1} \cdot \alpha_{\mathfrak{m}}^+ + \mu \cdot \alpha_{\mathfrak{m}}^-$ on (M, I), parameterized by $\mu \in \mathbb{C}^*$, is integrable; and furthermore, one can obtain a smooth map

$$F : M \times \mathbb{C}^* \to G^{\mathbb{C}}, \quad (p, \mu) \mapsto F_\mu(p),$$

from the integrability condition $d\alpha^\mu + (1/2) \cdot [\alpha^\mu \wedge \alpha^\mu] = 0$ and $F_\mu^{-1} \cdot dF_\mu = \alpha^\mu$ with $F_\mu(p_o) \equiv \mathrm{id}$. The above map $F = F_\mu : \mathbb{C}^* \to G^{\mathbb{C}}$ satisfies

(2.2.15) $\sigma(F_\mu) = F_{-\mu}$ for all $\mu \in \mathbb{C}^*$,
(2.2.16) $F_\lambda := F|_{S^1} : S^1 \to G^{\mathbb{C}}$, where $S^1 := \{\lambda \in \mathbb{C}^* \,|\, |\lambda| = 1\}$,
(2.2.17) $F_\theta := F|_{\mathbb{R}^+} : \mathbb{R}^+ \to G = \mathrm{Fix}(G^{\mathbb{C}}, \nu)$ $(\subset G^{\mathbb{C}})$, where $\mathbb{R}^+ := \{\theta \in \mathbb{R} \,|\, \theta > 0\}$.

Indeed, (2.2.15) follows from $d\sigma(\alpha^\mu) = \alpha^{-\mu}$ and $\sigma(F_\mu(p_o)) = F_{-\mu}(p_o)$; (2.2.16) is obvious; and (2.2.17) follows from α^θ being $\mathfrak{g}$-valued for any $\theta \in \mathbb{R}^+$.

Definition 2.2.6. The map F_θ is called the *extended framing* of the para-pluriharmonic map $f = \pi \circ F : (M, I) \to (G/H, \nabla^1)$; and $\{f_\theta\}_{\theta \in \mathbb{R}^+}$ is called an *associated family* of f, where $f_\theta := \pi \circ F_\theta$. Here, we remark that $f_1 = f$ and $F_1 = F$ are immediate from $\alpha^1 = \alpha$ and $F_1(p_o) = F(p_o)$.

REMARK 2.2.7. Throughout this paper we consider that for the extended framing F_θ of a para-pluriharmonic map, its variable θ varies in the whole $\mathbb{C}^*$ which contains not only $\mathbb{R}^+$ but also S^1.

2.3. **Pluriharmonic maps.**

2.3.1. In this subsection we will survey some basic facts and results about pluriharmonic maps. First, let us recall the notion of a pluriharmonic map:

DEFINITION 2.3.1. Let (M, J) be a real $2n$-dimensional complex manifold, and let N be a smooth manifold with a torsion-free affine connection ∇^N. Then a smooth map $f : (M, J) \to (N, \nabla^N)$ is called *pluriharmonic*, if it satisfies

$$(\nabla df)\big(\frac{\partial}{\partial \bar{z}^a}, \frac{\partial}{\partial z^b}\big) = 0 \quad \text{for all } 1 \leq a, b \leq n, \tag{H}$$

for any local holomorphic coordinate $(z^1, \cdots, z^n, \bar{z}^1, \cdots, \bar{z}^n)$ on (M, J). Here ∇ denotes the connection on $\mathrm{End}(TM, f^{-1}TN)$ which is induced from D and ∇^N, where D is any complex torsion-free affine connection on (M, J).

REMARK 2.3.2. (i) We utilize the terminology "pluriharmonic map," in a sense that is more general than the one originally given by Siu [26].

(ii) Any complex manifold admits a complex torsion-free affine connection (cf. [18, p. 145]).

(iii) The equation (H) in Definition 2.3.1 is independent of the choice of complex torsion-free affine connections D on (M, J) (ref. the proof of Lemma 2.2.3).

2.3.2. In Section 3 we will study the relation between pluriharmonic maps and the loop group method. For this we will use a result of Ohnita [23] about pluriharmonic maps (see Proposition 2.3.3). First, let us fix the setting for Proposition 2.3.3.

Let $(G/H, \sigma)$ denote the affine symmetric space defined in Subsection 2.2.2, and let F be a smooth map from a real $2n$-dimensional complex manifold (M, J) into G. Then, we consider:

(2.3.1) π: the same as in (2.2.1),
(2.3.2) ∇^1: the same as in (2.2.2),
(2.3.3) α: the same as in (2.2.3),
(2.3.4) $\mathfrak{g}$, $\mathfrak{h}$, $\mathfrak{m}$: the same as in (2.2.4),
(2.3.5) $\alpha_\mathfrak{h}$, $\alpha_\mathfrak{m}$: the same as in (2.2.5),
(2.3.6) $\alpha'_X := (-i/2) \cdot (i\alpha_X + {}^tJ(\alpha_X))$, $\quad \alpha''_X := (-i/2) \cdot (i\alpha_X - {}^tJ(\alpha_X))$ for $X = \mathfrak{h}, \mathfrak{m}$,
(2.3.7) $\overline{\partial}\alpha'_\mathfrak{m} + [\alpha''_\mathfrak{h} \wedge \alpha'_\mathfrak{m}] = 0$ as an abbreviation for $\overline{\partial}^a(\alpha_\mathfrak{m}(\partial^b)) + [\alpha_\mathfrak{h}(\overline{\partial}^a), \alpha_\mathfrak{m}(\partial^b)] = 0$ for all $1 \leq a, b \leq n$, where $(z^1, \cdots, z^n, \bar{z}^1, \cdots, \bar{z}^n)$ is any local holomorphic coordinate system on (M, J), and where $\partial^b := \partial/\partial z^b$ and $\overline{\partial}^a := \partial/\partial \bar{z}^a$,
(2.3.8) $[\alpha'_\mathfrak{m} \wedge \alpha'_\mathfrak{m}] = 0$ as an abbreviation for $[\alpha_\mathfrak{m}(\partial^a), \alpha_\mathfrak{m}(\partial^b)] = 0$ for all $1 \leq a, b \leq n$, where $(z^1, \cdots, z^n, \bar{z}^1, \cdots, \bar{z}^n)$ is any local holomorphic coordinate system on (M, J).

Proposition 2.3.3 (cf. Ohnita [23]). *With the above notation, a map* $f := \pi \circ F : (M, J) \to (G/H, \nabla^1)$ *is pluriharmonic if and only if* $\overline{\partial}\alpha'_{\mathfrak{m}} + [\alpha''_{\mathfrak{h}} \wedge \alpha'_{\mathfrak{m}}] = 0$. *Moreover, the following statements* (a) *and* (b) *are equivalent*:

(a) $f = \pi \circ F$ *is pluriharmonic and satisfies* $[\alpha'_{\mathfrak{m}} \wedge \alpha'_{\mathfrak{m}}] = 0$;
(b) $d\alpha^\mu + (1/2) \cdot [\alpha^\mu \wedge \alpha^\mu] = 0$ *for any* $\mu \in \mathbb{C}^*$, *where* $\alpha^\mu := \alpha_{\mathfrak{h}} + \mu^{-1} \cdot \alpha'_{\mathfrak{m}} + \mu \cdot \alpha''_{\mathfrak{m}}$.

2.3.3. We will first recall the notion of the extended framing of a pluriharmonic map (cf. Definition 2.3.4), and afterwards point out a crucial difference between the extended framings of pluriharmonic maps and para-pluriharmonic maps in view of the loop group method (cf. Remark 2.3.5).

The arguments below will be similar to those in Subsection 2.2.3. Let $G^{\mathbb{C}}$, $H^{\mathbb{C}}$, G and H denote the same Lie groups as in (2.2.14). Fix a base point p_o in a simply connected complex manifold (M, J). For a pluriharmonic map $f = \pi \circ F : (M, J) \to (G/H, \nabla^1)$ with $F(p_o) = \mathrm{id}$ and $[\alpha'_{\mathfrak{m}} \wedge \alpha'_{\mathfrak{m}}] = 0$, Proposition 2.3.3 shows that the $\mathfrak{g}^{\mathbb{C}}$-valued 1-form $\alpha^\mu = \alpha_{\mathfrak{h}} + \mu^{-1} \cdot \alpha'_{\mathfrak{m}} + \mu \cdot \alpha''_{\mathfrak{m}}$ on (M, J) parameterized by $\mu \in \mathbb{C}^*$ is integrable. Then there exists a unique map

$$F : M \times \mathbb{C}^* \to G^{\mathbb{C}}, \quad (p, \mu) \mapsto F_\mu(p),$$

such that $F_\mu^{-1} \cdot dF_\mu = \alpha^\mu$ and $F_\mu(p_o) \equiv \mathrm{id}$, by virtue of the integrability condition $d\alpha^\mu + (1/2) \cdot [\alpha^\mu \wedge \alpha^\mu] = 0$. Here we remark that $F = F_\mu : \mathbb{C}^* \to G^{\mathbb{C}}$ satisfies

(2.3.9) $\sigma(F_\mu) = F_{-\mu}$ for all $\mu \in \mathbb{C}^*$,
(2.3.10) $F_\lambda := F|_{S^1} : S^1 \to G = \mathrm{Fix}(G^{\mathbb{C}}, \nu)\ (\subset G^{\mathbb{C}})$.

Indeed, (2.3.10) follows from α^λ being $\mathfrak{g}$-valued for any $\lambda \in S^1$.

Definition 2.3.4. The map F_λ is called the *extended framing* of the pluriharmonic map $f = \pi \circ F : (M, J) \to (G/H, \nabla^1)$; and $\{f_\lambda\}_{\lambda \in S^1}$ is called an *associated family* of f, where $f_\lambda(p) := \pi \circ F_\lambda(p)$ for $(p, \lambda) \in M \times S^1$.

Remark 2.3.5. The map $F = F_\mu : \mathbb{C}^* \to G^{\mathbb{C}}$ defined above becomes G-valued if its variable μ varies in S^1; and $F_\lambda = F|_{S^1}$ is the extended framing of a pluriharmonic map. By contrast, the map $F = F_\mu : \mathbb{C}^* \to G^{\mathbb{C}}$ in Subsection 2.2.3 becomes G-valued if its variable μ varies in $\mathbb{R}^+$; and $F_\theta = F|_{\mathbb{R}^+}$ is the extended framing of a para-pluriharmonic map.

3. The loop group method

First, we introduce three kinds of loop groups $\Lambda G^{\mathbb{C}}_\sigma$, ΛG_σ and $\widetilde{\Lambda} G_\sigma$, and review their decomposition theorems. Next, we explain the relation between para-pluriharmonic maps and the loop group method, and interrelate para-pluriharmonic maps with para-pluriharmonic potentials. Finally, we treat the pluriharmonic case.

3.1. **Decomposition theorems of loop groups.**

3.1.1. Let $G^{\mathbb{C}}$ be a simply connected, simple, complex linear algebraic subgroup of $SL(m, \mathbb{C})$, and let σ be a holomorphic involution of $G^{\mathbb{C}}$. In this case the twisted loop group $\Lambda G^{\mathbb{C}}_{\sigma}$ is defined as follows:

$$\Lambda G^{\mathbb{C}}_{\sigma} := \left\{ A_\lambda : S^1 \to G^{\mathbb{C}} \;\middle|\; \begin{array}{r} A_\lambda = \sum_{k\in\mathbb{Z}} A_k \lambda^k,\ \sum ||A_k|| < \infty, \\ \sigma(A_\lambda) = A_{-\lambda} \text{ for all } \lambda \in S^1 \end{array} \right\},$$

where $||\cdot||$ denotes some matrix norm satisfying $||A \cdot B|| \le ||A|| \cdot ||B||$ and $||\,\mathrm{id}\,|| = 1$. Then $\Lambda G^{\mathbb{C}}_{\sigma}$, with this norm $||A_\lambda|| = \sum ||A_k||$, is a complex Banach Lie group (see [1], [12] and [24] for more details). Here, the Lie algebra $\Lambda \mathfrak{g}^{\mathbb{C}}_{\sigma}$ of $\Lambda G^{\mathbb{C}}_{\sigma}$ is given by

$$\Lambda \mathfrak{g}^{\mathbb{C}}_{\sigma} := \left\{ X_\lambda : S^1 \to \mathfrak{g}^{\mathbb{C}} \;\middle|\; \begin{array}{r} X_\lambda = \sum_{k\in\mathbb{Z}} X_k \lambda^k,\ \sum ||X_k|| < \infty, \\ d\sigma(X_\lambda) = X_{-\lambda} \text{ for all } \lambda \in S^1 \end{array} \right\}. \tag{3.1.1}$$

Define four subgroups $\Lambda^{\pm} G^{\mathbb{C}}_{\sigma}$ and $\Lambda^{\pm}_{*} G^{\mathbb{C}}_{\sigma}$ of $\Lambda G^{\mathbb{C}}_{\sigma}$ by

$$\Lambda^{\pm} G^{\mathbb{C}}_{\sigma} := \{ A_\lambda \in \Lambda G^{\mathbb{C}}_{\sigma} \,|\, A_\lambda \text{ has a holomorphic extension } \widehat{A}_z : \mathbb{D}_{\pm} \to G^{\mathbb{C}} \},$$
$$\Lambda^{+}_{*} G^{\mathbb{C}}_{\sigma} := \{ A_\lambda \in \Lambda^{+} G^{\mathbb{C}}_{\sigma} \,|\, \widehat{A}_0 = \mathrm{id} \}, \quad \Lambda^{-}_{*} G^{\mathbb{C}}_{\sigma} := \{ A_\lambda \in \Lambda^{-} G^{\mathbb{C}}_{\sigma} \,|\, \widehat{A}_\infty = \mathrm{id} \},$$

where $\mathbb{D}_+ := \{ z \in \mathbb{C} \,|\, |z| < 1 \}$ and $\mathbb{D}_- := \{ z \in \mathbb{C} \,|\, |z| > 1 \} \cup \{\infty\}$. With this notation, we can state the following two Theorems 3.1.1 and 3.1.2, which are called the *Iwasawa decomposition* of $\Lambda G^{\mathbb{C}}_{\sigma} \times \Lambda G^{\mathbb{C}}_{\sigma}$ and the *Birkhoff decomposition* of $\Lambda G^{\mathbb{C}}_{\sigma}$, respectively (see [1], [12], [24]):

Theorem 3.1.1 (Iwasawa decomposition of $\Lambda G^{\mathbb{C}}_{\sigma} \times \Lambda G^{\mathbb{C}}_{\sigma}$). *The multiplication maps*

$$\triangle(\Lambda G^{\mathbb{C}}_{\sigma} \times \Lambda G^{\mathbb{C}}_{\sigma}) \times (\Lambda^{-}_{*} G^{\mathbb{C}}_{\sigma} \times \Lambda^{+} G^{\mathbb{C}}_{\sigma}) \to \Lambda G^{\mathbb{C}}_{\sigma} \times \Lambda G^{\mathbb{C}}_{\sigma},$$
$$\triangle(\Lambda G^{\mathbb{C}}_{\sigma} \times \Lambda G^{\mathbb{C}}_{\sigma}) \times (\Lambda^{+}_{*} G^{\mathbb{C}}_{\sigma} \times \Lambda^{-} G^{\mathbb{C}}_{\sigma}) \to \Lambda G^{\mathbb{C}}_{\sigma} \times \Lambda G^{\mathbb{C}}_{\sigma}$$

are holomorphic diffeomorphisms onto open subsets of $\Lambda G^{\mathbb{C}}_{\sigma} \times \Lambda G^{\mathbb{C}}_{\sigma}$, *respectively. Here* $\triangle(\Lambda G^{\mathbb{C}}_{\sigma} \times \Lambda G^{\mathbb{C}}_{\sigma})$ *denotes the diagonal subgroup of* $\Lambda G^{\mathbb{C}}_{\sigma} \times \Lambda G^{\mathbb{C}}_{\sigma}$.

Theorem 3.1.2 (Birkhoff decomposition of $\Lambda G^{\mathbb{C}}_{\sigma}$). *The multiplication maps*

$$\Lambda^{-}_{*} G^{\mathbb{C}}_{\sigma} \times \Lambda^{+} G^{\mathbb{C}}_{\sigma} \to \Lambda G^{\mathbb{C}}_{\sigma}, \quad \Lambda^{+}_{*} G^{\mathbb{C}}_{\sigma} \times \Lambda^{-} G^{\mathbb{C}}_{\sigma} \to \Lambda G^{\mathbb{C}}_{\sigma}$$

are holomorphic diffeomorphisms onto the open subsets $\mathcal{B}^{\mathbb{C}}_{\mp} := \Lambda^{\mp}_{*} G^{\mathbb{C}}_{\sigma} \cdot \Lambda^{\pm} G^{\mathbb{C}}_{\sigma}$ *of* $\Lambda G^{\mathbb{C}}_{\sigma}$, *respectively. In particular, each element* $A_\lambda \in \mathcal{B}^{\mathbb{C}} := \mathcal{B}^{\mathbb{C}}_{-} \cap \mathcal{B}^{\mathbb{C}}_{+}$ *can be uniquely factorized*:

$$A_\lambda = A^{-}_{\lambda} \cdot B^{+}_{\lambda} = A^{+}_{\lambda} \cdot B^{-}_{\lambda}, \quad A^{\pm}_{\lambda} \in \Lambda^{\pm}_{*} G^{\mathbb{C}}_{\sigma}, \quad B^{\pm}_{\lambda} \in \Lambda^{\pm} G^{\mathbb{C}}_{\sigma}.$$

3.1.2. *Almost split real forms of* $\Lambda G^{\mathbb{C}}_{\sigma}$. Now, let ν be an antiholomorphic involution of $G^{\mathbb{C}}$ such that $[\sigma, \nu] = 0$ (i.e., $\sigma \circ \nu = \nu \circ \sigma$). Then one can define an antiholomorphic involution ν_S of $\Lambda G^{\mathbb{C}}_{\sigma}$ by setting

$$\nu_S(A_\lambda) := \nu(A_{\overline{\lambda}}) \quad \text{for } A_\lambda \in \Lambda G^{\mathbb{C}}_{\sigma}. \tag{3.1.2}$$

This involution ν_S is said to be of *the first kind*, and its fixed point set $\Lambda G_\sigma := \mathrm{Fix}(\Lambda G_\sigma^{\mathbb{C}}, \nu_S)$ is called an *almost split real form* of $\Lambda G_\sigma^{\mathbb{C}}$. Note that ν_S satisfies $\nu_S(\Lambda^\pm G_\sigma^{\mathbb{C}}) = \Lambda^\pm G_\sigma^{\mathbb{C}}$ and $\nu_S(\Lambda_*^\pm G_\sigma^{\mathbb{C}}) = \Lambda_*^\pm G_\sigma^{\mathbb{C}}$. That allows us to define four subgroups $\Lambda^\pm G_\sigma$ and $\Lambda_*^\pm G_\sigma$ as follows:

$$\Lambda^\pm G_\sigma := \mathrm{Fix}(\Lambda^\pm G_\sigma^{\mathbb{C}}, \nu_S), \quad \Lambda_*^\pm G_\sigma := \mathrm{Fix}(\Lambda_*^\pm G_\sigma^{\mathbb{C}}, \nu_S).$$

With this notation, one can state the following theorems (see [2], [3]):

THEOREM 3.1.3 (Iwasawa decomposition of $\Lambda G_\sigma \times \Lambda G_\sigma$). *The multiplication maps*

$$\triangle(\Lambda G_\sigma \times \Lambda G_\sigma) \times (\Lambda_*^- G_\sigma \times \Lambda^+ G_\sigma) \to \Lambda G_\sigma \times \Lambda G_\sigma,$$
$$\triangle(\Lambda G_\sigma \times \Lambda G_\sigma) \times (\Lambda_*^+ G_\sigma \times \Lambda^- G_\sigma) \to \Lambda G_\sigma \times \Lambda G_\sigma$$

are holomorphic diffeomorphisms onto open subsets of $\Lambda G_\sigma \times \Lambda G_\sigma$, respectively.

