Hyperbolic Schrödinger map and non-elliptic derivative Schrödinger equation in two spatial dimensions

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Abstract. Using the Gabor frame, we get some time-global dispersive estimates for the Schrödinger semi-group in anisotropic Lebesgue spaces, which include a time-global maximal function estimate in the space $L^2_{x_1}L^\infty_{x_2,t}$. By resorting to the smooth effect estimate together with the dispersive estimates in anisotropic Lebesgue spaces, we show that the hyperbolic Schrödinger map in 2D has a unique global solution if the initial data in Feichtinger algebra or in weighted Sobolev spaces are sufficiently small. Similar results hold for the non-elliptic derivative NLS.

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1 Introduction

A large amount of work has been devoted to the study of the Schrödinger map initial value problem together with their generalizations [1, 2, 3, 4, 6, 10, 11, 21, 22, 23, 24, 25, 33, 34, 42, 49, 53, 54],

$$s_t = s \times \Delta s, \quad s(0,x) = s_0(x). \tag{1.1}$$

The global well posedness of (1.1) in critical Sobolev spaces $\dot{H}^{n/2}$ was recently obtained in [4]. In this paper, we consider the Cauchy problem for the hyperbolic Schrödinger map equation in two spatial dimensions,

$$s_t = s \times \Box s, \quad s(0,x) = s_0(x), \tag{1.2}$$

where $s = (s_1, s_2, s_3)$, $s : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{S}^2$ is a real valued map of (t, x_1, x_2), $\Box = \partial^2_{x_1} - \partial^2_{x_2}$. Let u be the stereographic projection of s defined by

$$u = \frac{s_1 + is_2}{1 + s_3}.$$

It follows that the Cauchy problem for the hyperbolic Schrödinger map equation can be rewritten as

$$\mathrm{i}u_t + \Box u = \frac{2\bar{u}}{1+|u|^2}(u^2_{x_1} - u^2_{x_2}), \quad u(0,x) = u_0(x). \tag{1.3}$$

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On the other hand, if u satisfies (1.3), taking

$$s = \left(\frac{2Re\ u}{1+|u|^2}, \frac{2Im\ u}{1+|u|^2}, \frac{1-|u|^2}{1+|u|^2} \right),$$

we see that s is the solution of (1.2).

In this paper, we will use the smooth effect method to study the the global well posedness of (1.2) and the technique developed in this paper is also adapted to the derivative nonlinear Schrödinger equation (DNLS):

$$\mathrm{i}u_t - \Delta_{\pm} u = F(u, \bar{u}, \nabla u, \nabla \bar{u}), \quad u(0,x) = u_0(x), \tag{1.4}$$

where $\Delta_{\pm} = \partial^2_{x_1} \pm \partial^2_{x_2}$, $\nabla u = (u_{x_1}, u_{x_2})$,

$$F(u, \bar{u}, \nabla u, \nabla \bar{u}) := \overrightarrow{\lambda}_1 \cdot \nabla(|u|^2 u) + (\overrightarrow{\lambda}_2 \cdot \nabla u)|u|^2 + \mu |u|^{2\kappa} u, \tag{1.5}$$

$\vec{\lambda}_i \in \mathbb{C}^2$ and $\mu \in \mathbb{C}$. It is known that (1.4) with the nonlinearity (1.5) arises from the strongly interacting many-body systems near criticality as recently described in terms of nonlinear dynamics [9, 12, 45], where anisotropic interactions are manifested by the presence of the non-elliptic case, as well as additional residual terms which involve cross derivatives of the independent variables. More generally, $F(u, \bar{u}, \nabla u, \nabla \bar{u})$ can be taken as a polynomial series

$$F(z) = P(z_1, ..., z_6) = \sum_{3 \leqslant |\beta| < \infty} c_\beta z^\beta, \quad c_\beta \in \mathbb{C}. \tag{1.6}$$

In the elliptic case, many works have been devoted to the study of the local and global well posedness of (1.4), see for instance [7, 20, 29, 30, 35, 37]. When the nonlinear term F satisfies an energy structure condition $Re\partial F/\partial(\nabla u) = 0$ and the initial data are sufficiently smooth in weighted Sobolev spaces, Klainerman [29], Shatah [37] and Klainerman and Ponce [30] obtained the global existence of (1.4) in all spatial dimensions. Chihara [7] considered the initial data in sufficiently smooth weighted Sobolev spaces and removed the energy structure condition $Re\partial F/\partial(\nabla u) = 0$ for $n \geqslant 3$ and only assume that cubic terms $F_1(z)$ in $F(z)$ is modulation homogeneous (i.e., $F_1(e^{\mathrm{i}\theta} z) = e^{\mathrm{i}\theta} F_1(z)$) for $n = 2$. Ozawa and Zhai [35] was able to consider the initial data in H^s with $s > n/2 + 2$, $n \geqslant 3$ and $Re\partial F/\partial(\nabla u) = \nabla(\theta(|u|^2))$ for some real valued function $\theta \in C^2$ with $\theta(0) = 0$.