THEOREM 3.1.4 (Birkhoff decomposition of ΛG_σ). *The multiplication maps*

$$\Lambda_*^- G_\sigma \times \Lambda^+ G_\sigma \to \Lambda G_\sigma, \quad \Lambda_*^+ G_\sigma \times \Lambda^- G_\sigma \to \Lambda G_\sigma$$

are holomorphic diffeomorphisms onto the open subsets $\mathcal{B}_\mp := \Lambda_*^\mp G_\sigma \cdot \Lambda^\pm G_\sigma$ of ΛG_σ, respectively. In particular, each element $A_\lambda \in \mathcal{B} := \mathcal{B}_- \cap \mathcal{B}_+$ can be uniquely factorized:

$$A_\lambda = A_\lambda^- \cdot B_\lambda^+ = A_\lambda^+ \cdot B_\lambda^-, \quad A_\lambda^\pm \in \Lambda_*^\pm G_\sigma, \quad B_\lambda^\pm \in \Lambda^\pm G_\sigma.$$

3.1.3. For a general element $A_\lambda \in \Lambda G_\sigma^{\mathbb{C}}$, its variable λ only varies in S^1. However, for the framing F_λ of a para-pluriharmonic map, the variable λ of F_λ can vary in the whole $\mathbb{C}^*$ (cf. Subsection 2.2.3). Toda [27] has addressed this relevant point, since in her work λ is for all geometric purposes a positive real number. She proposed to consider the following subgroup $\widetilde{\Lambda} G_\sigma$ of ΛG_σ:

$$\widetilde{\Lambda} G_\sigma := \{A_\lambda \in \Lambda G_\sigma \mid A_\lambda \text{ has an analytic extension } \widetilde{A}_\mu : \mathbb{C}^* \to G^{\mathbb{C}}\}. \tag{3.1.3}$$

One equips $\widetilde{\Lambda} G_\sigma$ with the induced topology from ΛG_σ, where ΛG_σ is considered as a loop group with $\lambda \in S^1$; and in a similar way, one defines four subgroups $\widetilde{\Lambda}^\pm G_\sigma$ and $\widetilde{\Lambda}_*^\pm G_\sigma$ of $\Lambda^\pm G_\sigma$ and $\Lambda_*^\pm G_\sigma$, respectively. Then, the following two decomposition theorems hold (cf. [2], [9], [27]):

THEOREM 3.1.5 (Iwasawa decomposition of $\widetilde{\Lambda} G_\sigma \times \widetilde{\Lambda} G_\sigma$). *The multiplication maps*

$$\triangle(\widetilde{\Lambda} G_\sigma \times \widetilde{\Lambda} G_\sigma) \times (\widetilde{\Lambda}_*^- G_\sigma \times \widetilde{\Lambda}^+ G_\sigma) \to \widetilde{\Lambda} G_\sigma \times \widetilde{\Lambda} G_\sigma,$$
$$\triangle(\widetilde{\Lambda} G_\sigma \times \widetilde{\Lambda} G_\sigma) \times (\widetilde{\Lambda}_*^+ G_\sigma \times \widetilde{\Lambda}^- G_\sigma) \to \widetilde{\Lambda} G_\sigma \times \widetilde{\Lambda} G_\sigma$$

are real analytic diffeomorphisms onto open subsets of $\widetilde{\Lambda} G_\sigma \times \widetilde{\Lambda} G_\sigma$, respectively.

THEOREM 3.1.6 (Birkhoff decomposition of $\widetilde{\Lambda} G_\sigma$). *The multiplication maps*

$$\widetilde{\Lambda}_*^- G_\sigma \times \widetilde{\Lambda}^+ G_\sigma \to \widetilde{\Lambda} G_\sigma, \quad \widetilde{\Lambda}_*^+ G_\sigma \times \widetilde{\Lambda}^- G_\sigma \to \widetilde{\Lambda} G_\sigma$$

are real analytic diffeomorphisms onto the open subsets $\widetilde{\mathcal{B}}_{\mp} := \widetilde{\Lambda}^{\mp}_{*} G_\sigma \cdot \widetilde{\Lambda}^{\pm} G_\sigma$ *of* $\widetilde{\Lambda} G_\sigma$*, respectively. In particular, each element* $A_\lambda \in \widetilde{\mathcal{B}} := \widetilde{\mathcal{B}}_{-} \cap \widetilde{\mathcal{B}}_{+}$ *can be uniquely factorized*:

$$A_\lambda = A^{-}_{\lambda} \cdot B^{+}_{\lambda} = A^{+}_{\lambda} \cdot B^{-}_{\lambda}, \quad A^{\pm}_{\lambda} \in \widetilde{\Lambda}^{\pm}_{*} G_\sigma, \quad B^{\pm}_{\lambda} \in \widetilde{\Lambda}^{\pm} G_\sigma.$$

Remark 3.1.7. Throughout this paper, we consider that for $A_\lambda \in \widetilde{\Lambda} G_\sigma$, its variable λ varies not only in S^1 but also in $\mathbb{R}^+$ (or more generally in $\mathbb{C}^*$).

We end this subsection with showing the following lemma:

Lemma 3.1.8. *Each element* $C_\lambda \in \widetilde{\Lambda} G_\sigma$ *satisfies* $C_\theta \in G := \mathrm{Fix}(G^{\mathbb{C}}, \nu)$ *for all* $\theta \in \mathbb{R}^+$*.*

Proof. Since $C_\lambda \in \widetilde{\Lambda} G_\sigma \subset \Lambda G_\sigma = \mathrm{Fix}(\Lambda G^{\mathbb{C}}_{\sigma}, \nu_S)$, it satisfies $\nu(C_{\overline{\lambda}}) = \nu_S(C_\lambda) = C_\lambda$ for all $\lambda \in S^1$. Hence, one has $\nu(C_{\overline{\mu}}) = C_\mu$ for all $\mu \in \mathbb{C}^*$; and therefore $\nu(C_\theta) = C_\theta$ for all $\theta \in \mathbb{R}^+$. □

3.2. **Para-pluriharmonic maps and the loop group method.** In this subsection, we will study the relation between para-pluriharmonic maps and the loop group method.

3.2.1. Let $G^{\mathbb{C}}$ be a simply connected, simple, complex linear algebraic subgroup of $SL(m, \mathbb{C})$, let σ be a holomorphic involution of $G^{\mathbb{C}}$, and let ν be an antiholomorphic involution of $G^{\mathbb{C}}$ such that $[\sigma, \nu] = 0$. Define subgroups $H^{\mathbb{C}}$, G and H by the same conditions as in Subsection 2.2.3, respectively—that is,

$$H^{\mathbb{C}} := \mathrm{Fix}(G^{\mathbb{C}}, \sigma), \quad G := \mathrm{Fix}(G^{\mathbb{C}}, \nu), \quad H := \mathrm{Fix}(G, \sigma) = \mathrm{Fix}(H^{\mathbb{C}}, \nu).$$

We will conclude that the extended framing F_θ of a para-pluriharmonic map belongs to the loop group $\widetilde{\Lambda} G_\sigma$ (see (3.1.3) for $\widetilde{\Lambda} G_\sigma$). Let (M, I) be a simply connected para-complex manifold, and let F_θ be the extended framing of a para-pluriharmonic map $f = \pi \circ F : (M, I) \to (G/H, \nabla^1)$ with $F(p_o) = \mathrm{id}$ and $[\alpha^{\pm}_{\mathfrak{m}} \wedge \alpha^{\pm}_{\mathfrak{m}}] = 0$, where p_o is a base point in (M, I). Then it follows from (2.2.15) and (2.2.16) that F_λ belongs to $\Lambda G^{\mathbb{C}}_{\sigma}$. Moreover, the variable λ of F_λ can vary in all of $\mathbb{C}^*$ (cf. Subsection 2.2.3). Accordingly one can assert that the framing F_λ belongs to $\widetilde{\Lambda} G_\sigma$, if it satisfies

$$\nu_S(F_\lambda) = F_\lambda \tag{3.2.1}$$

(see (3.1.2) for ν_S). Let us show (3.2.1). From (2.2.17) we know $\nu(F_\theta) = F_\theta$ for any $\theta \in \mathbb{R}^+$. This yields that $\nu_S(F_\lambda) = \nu(F_{\overline{\lambda}}) = F_\lambda$ for any $\lambda \in S^1$ because $\nu(F_{\overline{\mu}}) = F_\mu$ for any $\mu \in \mathbb{C}^*$ follows from $\nu(F_\theta) = F_\theta$ for any $\theta \in \mathbb{R}^+$. Hence, we have shown (3.2.1). Consequently the framing F_λ belongs to $\widetilde{\Lambda} G_\sigma$.

3.2.2. *Para-pluriharmonic potentials.* We have just shown that F_λ belongs to $\widetilde{\Lambda} G_\sigma$, where F_λ is the extended framing of a para-pluriharmonic map $f = \pi \circ F : (M, I) \to (G/H, \nabla^1)$ with $F(p_o) = \mathrm{id}$ and $[\alpha^{\pm}_{\mathfrak{m}} \wedge \alpha^{\pm}_{\mathfrak{m}}] = 0$. To $F_\lambda \in \widetilde{\Lambda} G_\sigma$, one can apply the Birkhoff decomposition theorem (cf. Theorem 3.1.6). We will obtain a pair of $\mathfrak{m}$-valued 1-forms η_θ and τ_θ on (M, I) parameterized $\theta \in \mathbb{R}^+$, from the framing F_θ.

Since $F_\lambda(p_o) \equiv \mathrm{id} \in \widetilde{\mathcal{B}}$, one can perform a Birkhoff decomposition of the framing $F_\lambda \in \widetilde{\Lambda} G_\sigma$:

$$F_\lambda = F_\lambda^- \cdot L_\lambda^+ = F_\lambda^+ \cdot L_\lambda^-, \quad F_\lambda^\pm \in \widetilde{\Lambda}_*^\pm G_\sigma, \quad L_\lambda^\pm \in \widetilde{\Lambda}^\pm G_\sigma,$$

on an open neighborhood U of M at p_o (cf. Theorem 3.1.6). Define η_θ and τ_θ by

$$\eta_\theta := (F_\theta^-)^{-1} \cdot dF_\theta^-, \quad \tau_\theta := (F_\theta^+)^{-1} \cdot dF_\theta^+,$$

respectively. Then for any $\theta \in \mathbb{R}^+$, both η_θ and τ_θ become $\mathfrak{m}$-valued 1-forms on the para-complex manifold (U, I); and furthermore, η_θ is para-holomorphic and τ_θ is para-antiholomorphic. Indeed, it is immediate from $F_\theta^{-1} \cdot dF_\theta = \alpha^\theta$ that

$$\begin{aligned} \alpha_\mathfrak{h} + \theta^{-1} \cdot \alpha_\mathfrak{m}^+ + \theta \cdot \alpha_\mathfrak{m}^- = \alpha^\theta &= (L_\theta^+)^{-1} \cdot ((F_\theta^-)^{-1} \cdot dF_\theta^-) \cdot L_\theta^+ + (L_\theta^+)^{-1} \cdot dL_\theta^+ \\ &= (L_\theta^-)^{-1} \cdot ((F_\theta^+)^{-1} \cdot dF_\theta^+) \cdot L_\theta^- + (L_\theta^-)^{-1} \cdot dL_\theta^-, \end{aligned}$$

and that $\eta_\theta = \theta^{-1} \cdot \mathrm{Ad}(L_0^+)\alpha_\mathfrak{m}^+$ and $\tau_\theta = \theta \cdot \mathrm{Ad}(L_0^-)\alpha_\mathfrak{m}^-$, where $L_\lambda^\pm = \sum_{\pm k \geq 0} L_k^\pm \lambda^k$. Here, we remark that $L_0^\pm \in H$ by Lemma 3.1.8.

From the extended framing F_θ, we have obtained the pair $(\eta_\theta, \tau_\theta)$ of an $\mathfrak{m}$-valued para-holomorphic 1-form and an $\mathfrak{m}$-valued para-antiholomorphic 1-form on (U, I) parameterized by $\theta \in \mathbb{R}^+$. In the next subsection, we will see that the pair $(\eta_\theta, \tau_\theta)$ is a *para-pluriharmonic potential* (cf. Definition 3.2.1).

3.2.3. We are going to introduce the notion of a para-pluriharmonic potential. Consider two linear subspaces $\widetilde{\Lambda}_{-1,\infty}\mathfrak{g}_\sigma$ and $\widetilde{\Lambda}_{-\infty,1}\mathfrak{g}_\sigma$ of $\widetilde{\Lambda}\mathfrak{g}_\sigma$:

$$\begin{aligned} \widetilde{\Lambda}_{-1,\infty}\mathfrak{g}_\sigma &:= \{X_\lambda \in \widetilde{\Lambda}\mathfrak{g}_\sigma \mid X_\lambda = \textstyle\sum_{i=-1}^\infty X_i \lambda^i\}, \\ \widetilde{\Lambda}_{-\infty,1}\mathfrak{g}_\sigma &:= \{Y_\lambda \in \widetilde{\Lambda}\mathfrak{g}_\sigma \mid Y_\lambda = \textstyle\sum_{j=-\infty}^1 Y_j \lambda^j\}, \end{aligned}$$

where $\widetilde{\Lambda}\mathfrak{g}_\sigma$ denotes the Lie algebra of $\widetilde{\Lambda} G_\sigma$ (see (3.1.3) for $\widetilde{\Lambda} G_\sigma$). Let $\widetilde{\mathcal{P}}_+ = \widetilde{\mathcal{P}}_+(\mathfrak{g})$ and $\widetilde{\mathcal{P}}_- = \widetilde{\mathcal{P}}_-(\mathfrak{g})$ denote the sets of all $\widetilde{\Lambda}_{-1,\infty}\mathfrak{g}_\sigma$-valued para-holomorphic and $\widetilde{\Lambda}_{-\infty,1}\mathfrak{g}_\sigma$-valued para-antiholomorphic 1-forms on a simply connected para-complex manifold (M, I), respectively.

Definition 3.2.1. An element $(\eta_\lambda, \tau_\lambda) \in \widetilde{\mathcal{P}}_+ \times \widetilde{\mathcal{P}}_-$ is called a *para-pluriharmonic potential* (or a *potential*, for short) on (M, I).

Remark 3.2.2. (1) For each potential $(\eta_\lambda, \tau_\lambda) \in \widetilde{\mathcal{P}}_+ \times \widetilde{\mathcal{P}}_-$, one may assume that the variable λ of η_λ (resp. τ_λ) varies in $\mathbb{R}^+$ by virtue of $\eta_\lambda \in \widetilde{\Lambda}\mathfrak{g}_\sigma$ (resp. $\tau_\lambda \in \widetilde{\Lambda}\mathfrak{g}_\sigma$).

(2) Note that we has just obtained a para-pluriharmonic potential $(\eta_\theta, \tau_\theta)$ from the extended framing F_θ of a para-pluriharmonic map $f : (M, I) \to (G/H, \nabla^1)$ with $F(p_o) = \mathrm{id}$ and $[\alpha_\mathfrak{m}^\pm \wedge \alpha_\mathfrak{m}^\pm] = 0$.

(3) It is unfortunate that the condition $[\alpha_\mathfrak{m}^\pm \wedge \alpha_\mathfrak{m}^\pm] = 0$ for the applicability of the loop group method is necessary, but not always satisfied as shown by Krahe [19]. The condition is always true for surfaces and if the pseudo metric of the target space is positive definite. Other natural conditions for the existence of $[\alpha_\mathfrak{m}^\pm \wedge \alpha_\mathfrak{m}^\pm] = 0$ are not known.

We has just obtained a para-pluriharmonic potential $(\eta_\theta, \tau_\theta)$ from the extended framing F_θ of a para-pluriharmonic map $f : (M, I) \to (G/H, \nabla^1)$ with $F(p_o) = \mathrm{id}$ and $[\alpha^{\pm}_{\mathfrak{m}} \wedge \alpha^{\pm}_{\mathfrak{m}}] = 0$. The converse statement is also true—that is, one can obtain a para-pluriharmonic map and its extended framing from any para-pluriharmonic potential and this framing satisfies $[\alpha^{\pm}_{\mathfrak{m}} \wedge \alpha^{\pm}_{\mathfrak{m}}] = 0$:

Proposition 3.2.3. *Let $(\eta_\theta, \tau_\theta) \in \widetilde{\mathcal{P}}_+(\mathfrak{g}) \times \widetilde{\mathcal{P}}_-(\mathfrak{g})$ be a para-pluriharmonic potential on the para-complex manifold (M, I). Then, the following steps provide an $\mathbb{R}^+$-family $\{f_\theta\}_{\theta \in \mathbb{R}^+}$ of para-pluriharmonic maps*:

(S1) *Solve the two initial value problems*: $(A^-_\theta)^{-1} \cdot dA^-_\theta = \eta_\theta$ *and* $(A^+_\theta)^{-1} \cdot dA^+_\theta = \tau_\theta$ *with* $A^{\pm}_\theta(p_o) \equiv \mathrm{id}$, *where p_o is a base point in (M, I).*

(S2) *Factorize $(A^-_\theta, A^+_\theta) \in \widetilde{\Lambda} G_\sigma \times \widetilde{\Lambda} G_\sigma$ in the Iwasawa decomposition* (*cf. Theorem* 3.1.5) : $(A^-_\theta, A^+_\theta) = (C_\theta, C_\theta) \cdot (B^+_\theta, B^-_\theta)$, *where $C_\theta \in \widetilde{\Lambda} G_\sigma$, $B^+_\theta \in \widetilde{\Lambda}^+_* G_\sigma$ and $B^-_\theta \in \widetilde{\Lambda}^- G_\sigma$.*

(S3) *Then, $f_\theta := \pi \circ C_\theta : (W, I) \to (G/H, \nabla^1)$ becomes a para-pluriharmonic map for every $\theta \in \mathbb{R}^+$. Here, W is any open neighborhood of M at p_o such that both* (S1) *and* (S2) *are solved on W.*

In particular, $C_\theta(p_o) \equiv \mathrm{id}$ and C_θ is the extended framing of the para-pluriharmonic map $f_1 = \pi \circ C_1 : (W, I) \to (G/H, \nabla^1)$.

Proof. (S1), (S2): The solution (A^-_θ, A^+_θ) to (S1) satisfies $A^{\mp}_\theta \in \widetilde{\Lambda} G_\sigma$ and $A^{\mp}_\theta(p_o) \equiv \mathrm{id}$. Therefore, it belongs to the open subset of $\widetilde{\Lambda} G_\sigma \times \widetilde{\Lambda} G_\sigma$ locally. Hence, one can factorize (A^-_θ, A^+_θ) by means of (S2).

(S3): Let W be any open neighborhood of M at p_o such that both (S1) and (S2) are solved on W. First, let us show $C_\theta(p_o) \equiv \mathrm{id}$. Since $A^{\mp}_\theta(p_o) \equiv \mathrm{id}$ we have $\widetilde{\Lambda}^+_* G_\sigma \ni B^+_\theta(p_o) = C_\theta(p_o)^{-1} = B^-_\theta(p_o) \in \widetilde{\Lambda}^- G_\sigma$. Hence, $C_\theta(p_o) \in (\widetilde{\Lambda}^+_* G_\sigma \cap \widetilde{\Lambda}^- G_\sigma) = \{\mathrm{id}\}$. Now, let $\beta^\mu := C_\mu^{-1} \cdot dC_\mu$ for $\mu \in \mathbb{C}^*$. Lemma 3.1.8 implies that β^θ is a $\mathfrak{g}$-valued 1-form on W for any $\theta \in \mathbb{R}^+$. Therefore, one can express it as $\beta^\theta = (\beta^\theta)_{\mathfrak{h}} + (\beta^\theta)_{\mathfrak{m}} = (\beta^\theta)_{\mathfrak{h}} + (\beta^\theta)^+_{\mathfrak{m}} + (\beta^\theta)^-_{\mathfrak{m}}$ by taking $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ into consideration (see (2.2.5) and (2.2.6) for $(\beta^\theta)_{\mathfrak{h}}$ and $(\beta^\theta)^{\pm}_{\mathfrak{m}}$). Then we obtain the conclusion, if one has

$$\beta^\theta = (\beta^1)_{\mathfrak{h}} + \theta^{-1} \cdot (\beta^1)^+_{\mathfrak{m}} + \theta \cdot (\beta^1)^-_{\mathfrak{m}} \tag{3.2.2}$$

because $\beta^\theta = C_\theta^{-1} \cdot dC_\theta$ satisfies $d\beta^\theta + (1/2) \cdot [\beta^\theta \wedge \beta^\theta] = 0$ for any $\theta \in \mathbb{R}^+$, and thus the proof of Proposition 2.2.4 and (3.2.2) allow us to conclude that $f_\theta = \pi \circ C_\theta : (W, I) \to (G/H, \nabla^1)$ is a para-pluriharmonic map for every $\theta \in \mathbb{R}^+$. Hence, it suffices to prove (3.2.2). Direct computation, together with $C_\theta = A^-_\theta \cdot (B^+_\theta)^{-1} = A^+_\theta \cdot (B^-_\theta)^{-1}$, gives us

$$\begin{aligned}(\beta^\theta, \beta^\theta) &= \left(C_\theta^{-1} \cdot dC_\theta,\ C_\theta^{-1} \cdot dC_\theta\right) \\ &= \left(B^+_\theta \cdot \eta_\theta \cdot (B^+_\theta)^{-1} + B^+_\theta \cdot d(B^+_\theta)^{-1},\ B^-_\theta \cdot \tau_\theta \cdot (B^-_\theta)^{-1} + B^-_\theta \cdot d(B^-_\theta)^{-1}\right).\end{aligned}$$

Therefore, the Fourier series $\beta^\lambda = \sum_{k\in\mathbb{Z}} \beta_k \lambda^k$ has actually the simple form:

$$\beta^\lambda = \lambda^{-1} \cdot \beta_{-1} + \beta_0 + \lambda \cdot \beta_{+1} \tag{a}$$

because the n-th and m-th Fourier coefficients of $B^+_\lambda \cdot \eta_\lambda \cdot (B^+_\lambda)^{-1} + B^+_\lambda \cdot d(B^+_\lambda)^{-1}$ and $B^-_\lambda \cdot \tau_\lambda \cdot (B^-_\lambda)^{-1} + B^-_\lambda \cdot d(B^-_\lambda)^{-1}$ are zero for all $n \le -2$ and $2 \le m$, respectively. Let us denote by $(\beta_j)^+$ and $(\beta_j)^-$ the para-holomorphic component and the para-antiholomorphic component of β_j, respectively (i.e., $(\beta_j)^\pm := (1/2) \cdot (\beta_j \pm {}^tI(\beta_j))$) for $j = \pm 1$, and rewrite the above (a) as

$$\beta^\lambda = \lambda^{-1} \cdot ((\beta_{-1})^+ + (\beta_{-1})^-) + \beta_0 + \lambda \cdot ((\beta_{+1})^+ + (\beta_{+1})^-). \tag{a$'$}$$

Then, (a$'$) simplifies to

$$\beta^\lambda = \lambda^{-1} \cdot (\beta_{-1})^+ + \beta_0 + \lambda \cdot (\beta_{+1})^- \tag{a$''$}$$

because the -1st and $+1$st Fourier coefficients of $B^+_\lambda \cdot \eta_\lambda \cdot (B^+_\lambda)^{-1} + B^+_\lambda \cdot d(B^+_\lambda)^{-1}$ and $B^-_\lambda \cdot \tau_\lambda \cdot (B^-_\lambda)^{-1} + B^-_\lambda \cdot d(B^-_\lambda)^{-1}$ are para-holomorphic and para-antiholomorphic, respectively. From (a$''$) and $\beta^\lambda \in \widetilde{\Lambda}\mathfrak{g}_\sigma$ we see that $(\beta^1)_\mathfrak{h} = \beta_0$ and $(\beta^1)_\mathfrak{m} = (\beta_{-1})^+ + (\beta_{+1})^-$. This implies $(\beta^1)^+_\mathfrak{m} = (\beta_{-1})^+$, $(\beta^1)^-_\mathfrak{m} = (\beta_{+1})^-$ and $\beta^\lambda = \lambda^{-1} \cdot (\beta^1)^+_\mathfrak{m} + (\beta^1)_\mathfrak{h} + \lambda \cdot (\beta^1)^-_\mathfrak{m}$; and (3.2.2) follows. □

3.3. **Pluriharmonic maps and the loop group method.** We have explained the relation between para-pluriharmonic maps and the loop group method in Subsection 3.2. In this subsection, we will explain the relation between pluriharmonic maps and the loop group method. The arguments below will be similar to those in Subsection 3.2.

3.3.1. In Subsection 3.2.1 we have learned that the extended framing F'_θ of a para-pluriharmonic map belongs to the almost split real form ΛG_σ—that is, it satisfies $\nu_S(F'_\lambda) = F'_\lambda$ for the involution ν_S of *the first kind* (cf. (3.1.2) for ν_S). In this subsection, we will first confirm that the extended framing F_λ of a pluriharmonic map satisfies $\nu_C(F_\lambda) = F_\lambda$ for the involution ν_C of *the second kind* defined below.

Let $G^\mathbb{C}$ be a simply connected, simple, complex linear algebraic subgroup of $SL(m, \mathbb{C})$, let σ be a holomorphic involution of $G^\mathbb{C}$, and let ν be an antiholomorphic involution of $G^\mathbb{C}$ such that $[\sigma, \nu] = 0$. Denote by $H^\mathbb{C}$, G and H, the subgroups defined in Subsection 2.2.3, respectively (cf. (2.2.14)). Now, let us define an antiholomorphic involution ν_C of $\Lambda G^\mathbb{C}_\sigma$ by

$$\nu_C(A_\lambda) := \nu(A_{1/\bar{\lambda}}) \quad \text{for } A_\lambda \in \Lambda G^\mathbb{C}_\sigma. \tag{3.3.1}$$

This involution ν_C is said to be of *the second kind*, and satisfies the following:

$$\nu_C(\Lambda^\pm G^\mathbb{C}_\sigma) = \Lambda^\mp G^\mathbb{C}_\sigma, \quad \nu_C(\Lambda^\pm_* G^\mathbb{C}_\sigma) = \Lambda^\mp_* G^\mathbb{C}_\sigma. \tag{3.3.2}$$

Let p_o be a base point in a simply connected complex manifold (M, J), and let F_λ be the extended framing of a pluriharmonic map $f = \pi \circ F : (M, J) \to (G/H, \nabla^1)$ with

$F(p_o) = \mathrm{id}$ and $[\alpha'_{\mathfrak{m}} \wedge \alpha'_{\mathfrak{m}}] = 0$. From (2.3.9) it follows that $F_\lambda \in \Lambda G^{\mathbb{C}}_\sigma$. In particular, (2.3.10) implies that F_λ satisfies

$$\nu_C(F_\lambda) = F_\lambda. \tag{3.3.3}$$

3.3.2. *Pluriharmonic potentials.* Since $F_\lambda(p_o) \equiv \mathrm{id}$ we perform a Birkhoff decomposition of the framing F_λ. Therefore we obtain a pair of $\mathfrak{m}^{\mathbb{C}}$-valued 1-forms η_λ and τ_λ on (M, J) parameterized by $\lambda \in S^1$. Here $\mathfrak{m}^{\mathbb{C}} := \mathrm{Fix}(\mathfrak{g}^{\mathbb{C}}, -d\sigma)$. We will see later that the pair $(\eta_\lambda, \tau_\lambda)$ is a pluriharmonic potential (cf. Definition 3.3.1).

Since $F_\lambda(p_o) \equiv \mathrm{id} \in \mathcal{B}^{\mathbb{C}}$, we factorize the framing $F_\lambda \in \Lambda G^{\mathbb{C}}_\sigma$ in the Birkhoff decomposition:

$$F_\lambda = F^-_\lambda \cdot L^+_\lambda = F^+_\lambda \cdot L^-_\lambda, \quad F^\pm_\lambda \in \Lambda^\pm_* G^{\mathbb{C}}_\sigma, \quad L^\pm_\lambda \in \Lambda^\pm G^{\mathbb{C}}_\sigma,$$

on an open neighborhood U of M at p_o (cf. Theorem 3.1.2). Define η_λ and τ_λ by

$$\eta_\lambda := (F^-_\lambda)^{-1} \cdot dF^-_\lambda, \quad \tau_\lambda := (F^+_\lambda)^{-1} \cdot dF^+_\lambda,$$

respectively. Then for any $\lambda \in S^1$, both η_λ and τ_λ become $\mathfrak{m}^{\mathbb{C}}$-valued 1-forms on the complex manifold (U, J). In addition, η_λ is holomorphic and τ_λ is antiholomorphic. Indeed, $F^{-1}_\lambda \cdot dF_\lambda = \alpha^\lambda$ yields

$$\begin{aligned} \alpha_{\mathfrak{h}} + \lambda^{-1} \cdot \alpha'_{\mathfrak{m}} + \lambda \cdot \alpha''_{\mathfrak{m}} = \alpha^\lambda &= (L^+_\lambda)^{-1} \cdot ((F^-_\lambda)^{-1} \cdot dF^-_\lambda) \cdot L^+_\lambda + (L^+_\theta)^{-1} \cdot dL^+_\theta \\ &= (L^-_\lambda)^{-1} \cdot ((F^+_\lambda)^{-1} \cdot dF^+_\lambda) \cdot L^-_\lambda + (L^-_\lambda)^{-1} \cdot dL^-_\lambda, \end{aligned}$$

and $\eta_\lambda = \lambda^{-1} \cdot \mathrm{Ad}(L^+_0)\alpha'_{\mathfrak{m}}$ and $\tau_\lambda = \lambda \cdot \mathrm{Ad}(L^-_0)\alpha''_{\mathfrak{m}}$, where $L^\pm_\lambda = \sum_{\pm k \geq 0} L^\pm_k \lambda^k$. Now, it follows from (3.3.2) and (3.3.3) that $\nu_C(F^-_\lambda) = F^+_\lambda$. This implies that η_λ is related with τ_λ by the formula $d\nu_C(\eta_\lambda) = \tau_\lambda$. Consequently we obtain from the extended framing F_λ of a pluriharmonic map the pair $(\eta_\lambda, \tau_\lambda)$ of an $\mathfrak{m}^{\mathbb{C}}$-valued holomorphic 1-form and an $\mathfrak{m}^{\mathbb{C}}$-valued antiholomorphic 1-form on (U, J) satisfying $d\nu_C(\eta_\lambda) = \tau_\lambda$.

3.3.3. Let us introduce the following subspaces $\Lambda_{-1,\infty}\mathfrak{g}^{\mathbb{C}}_\sigma$ and $\Lambda_{-\infty,1}\mathfrak{g}^{\mathbb{C}}_\sigma$ of $\Lambda\mathfrak{g}^{\mathbb{C}}_\sigma$, in order to recall the notion of a pluriharmonic potential:

$$\begin{aligned} \Lambda_{-1,\infty}\mathfrak{g}^{\mathbb{C}}_\sigma &:= \{X_\lambda \in \Lambda\mathfrak{g}^{\mathbb{C}}_\sigma \,|\, X_\lambda = \textstyle\sum_{i=-1}^{\infty} X_i \lambda^i\}, \\ \Lambda_{-\infty,1}\mathfrak{g}^{\mathbb{C}}_\sigma &:= \{Y_\lambda \in \Lambda\mathfrak{g}^{\mathbb{C}}_\sigma \,|\, Y_\lambda = \textstyle\sum_{j=-\infty}^{1} Y_j \lambda^j\} \end{aligned}$$

(cf. (3.1.1) for $\Lambda\mathfrak{g}^{\mathbb{C}}_\sigma$). Let $\mathcal{P}' = \mathcal{P}'(\mathfrak{g}^{\mathbb{C}})$ and $\mathcal{P}'' = \mathcal{P}''(\mathfrak{g}^{\mathbb{C}})$ denote the set of all $\Lambda_{-1,\infty}\mathfrak{g}^{\mathbb{C}}_\sigma$-valued holomorphic and $\Lambda_{-\infty,1}\mathfrak{g}^{\mathbb{C}}_\sigma$-valued antiholomorphic 1-forms on a simply connected complex manifold (M, J), respectively.

Definition 3.3.1. An element $(\eta_\lambda, \tau_\lambda) \in \mathcal{P}' \times \mathcal{P}''$ is called a *pluriharmonic potential* (or a *potential*, for short) on (M, J), if it satisfies $d\nu_C(\eta_\lambda) = \tau_\lambda$ (cf. (3.3.1) for ν_C).

In Subsection 3.3.2 one has obtained a pluriharmonic potential $(\eta_\lambda, \tau_\lambda)$ from the extended framing F_λ of a pluriharmonic map $f = \pi \circ F : (M, J) \to (G/H, \nabla^1)$ with $F(p_o) = \mathrm{id}$ and $[\alpha'_{\mathfrak{m}} \wedge \alpha'_{\mathfrak{m}}] = 0$. Next we recall from [7] that one can obtain a pluriharmonic map and its extended framing from a pluriharmonic potential:

Proposition 3.3.2. *Let* $(\eta_\lambda, \tau_\lambda) = (\eta_\lambda, d\nu_C(\eta_\lambda)) \in \mathcal{P}'(\mathfrak{g}^{\mathbb{C}}) \times \mathcal{P}''(\mathfrak{g}^{\mathbb{C}})$ *be any pluriharmonic potential on the complex manifold* (M, J). *Then, the following steps provide an* S^1*-family* $\{f_\lambda\}_{\lambda \in S^1}$ *of pluriharmonic maps*:

(S1) *Solve the two initial value problems*: $A_\lambda^{-1} \cdot dA_\lambda = \eta_\lambda$, $B_\lambda^{-1} \cdot dB_\lambda = \tau_\lambda$ *with* $A_\lambda(p_o) \equiv \mathrm{id} \equiv B_\lambda(p_o)$, *where* p_o *is a base point in* (M, J).

(S2) *Factorize* $(A_\lambda, B_\lambda) \in \Lambda G_\sigma^{\mathbb{C}} \times \Lambda G_\sigma^{\mathbb{C}}$ *in the Iwasawa decomposition* (*cf. Theorem 3.1.1*) : $(A_\lambda, B_\lambda) = (C_\lambda, C_\lambda) \cdot (B_\lambda^+, B_\lambda^-)$, *where* $C_\lambda \in \Lambda G_\sigma^{\mathbb{C}}$, $B_\lambda^+ \in \Lambda_*^+ G_\sigma^{\mathbb{C}}$ *and* $B_\lambda^- \in \Lambda^- G_\sigma^{\mathbb{C}}$.

(S3) *Take an open neighborhood* V *of* M *at* p_o *and a smooth map* $h^{\mathbb{C}} = h^{\mathbb{C}}(p) : V \to H^{\mathbb{C}}$ *such that*
  (1) $C'_\lambda(p) \in G$ *for all* $(p, \lambda) \in V \times S^1$,
  (2) $C'_\lambda(p_o) \equiv \mathrm{id}$, *where* $C'_\lambda := C_\lambda \cdot h^{\mathbb{C}}$.

(S4) *Then,* $f_\lambda := \pi \circ C'_\lambda : (V, J) \to (G/H, \nabla^1)$ *becomes an* S^1*-family of pluriharmonic maps.*

Proof. (S1), (S2): For the solutions A_λ and B_λ to (S1), we deduce that they satisfy

$$\nu_C(A_\lambda) = B_\lambda \tag{3.3.4}$$

in terms of $d\nu_C(\eta_\lambda) = \tau_\lambda$. Since $A_\lambda(p_o) \equiv \mathrm{id} \equiv B_\lambda(p_o)$ and $(A_\lambda(p_o), B_\lambda(p_o))$ belongs to a suitable open subset of $\Lambda G_\sigma^{\mathbb{C}} \times \Lambda G_\sigma^{\mathbb{C}}$, one can factorize (A_λ, B_λ) by means of (S2).

(S3): Let us assume that both (S1) and (S2) hold on an open neighborhood W of M at p_o. We will confirm that there exist an open neighborhood V $(\subset W)$ of M at p_o and a smooth map $h^{\mathbb{C}} = h^{\mathbb{C}}(p) : V \to H^{\mathbb{C}}$ such that

(1) $C_\lambda(p) \cdot h^{\mathbb{C}}(p) \in G = \mathrm{Fix}(G^{\mathbb{C}}, \nu)$ for all $(p, \lambda) \in V \times S^1$;
(2) $C_\lambda(p_o) \cdot h^{\mathbb{C}}(p_o) \equiv \mathrm{id}$

—that is, we want to assert that (S3) holds. First, let us verify

$$C_\lambda(p_o) \equiv \mathrm{id}\,.$$

By $A_\lambda(p_o) \equiv \mathrm{id} \equiv B_\lambda(p_o)$ we conclude $\Lambda_*^+ G_\sigma^{\mathbb{C}} \ni B_\lambda^+(p_o) = C_\lambda(p_o)^{-1} = B_\lambda^-(p_o) \in \Lambda^- G_\sigma^{\mathbb{C}}$; so that $C_\lambda(p_o) \in (\Lambda_*^+ G_\sigma^{\mathbb{C}} \cap \Lambda^- G_\sigma^{\mathbb{C}}) = \{\mathrm{id}\}$, and $C_\lambda(p_o) \equiv \mathrm{id}$. Next, we will deduce that

$$(C_\lambda(q))^{-1} \cdot \nu(C_\lambda(q)) \in H^{\mathbb{C}} \quad \text{for any point } (q, \lambda) \in W \times S^1. \tag{3.3.5}$$

Since (3.3.4), (3.3.2) and $C_\lambda = A_\lambda \cdot (B_\lambda^+)^{-1} = B_\lambda \cdot (B_\lambda^-)^{-1}$, we obtain

$$(C_\lambda)^{-1} \cdot \nu_C(C_\lambda) = \big(B_\lambda \cdot (B_\lambda^-)^{-1}\big)^{-1} \cdot \nu_C\big(A_\lambda \cdot (B_\lambda^+)^{-1}\big) = B_\lambda^- \cdot \nu_C((B_\lambda^+)^{-1}) \in \Lambda^- G_\sigma^{\mathbb{C}}.$$

The above also leads to $(C_\lambda)^{-1} \cdot \nu_C(C_\lambda) = \nu_C\big(\{(C_\lambda)^{-1} \cdot \nu_C(C_\lambda)\}^{-1}\big) \in \nu_C(\Lambda^- G_\sigma^{\mathbb{C}}) = \Lambda^+ G_\sigma^{\mathbb{C}}$. Therefore we have $(C_\lambda)^{-1} \cdot \nu_C(C_\lambda) \in (\Lambda^- G_\sigma^{\mathbb{C}} \cap \Lambda^+ G_\sigma^{\mathbb{C}}) = H^{\mathbb{C}}$, and so (3.3.5) follows. It remains to show that there exist an open neighborhood V of M at p_o and a smooth map $h^{\mathbb{C}} = h^{\mathbb{C}}(p) : V \to H^{\mathbb{C}}$ satisfying the equations (1) and (2) above. Let U_H and $O_{\mathfrak{h}}$ denote open neighborhoods of $H^{\mathbb{C}}$ at id and of $\mathfrak{h}^{\mathbb{C}}$ at 0 such that $\exp : O_{\mathfrak{h}} \to U_H$ is a diffeomorphism and $\nu(U_H) \subset U_H$. Since (3.3.5) and $(C_\lambda(p_o))^{-1} \cdot \nu(C_\lambda(p_o)) = \mathrm{id} \in U_H$, there exists an open neighborhood V $(\subset W)$ of p_o in M such that $(C_\lambda(p))^{-1} \cdot \nu(C_\lambda(p)) \in U_H$ for all $p \in V$. Hence,

$$(C_\lambda(p))^{-1} \cdot \nu(C_\lambda(p)) = \exp X(p) \quad \text{on } V, \tag{3.3.6}$$

where $X = X(p) : V \to O_{\mathfrak{h}}$ is a smooth map with $X(p_o) = 0$. This yields

$$\exp d\nu(X(p)) = \nu\big((C_\lambda(p))^{-1} \cdot \nu(C_\lambda(p))\big) = \nu(C_\lambda(p))^{-1} \cdot (C_\lambda(p)) = \exp(-X(p))$$

and $d\nu(X(p)) = -X(p)$. Accordingly we conclude that (1) $\nu(C_\lambda(p) \cdot h^{\mathbb{C}}(p)) = C_\lambda(p) \cdot h^{\mathbb{C}}(p)$ for all $(p, \lambda) \in V \times S^1$ and (2) $C_\lambda(p_o) \cdot h^{\mathbb{C}}(p_o) \equiv \mathrm{id}$, by setting $h^{\mathbb{C}}(p) := \exp((1/2) \cdot X(p))$ (cf. (3.3.6)).