For the non-elliptic case, the smooth effect estimates seem to be useful tools for the well posedness of (1.4). Roughly speaking, this method relies upon the dispersive structure for the Schrödinger semi-group and the energy structure conditions for the nonlinear terms are not necessary for the local well posedness and for the global well posedness with small data. By setting up the local smooth effects for the solutions of the linear Schrödinger equation, Kenig, Ponce and Vega [26, 27] were able to deal with the non-elliptical case and they established the local well posedness of Eq. (1.4) in H^s with $s \gg n/2$. The global existence and scattering of solutions of (1.4) with small data in $B^{(n+3)/2}_{2,1}$ (so in H^s with $s > n/2 + 3/2$, $n \geqslant 3$) were recently obtained in [49, 50, 51], where the initial data can belong to a class of modulation spaces $M^{3/2}_{2,1}$ at lower regularity. Moreover, the results in [49, 51] contains non-elliptic Schrödinger map equation as a special case if $n \geqslant 3$.

In this paper we study the global well posedness of solutions of (1.4) in 2D and show that (1.4) is globally well posed for the small Cauchy data in Feichtinger's algebra $M^2_{1,1}(\mathbb{R}^n)$. On the basis of the Gabor frame expression for the initial data, we will establish a class of linear estimates in anisotropic Lebesgue spaces. These estimates together with the smooth effect estimates for the linear Schrödinger equation (cf. [8, 26, 27, 32, 39, 49, 52]) and frequency-uniform decomposition techniques yield the existence and uniqueness of global solutions for small initial data in Feichtinger's algebra $M^2_{1,1}$ and in weighted Sobolev spaces $H^{s,b}$ with $s > 3, b > 1$.

1.1 Notation

In the sequel C, C_i will denote universal positive constants which can be different at each appearance. $x \lesssim y$ for $x, y > 0$ means that $x \leqslant Cy$, and $x \sim y$ stands for $x \lesssim y$ and $y \lesssim x$; $x \vee y = \max(x, y)$. For any $p \in [1, \infty]$, p' denotes the dual number of p, i.e., $1/p + 1/p' = 1$.

Let $\mathscr{S}$ be Schwartz space and $\mathscr{S}'$ be its dual space. All of the function spaces used in this paper are subspaces of $\mathscr{S}'$. We will use the Lebesgue spaces $L^p := L^p(\mathbb{R}^n)$ with the norm $\|\cdot\|_p := \|\cdot\|_{L^p}$, the function spaces $L^q_t L^p_x$ and $L^p_x L^q_t$ for which the norms are defined by:

$$\|f\|_{L^q_t L^p_x} = \Big\| \|f\|_{L^p_x(\mathbb{R}^n)} \Big\|_{L^q_t(\mathbb{R})}, \quad \|f\|_{L^p_x L^q_t} = \Big\| \|f\|_{L^q_t(\mathbb{R})} \Big\|_{L^p_x(\mathbb{R}^n)}.$$

We denote by $L^{p_1}_{x_1} L^{p_2}_{\bar{x},t}$ for $\bar{x} = (x_2, ..., x_n)$ the anisotropic Lebesgue space for which the following norm is finite:

$$\|f\|_{L^{p_1}_{x_1} L^{p_2}_{\bar{x},t}} = \Big\| \|f\|_{L^{p_2}_{\bar{x},t}(\mathbb{R}^{1+n-1})} \Big\|_{L^{p_1}_{x_1}(\mathbb{R})}. \tag{1.7}$$

For simplicity, we will write $D^s_{x_i} = (-\partial^2_{x_i})^{s/2} = \mathscr{F}^{-1}_{\xi_i} |\xi_i|^s \mathscr{F}_{x_i}$ denotes the partial Rieze potential in the x_i direction. The homogeneous Sobolev space $\dot{H}^s$ is defined by $(-\Delta)^{-s/2} L^2$, $H^s = L^2 \cap \dot{H}^s$. Recall that the weighted Sobolev space $H^{s,b}(\mathbb{R}^n)$ is defined by

$$\|u\|_{H^{s,b}} = \|\langle x\rangle^b \mathscr{F}^{-1} \langle \xi \rangle^s \mathscr{F} u\|_2.$$