(S4): The arguments below will be similar to those of the proof of (S3) in Proposition 3.2.3. Define a $\mathfrak{g}$-valued 1-form β^λ on (V, J) by $\beta^\lambda := (C'_\lambda)^{-1} \cdot dC'_\lambda$, and express it as $\beta^\lambda = (\beta^\lambda)_{\mathfrak{h}} + (\beta^\lambda)_{\mathfrak{m}} = (\beta^\lambda)_{\mathfrak{h}} + (\beta^\lambda)'_{\mathfrak{m}} + (\beta^\lambda)''_{\mathfrak{m}}$, where $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ (see (2.3.6) for $(\beta^\lambda)'_{\mathfrak{m}}$ and $(\beta^\lambda)''_{\mathfrak{m}}$). Then, it suffices to verify (3.3.7):

$$\beta^\lambda = (\beta^1)_{\mathfrak{h}} + \lambda^{-1} \cdot (\beta^1)'_{\mathfrak{m}} + \lambda \cdot (\beta^1)''_{\mathfrak{m}}. \tag{3.3.7}$$

Indeed, $\beta^\lambda = (C'_\lambda)^{-1} \cdot dC'_\lambda$ satisfies $d\beta^\lambda + (1/2) \cdot [\beta^\lambda \wedge \beta^\lambda] = 0$ for any $\lambda \in S^1$, and so Proposition 2.3.3 and (3.3.7) allow us to conclude that $f_\lambda = \pi \circ C'_\lambda : (V, J) \to (G/H, \nabla^1)$ is a pluriharmonic map for every $\lambda \in S^1$. Direct computation, together with $C_\lambda = A_\lambda \cdot (B^+_\lambda)^{-1} = B_\lambda \cdot (B^-_\lambda)^{-1}$ and $C'_\lambda = C_\lambda \cdot h^{\mathbb{C}}$, gives us

$$\begin{aligned}(\beta^\lambda, \beta^\lambda) &= \big((C'_\lambda)^{-1} \cdot dC'_\lambda,\ (C'_\lambda)^{-1} \cdot dC'_\lambda\big) \\ &= \big(D^+_\lambda \cdot \eta_\lambda \cdot (D^+_\lambda)^{-1} + D^+_\lambda \cdot d(D^+_\lambda)^{-1},\ D^-_\lambda \cdot \tau_\lambda \cdot (D^-_\lambda)^{-1} + D^-_\lambda \cdot d(D^-_\lambda)^{-1}\big),\end{aligned}$$

where $(D^\pm_\lambda)^{-1} := (B^\pm_\lambda)^{-1} \cdot h^{\mathbb{C}}$. It follows from $h^{\mathbb{C}} \in H^{\mathbb{C}}$ that $D^\pm_\lambda \in \Lambda^\pm G^{\mathbb{C}}_\sigma$. Therefore, the Fourier series $\beta^\lambda = \sum_{k \in \mathbb{Z}} \beta_k \lambda^k$ is actually a Laurent polynomial of the form

$$\text{(a)} \quad \beta^\lambda = \lambda^{-1} \cdot \beta_{-1} + \beta_0 + \lambda \cdot \beta_{+1} = \lambda^{-1} \cdot ((\beta_{-1})' + (\beta_{-1})'') + \beta_0 + \lambda \cdot ((\beta_{+1})' + (\beta_{+1})'')$$

because the n-th and m-th Fourier coefficients of $D^+_\lambda \cdot \eta_\lambda \cdot (D^+_\lambda)^{-1} + D^+_\lambda \cdot d(D^+_\lambda)^{-1}$ and $D^-_\lambda \cdot \tau_\lambda \cdot (D^-_\lambda)^{-1} + D^-_\lambda \cdot d(D^-_\lambda)^{-1}$ are zero for all $n \leq -2$ and $2 \leq m$, respectively. Moreover, (a) simplifies to

$$\text{(a}'\text{)} \qquad \beta^\lambda = \lambda^{-1} \cdot (\beta_{-1})' + \beta_0 + \lambda \cdot (\beta_{+1})''$$

because the -1st and $+1$st Fourier coefficients of $D^+_\lambda \cdot \eta_\lambda \cdot (D^+_\lambda)^{-1} + D^+_\lambda \cdot d(D^+_\lambda)^{-1}$ and $D^-_\lambda \cdot \tau_\lambda \cdot (D^-_\lambda)^{-1} + D^-_\lambda \cdot d(D^-_\lambda)^{-1}$ are holomorphic and antiholomorphic, respectively. In view of (a′) and $\beta^\lambda \in \Lambda\mathfrak{g}^{\mathbb{C}}_\sigma$ it turns out that $(\beta^1)_{\mathfrak{h}} = \beta_0$ and $(\beta^1)_{\mathfrak{m}} = (\beta_{-1})' + (\beta_{+1})''$. Therefore (3.3.7) follows from $(\beta^1)'_{\mathfrak{m}} = (\beta_{-1})'$ and $(\beta^1)''_{\mathfrak{m}} = (\beta_{+1})''$. □

4. **Relation between pluriharmonic maps and para-pluriharmonic maps**

In this section, by utilizing the loop group method, we interrelate pluriharmonic maps with para-pluriharmonic maps. We consider two real subspaces $\mathbb{A}^{2n}$ and $\mathbb{B}^{2n}$ of $\mathbb{C}^{2n}$ (cf. Subsection 4.1), and two symmetric closed subspaces G_1/H_1 and G_2/H_2 of $G^{\mathbb{C}}/H^{\mathbb{C}}$ (cf. Subsection 4.2), and we investigate the relation between certain pluriharmonic maps $f_1 : \mathbb{A}^{2n} \to G_1/H_1$ and certain para-pluriharmonic maps $f_2 : \mathbb{B}^{2n} \to G_2/H_2$ (cf. Subsection 4.3).

$$\begin{array}{ccccl} f_1 : \mathbb{A}^{2n} & \longrightarrow & G_1/H_1, & \text{pluriharmonic} \\ \cap & & \cap & \\ \mathbb{C}^{2n} & & G^{\mathbb{C}}/H^{\mathbb{C}} & \\ \cup & & \cup & \\ f_2 : \mathbb{B}^{2n} & \longrightarrow & G_2/H_2, & \text{para-pluriharmonic} \end{array}$$

4.1. **The real subspaces $\mathbb{A}^{2n}$ and $\mathbb{B}^{2n}$ of $\mathbb{C}^{2n}$.** Let $\mathbb{A}^{2n}$ and $\mathbb{B}^{2n}$ be the real subspaces of $\mathbb{C}^{2n}$ given by

$$\begin{aligned} \mathbb{A}^{2n} :&= \{(z^1, \cdots, z^n, w^1, \cdots, w^n) \in \mathbb{C}^n \times \mathbb{C}^n \,|\, \bar{z}^a = w^a \text{ for all } 1 \leq a \leq n\} \\ &= \{(z, w) \in \mathbb{C}^n \times \mathbb{C}^n \,|\, w = \bar{z}\}, \\ \mathbb{B}^{2n} :&= \{(z^1, \cdots, z^n, w^1, \cdots, w^n) \in \mathbb{C}^n \times \mathbb{C}^n \,|\, z^a = \bar{z}^a \text{ and } w^a = \bar{w}^a \text{ for all } 1 \leq a \leq n\} \\ &= \mathbb{R}^n \times \mathbb{R}^n. \end{aligned}$$

Let $(x^1, \cdots, x^n, y^1, \cdots, y^n)$ denote the global coordinate system on $\mathbb{B}^{2n}$ defined by $x^a := \mathrm{Re}(z^a)$ and $y^a := \mathrm{Re}(w^a)$ for $1 \leq a \leq n$. Define smooth $(1,1)$-tensor fields J on $\mathbb{A}^{2n}$ and I on $\mathbb{B}^{2n}$ by

$$J\Big(\frac{\partial}{\partial z^a}\Big) := i\frac{\partial}{\partial z^a},\ J\Big(\frac{\partial}{\partial \bar{z}^a}\Big) := -i\frac{\partial}{\partial \bar{z}^a} \text{ and } I\Big(\frac{\partial}{\partial x^a}\Big) := \frac{\partial}{\partial x^a},\ I\Big(\frac{\partial}{\partial y^a}\Big) := -\frac{\partial}{\partial y^a}.$$

Then $(\mathbb{A}^{2n}, J)$ and $(\mathbb{B}^{2n}, I)$ are simply connected complex and para-complex manifolds, respectively. Henceforth, for the natural coordinate systems $(z^1, \cdots, z^n, \bar{z}^1, \cdots, \bar{z}^n)$ on $\mathbb{A}^{2n}$, $(x^1, \cdots, x^n, y^1, \cdots, y^n)$ on $\mathbb{B}^{2n}$ and $(z^1, \cdots, z^n, w^1, \cdots, w^n)$ on $\mathbb{C}^{2n}$, we will use the notation $(\mathbf{z}, \bar{\mathbf{z}})$, $(\mathbf{x}, \mathbf{y})$ and $(\mathbf{z}, \mathbf{w})$, respectively.

4.2. **The symmetric subspaces G_1/H_1 and G_2/H_2 of $G^{\mathbb{C}}/H^{\mathbb{C}}$.** In this subsection, we introduce two symmetric subspaces G_1/H_1 and G_2/H_2 of $G^{\mathbb{C}}/H^{\mathbb{C}}$. Let $G^{\mathbb{C}}$ be a simply connected, simple, complex linear algebraic subgroup of $SL(m, \mathbb{C})$, let σ be a holomorphic involution of $G^{\mathbb{C}}$, and let ν_1 and ν_2 be antiholomorphic involutions of $G^{\mathbb{C}}$ satisfying $[\sigma, \nu_1] = [\sigma, \nu_2] = [\nu_1, \nu_2] = 0$. Then we define $H^{\mathbb{C}}$, G_i, H_i, π_i $(i = 1, 2)$ and $\mathfrak{g}_2$ as follows:

(4.2.1) $H^{\mathbb{C}} := \mathrm{Fix}(G^{\mathbb{C}}, \sigma)$,
(4.2.2) $G_i := \mathrm{Fix}(G^{\mathbb{C}}, \nu_i)$,
(4.2.3) $H_i := \mathrm{Fix}(G_i, \sigma) = \mathrm{Fix}(H^{\mathbb{C}}, \nu_i)$,
(4.2.4) π_i: the projection from G_i onto G_i/H_i,
(4.2.5) $\mathfrak{g}_2 := \mathrm{Lie}\, G_2$.

Clearly, $(G^{\mathbb{C}}/H^{\mathbb{C}}, \sigma)$ is an affine symmetric space, and both G_1/H_1 and G_2/H_2 are symmetric closed subspaces of $(G^{\mathbb{C}}/H^{\mathbb{C}}, \sigma)$ (ref. [18, p. 227] for the definition of symmetric closed subspace). In particular, $(G_i/H_i, \sigma|_{G_i})$, $i = 1, 2$, are affine symmetric spaces.

4.3. **The main result.** With the notation in Subsections 4.1 and 4.2 we assert the following (see (3.3.1) for $(\nu_1)_C$):

THEOREM 4.3.1. *Let* $(\eta_\theta, \tau_\theta) = (\eta_\theta(\mathbf{x}), \tau_\theta(\mathbf{y})) \in \widetilde{\mathcal{P}}_+(\mathfrak{g}_2) \times \widetilde{\mathcal{P}}_-(\mathfrak{g}_2)$ *be a real analytic, para-pluriharmonic potential on* $(\mathbb{B}^{2n}, I)$*, and let* $(f_2)_\theta = \pi_2 \circ C_\theta(\mathbf{x}, \mathbf{y}) : (W, I) \to (G_2/H_2, \nabla^1)$ *denote the* $\mathbb{R}^+$*-family of para-pluriharmonic maps constructed from* $(\eta_\theta, \tau_\theta)$ *in the neighborhood* W *of* $\mathbb{B}^{2n}$ *at* $(\mathbf{0}, \mathbf{0})$ *in Proposition 3.2.3. Suppose that* $(\eta_\theta, \tau_\theta)$ *satisfies the morphing condition*

$$d(\nu_1)_C(\eta_\lambda(\mathbf{z})) = \tau_\lambda(\bar{\mathbf{z}}). \tag{M}$$

Then, there exist an open neighborhood V *of* $\mathbb{A}^{2n}$ *at* $(\mathbf{0}, \mathbf{0})$ *and a smooth map* $h^{\mathbb{C}}(\mathbf{z}, \bar{\mathbf{z}}) : V \to H^{\mathbb{C}}$ *such that*

(1) $C'_\lambda(\mathbf{z}, \bar{\mathbf{z}}) \in G_1$ *for all* $(\mathbf{z}, \bar{\mathbf{z}}; \lambda) \in V \times S^1$;

(2) $(f_1)_\lambda := \pi_1 \circ C'_\lambda(\mathbf{z}, \bar{\mathbf{z}}) : (V, J) \to (G_1/H_1, \nabla^1)$ *is an* S^1*-family of pluriharmonic maps with* $C'_\lambda(\mathbf{0}, \mathbf{0}) \equiv \mathrm{id}$*, where* $C'_\lambda(\mathbf{z}, \bar{\mathbf{z}}) := C_\lambda(\mathbf{z}, \bar{\mathbf{z}}) \cdot h^{\mathbb{C}}(\mathbf{z}, \bar{\mathbf{z}})$.

REMARK 4.3.2. (i) Since both $\eta_\theta(\mathbf{x})$ and $\tau_\theta(\mathbf{y})$ are analytic on $\mathbb{B}^{2n}$ and $\mathbb{B}^{2n}$ is a totally real submanifold of $\mathbb{C}^{2n}$, one can uniquely extend them as holomorphic 1-forms $\eta_\theta(\mathbf{z})$ and $\tau_\theta(\mathbf{w})$ to an open subset $\widetilde{W}$ of $\mathbb{C}^{2n}$ such that $\mathbb{B}^{2n} \subset \widetilde{W}$. For this reason, the notation $\eta_\lambda(\mathbf{z})$ and $\tau_\lambda(\bar{\mathbf{z}})$ in Theorem 4.3.1 makes sense.

(ii) Similarly, one can verify that the notation $C_\lambda(\mathbf{z}, \bar{\mathbf{z}})$, used in Theorem 4.3.1, makes sense.

Proof of Theorem 4.3.1. Let $(A_\lambda(\mathbf{x}), B_\lambda(\mathbf{y})) = (C_\lambda(\mathbf{x}, \mathbf{y}), C_\lambda(\mathbf{x}, \mathbf{y})) \cdot (B^+_\lambda(\mathbf{x}, \mathbf{y}), B^-_\lambda(\mathbf{x}, \mathbf{y}))$ denote the Iwasawa decomposition in (S2) of Proposition 3.2.3. Note that $A_\lambda(\mathbf{x})$ and $B_\lambda(\mathbf{y})$ satisfy

$$(A_\lambda^{-1} \cdot dA_\lambda)(\mathbf{x}) = \eta_\lambda(\mathbf{x}), \quad (B_\lambda^{-1} \cdot dB_\lambda)(\mathbf{y}) = \tau_\lambda(\mathbf{y}), \quad A_\lambda(\mathbf{0}) \equiv \mathrm{id} \equiv B_\lambda(\mathbf{0}).$$

Since $(\eta_\lambda(\mathbf{x}), \tau_\lambda(\mathbf{y}))$ is analytic, we deduce that $A_\lambda(\mathbf{x})$, $B_\lambda(\mathbf{y})$, $C_\lambda(\mathbf{x}, \mathbf{y})$ and $B^\pm_\lambda(\mathbf{x}, \mathbf{y})$ are analytic with respect to the variables $\mathbf{x}$ and $\mathbf{y}$. Therefore these matrices have unique analytic extensions $A_\lambda(\mathbf{z})$, $B_\lambda(\mathbf{w})$, $C_\lambda(\mathbf{z}, \mathbf{w})$ and $B^\pm_\lambda(\mathbf{z}, \mathbf{w})$ to an open neighborhood $\widetilde{W}$ of $\mathbb{C}^{2n}$ at $(\mathbf{0}, \mathbf{0})$, respectively, because $\mathbb{B}^{2n}$ is a totally real submanifold of $\mathbb{C}^{2n}$. Then on the neighborhood $\widetilde{W} \cap \mathbb{A}^{2n}$ of $\mathbb{A}^{2n}$ at $(\mathbf{0}, \mathbf{0})$, we confirm that $A_\lambda(\mathbf{z})$ and $B_\lambda(\bar{\mathbf{z}})$ satisfy $(A_\lambda^{-1} \cdot dA_\lambda)(\mathbf{z}) = \eta_\lambda(\mathbf{z})$, $(B_\lambda^{-1} \cdot dB_\lambda)(\bar{\mathbf{z}}) = \tau_\lambda(\bar{\mathbf{z}})$ and $A_\lambda(\mathbf{0}) \equiv \mathrm{id} \equiv B_\lambda(\mathbf{0})$; and furthermore, $(A_\lambda(\mathbf{z}), B_\lambda(\bar{\mathbf{z}})) = (C_\lambda(\mathbf{z}, \bar{\mathbf{z}}), C_\lambda(\mathbf{z}, \bar{\mathbf{z}})) \cdot (B^+_\lambda(\mathbf{z}, \bar{\mathbf{z}}), B^-_\lambda(\mathbf{z}, \bar{\mathbf{z}}))$ becomes the Iwasawa decomposition in (S2) of Proposition 3.3.2, where we remark that $(\eta_\lambda(\mathbf{z}), \tau_\lambda(\bar{\mathbf{z}}))$ satisfy $(\eta_\lambda(\mathbf{z}), \tau_\lambda(\bar{\mathbf{z}})) \in \mathcal{P}'(\mathfrak{g}^{\mathbb{C}}) \times \mathcal{P}''(\mathfrak{g}^{\mathbb{C}})$ and $d(\nu_1)_C(\eta_\lambda(\mathbf{z})) = \tau_\lambda(\bar{\mathbf{z}})$. Consequently, the proof of

Proposition 3.3.2 assures that there exist an open neighborhood $V \subset \widetilde{W} \cap \mathbb{A}^{2n}$ of $\mathbb{A}^{2n}$ at $(\mathbf{0}, \mathbf{0})$ and a smooth map $h^{\mathbb{C}}(\mathbf{z}, \bar{\mathbf{z}}) : V \to H^{\mathbb{C}}$ satisfying the conditions (1) and (2). □

5. Appendix

We will interrelate concretely some pluriharmonic maps with para-pluriharmonic maps by means of Theorem 4.3.1. In Subsection 5.2 we will focus on harmonic maps and Lorentz harmonic maps. This will yield a relation between CMC-surfaces in $\mathbb{R}^3$ and CMC-surface in $\mathbb{R}^3_1$.

5.1. **A relation between certain pluriharmonic maps and certain para-pluriharmonic maps.**

5.1.1. $f_1 : \mathbb{A}^4 \to Gr_{2,4}(\mathbb{C}) \iff f_2 : \mathbb{B}^4 \to Gr_{2,4}(\mathbb{C}')$. Following the main result of this paper, we construct in this subsection a pluriharmonic map $f_1(z^1, z^2, \bar{z}^1, \bar{z}^2) : \mathbb{A}^4 \to Gr_{2,4}(\mathbb{C})$ and a para-pluriharmonic map $f_2(x^1, x^2, y^1, y^2) : \mathbb{B}^4 \to Gr_{2,4}(\mathbb{C}')$ from one potential (5.1.10) below, where $Gr_{2,4}(\mathbb{C})$ (resp. $Gr_{2,4}(\mathbb{C}')$) denotes a complex (resp. para-complex) Grassmann manifold. In this subsection, we will use the following notation:

(5.1.1) $G^{\mathbb{C}} = SL(4, \mathbb{C})$,
(5.1.2) $\sigma(A) := I_{2,2} \cdot A \cdot I_{2,2}$ for $A \in G^{\mathbb{C}}$, where $I_{2,2} := \mathrm{diag}(-1, -1, 1, 1)$,
(5.1.3) $\nu_1(A) := {}^t(\overline{A})^{-1}$ for $A \in G^{\mathbb{C}}$,
(5.1.4) $\nu_2(A) := \overline{A}$ for $A \in G^{\mathbb{C}}$,
(5.1.5) $G^{\mathbb{C}}/H^{\mathbb{C}} = SL(4, \mathbb{C})/S(GL(2, \mathbb{C}) \times GL(2, \mathbb{C}))$,
(5.1.6) $G_1/H_1 = SU(4)/S(U(2) \times U(2)) \simeq Gr_{2,4}(\mathbb{C})$,
(5.1.7) $G_2/H_2 = SL(4, \mathbb{R})/S(GL(2, \mathbb{R}) \times GL(2, \mathbb{R})) \simeq Gr_{2,4}(\mathbb{C}')$,
(5.1.8) π_i: the projection from G_i onto G_i/H_i $(i = 1, 2)$,
(5.1.9) $\mathfrak{g}_2 := \mathrm{Lie}\, G_2 = \mathfrak{sl}(4, \mathbb{R})$.

First, we define a $\widetilde{\Lambda}_{-1,\infty}(\mathfrak{g}_2)_\sigma$-valued, real analytic para-holomorphic 1-form $\eta_\theta(x^1, x^2)$ on $(\mathbb{B}^4, I)$ by

$$\eta_\theta(x^1, x^2) := \theta^{-1} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} dx^1 + \theta^{-1} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} dx^2. \tag{5.1.10}$$

Taking the morphing condition (M) of Theorem 4.3.1 into consideration, we define a $\widetilde{\Lambda}_{-\infty,1}(\mathfrak{g}_2)_\sigma$-valued, real analytic para-antiholomorphic 1-form $\tau_\theta(y^1, y^2)$ on $(\mathbb{B}^4, I)$ by setting

$$\tau_\theta(y^1, y^2) := \theta \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} dy^1 + \theta \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} dy^2.$$

Hence we obtain the real analytic, para-pluriharmonic potential $(\eta_\theta(x^1,x^2),\tau_\theta(y^1,y^2))$.

From Proposition 3.2.3 we obtain a para-pluriharmonic map $f_2 : (\mathbb{B}^4, I) \to G_2/H_2 \simeq Gr_{2,4}(\mathbb{C}')$.

(S1): Solve the two initial value problems:

$$(A^-_\theta)^{-1}\cdot dA^-_\theta = \eta_\theta, \quad (A^+_\theta)^{-1}\cdot dA^+_\theta = \tau_\theta, \quad A^\pm_\theta(0,0) \equiv \mathrm{id}\,.$$

The solutions are

$$A^-_\theta(x^1,x^2) = \begin{pmatrix} \cos(x^1/\theta) & 0 & 0 & \sin(x^1/\theta) \\ 0 & \cosh(x^2/\theta) & \sinh(x^2/\theta) & 0 \\ 0 & \sinh(x^2/\theta) & \cosh(x^2/\theta) & 0 \\ -\sin(x^1/\theta) & 0 & 0 & \cos(x^1/\theta) \end{pmatrix},$$

$$A^+_\theta(y^1,y^2) = \begin{pmatrix} \cos(\theta y^1) & 0 & 0 & \sin(\theta y^1) \\ 0 & \cosh(-\theta y^2) & \sinh(-\theta y^2) & 0 \\ 0 & \sinh(-\theta y^2) & \cosh(-\theta y^2) & 0 \\ -\sin(\theta y^1) & 0 & 0 & \cos(\theta y^1) \end{pmatrix}.$$

(S2): Factorize $(A^-_\theta, A^+_\theta) \in \widetilde{\Lambda}^-_*(G_2)_\sigma \times \widetilde{\Lambda}^+_*(G_2)_\sigma$ in the Iwasawa decomposition Theorem 3.1.5:

$$(A^-_\theta, A^+_\theta) = (C_\theta, C_\theta)\cdot(B^+_\theta, B^-_\theta),$$

where $C_\theta \in \widetilde{\Lambda}(G_2)_\sigma$, $B^+_\theta \in \widetilde{\Lambda}^+_*(G_2)_\sigma$ and $B^-_\theta \in \widetilde{\Lambda}^-(G_2)_\sigma$. Here, $B^\pm_\theta$ and C_θ are given by $B^\pm_\theta = (A^\pm_\theta)^{-1}$ and

$$C_\theta(x^1,x^2,y^1,y^2)$$
$$= \begin{pmatrix} \cos(x^1/\theta+\theta y^1) & 0 & 0 & \sin(x^1/\theta+\theta y^1) \\ 0 & \cosh(x^2/\theta-\theta y^2) & \sinh(x^2/\theta-\theta y^2) & 0 \\ 0 & \sinh(x^2/\theta-\theta y^2) & \cosh(x^2/\theta-\theta y^2) & 0 \\ -\sin(x^1/\theta+\theta y^1) & 0 & 0 & \cos(x^1/\theta+\theta y^1) \end{pmatrix}.$$

(S3): The last step of Proposition 3.2.3 assures

$$(f_2)_\theta := \pi_2 \circ C_\theta(x^1,x^2,y^1,y^2) : (\mathbb{B}^4, I) \to Gr_{2,4}(\mathbb{C}') \text{ is para-pluriharmonic}$$

for every $\theta \in \mathbb{R}^+$.

We will construct a pluriharmonic map $f_1 : (\mathbb{A}^4, J) \to G_1/H_1 \simeq Gr_{2,4}(\mathbb{C})$ from $C_\theta(x^1,x^2,y^1,y^2)$ given above. Substituting λ, z^i and $\bar{z}^i$ for θ, x^i and y^i, respectively $(i=1,2)$ we obtain

$$C_\lambda(z^1,z^2,\bar{z}^1,\bar{z}^2)$$
$$= \begin{pmatrix} \cos(z^1/\lambda+\lambda\bar{z}^1) & 0 & 0 & \sin(z^1/\lambda+\lambda\bar{z}^1) \\ 0 & \cosh(z^2/\lambda-\lambda\bar{z}^2) & \sinh(z^2/\lambda-\lambda\bar{z}^2) & 0 \\ 0 & \sinh(z^2/\lambda-\lambda\bar{z}^2) & \cosh(z^2/\lambda-\lambda\bar{z}^2) & 0 \\ -\sin(z^1/\lambda+\lambda\bar{z}^1) & 0 & 0 & \cos(z^1/\lambda+\lambda\bar{z}^1) \end{pmatrix}$$

for $C_\theta(x^1, x^2, y^1, y^2)$. Then $C_\lambda(z^1, z^2, \bar{z}^1, \bar{z}^2) \in G_1 = SU(4)$ for all $(z^1, z^2, \bar{z}^1, \bar{z}^2; \lambda) \in \mathbb{A}^4 \times S^1$ because $z^1/\lambda + \lambda\bar{z}^1$ is a real number and $z^2/\lambda - \lambda\bar{z}^2$ is a purely imaginary number. Hence, we conclude that

$$(f_1)_\lambda := \pi_1 \circ C_\lambda(z^1, z^2, \bar{z}^1, \bar{z}^2) : (\mathbb{A}^4, J) \to Gr_{2,4}(\mathbb{C}) \text{ is a pluriharmonic map}$$

for every $\lambda \in S^1$. Consequently, we have constructed a pluriharmonic map $f_1 : \mathbb{A}^4 \to Gr_{2,4}(\mathbb{C})$ and a para-pluriharmonic map $f_2 : \mathbb{B}^4 \to Gr_{2,4}(\mathbb{C}')$ from the potential (5.1.10).

$$\boxed{\begin{array}{c} (f_1)_\lambda = \pi_1 \circ C_\lambda(z^1, z^2, \bar{z}^1, \bar{z}^2) : (\mathbb{A}^4, J) \to Gr_{2,4}(\mathbb{C}) \text{ is pluriharmonic} \\ \Updownarrow \\ (f_2)_\theta = \pi_2 \circ C_\theta(x^1, x^2, y^1, y^2) : (\mathbb{B}^4, I) \to Gr_{2,4}(\mathbb{C}') \text{ is para-pluriharmonic} \end{array}}$$

5.1.2. $f_1 : \mathbb{A}^4 \to S^4 \iff f_2 : \mathbb{B}^4 \to H^4$. In this subsection, we will construct a pluriharmonic map $f_1(z^1, z^2, \bar{z}^1, \bar{z}^2) : \mathbb{A}^4 \to S^4$ and a para-pluriharmonic map $f_2(x^1, x^2, y^1, y^2) : \mathbb{B}^4 \to H^4$ by arguments similar to those in Subsection 5.1.1. Here S^4 and H^4 denote a sphere and a upper half space of dimension 4, respectively. Henceforth, we will use the following notation:

(5.1.11) $G^{\mathbb{C}} = Sp(2, \mathbb{C})$ (see [14, p. 445] for $Sp(2, \mathbb{C})$),
(5.1.12) $\sigma(A) := K_{1,1} \cdot A \cdot K_{1,1}$ for $A \in G^{\mathbb{C}}$, where $K_{1,1} := \operatorname{diag}(-1, 1, -1, 1)$,
(5.1.13) $\nu_1(A) := {}^t(\overline{A})^{-1}$ for $A \in G^{\mathbb{C}}$,
(5.1.14) $\nu_2(A) := K_{1,1} \cdot {}^t(\overline{A})^{-1} \cdot K_{1,1}$ for $A \in G^{\mathbb{C}}$,
(5.1.15) $G^{\mathbb{C}}/H^{\mathbb{C}} = Sp(2, \mathbb{C})/(Sp(1, \mathbb{C}) \times Sp(1, \mathbb{C}))$,
(5.1.16) $G_1/H_1 = Sp(2)/(Sp(1) \times Sp(1)) \simeq S^4$,
(5.1.17) $G_2/H_2 = Sp(1,1)/(Sp(1) \times Sp(1)) \simeq H^4$,
(5.1.18) π_i: the projection from G_i onto G_i/H_i $(i = 1, 2)$,
(5.1.19) $\mathfrak{g}_2 := \operatorname{Lie} G_2 = \mathfrak{sp}(1, 1)$.

Define a $\widetilde{\Lambda}_{-1,\infty}(\mathfrak{g}_2)_\sigma$-valued para-holomorphic 1-form $\eta_\theta(x^1, x^2)$ on $(\mathbb{B}^4, I)$ by

$$\eta_\theta(x^1, x^2) := \theta^{-1} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} dx^1 + \theta \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} dx^2.$$

In view of the morphing condition (M), it is natural that one defines a $\widetilde{\Lambda}_{-\infty,1}(\mathfrak{g}_2)_\sigma$-valued para-antiholomorphic 1-form $\tau_\theta(y^1, y^2)$ as follows:

$$\tau_\theta(y^1, y^2) := \theta \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} dy^1 + \theta^{-1} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} dy^2.$$

Let us solve the two initial value problems: $(A_\theta^-)^{-1} \cdot dA_\theta^- = \eta_\theta$ and $(A_\theta^+)^{-1} \cdot dA_\theta^+ = \tau_\theta$ with $A_\theta^\pm(0, 0) \equiv \mathrm{id}$, and factorize $(A_\theta^-, A_\theta^+) \in \widetilde{\Lambda}(G_2)_\sigma \times \widetilde{\Lambda}(G_2)_\sigma$ in the Iwasawa decomposition (cf. Theorem 3.1.5): $(A^-_\theta, A^+_\theta) = (C_\theta, C_\theta)\cdot(B^+_\theta, B^-_\theta)$, where $C_\theta \in \widetilde{\Lambda}(G_2)_\sigma$, $B^+_\theta \in \widetilde{\Lambda}^+_*(G_2)_\sigma$ and $B^-_\theta \in \widetilde{\Lambda}^-(G_2)_\sigma$. Then, it follows that

$$A^-_\theta(x^1,x^2) = \begin{pmatrix} \cosh(\frac{x^1-\theta^2x^2}{\theta}) & \sinh(\frac{x^1-\theta^2x^2}{\theta}) & 0 & 0 \\ \sinh(\frac{x^1-\theta^2x^2}{\theta}) & \cosh(\frac{x^1-\theta^2x^2}{\theta}) & 0 & 0 \\ 0 & 0 & \cosh(\frac{x^1-\theta^2x^2}{\theta}) & -\sinh(\frac{x^1-\theta^2x^2}{\theta}) \\ 0 & 0 & -\sinh(\frac{x^1-\theta^2x^2}{\theta}) & \cosh(\frac{x^1-\theta^2x^2}{\theta}) \end{pmatrix},$$

$$A^+_\theta(y^1,y^2) = \begin{pmatrix} \cosh(\frac{\theta^2y^1-y^2}{\theta}) & -\sinh(\frac{\theta^2y^1-y^2}{\theta}) & 0 & 0 \\ -\sinh(\frac{\theta^2y^1-y^2}{\theta}) & \cosh(\frac{\theta^2y^1-y^2}{\theta}) & 0 & 0 \\ 0 & 0 & \cosh(\frac{\theta^2y^1-y^2}{\theta}) & \sinh(\frac{\theta^2y^1-y^2}{\theta}) \\ 0 & 0 & \sinh(\frac{\theta^2y^1-y^2}{\theta}) & \cosh(\frac{\theta^2y^1-y^2}{\theta}) \end{pmatrix},$$

$$B^+_\theta(x^2,y^1)$$
$$= \begin{pmatrix} \cosh(\theta(x^2-y^1)) & -\sinh(\theta(x^2-y^1)) & 0 & 0 \\ -\sinh(\theta(x^2-y^1)) & \cosh(\theta(x^2-y^1)) & 0 & 0 \\ 0 & 0 & \cosh(\theta(x^2-y^1)) & \sinh(\theta(x^2-y^1)) \\ 0 & 0 & \sinh(\theta(x^2-y^1)) & \cosh(\theta(x^2-y^1)) \end{pmatrix},$$

$$B^-_\theta(x^1,y^2) = \begin{pmatrix} \cosh(\frac{x^1-y^2}{\theta}) & -\sinh(\frac{x^1-y^2}{\theta}) & 0 & 0 \\ -\sinh(\frac{x^1-y^2}{\theta}) & \cosh(\frac{x^1-y^2}{\theta}) & 0 & 0 \\ 0 & 0 & \cosh(\frac{x^1-y^2}{\theta}) & \sinh(\frac{x^1-y^2}{\theta}) \\ 0 & 0 & \sinh(\frac{x^1-y^2}{\theta}) & \cosh(\frac{x^1-y^2}{\theta}) \end{pmatrix},$$

$$C_\theta(x^1,x^2,y^1,y^2) = \begin{pmatrix} \cosh(\frac{x^1-\theta^2y^1}{\theta}) & \sinh(\frac{x^1-\theta^2y^1}{\theta}) & 0 & 0 \\ \sinh(\frac{x^1-\theta^2y^1}{\theta}) & \cosh(\frac{x^1-\theta^2y^1}{\theta}) & 0 & 0 \\ 0 & 0 & \cosh(\frac{x^1-\theta^2y^1}{\theta}) & -\sinh(\frac{x^1-\theta^2y^1}{\theta}) \\ 0 & 0 & -\sinh(\frac{x^1-\theta^2y^1}{\theta}) & \cosh(\frac{x^1-\theta^2y^1}{\theta}) \end{pmatrix}.$$

Substitute λ, z^i and $\bar{z}^i$ for θ, x^i and y^i $(i = 1,2)$, respectively:

$$C_\lambda(z^1,z^2,\bar{z}^1,\bar{z}^2) = \begin{pmatrix} \cosh(\frac{z^1-\lambda^2\bar{z}^1}{\lambda}) & \sinh(\frac{z^1-\lambda^2\bar{z}^1}{\lambda}) & 0 & 0 \\ \sinh(\frac{z^1-\lambda^2\bar{z}^1}{\lambda}) & \cosh(\frac{z^1-\lambda^2\bar{z}^1}{\lambda}) & 0 & 0 \\ 0 & 0 & \cosh(\frac{z^1-\lambda^2\bar{z}^1}{\lambda}) & -\sinh(\frac{z^1-\lambda^2\bar{z}^1}{\lambda}) \\ 0 & 0 & -\sinh(\frac{z^1-\lambda^2\bar{z}^1}{\lambda}) & \cosh(\frac{z^1-\lambda^2\bar{z}^1}{\lambda}) \end{pmatrix}$$

for $C_\theta(x^1,x^2,y^1,y^2)$. Since $(z^1-\lambda^2\bar{z}^1)/\lambda$ is a purely imaginary number, one sees that $C_\lambda(z^1,z^2,\bar{z}^1,\bar{z}^2) \in G_1 = Sp(2)$ for all $(z^1,z^2,\bar{z}^1,\bar{z}^2;\lambda) \in \mathbb{A}^4\times S^1$. Accordingly, we obtain a pluriharmonic map f_1 and a para-pluriharmonic map f_2,

$$(f_1)_\lambda = \pi_1\circ C_\lambda(z^1,z^2,\bar{z}^1,\bar{z}^2) : (\mathbb{A}^4, J) \longrightarrow G_1/H_1 \simeq S^4, \quad \lambda \in S^1,$$
$$(f_2)_\theta = \pi_2\circ C_\theta(x^1,x^2,y^1,y^2) : (\mathbb{B}^4, I) \longrightarrow G_2/H_2 \simeq H^4, \quad \theta \in \mathbb{R}^+$$

(ref. Subsection 5.1.1).

$$(f_1)_\lambda = \pi_1 \circ C_\lambda(z^1, z^2, \bar{z}^1, \bar{z}^2) : (\mathbb{A}^4, J) \to S^4 \text{ is pluriharmonic}$$
$$\Updownarrow$$
$$(f_2)_\theta = \pi_2 \circ C_\theta(x^1, x^2, y^1, y^2) : (\mathbb{B}^4, I) \to H^4 \text{ is para-pluriharmonic}$$

5.2. **Harmonic maps, Lorentz harmonic maps and CMC-surfaces.** In this subsection we will interrelate some harmonic maps $f_1(z, \bar{z}) : \mathbb{A}^2 \to G_1/H_1$ with Lorentz harmonic maps $f_2(x, y) : \mathbb{B}^2 \to G_2/H_2$ by means of Theorem 4.3.1; and in addition, we will interrelate CMC-surfaces with other CMC-surfaces in $\mathbb{R}^3$ or $\mathbb{R}^3_1$, by use of $f_1(z, \bar{z})$ and $f_2(x, y)$. More precisely, we interrelate a cylinder in $\mathbb{R}^3$ with a hyperbolic cylinder in $\mathbb{R}^3_1$ (cf. Subsection 5.2.1), a two sheeted hyperboloid in $\mathbb{R}^3_1$ with a one sheeted hyperboloid in $\mathbb{R}^3_1$ (cf. Subsection 5.2.2), a sphere in $\mathbb{R}^3$ with a one sheeted hyperboloid in $\mathbb{R}^3_1$ (cf. Subsection 5.2.3), a Smyth surface in $\mathbb{R}^3$ with a timelike Smyth surface in $\mathbb{R}^3_1$ (cf. Subsection 5.2.4), and a Delaunay surface in $\mathbb{R}^3$ with a K-surface of revolution in $\mathbb{R}^3$ (cf. Subsection 5.2.5).

5.2.1. *Cylinder in* $\mathbb{R}^3$ $\Leftrightarrow$ *Hyperbolic cylinder in* $\mathbb{R}^3_1$. In this subsection we will use the following notation:

(5.2.1) $G^{\mathbb{C}} = SL(2, \mathbb{C})$,
(5.2.2) $\sigma(A) := I_{1,1} \cdot A \cdot I_{1,1}$ for $A \in G^{\mathbb{C}}$, where $I_{1,1} := \mathrm{diag}(-1, 1)$,
(5.2.3) $\nu_1(A) := {}^t(\overline{A})^{-1}$ for $A \in G^{\mathbb{C}}$,
(5.2.4) $\nu_2(A) := \overline{A}$ for $A \in G^{\mathbb{C}}$,
(5.2.5) $G^{\mathbb{C}}/H^{\mathbb{C}} = SL(2, \mathbb{C})/S(GL(1, \mathbb{C}) \times GL(1, \mathbb{C}))$,
(5.2.6) $G_1/H_1 = SU(2)/S(U(1) \times U(1)) \simeq S^2$,
(5.2.7) $G_2/H_2 = SL(2, \mathbb{R})/S(GL(1, \mathbb{R}) \times GL(1, \mathbb{R})) \simeq S^2_1$,
(5.2.8) π_i: the projection from G_i onto G_i/H_i $(i = 1, 2)$,
(5.2.9) $\mathfrak{g}_2 := \mathrm{Lie}\, G_2 = \mathfrak{sl}(2, \mathbb{R})$.

We will construct a harmonic map $f_1 : (\mathbb{A}^2, J) \to S^2$ and a Lorentz harmonic map $f_2 : (\mathbb{B}^2, I) \to S^2_1$ by means of Theorem 4.3.1; and moreover, a cylinder in $\mathbb{R}^3$ and a hyperbolic cylinder in $\mathbb{R}^3_1$ from f_1 and f_2, respectively.