Let $\{\sigma_k\}$ be a smooth cut-off function sequence satisfying

$$\mathrm{supp}\sigma_0 \subset [-1, 1]^n, \quad \sigma_k = \sigma(\cdot - k), \quad \sum_{k\in\mathbb{Z}^n} \sigma_k(\xi) \equiv 1. \tag{1.8}$$

Then we can define the frequency-uniform decomposition operators $\Box_k$ as:

$$\Box_k := \mathscr{F}^{-1} \sigma_k \mathscr{F}, \quad k \in \mathbb{Z}^n, \tag{1.9}$$

and we write

$$\|f\|_{M^s_{p,q}(\mathbb{R}^n)} = \left(\sum_{k\in\mathbb{Z}^n} \langle k\rangle^{sq} \|\Box_k f\|^q_{L^p(\mathbb{R}^n)} \right)^{1/q}, \tag{1.10}$$

which is said to be a modulation space. Modulation spaces $M^s_{p,q}$ were introduced by Feichtinger with the following equivalent norm (cf. [17, 18])

$$\|f\|_{M^s_{p,q}} = \left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |V_g f(x,\omega)|^p dx \right)^{q/p} \langle \omega\rangle^{sq} d\omega \right)^{1/q}, \tag{1.11}$$

where, for a given window function $g \in \mathscr{S}$, V_g is the short-time Fourier transform:

$$V_g f(x,\omega) = \int_{\mathbb{R}^n} e^{-\mathrm{i}t\omega} \overline{g(t-x)} f(t) dt.$$

The space $M^s_{1,1}$ with $s \geqslant 0$, so called Feichtinger algebra (or Feichtinger-Segal algebra), is one of the most important modulation spaces which enjoys the following interesting properties ([13, 14]):

- $M^s_{1,1}(\mathbb{R}^n)$ with $s \geqslant 0$ is a Banach algebra.
- $\mathscr{F}$ $(\mathscr{F}^{-1}) : M^0_{1,1}(\mathbb{R}^n) \to M^0_{1,1}(\mathbb{R}^n)$ is an isometric mapping.

On the other hand, it is known that $B^{s+n}_{1,1}(\mathbb{R}^n) \subset M^s_{1,1}(\mathbb{R}^n) \subset B^s_{1,1}(\mathbb{R}^n)$ are sharp inclusions (cf. [41, 44, 48])[1]. $M^0_{1,1}$ can be regarded as an analogue of Schwartz space which preserves Fourier transform but has little smoothness.

1.2 Main results

For convenience, we write

$$\begin{aligned} \|u\|^{\mathrm{sm}} = & \sum_{k\in\mathbb{Z}^2,\, |k_1|\geqslant |k_2|\vee 10} \langle k_1\rangle^{3/2} \|\Box_k u\|_{L^\infty_{x_1} L^2_{x_2,t}} \\ & + \sum_{k\in\mathbb{Z}^2,\, |k_2|\geqslant |k_1|\vee 10} \langle k_2\rangle^{3/2} \|\Box_k u\|_{L^\infty_{x_2} L^2_{x_1,t}}, \end{aligned} \tag{1.12}$$

$$\|u\|^{\max} = \sum_{k\in\mathbb{Z}^2} \left(\|\Box_k u\|_{L^2_{x_1} L^\infty_{x_2,t}} + \|\Box_k u\|_{L^2_{x_2} L^\infty_{x_1,t}} \right), \tag{1.13}$$

$$\|u\|^{\mathrm{ant}} = \sum_{k\in\mathbb{Z}^2} \left(\|\Box_k u\|_{L^2_{x_1} L^4_{x_2,t}} + \|\Box_k u\|_{L^2_{x_2} L^4_{x_1,t}} \right), \tag{1.14}$$

$$\|u\|^{\mathrm{str}} = \sum_{k\in\mathbb{Z}^2} \langle k\rangle \|\Box_k u\|_{L^\infty_t L^2_x \cap L^4_{x,t}}, \tag{1.15}$$

$$\|u\|^{\mathrm{gstr}} = \sum_{k\in\mathbb{Z}^2} \|\Box_k u\|_{L^3_{x,t}}. \tag{1.16}$$

We denote by $\|u\|^{\Upsilon\cap\Gamma} = \|u\|^{\Upsilon} + \|u\|^{\Gamma}$.