In the first place, we define a $\widetilde{\Lambda}_{-1,\infty}(\mathfrak{g}_2)_\sigma$-valued, real analytic para-holomorphic 1-form $\eta_\theta(x)$ on $(\mathbb{B}^2, I)$ by

$$\eta_\theta(x) := \theta^{-1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} dx. \tag{5.2.10}$$

In the second place, we define a $\widetilde{\Lambda}_{-\infty,1}(\mathfrak{g}_2)_\sigma$-valued para-antiholomorphic 1-form $\tau_\theta(y)$ on $(\mathbb{B}^2, I)$ by taking the morphing condition (M) in Theorem 4.3.1 into consideration, i.e.,

$$\tau_\theta(y) := \theta \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} dy.$$

In the third place, let us solve the two initial value problems: $A_\theta^{-1}\cdot dA_\theta = \eta_\theta(x)$, $B_\theta^{-1}\cdot dB_\theta = \tau_\theta(y)$ and $A_\theta(0) \equiv \mathrm{id} \equiv B_\theta(0)$. In this case, one can obtain

$$A_\theta(x) = \begin{pmatrix} \cosh(\theta^{-1}x) & \sinh(\theta^{-1}x) \\ \sinh(\theta^{-1}x) & \cosh(\theta^{-1}x) \end{pmatrix}, \quad B_\theta(y) = \begin{pmatrix} \cosh(-\theta y) & \sinh(-\theta y) \\ \sinh(-\theta y) & \cosh(-\theta y) \end{pmatrix}$$

and the Iwasawa decomposition: $(A_\theta(x), B_\theta(y)) = (C_\theta(x,y), C_\theta(x,y))\cdot(B_\theta^+(x,y), B_\theta^-(x,y))$, where $B_\theta^+(x,y) := B_\theta(y)^{-1} \in \widetilde{\Lambda}_*^+(G_2)_\sigma$ and $B_\theta^-(x,y) := A_\theta(x)^{-1} \in \widetilde{\Lambda}_*^-(G_2)_\sigma$. Here $C_\theta(x,y)$ is given as follows:

$$C_\theta(x,y) = \begin{pmatrix} \cosh(\theta^{-1}x - \theta y) & \sinh(\theta^{-1}x - \theta y) \\ \sinh(\theta^{-1}x - \theta y) & \cosh(\theta^{-1}x - \theta y) \end{pmatrix}. \tag{5.2.11}$$

This $C_\theta(x,y)$ provides us with an $\mathbb{R}^+$-family of Lorentz harmonic maps

$$(f_2)_\theta = \pi_2 \circ C_\theta(x,y) : (\mathbb{B}^2, I) \longrightarrow G_2/H_2 \simeq S_1^2, \quad \theta \in \mathbb{R}^+$$

(cf. Proposition 3.2.3). In the fourth place, we substitute λ, z and $\bar{z}$ for θ, x and y, respectively:

$$C_\lambda(z,\bar{z}) = \begin{pmatrix} \cosh(\lambda^{-1}z - \lambda\bar{z}) & \sinh(\lambda^{-1}z - \lambda\bar{z}) \\ \sinh(\lambda^{-1}z - \lambda\bar{z}) & \cosh(\lambda^{-1}z - \lambda\bar{z}) \end{pmatrix} \tag{5.2.12}$$

for $C_\theta(x,y)$. Remark that $C_\lambda(z,\bar{z}) \in G_1 = SU(2)$ for all $(z,\bar{z};\lambda) \in \mathbb{A}^2 \times S^1$ because $(\lambda^{-1}z - \lambda\bar{z})$ is a purely imaginary number. As a consequence, one can construct a harmonic map f_1 and a Lorentz harmonic map f_2,

$$\begin{aligned} (f_1)_\lambda &= \pi_1 \circ C_\lambda(z,\bar{z}) : (\mathbb{A}^2, J) \longrightarrow G_1/H_1 \simeq S^2, \quad \lambda \in S^1, \\ (f_2)_\theta &= \pi_2 \circ C_\theta(x,y) : (\mathbb{B}^2, I) \longrightarrow G_2/H_2 \simeq S_1^2, \quad \theta \in \mathbb{R}^+, \end{aligned}$$

from the potential (5.2.10) $\eta_\theta(x)$.

$$\boxed{\begin{array}{c} (f_1)_\lambda = \pi_1 \circ C_\lambda(z,\bar{z}) : (\mathbb{A}^2, J) \to S^2 \text{ is harmonic} \\ \Updownarrow \\ (f_2)_\theta = \pi_2 \circ C_\theta(x,y) : (\mathbb{B}^2, I) \to S_1^2 \text{ is Lorentz harmonic} \end{array}}$$

Here $C_\lambda(z,\bar{z})$ and $C_\theta(x,y)$ are given by (5.2.12) and (5.2.11), respectively.

Note that we have constructed the extended framing $C_\lambda(z,\bar{z}) : \mathbb{A}^2 \to S^2$ of a harmonic map and the extended framing $C_\theta(x,y) : \mathbb{B}^2 \to S_1^2$ of a Lorentz harmonic map. For this reason, the Sym-Bobenko formula will enable us to obtain a CMC-surface $\phi_1(z,\bar{z}) : \mathbb{A}^2 \to \mathbb{R}^3$ and a timelike CMC-surface $\phi_2(x,y) : \mathbb{B}^2 \to \mathbb{R}_1^3$ from them.

For $C_\lambda(z,\bar{z})$, the Sym-Bobenko formula in [11, p. 30] yields

$$\begin{aligned}\phi_1(z,\bar{z}) :&= -\left\{ i\cdot\lambda\cdot\frac{\partial C_\lambda}{\partial\lambda}\cdot C_\lambda^{-1} + \frac{1}{2}\cdot \mathrm{Ad}(C_\lambda)\cdot\begin{pmatrix} i & 0\\ 0 & -i\end{pmatrix}\right\}\Big|_{\lambda=1}\\ &= \frac{-i}{2}\begin{pmatrix} \cosh 2(z-\bar{z}) & -2(z+\bar{z})-\sinh 2(z-\bar{z})\\ -2(z+\bar{z})+\sinh 2(z-\bar{z}) & -\cosh 2(z-\bar{z})\end{pmatrix}\\ &\simeq (-2(z+\bar{z}), i\cdot\sinh 2(z-\bar{z}), -\cosh 2(z-\bar{z})).\end{aligned}$$

This CMC-surface $\phi_1(z,\bar{z}) : \mathbb{A}^2 \to \mathbb{R}^3$ is a cylinder. For $C_\theta(x,y)$, the Sym-Bobenko formula in [9][1] is given as follows:

$$\begin{aligned}\phi_2(x,y) :&= -2\left\{ -\theta\cdot\frac{\partial C_\theta}{\partial\theta}\cdot C_\theta^{-1} + \frac{1}{2}\cdot \mathrm{Ad}(C_\theta)\cdot\begin{pmatrix} -1 & 0\\ 0 & 1\end{pmatrix}\right\}\Big|_{\theta=1}\\ &= \begin{pmatrix} \cosh 2(x-y) & -2(x+y)-\sinh 2(x-y)\\ -2(x+y)+\sinh 2(x-y) & -\cosh 2(x-y)\end{pmatrix}\\ &\simeq (\sinh 2(x-y), -2(x+y), -\cosh 2(x-y)).\end{aligned}$$

This timelike CMC-surface $\phi_2(x,y) : \mathbb{B}^2 \to \mathbb{R}^3_1$ is a hyperbolic cylinder, because $-(\sinh 2(x-y))^2 + (-2(x+y))^2 + (-\cosh 2(x-y))^2 = 4(x+y)^2 + 1$ (see Section 1.1 in [9] for the metric on $\mathbb{R}^3_1$).

CMC-surface in $\mathbb{R}^3$: Cylinder
$\phi_1(z,\bar{z}) = (-2(z+\bar{z}), i\cdot\sinh 2(z-\bar{z}), -\cosh 2(z-\bar{z}))$
$\Updownarrow$
Timelike CMC-surface in $\mathbb{R}^3_1$: Hyperbolic cylinder
$\phi_2(x,y) = (\sinh 2(x-y), -2(x+y), -\cosh 2(x-y))$

5.2.2. *Two sheeted hyperboloid in* $\mathbb{R}^3_1 \Leftrightarrow$ *One sheeted hyperboloid in* $\mathbb{R}^3_1$. In this subsection we will use the following notation:

(5.2.13) $G^{\mathbb{C}}$: the same notation (5.2.1) as in Subsection 5.2.1,
(5.2.14) σ: the same notation (5.2.2) as in Subsection 5.2.1,
(5.2.15) $\nu_1(A) := I_{1,1}\cdot {}^t(\overline{A})^{-1}\cdot I_{1,1}$ for $A\in G^{\mathbb{C}}$,
(5.2.16) $\nu_2(A) := I_{1,1}\cdot \overline{A}\cdot I_{1,1}$ for $A\in G^{\mathbb{C}}$,
(5.2.17) $G^{\mathbb{C}}/H^{\mathbb{C}}$: the same notation (5.2.5) as in Subsection 5.2.1,
(5.2.18) $G_1/H_1 = SU(1,1)/S(U(1)\times U(1)) \simeq H^2$,
(5.2.19) $G_2/H_2 = SL_*(2,\mathbb{R})/S(GL(1,\mathbb{R})\times GL(1,\mathbb{R})) \simeq S^2_1$,
(5.2.20) π_i: the projection from G_i onto G_i/H_i $(i=1,2)$,
(5.2.21) $\mathfrak{g}_2 := \mathrm{Lie}\, G_2 = \mathfrak{sl}_*(2,\mathbb{R})$.

[1] We must change $(\partial\Phi/\partial t)$ into $-(\partial\Phi/\partial t)$ in the Sym-Bobenko formula [9, Proposition 5.1] because the parameter λ in this paper corresponds to the parameter λ^{-1} in [9].

where the above notation $SL_*(2,\mathbb{R})$ and $\mathfrak{sl}_*(2,\mathbb{R})$ are the same as those in [17].

The arguments below are similar to those in Subsection 5.2.1. Define a $\widetilde{\Lambda}_{-1,\infty}(\mathfrak{g}_2)_\sigma$-valued analytic para-holomorphic 1-form $\eta_\theta(x)$ on $(\mathbb{B}^2, I)$ by

$$\eta_\theta(x) := \theta^{-1}\begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix} dx. \tag{5.2.22}$$

We want $\tau_\theta(y) \in \widetilde{\mathcal{P}}^-(\mathfrak{g}_2)$ to satisfy the morphing condition (M) in Theorem 4.3.1; and therefore we define $\tau_\theta(y)$ as follows:

$$\tau_\theta(y) := \theta\begin{pmatrix} 0 & 0 \\ -i & 0 \end{pmatrix} dy.$$

Solve the two initial value problems: $A_\theta^{-1}\cdot dA_\theta = \eta_\theta(x)$, $B_\theta^{-1}\cdot dB_\theta = \tau_\theta(y)$ and $A_\theta(0) \equiv \mathrm{id} \equiv B_\theta(0)$. Then one has

$$A_\theta(x) = \begin{pmatrix} 1 & i\theta^{-1}x \\ 0 & 1 \end{pmatrix}, \quad B_\theta(y) = \begin{pmatrix} 1 & 0 \\ -i\theta y & 1 \end{pmatrix}.$$

Let us factorize $(A_\theta, B_\theta) \in \widetilde{\Lambda}(G_2)_\sigma \times \widetilde{\Lambda}(G_2)_\sigma$ in the Iwasawa decomposition around $(0,0)$: $(A_\theta, B_\theta) = (C_\theta, C_\theta)\cdot(B_\theta^+, B_\theta^-)$, $C_\theta \in \widetilde{\Lambda}(G_2)_\sigma$ and $B_\theta^\pm \in \widetilde{\Lambda}^\pm(G_2)_\sigma$ (cf. Theorem 3.1.5). Here $B_\theta^\pm$ and C_θ are given as follows:

$$B_\theta^+(x,y) = \frac{1}{\sqrt{1-xy}}\begin{pmatrix} 1 & 0 \\ i\theta y & 1-xy \end{pmatrix}, \quad B_\theta^-(x,y) = \frac{1}{\sqrt{1-xy}}\begin{pmatrix} 1-xy & -i\theta^{-1}x \\ 0 & 1 \end{pmatrix},$$

$$C_\theta(x,y) = \frac{1}{\sqrt{1-xy}}\begin{pmatrix} 1 & i\theta^{-1}x \\ -i\theta y & 1 \end{pmatrix}. \tag{5.2.23}$$

From $C_\theta(x,y)$ one obtains an $\mathbb{R}^+$-family of Lorentz harmonic maps

$$(f_2)_\theta = \pi_2 \circ C_\theta(x,y) : (W, I) \longrightarrow G_2/H_2 \simeq S_1^2,$$

where $W := \{(x,y) \in \mathbb{B}^2 \,|\, xy \neq 1\}$. Substituting λ, z and $\bar{z}$ for θ, x and y, respectively, we have

$$C_\lambda(z,\bar{z}) = \frac{1}{\sqrt{1-|z|^2}}\begin{pmatrix} 1 & i\lambda^{-1}z \\ -i\lambda\bar{z} & 1 \end{pmatrix} \tag{5.2.24}$$

for $C_\theta(x,y)$. It is obvious that $C_\lambda(z,\bar{z}) \in G_1 = SU(1,1)$ for all $(z,\bar{z};\lambda) \in V \times S^1$, where $V := \mathbb{A}^2 \setminus S^1$. Consequently, one can get a harmonic map $f_1(z,\bar{z})$ and a Lorentz harmonic map $f_2(x,y)$,

$$(f_1)_\lambda = \pi_1 \circ C_\lambda(z,\bar{z}) : (V, J) \longrightarrow G_1/H_1 \simeq H^2, \quad \lambda \in S^1,$$
$$(f_2)_\theta = \pi_2 \circ C_\theta(x,y) : (W, I) \longrightarrow G_2/H_2 \simeq S_1^2, \quad \theta \in \mathbb{R}^+,$$

from the potential (5.2.22).

$$\boxed{\begin{array}{c}(f_1)_\lambda = \pi_1 \circ C_\lambda(z,\bar{z}) : (V,J) \to H^2 \text{ is harmonic} \\ \Updownarrow \\ (f_2)_\theta = \pi_2 \circ C_\theta(x,y) : (W,I) \to S^2_1 \text{ is Lorentz harmonic}\end{array}}$$

Here $C_\lambda(z,\bar{z})$ and $C_\theta(x,y)$ are given by (5.2.24) and (5.2.23), respectively; and $V = \mathbb{A}^2 \setminus S^1$ and $W = \{(x,y) \in \mathbb{B}^2 \,|\, xy \neq 1\}$.

Now, let us obtain a spacelike CMC-surface $\phi_1(z,\bar{z}) : V \to \mathbb{R}^3_1$ and a timelike CMC-surface $\phi_2(x,y) : W \to \mathbb{R}^3_1$ from the above $f_1(z,\bar{z})$ and $f_2(x,y)$, respectively.

On the one hand, the Sym-Bobenko formula in [4], together with (5.2.24), gives us

$$\begin{aligned}\phi_1(z,\bar{z}) :&= -\left\{ i \cdot \lambda \cdot \frac{\partial C_\lambda}{\partial \lambda} \cdot C_\lambda^{-1} + \frac{1}{2} \cdot \mathrm{Ad}(C_\lambda) \cdot \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \right\} \Big|_{\lambda=1} \\ &= \begin{pmatrix} -i(1+3|z|^2)/2(1-|z|^2) & -2z/(1-|z|^2) \\ -2\bar{z}/(1-|z|^2) & i(1+3|z|^2)/2(1-|z|^2) \end{pmatrix}.\end{aligned}$$

Thus we have a spacelike CMC-surface in $\mathbb{R}^3_1$,

$$\phi_1 : V \longrightarrow \mathbb{R}^3_1, \quad (z,\bar{z}) \mapsto \Big(-\frac{z+\bar{z}}{1-|z|^2}, \frac{i(z-\bar{z})}{1-|z|^2}, -\frac{1+3|z|^2}{2(1-|z|^2)} \Big)$$

(cf. Subsection 3.2.1 in [4]). This $\phi_1(z,\bar{z})$ is a two sheeted hyperboloid centered at $(0,0,1/2)$ because

$$\Big(-\frac{z+\bar{z}}{1-|z|^2} \Big)^2 + \Big(\frac{i(z-\bar{z})}{1-|z|^2} \Big)^2 - \Big(-\frac{1+3|z|^2}{2(1-|z|^2)} - \frac{1}{2} \Big)^2 = -1$$

(see Subsection 3.2.1 in [4] for the metric on $\mathbb{R}^3_1$). One the other hand, the Sym-Bobenko formula in [17], combined with (5.2.23), gives us

$$\begin{aligned}\phi_2(x,y) :&= -\frac{1}{2} \left\{ \theta \cdot \frac{\partial C_\theta}{\partial \theta} \cdot C_\theta^{-1} + \frac{1}{2} \cdot \mathrm{Ad}(C_\theta) \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\} \Big|_{\theta=1} \\ &= \begin{pmatrix} -(1+3xy)/4(1-xy) & ix/(1-xy) \\ iy/(1-xy) & (1+3xy)/4(1-xy) \end{pmatrix}\end{aligned}$$

(ref. Proof of Corollary 3.4 in [17]). Then it turns out that

$$\phi_2 : W \longrightarrow \mathbb{R}^3_1, \quad (x,y) \mapsto \Big(-\frac{x+y}{1-xy}, -\frac{x-y}{1-xy}, -\frac{1+3xy}{2(1-xy)} \Big)$$

(cf. Subsection 3.1 in [17]). This $\phi_2(x,y)$ is a one sheeted hyperboloid centered at $(0,0,1/2)$. Indeed, we deduce

$$\Big(-\frac{x+y}{1-xy} \Big)^2 - \Big(-\frac{x-y}{1-xy} \Big)^2 - \Big(-\frac{1+3xy}{2(1-xy)} - \frac{1}{2} \Big)^2 = -1$$

by a direct computation (see Remark 3.2 in [17] for the metric on $\mathbb{R}^3_1$).

Spacelike CMC-surface in $\mathbb{R}^3_1$: Two sheeted hyperboloid
$\phi_1(z,\bar{z}) = \left(-\frac{z+\bar{z}}{1-\lvert z\rvert^2}, \frac{i(z-\bar{z})}{1-\lvert z\rvert^2}, -\frac{1+3\lvert z\rvert^2}{2(1-\lvert z\rvert^2)}\right)$
$\Updownarrow$
Timelike CMC-surface in $\mathbb{R}^3_1$: One sheeted hyperboloid
$\phi_2(x,y) = \left(-\frac{x+y}{1-xy}, -\frac{x-y}{1-xy}, -\frac{1+3xy}{2(1-xy)}\right)$

5.2.3. *Sphere in $\mathbb{R}^3 \Leftrightarrow$ One sheeted hyperboloid in $\mathbb{R}^3_1$.* In this subsection, we utilize the same potential as in Subsection 5.2.2, but we will obtain other CMC-surfaces. For this we will use the following notation:

(5.2.25) $G^{\mathbb{C}}$: the same notation (5.2.1) as in Subsection 5.2.1,
(5.2.26) σ: the same notation (5.2.2) as in Subsection 5.2.1,
(5.2.27) ν_1: the same notation (5.2.3) as in Subsection 5.2.1,
(5.2.28) ν_2: the same notation (5.2.16) as in Subsection 5.2.2,
(5.2.29) $G^{\mathbb{C}}/H^{\mathbb{C}}$: the same notation (5.2.5) as in Subsection 5.2.1,
(5.2.30) G_1/H_1: the same notation (5.2.6) as in Subsection 5.2.1,
(5.2.31) G_2/H_2: the same notation (5.2.19) as in Subsection 5.2.2,
(5.2.32) π_i: the projection from G_i onto G_i/H_i $(i = 1, 2)$,
(5.2.33) $\mathfrak{g}_2$: the same notation (5.2.21) as in Subsection 5.2.2.

Let $\eta_\theta(x)$ denote the potential (5.2.22). Define $\tau_\theta(y) \in \widetilde{\mathcal{P}}^-(\mathfrak{g}_2)$ by

$$\tau_\theta(y) := \theta \begin{pmatrix} 0 & 0 \\ i & 0 \end{pmatrix} dy.$$

Here we remark that $(\eta_\theta(x), \tau_\theta(y))$ is a real analytic para-pluriharmonic potential on $(\mathbb{B}^2, I)$ satisfying the morphing condition (M). Solve the two initial value problems: $A_\theta^{-1} \cdot dA_\theta = \eta_\theta(x)$, $B_\theta^{-1} \cdot dB_\theta = \tau_\theta(y)$ and $A_\theta(0) \equiv \mathrm{id} \equiv B_\theta(0)$; and factorize $(A_\theta, B_\theta) \in \widetilde{\Lambda}(G_2)_\sigma \times \widetilde{\Lambda}(G_2)_\sigma$ in the Iwasawa decomposition (cf. Theorem 3.1.5): $(A_\theta, B_\theta) = (C_\theta, C_\theta) \cdot (B_\theta^+, B_\theta^-)$, $C_\theta \in \widetilde{\Lambda}(G_2)_\sigma$ and $B_\theta^\pm \in \widetilde{\Lambda}^\pm(G_2)_\sigma$. In this case it follows that

$$A_\theta(x) = \begin{pmatrix} 1 & i\theta^{-1}x \\ 0 & 1 \end{pmatrix}, \qquad B_\theta(y) = \begin{pmatrix} 1 & 0 \\ i\theta y & 1 \end{pmatrix};$$
$$B_\theta^+(x,y) = \frac{1}{\sqrt{1+xy}} \begin{pmatrix} 1 & 0 \\ -i\theta y & 1+xy \end{pmatrix}, \quad B_\theta^-(x,y) = \frac{1}{\sqrt{1+xy}} \begin{pmatrix} 1+xy & -i\theta^{-1}x \\ 0 & 1 \end{pmatrix};$$
$$C_\theta(x,y) = \frac{1}{\sqrt{1+xy}} \begin{pmatrix} 1 & i\theta^{-1}x \\ i\theta y & 1 \end{pmatrix}.$$

It is easy to see that $C_\lambda(z,\bar{z}) \in G_1 = SU(2)$ for all $(z,\bar{z};\lambda) \in \mathbb{A}^2 \times S^1$. Accordingly, we obtain a harmonic map f_1 and a Lorentz harmonic map f_2,

$$(f_1)_\lambda = \pi_1 \circ C_\lambda(z,\bar{z}) : (\mathbb{A}^2, J) \longrightarrow G_1/H_1 \simeq S^2, \quad \lambda \in S^1,$$
$$(f_2)_\theta = \pi_2 \circ C_\theta(x,y) : (W, I) \longrightarrow G_2/H_2 \simeq S^2_1, \quad \theta \in \mathbb{R}^+,$$

from (5.2.22). Here $W := \{(x,y) \in \mathbb{B}^2 \,|\, xy \neq -1\}$. The above maps will provide us with a CMC-surface $\phi_1 : \mathbb{A}^2 \to \mathbb{R}^3$ and a timelike CMC-surface $\phi_2 : W \to \mathbb{R}^3_1$. The Sym-Bobenko formula in [11], combined with $C_\lambda(z,\bar{z})$, gives

$$\begin{aligned}\phi_1(z,\bar{z}) :&= -\left\{ i \cdot \lambda \cdot \frac{\partial C_\lambda}{\partial \lambda} \cdot C_\lambda^{-1} + \frac{1}{2} \cdot \mathrm{Ad}(C_\lambda) \cdot \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \right\} \Big|_{\lambda=1} \\ &= \frac{-i}{2} \begin{pmatrix} (1-3|z|^2)/(1+|z|^2) & -4iz/(1+|z|^2) \\ 4i\bar{z}/(1+|z|^2) & -(1-3|z|^2)/(1+|z|^2) \end{pmatrix} \\ &\simeq \Big(\frac{-2i(z-\bar{z})}{1+|z|^2}, \frac{-2(z+\bar{z})}{1+|z|^2}, \frac{-1+3|z|^2}{1+|z|^2} \Big).\end{aligned}$$

This CMC-surface $\phi_1(z,\bar{z}) : \mathbb{A}^2 \to \mathbb{R}^3$ is a sphere centered at $(0,0,1)$. By the above $C_\theta(x,y)$ and the Sym-Bobenko formula in [17], we obtain

$$\begin{aligned}\phi_2(x,y) :&= -\frac{1}{2}\left\{ \theta \cdot \frac{\partial C_\theta}{\partial \theta} \cdot C_\theta^{-1} + \frac{1}{2} \cdot \mathrm{Ad}(C_\theta) \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\} \Big|_{\theta=1} \\ &= \begin{pmatrix} -(1-3xy)/4(1+xy) & ix/(1+xy) \\ -iy/(1+xy) & (1-3xy)/4(1+xy) \end{pmatrix} \\ &\simeq \Big(-\frac{x-y}{1+xy}, -\frac{x+y}{1+xy}, -\frac{1-3xy}{2(1+xy)} \Big).\end{aligned}$$

This timelike CMC-surface $\phi_2(x,y) : W \to \mathbb{R}^3_1$ is a one sheeted hyperboloid centered at $(0,0,1/2)$ because

$$\Big(-\frac{x-y}{1+xy} \Big)^2 - \Big(-\frac{x+y}{1+xy} \Big)^2 - \Big(-\frac{1-3xy}{2(1+xy)} - \frac{1}{2} \Big)^2 = -1$$

(see Remark 3.2 in [17] for the metric on $\mathbb{R}^3_1$).

CMC-surface in $\mathbb{R}^3$: Sphere
$\phi_1(z,\bar{z}) = \big(\frac{-2i(z-\bar{z})}{1+\|z\|^2}, \frac{-2(z+\bar{z})}{1+\|z\|^2}, \frac{-1+3\|z\|^2}{1+\|z\|^2} \big)$
$\Updownarrow$
Timelike CMC-surface in $\mathbb{R}^3_1$: One sheeted hyperboloid
$\phi_2(x,y) = \big(-\frac{x-y}{1+xy}, -\frac{x+y}{1+xy}, -\frac{1-3xy}{2(1+xy)} \big)$

5.2.4. *Smyth surface in* $\mathbb{R}^3 \Leftrightarrow$ *Timelike Smyth surface in* $\mathbb{R}^3_1$. In this subsection we construct a timelike CMC-surface, $\phi_2(x,y) : W \to \mathbb{R}^3_1$, from the potential of Smyth surface in $\mathbb{R}^3$ (cf. (5.2.34)); and we study the relation between the Gauß equation for $\phi_2(x,y) : W \to \mathbb{R}^3_1$ and the Painlevé equation of type (III). Henceforth we will use the same notation as in Subsection 5.2.1.

Define a $\widetilde{\Lambda}_{-1,\infty}(\mathfrak{g}_2)_\sigma$-valued, real analytic para-holomorphic 1-form $\eta_\theta(x)$ on $(\mathbb{B}^2, I)$ by

$$\eta_\theta(x) := \theta^{-1} \begin{pmatrix} 0 & 1 \\ x^m & 0 \end{pmatrix} dx, \tag{5.2.34}$$

where $m \in \mathbb{N}$. Taking the morphing condition (M) into consideration, we define $\tau_\theta(y)$ as follows:

$$\tau_\theta(x) := \theta \begin{pmatrix} 0 & -y^m \\ -1 & 0 \end{pmatrix} dy.$$

Solve the two initial value problems: $A_\theta^{-1} \cdot dA_\theta = \eta_\theta(x)$, $B_\theta^{-1} \cdot dB_\theta = \tau_\theta(y)$ and $A_\theta(0) \equiv \mathrm{id} \equiv B_\theta(0)$. In terms of Theorem 3.1.5 we factorize $(A_\theta, B_\theta) \in \widetilde{\Lambda}_*^-(G_2)_\sigma \times \widetilde{\Lambda}_*^+(G_2)_\sigma$ as follows: $(A_\theta, B_\theta) = (C_\theta, C_\theta) \cdot (B_\theta^+, B_\theta^-)$, $C_\theta \in \widetilde{\Lambda}(G_2)_\sigma$ and $B_\theta^+ \in \widetilde{\Lambda}_*^+(G_2)_\sigma$ and $B_\theta^- \in \widetilde{\Lambda}^-(G_2)_\sigma$. Then Proposition 3.2.3 enables us to obtain an $\mathbb{R}^+$-family of Lorentz harmonic maps

$$(f_2)_\theta = \pi_2 \circ C_\theta(x, y) : (W, I) \longrightarrow G_2/H_2 \simeq S_1^2,$$

where W is an open neighborhood of $\mathbb{B}^2$ at $(0, 0)$. Furthermore, Theorem 4.3.1 tells us that there is an S^1-family of harmonic maps

$$(f_1)_\lambda = \pi_1 \circ C'_\lambda(z, \bar{z}) : (V, J) \longrightarrow G_1/H_1 \simeq S^2, \quad C'_\lambda(0, 0) \equiv \mathrm{id},$$

where $C'_\lambda(z, \bar{z}) := C_\lambda(z, \bar{z}) \cdot h^{\mathbb{C}}(z, \bar{z})$ (see Theorem 4.3.1 for V and $h^{\mathbb{C}}(z, \bar{z})$). From the above harmonic map $f_1(z, \bar{z})$, the Sym-Bobenko formula enables us to obtain a CMC-surface $\phi_1(z, \bar{z}) : V \to \mathbb{R}^3$ (ref. Subsection 5.2.1), which is called the *Smyth surface* (cf. [10, p. 662]). In addition, one can obtain a timelike CMC-surface $\phi_2(x, y) : W \to \mathbb{R}_1^3$, from the above Lorentz harmonic map $f_2(x, y)$. We end this subsection with clarifying an important property of $\phi_2(x, y) : W \to \mathbb{R}_1^3$:

PROPOSITION 5.2.1. *With the above setting and notation, the Gauß equation for $\phi_2(x, y)$: $W \to \mathbb{R}_1^3$ is the Painlevé equation of type* (III).

PROOF. Our first aim is to deduce (5.2.36) below. For $k = \mathrm{diag}(s, 1/s) \in H_2 = S(GL(1, \mathbb{R}) \times GL(1, \mathbb{R}))$, let us define real numbers $a = a(k)$ and $b = b(k)$ by $a(k) := s^{-4/m}$ and $b(k) := s^{-(4+2m)/m}$, respectively. Since $k \cdot \eta_\lambda(x) \cdot k^{-1} = \eta_{(b\cdot\lambda)}(a \cdot x)$, $k \cdot \tau_\lambda(y) \cdot k^{-1} = \tau_{(b\cdot\lambda)}(a^{-1} \cdot y)$ and $A_\lambda(0) \equiv \mathrm{id} \equiv B_\lambda(0)$, we understand that

$$k \cdot A_\lambda(x) \cdot k^{-1} = A_{b\cdot\lambda}(a \cdot x), \quad k \cdot B_\lambda(y) \cdot k^{-1} = B_{b\cdot\lambda}(a^{-1} \cdot y). \tag{5.2.35}$$

It is immediate from $B_\lambda^{-1} \cdot A_\lambda = (B_\lambda^-)^{-1} \cdot B_\lambda^+$ and (5.2.35) that $(k \cdot B_\lambda^-(x, y) \cdot k^{-1})^{-1} \cdot (k \cdot B_\lambda^+(x, y) \cdot k^{-1}) = B_{b\cdot\lambda}^-(a \cdot x, a^{-1} \cdot y)^{-1} \cdot B_{b\cdot\lambda}^+(a \cdot x, a^{-1} \cdot y)$ for any $k \in H_2$ and $\lambda \in S^1$. Therefore, the uniqueness of the Birkhoff decomposition allows us to conclude

$$k \cdot B_\lambda^+(x, y) \cdot k^{-1} = B_{b\cdot\lambda}^+(a \cdot x, a^{-1} \cdot y) \quad \text{for any } k \in H_2 \text{ and } \lambda \in S^1.$$

The above and (5.2.35) imply that

$$\begin{aligned} k \cdot C_\lambda(x, y) \cdot k^{-1} &= k \cdot A_\lambda(x) \cdot B_\lambda^+(x, y)^{-1} \cdot k^{-1} \\ &= A_{b\cdot\lambda}(a \cdot x) \cdot B_{b\cdot\lambda}^+(a \cdot x, a^{-1} \cdot y)^{-1} = C_{b\cdot\lambda}(a \cdot x, a^{-1} \cdot y) \end{aligned}$$

—that is, they imply that

$$k \cdot C_\lambda(x, y) \cdot k^{-1} = C_{b\cdot\lambda}(a \cdot x, a^{-1} \cdot y) \quad \text{for any } k \in H_2 \text{ and } \lambda \in S^1. \tag{5.2.36}$$

Now, let $U_\lambda(x,y) := C_\lambda(x,y)^{-1} \cdot \partial_x C_\lambda(x,y)$ and $V_\lambda(x,y) := C_\lambda(x,y)^{-1} \cdot \partial_y C_\lambda(x,y)$. We express these Maurer-Cartan forms explicitly as follows:

$$(5.2.37)\qquad \begin{aligned} U_\lambda(x,y) &= \begin{pmatrix} u_x(x,y)/4 & -(\lambda^{-1}/2)\cdot H \cdot e^{u(x,y)/2} \\ \lambda^{-1}\cdot Q(x)\cdot e^{-u(x,y)/2} & -u_x(x,y)/4 \end{pmatrix}, \\ V_\lambda(x,y) &= \begin{pmatrix} -u_y(x,y)/4 & -\lambda \cdot R(y)\cdot e^{-u(x,y)/2} \\ (\lambda/2)\cdot H\cdot e^{u(x,y)/2} & u_y(x,y)/4, \end{pmatrix} \end{aligned}$$

where H ($\neq 0$) is constant (cf. (2.1.5) in [17]). Then, the Gauß equation for $\phi_2(x,y) : W \to \mathbb{R}^3_1$ is

$$(5.2.38)\qquad u_{xy}(x,y) - 2\cdot Q(x)\cdot R(y)\cdot e^{-u(x,y)} + \frac{1}{2}\cdot H^2 \cdot e^{u(x,y)} = 0$$

(cf. (2.1.7) in [17]). This equation will become the Painlevé equation of type (III) later (cf. (5.2.38″)). It follows from (5.2.36) that $\alpha^\lambda(x,y) := C_\lambda(x,y)^{-1}\cdot dC_\lambda(x,y)$ satisfies $\alpha^\lambda(x,y) = U_\lambda(x,y)dx + V_\lambda(x,y)dy$ and $k\cdot\alpha^\lambda(x,y)\cdot k^{-1} = \alpha^{b\cdot\lambda}(a\cdot x, a^{-1}\cdot y)$. Hence

$$k\cdot U_\lambda(x,y)\cdot k^{-1} = a\cdot U_{b\cdot\lambda}(a\cdot x, a^{-1}\cdot y), \quad k\cdot V_\lambda(x,y)\cdot k^{-1} = a^{-1}\cdot V_{b\cdot\lambda}(a\cdot x, a^{-1}\cdot y).$$

Accordingly we obtain

$$u(a\cdot x, a^{-1}\cdot y) = u(x,y), \quad Q(a\cdot x) = a^m\cdot Q(x), \quad R(a^{-1}\cdot y) = a^{-m}\cdot R(y)$$

from (5.2.37). Let $\Omega(x\cdot y) := u(1, x\cdot y)$. Then $\Omega(x\cdot y) = u(x,y)$ follows from $u(a\cdot x, a^{-1}\cdot y) = u(x,y)$ and $a := x^{-1}$. Hence we conclude that

$$(5.2.39)\qquad u_{xy} = \partial_x\partial_y\Omega(x\cdot y) = \partial_x(\Omega(x\cdot y)'\cdot x) = \Omega(x\cdot y)''\cdot x\cdot y + \Omega(x\cdot y)'.$$

Since $Q(a\cdot x) = a^m\cdot Q(x)$ and $R(a^{-1}\cdot y) = a^{-m}\cdot R(y)$ one can express $Q(x)$ and $R(x)$ as $Q(x) = Q_0\cdot x^m$ and $R(y) = R_0\cdot y^m$, respectively, where both Q_0 and R_0 are constant. Therefore we show

$$(5.2.40)\qquad \begin{aligned} &-2\cdot Q(x)\cdot R(y)\cdot e^{-u(x,y)} + \frac{1}{2}\cdot H^2\cdot e^{u(x,y)} \\ &= -2\cdot Q_0\cdot R_0\cdot (x\cdot y)^m\cdot e^{-\Omega(x\cdot y)} + \frac{1}{2}\cdot H^2\cdot e^{\Omega(x\cdot y)}. \end{aligned}$$

In terms of (5.2.39) and (5.2.40) we rewrite (5.2.38) as follows:

$$(5.2.38')\qquad \Omega(t)''\cdot t + \Omega(t)' - 2\cdot Q_0\cdot R_0\cdot t^m\cdot e^{-\Omega(t)} + \frac{1}{2}\cdot H^2\cdot e^{\Omega(t)} = 0,$$

where $t := x\cdot y$. Furthermore, one can rewrite (5.2.38′) as follows:

$$(5.2.38'')\qquad \frac{d^2v}{du^2} = \frac{1}{v}\Big(\frac{dv}{du}\Big)^2 - \frac{1}{u}\frac{dv}{du} + \frac{1}{u}\Big(-\frac{H^2}{2+m}v^2 + 4\frac{R_0\cdot Q_0}{2+m}\Big)$$

by setting $u := (2t^{(2+m)/2})/(2+m)$ and $v := e^{\Omega(t)}\cdot t^{-m/2}$. Consequently we assert that the Gauß equation (5.2.38) for $\phi_2(x,y) : W \to \mathbb{R}^3_1$ is the Painlevé equation (5.2.38″) of type (III). □

5.2.5. *Delaunay surface in* $\mathbb{R}^3 \Leftrightarrow$ *K-surface of revolution in* $\mathbb{R}^3$. In this subsection we will use the following notation:

(5.2.41) $G^{\mathbb{C}}$: the same notation (5.2.1) as in Subsection 5.2.1,
(5.2.42) σ: the same notation (5.2.2) as in Subsection 5.2.1,
(5.2.43) ν_1: the same notation (5.2.3) as in Subsection 5.2.1,
(5.2.44) $\nu_2 := \nu_1$,
(5.2.45) $G^{\mathbb{C}}/H^{\mathbb{C}}$: the same notation (5.2.5) as in Subsection 5.2.1,
(5.2.46) G_1/H_1: the same notation (5.2.6) as in Subsection 5.2.1,
(5.2.47) $G_2/H_2 = G_1/H_1 = SU(2)/S(U(1)\times U(1)) \simeq S^2$,
(5.2.48) π_i: the projection from G_i onto G_i/H_i $(i = 1, 2)$.

The main purpose in this subsection is to interrelate a surface of revolution in $\mathbb{R}^3$ (i.e., a Delaunay surface in $\mathbb{R}^3$) with a K-surface of revolution in $\mathbb{R}^3$ by means of Theorem 4.3.1 (see Theorem 5.2.7). Here, a *K-surface* means a surface of constant negative curvature $K = -1$. Such a surface is sometimes called a *pseudospherical surface*.

According to Toda [27] (see [28] also), one can characterize each K-surface M in $\mathbb{R}^3$ by an arc length asymptotic line coordinate system (x, y) on M and the angle function $\omega(x, y)$ with respect to (x, y) by the loop group method. For our purpose, we need to specialize her way concretely to surfaces of revolution. First we recall

Lemma 5.2.2. *Let $f_{pseud}(u, v)$, $f_{hyper}(u, v)$ and $f_{conic}(u, v)$ denote the K-surfaces of revolution given in Gray* [13, Chapter 19.3][2], *respectively*:

$$\begin{aligned} f_{pseud}(u,v) &= \big(\cos u\sin v, \sin u\sin v, \cos v + \log(\tan v/2)\big);\\ f_{hyper}(u,v) &= \Big(b\cos u\cosh v, b\sin u\cosh v, \int_0^v \sqrt{1-b^2\sinh^2(t)}dt\Big), \quad 0<b;\\ f_{conic}(u,v) &= \Big(b\cos u\sinh v, b\sin u\sinh v, \int_0^v \sqrt{1-b^2\cosh^2(t)}dt\Big), \quad 0<b<1. \end{aligned}$$

Then in each case, an arc length asymptotic line parametrization (x, y) is given by

$$\begin{aligned} &\textit{Pseudosphere}: && u = x+y, && v = 2\tan^{-1}(\exp(x-y));\\ &\textit{Hyperboloid type}: && u = \frac{x+y}{\sqrt{1+b^2}}, && v = -i\cdot \mathrm{am}\Big(\frac{i(x-y)}{\sqrt{1+b^2}}, ib\Big);\\ &\textit{Conic type}: && u = \frac{x+y}{\sqrt{1-b^2}}, && v = -i\cdot \mathrm{am}\Big(i(x-y), \frac{ib}{\sqrt{1-b^2}}\Big) \end{aligned}$$

[2]Erratum: p. 381, the equation (19.4) in [13], should be $-i\sqrt{a^2-b^2}E(iv/a, -b^2/(a^2-b^2))$ instead of $-i\sqrt{a^2-b^2}E(iv/a, b^2/(a^2-b^2))$.

and the first fundamental form $I^{\rm st}$ is expressed as

$$
\begin{aligned}
&\textit{Pseudosphere:} && I^{\rm st}_{pseud} = dx^2 + 2\big(-1 + \frac{2}{\cosh^2(x-y)}\big)dxdy + dy^2;\\
&\textit{Hyperboloid type:} && I^{\rm st}_{hyper} = dx^2 + 2\Big(1 - \frac{2}{1+b^2}\cdot \mathrm{dn}^2\big(\frac{i(x-y)}{\sqrt{1+b^2}}, ib\big)\Big)dxdy + dy^2;\\
&\textit{Conic type:} && I^{\rm st}_{conic} = dx^2 + 2\Big(1 - 2\cdot \mathrm{dn}^2\big(i(x-y), \frac{ib}{\sqrt{1-b^2}}\big)\Big)dxdy + dy^2
\end{aligned}
$$

(*see Remark 5.2.3 for* $\mathrm{am}(u,k)$ *and* $\mathrm{dn}(u,k)$).

Proof. Gray [13] presents a method of computing asymptotic line parametrizations by the software *Mathematica*. His arguments [13, p. 329–330], together with the program in [13, p. 328], enable us to obtain the arc length asymptotic line parametrizations (x,y) for the K-surfaces $f_{\rm pseud}(u,v)$, $f_{\rm hyper}(u,v)$ and $f_{\rm conic}(u,v)$, respectively. □

Remark 5.2.3. Throughout this paper, we use the notation $\mathrm{am}(u,k)$, $\mathrm{sn}(u,k)$, $\mathrm{cn}(u,k)$, $\mathrm{dn}(u,k)$ and $\mathrm{sd}(u,k)$ as in Byrd-Friedman [6] for the Jacobi functions.

Lemma 5.2.4. *Let $\omega_{pseud}(x,y)$, $\omega_{hyper}(x,y)$ and $\omega_{conic}(x,y)$ be the real analytic functions around $(0,0)$ defined by*

$$
\begin{aligned}
&\omega_{pseud}(x,y) := 2\sin^{-1}(\tanh(x-y));\\
&\omega_{hyper}(x,y) := 2\sin^{-1}\Big(\frac{1}{\sqrt{1+b^2}}\cdot \mathrm{dn}\big(\frac{i(x-y)}{\sqrt{1+b^2}}, ib\big)\Big), \qquad 0<b;\\
&\omega_{conic}(x,y) := -2\sin^{-1}\Big(b\cdot \mathrm{sd}\big(\frac{x-y}{\sqrt{1-b^2}}, \sqrt{1-b^2}\big)\Big) + \pi, \quad 0<b<1.
\end{aligned}
$$

Then, they satisfy

$$
\begin{aligned}
&\text{(i.1)}\quad \textit{Pseudosphere:} && I^{\rm st}_{pseud} = dx^2 + 2\cos(\omega_{pseud}(x,y))dxdy + dy^2;\\
&\text{(i.2)}\quad \textit{Hyperboloid type:} && I^{\rm st}_{hyper} = dx^2 + 2\cos(\omega_{hyper}(x,y))dxdy + dy^2;\\
&\text{(i.3)}\quad \textit{Conic type:} && I^{\rm st}_{conic} = dx^2 + 2\cos(\omega_{conic}(x,y))dxdy + dy^2;
\end{aligned}
$$

and furthermore, they are solutions to the sine-Gordon equation $\partial_x\partial_y\omega = \sin\omega$.

Proof. Both (i.1) and (i.2) are immediate from $\cos\omega = 1 - 2\sin^2(\omega/2)$. Let us show (i.3). By direct computations we have $\cos(\omega_{\rm conic}/2) = \sin\big((\pi/2) - (\omega_{\rm conic}/2)\big) = b\cdot \mathrm{sd}\big((x-y)/\sqrt{1-b^2}, \sqrt{1-b^2}\big)$, and

$$
\cos^2\frac{\omega_{\rm conic}}{2} = b^2\cdot \mathrm{sd}^2\big(\frac{x-y}{\sqrt{1-b^2}}, \sqrt{1-b^2}\big). \tag{5.2.49}
$$

Transformation formulas in [6, p. 38] lead to

$$
\mathrm{sn}(iu, ib/\sqrt{1-b^2}) = i\sqrt{1-b^2}\cdot \mathrm{sd}(u/\sqrt{1-b^2}, \sqrt{1-b^2}).
$$

Therefore, (5.2.49) and $k^2\cdot \mathrm{sn}^2(u,k) + \mathrm{dn}^2(u,k) = 1$ yield that

$$
\frac{1+\cos\omega_{\rm conic}}{2} = \cos^2\frac{\omega_{\rm conic}}{2} = -\frac{b^2}{1-b^2}\cdot \mathrm{sn}^2\big(i(x-y), \frac{ib}{\sqrt{1-b^2}}\big) = 1 - \mathrm{dn}^2\big(i(x-y), \frac{ib}{\sqrt{1-b^2}}\big).
$$

Accordingly one deduces $\cos(\omega_{\mathrm{conic}}(x,y)) = 1-2\cdot \mathrm{dn}^2(i(x-y), ib/\sqrt{1-b^2})$, and thus (i.3) follows. Now, the rest of this proof is to demonstrate that ω_{pseud}, ω_{hyper} and ω_{conic} are solutions to the sine-Gordon equation, respectively. We will only prove that ω_{conic} is a solution to the equation, because one can consider the other cases in a similar way. Note that $\mathrm{dn}((x-y)/\sqrt{1-b^2}, \sqrt{1-b^2})$ is positive around $(0,0)$ because of $0<b<1$. On the one hand, direct computations show

$$\partial_x\omega_{\mathrm{conic}} = -\frac{2}{\sqrt{1-b^2}}\cdot\frac{1}{\mathrm{dn}((x-y)/\sqrt{1-b^2}, \sqrt{1-b^2})};$$
$$\partial_x\partial_y\omega_{\mathrm{conic}} = \frac{2}{\sqrt{1-b^2}}\cdot\frac{\mathrm{sn}((x-y)/\sqrt{1-b^2}, \sqrt{1-b^2})\cdot\mathrm{cn}((x-y)/\sqrt{1-b^2}, \sqrt{1-b^2})}{\mathrm{dn}^2((x-y)/\sqrt{1-b^2}, \sqrt{1-b^2})}.$$

One the other hand, it follows from (5.2.49) that

$$\begin{aligned}\partial_x\omega_{\mathrm{conic}}\cdot\sin\omega_{\mathrm{conic}} &= -2\cdot\frac{\partial}{\partial x}\cos^2\frac{\omega_{\mathrm{conic}}}{2} = -2b^2\cdot\frac{\partial}{\partial x}\mathrm{sd}^2\Big(\frac{x-y}{\sqrt{1-b^2}}, \sqrt{1-b^2}\Big)\\ &= -\frac{4b^2}{\sqrt{1-b^2}}\cdot\frac{\mathrm{sn}((x-y)/\sqrt{1-b^2}, \sqrt{1-b^2})\cdot\mathrm{cn}((x-y)/\sqrt{1-b^2}, \sqrt{1-b^2})}{\mathrm{dn}^3((x-y)/\sqrt{1-b^2}, \sqrt{1-b^2})}\\ &= \partial_x\omega_{\mathrm{conic}}\cdot\partial_x\partial_y\omega_{\mathrm{conic}}.\end{aligned}$$

Therefore, one has $\partial_x\partial_y\omega_{\mathrm{conic}} = \sin\omega_{\mathrm{conic}}$ by virtue of $\partial_x\omega_{\mathrm{conic}}<0$. □

REMARK 5.2.5. (i) The solution $\omega_{\mathrm{pseud}}(x,y)$ to the sine-Gordon equation in Lemma 5.2.4 can be rewritten as follows:

$$\omega_{\mathrm{pseud}}(x,y) = 4\tan^{-1}(\exp(x-y)) - \pi. \tag{5.2.50}$$

Indeed, $f(x,y) := 4\tan^{-1}(\exp(x-y)) - \pi$ is analytic and satisfies $\omega_{\mathrm{pseud}}(0,0) = f(0,0)$, $\partial_x\omega_{\mathrm{pseud}} = \partial_x f$ and $\partial_y\omega_{\mathrm{pseud}} = \partial_y f$. (ii) From every solution $\omega(x,y)$ to the sine-Gordon equation, one can construct another solution $\omega'(x,y)$ to the sine-Gordon equation by setting

$$\omega'(x,y) := \omega(x,-y) + \pi.$$

Consequently, $\omega'_{\mathrm{pseud}}(x,y) = 4\tan^{-1}(\exp(x+y))$ becomes a solution to the sine-Gordon equation by virtue of (5.2.50). Toda [27] uses this solution to study pseudospheres.

For the K-surfaces f_{pseud}, f_{hyper} and f_{conic} in Lemma 5.2.2, we have obtained arc length asymptotic line parametrizations (x,y) and the angle functions $\omega(x,y)$ with respect to (x,y), respectively (cf. Lemmas 5.2.2 and 5.2.4). According to Toda [27] one can, up to an isometry of $\mathbb{R}^3$, reconstruct the K-surface from $\omega(x,y)$ and the following potential

$(\eta_\theta(x), \tau_\theta(y))$:

$$\begin{aligned} \eta_\theta(x) &:= \frac{i\theta^{-1}}{2}\begin{pmatrix} 0 & e^{i(\omega(x,0)-\omega(0,0))} \\ e^{-i(\omega(x,0)-\omega(0,0))} & 0 \end{pmatrix} dx, \\ \tau_\theta(y) &:= \frac{-i\theta}{2}\begin{pmatrix} 0 & e^{-i\omega(0,y)} \\ e^{i\omega(0,y)} & 0 \end{pmatrix} dy. \end{aligned} \tag{5.2.51}$$

REMARK 5.2.6. This is not difficult to verify that for $\omega(x,y) = \omega_{\text{conic}}(x,y)$ the above potential $(\eta_\theta(x), \tau_\theta(y))$ satisfies the morphing condition (M) in Theorem 4.3.1, while this is not true for the angle functions ω_{pseud} and ω_{hyper} in Lemma 5.2.4.

By means of Theorem 4.3.1, we will construct a harmonic map $f_1(z,\bar{z}) : \mathbb{A}^2 \to G_1/H_1 \simeq S^2$ and a Lorentz harmonic map $f_2(x,y) : \mathbb{B}^2 \to G_2/H_2 \simeq S^2$ from the angle function ω_{conic}; and interrelate the associated CMC-surfaces $\phi_1(z,\bar{z}) : \mathbb{A}^2 \to \mathbb{R}^3$ and K-surfaces $\phi_2(x,y) : \mathbb{B}^2 \to \mathbb{R}^3$ using $f_1(z,\bar{z})$ and $f_2(x,y)$. One will see that $\phi_1(z,\bar{z})$ is a Delaunay surface and $\phi_2(x,y)$ is a conic K-surface of revolution (cf. Theorem 5.2.7).

First, we define a real analytic, para-pluriharmonic potential $(\eta_\theta(x), \tau_\theta(y))$ on (U, I) by (5.2.51) with $\omega(x,y) = \omega_{\text{conic}}(x,y)$ as given in Lemma 5.2.4. Here U denotes any open neighborhood of $\mathbb{B}^2$ at $(0,0)$ such that $\omega_{\text{conic}}(x,y)$ is analytic on U. As remarked above, $(\eta_\theta(x), \tau_\theta(y))$ satisfies the morphing condition (M). Next, let us solve the two initial value problems: $(A_\theta)^{-1} \cdot dA_\theta = \eta_\theta$ and $(B_\theta)^{-1} \cdot dB_\theta = \tau_\theta$ with $A_\theta(0,0) \equiv \text{id} \equiv B_\theta(0,0)$; and factorize $(A_\theta, B_\theta) \in \widetilde{\Lambda}(G_2)_\sigma \times \widetilde{\Lambda}(G_2)_\sigma$ in the Iwasawa decomposition (cf. Theorem 3.1.5):

$$(A_\theta, B_\theta) = (C_\theta, C_\theta) \cdot (B_\theta^+, B_\theta^-), \quad C_\theta \in \widetilde{\Lambda}(G_2)_\sigma, \quad B_\theta^+ \in \widetilde{\Lambda}_*^+(G_2)_\sigma, \quad B_\theta^- \in \widetilde{\Lambda}^-(G_2)_\sigma.$$

Then Proposition 3.2.3 assures that there exists an open neighborhood W of U at $(0,0)$, and $(f_2)_\theta := \pi_2 \circ C_\theta(x,y) : (W, I) \to G_2/H_2 \simeq S^2$ is a Lorentz harmonic map for any $\theta \in \mathbb{R}^+$. Moreover, by Theorem 4.3.1 there exist an open neighborhood V of $\mathbb{A}^2$ at $(0,0)$ and a smooth map $h^{\mathbb{C}} : V \to H^{\mathbb{C}}$ such that $(f_1)_\lambda := \pi_1 \circ C'_\lambda(z,\bar{z}) : (V, J) \to G_1/H_1 \simeq S^2$ is a harmonic map for any $\lambda \in S^1$, where $C'_\lambda := C_\lambda \cdot h^{\mathbb{C}}$. Accordingly we have obtained a harmonic map $f_1(z,\bar{z})$ and a Lorentz harmonic map $f_2(x,y)$ from the potential (5.2.51):

$$\begin{aligned} (f_1)_\lambda &= \pi_1 \circ C'_\lambda(z,\bar{z}) : (V, J) \to G_1/H_1 \simeq S^2, \quad C'_\lambda(0,0) \equiv \text{id}, \quad \lambda \in S^1; \\ (f_2)_\theta &= \pi_2 \circ C_\theta(x,y) : (W, I) \to G_2/H_2 \simeq S^2, \quad C_\theta(0,0) \equiv \text{id}, \quad \theta \in \mathbb{R}^+. \end{aligned}$$

From $f_1(z,\bar{z})_\lambda$ and $f_2(x,y)_\theta$ one obtains a Delaunay surface and a conic K-surface of revolution, respectively:

THEOREM 5.2.7. *Let $(f_1)_\lambda = \pi_1 \circ C'_\lambda(z,\bar{z}) : (V, J) \to S^2$ and $(f_2)_\theta = \pi_2 \circ C_\theta(x,y) : (W, I) \to S^2$ be the above harmonic map and Lorentz harmonic map. Let $\phi_1(z,\bar{z})_\lambda : V \to \mathbb{R}^3$ (resp. $\phi_2(x,y)_\theta : W \to \mathbb{R}^3$) denote the CMC-surface (resp. K-surface) determined by* *the Sym-Bobenko formula (resp. the Sym formula) :*

$$\phi_1(z,\bar{z})_\lambda := i\cdot\lambda\cdot\frac{\partial C'_\lambda}{\partial\lambda}\cdot {C'_\lambda}^{-1} + \frac{1}{2}\cdot \mathrm{Ad}(C'_\lambda)\cdot\begin{pmatrix} i & 0 \\ 0 & -i\end{pmatrix},$$
$$\phi_2(x,y)_\theta := \theta\cdot\frac{\partial C_\theta}{\partial\theta}\cdot C_\theta^{-1}.$$

Then, $\phi_1(z,\bar{z})_\lambda : V \to \mathbb{R}^3$ is a Delaunay surface, and $\phi_2(x,y)_\theta : W \to \mathbb{R}^3$ is a conic K-surface of revolution.

Proof. The K-surface $\phi_2(x,y)_\theta : W \to \mathbb{R}^3$ is endowed with the angle function $\omega_{\mathrm{conic}}(x,y)$ (cf. (5.2.51)). Therefore, Toda [27] assures that $\phi_2(x,y)_\theta : W \to \mathbb{R}^3$ coincides, up to an isometry of $\mathbb{R}^3$, with the K-surface f_{conic} given in Lemma 5.2.2. Consequently, the rest of proof is to conclude that $\phi_1(z,\bar{z})_\lambda : V \to \mathbb{R}^3$ is a Delaunay surface. First, let us verify that

(5.2.52) $$C_\theta(x+t,y+t) = \chi_\theta(t)\cdot C_\theta(x,y), \quad \text{for any } t\in\mathbb{R} \text{ with } (x+t,y+t)\in W,$$

where $\chi_\theta(t) := C_\theta(t,t)$. By the proof of Lemma 5.2.4 we have $(\partial_x\omega_{\mathrm{conic}})(x+t,y+t) = (\partial_x\omega_{\mathrm{conic}})(x,y)$ and $\omega_{\mathrm{conic}}(x+t,y+t) = \omega_{\mathrm{conic}}(x,y)$. Therefore $(C_\theta^{-1}\cdot dC_\theta)(x+t,y+t) = (C_\theta^{-1}\cdot dC_\theta)(x,y)$ follows from the equation (6) in [28]; and thus

$$(C_\theta^{-1}\cdot dC_\theta)(x+t,y+t) = (C_\theta^{-1}\cdot dC_\theta)(x,y) = \big((\chi_\theta(t)\cdot C_\theta)^{-1}\cdot d(\chi_\theta(t)\cdot C_\theta)\big)(x,y).$$

In view of $C_\theta(0,0)\equiv \mathrm{id}$ one sees that $C_\theta(0+t,0+t) = \chi_\theta(t)\cdot C_\theta(0,0) = \chi_\theta(t)$. Hence, one concludes (5.2.52). From (5.2.52) it follows that

$$C_\lambda(z+t,\bar{z}+t) = \chi_\lambda(t)\cdot C_\lambda(z,\bar{z}), \quad \lambda\in S^1,$$

where we remark that the variable θ of $\chi_\theta(t)$ can vary in the whole $\mathbb{C}^*$ because of $\chi_\theta(t) = C_\theta(t,t)$. Since $C'_\lambda(z,\bar{z}) = C_\lambda(z,\bar{z})\cdot h^{\mathbb{C}}(z,\bar{z})$, we deduce that

(5.2.53) $$C'_\lambda(z+t,\bar{z}+t) = \chi_\lambda(t)\cdot C'_\lambda(z,\bar{z})\cdot k^{\mathbb{C}}(t,z,\bar{z}),$$

where $k^{\mathbb{C}}(t,z,\bar{z}) := h^{\mathbb{C}}(z,\bar{z})^{-1}\cdot h^{\mathbb{C}}(z+t,\bar{z}+t)$. If $k^{\mathbb{C}}(t,z,\bar{z})$ belongs to H_1 ($\subset G_1$), then it is immediate from $C'_\lambda(z+t,\bar{z}+t), C'_\lambda(z,\bar{z})\in G_1$ that $\chi_\lambda(t)\in G_1$; so that $\phi_1(z,\bar{z})_\lambda : V\to\mathbb{R}^3$ admits a one-parameter group of isometries, which implies that $\phi_1(z,\bar{z})_\lambda : V\to\mathbb{R}^3$ is a Delaunay surface (cf. Theorem [8, p. 127]). Thus it suffices to confirm

$$k^{\mathbb{C}}(t,z,\bar{z})\in H_1.$$

For the extended framing $C'_\lambda(z,\bar{z})$ of the harmonic map $(f_1)_\lambda : (V,J)\to S^2$, we have the Maurer-Cartan form:

$$ {C'_\lambda}^{-1}\cdot\partial_z C'_\lambda = U, \qquad {C'_\lambda}^{-1}\cdot\partial_{\bar{z}} C'_\lambda = V,$$
$$U = \begin{pmatrix} u_z/4 & -(\lambda^{-1}/2)\cdot H\cdot e^{u/2} \\ \lambda^{-1}\cdot Q\cdot e^{-u/2} & -u_z/4\end{pmatrix}, \quad V = \begin{pmatrix} -u_{\bar{z}}/4 & -\lambda\cdot R\cdot e^{-u/2} \\ (\lambda/2)\cdot H\cdot e^{u/2} & u_{\bar{z}}/4\end{pmatrix},$$

where H $(\neq 0)$ is constant. Since $k^{\mathbb{C}}(t,z,\bar{z}) = h^{\mathbb{C}}(z,\bar{z})^{-1} \cdot h^{\mathbb{C}}(z+t,\bar{z}+t) \in H^{\mathbb{C}} = S(GL(1,\mathbb{C}) \times GL(1,\mathbb{C}))$ is a diagonal matrix, we can express it as

$$k^{\mathbb{C}}(t,z,\bar{z}) = \begin{pmatrix} d(t,z,\bar{z}) & 0 \\ 0 & d(t,z,\bar{z})^{-1} \end{pmatrix}.$$

Then, the Maurer-Cartan form on the right hand side of (5.2.53) is

$$(5.2.54) \qquad \begin{aligned} U(z,\bar{z}) &= \begin{pmatrix} u_z(z,\bar{z})/4 + d^{-1} \cdot d_z & -(\lambda^{-1}/2) \cdot H \cdot e^{u(z,\bar{z})/2} \cdot d^{-2} \\ \lambda^{-1} \cdot Q(z) \cdot e^{-u(z,\bar{z})/2} \cdot d^2 & -u_z(z,\bar{z})/4 - d^{-1} \cdot d_z \end{pmatrix}, \\ V(z,\bar{z}) &= \begin{pmatrix} -u_{\bar{z}}(z,\bar{z})/4 + d^{-1} \cdot d_{\bar{z}} & -\lambda \cdot R(\bar{z}) \cdot e^{-u(z,\bar{z})/2} \cdot d^{-2} \\ (\lambda/2) \cdot H \cdot e^{u(z,\bar{z})/2} \cdot d^2 & u_{\bar{z}}(z,\bar{z})/4 - d^{-1} \cdot d_{\bar{z}} \end{pmatrix}. \end{aligned}$$

The Maurer-Cartan form on the left hand side of (5.2.53) is

$$(5.2.55) \qquad \begin{aligned} U(z+t,\bar{z}+t) &= \begin{pmatrix} u_z(z+t,\bar{z}+t)/4 & -(\lambda^{-1}/2) \cdot H \cdot e^{u(z+t,\bar{z}+t)/2} \\ \lambda^{-1} \cdot Q(z+t) \cdot e^{-u(z+t,\bar{z}+t)/2} & -u_z(z+t,\bar{z}+t)/4 \end{pmatrix}, \\ V(z+t,\bar{z}+t) &= \begin{pmatrix} -u_{\bar{z}}(z+t,\bar{z}+t)/4 & -\lambda \cdot R(\bar{z}+t) \cdot e^{-u(z+t,\bar{z}+t)/2} \\ (\lambda/2) \cdot H \cdot e^{u(z+t,\bar{z}+t)/2} & u_{\bar{z}}(z+t,\bar{z}+t)/4 \end{pmatrix}. \end{aligned}$$

Let us compare the 12-entry of U with the 21-entry of V in (5.2.54) and (5.2.55). Then one has $d^4 = 1$, whence $d = \pm i, \pm 1$. This means that $k^{\mathbb{C}}(t,z,\bar{z}) \in S(U(1) \times U(1)) = H_1$. $\square$

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