[1] $B^s_{p,q}$ denotes Besov space.

Theorem 1.1 *Let $u_0 \in M^2_{1,1}$ and there exists a suitably small $\delta > 0$ such that $\|u_0\|_{M^2_{1,1}} \leqslant \delta$. Then* (1.3) *has a unique solution $u \in C(\mathbb{R}, M^2_{2,1}) \cap C_{\rm loc}(\mathbb{R}, M^{3/2}_{1,1}) \cap X$, where*

$$X = \left\{ u \in \mathscr{S}' : \sum_{\alpha=0,1} \sum_{i=1,2} \|\partial^\alpha_{x_i} u\|^{\rm sm \cap max \cap ant \cap str \cap gstr} \lesssim \delta \right\}. \tag{1.17}$$

Moreover, there exists $u^\pm \in M^2_{2,1}$ such that

$$\lim_{t\to\pm\infty} \|u(t) - e^{{\rm i}t\Box} u^\pm\|_{M^2_{2,1}} \to 0. \tag{1.18}$$

Corollary 1.2 *Let $s_0 = (s_1(0), s_2(0), s_3(0)) \in \mathbb{S}^2$ with $s_1(0), s_2(0) \in M^2_{1,1}$ and there exists a suitably small $\delta > 0$ such that $\|s_i(0)\|_{M^2_{1,1}} \leqslant \delta$ for $i = 1, 2$. Then* (1.2) *has a unique solution $s = (s_1, s_2, s_3) \in \mathbb{S}^2$ with $s_1, s_2, s_3 - 1 \in C(\mathbb{R}, M^2_{2,1}) \cap C_{\rm loc}(\mathbb{R}, M^{3/2}_{1,1}) \cap X$, where X is as in* (1.17)*.*

Theorem 1.3 *Let $F(z)$ be as in* (1.6) *with $|c_\beta| \leqslant C^{|\beta|}$. Let $u_0 \in M^2_{1,1}$ and there exists a suitably small $\delta > 0$ such that $\|u_0\|_{M^2_{1,1}} \leqslant \delta$. Then* (1.3) *has a unique solution $u \in C(\mathbb{R}, M^2_{2,1}) \cap C_{\rm loc}(\mathbb{R}, M^{3/2}_{1,1}) \cap X$, where X is as in* (1.17)*. Moreover,* (1.18) *holds if one replaces $e^{{\rm i}t\Box}$ by $e^{{\rm i}t\Delta_\pm}$.*

Corollary 1.4 *Let $s > 3$, $b > 1$, $u_0 \in H^{s,b}$ and $\|u_0\|_{H^{s,b}} \leqslant \delta \ll 1$. Then the same result as in Theorem 1.1 holds.*

Corollary 1.5 *Let $F(z)$ be as in* (1.6) *with $|c_\beta| \leqslant C^{|\beta|}$. Let $s > 3$, $b > 1$, $u_0 \in H^{s,b}$ and $\|u_0\|_{H^{s,b}} \leqslant \delta \ll 1$. Then the same result as in Theorem 1.3 holds.*

1.3 Strategy of the proof

We now sketch the proof for the main results. We consider the following equivalent integral equation,

$$u(t) = S(t)u_0 - {\rm i} \int_0^t S(t-\tau) F(u(\tau)) d\tau, \quad S(t) := e^{{\rm i}t\Box}, \tag{1.19}$$

where we assume for simply that $F(u) = \partial_{x_1}(|u|^2 u)$. By following the smooth effects in 1D as in Kenig-Ponce-Vega [26], the global smooth effects for the solutions of the Schrödinger equation in 2D were essentially obtained by Linares and Ponce [32],

$$\|D^{1/2}_{x_1} S(t)\phi\|_{L^\infty_{x_1} L^2_{x_2,t}(\mathbb{R}^{1+2})} \lesssim \|\phi\|_{L^2(\mathbb{R}^2)},$$
$$\left\| \partial_{x_1} \int_0^t S(t-\tau) F(\tau) d\tau \right\|_{L^\infty_{x_1} L^2_{x_2,t}(\mathbb{R}^3)} \lesssim \|F\|_{L^1_{x_1} L^2_{x_2,t}(\mathbb{R}^3)}.$$

The second estimate from $L^1_{x_1}L^2_{x_2,t}$ to $L^\infty_{x_1}L^2_{x_2,t}$ has absorbed one order derivative, which enables us dealing with the derivative in the nonlinearity $F = \partial_{x_1}(|u|^2u)$. By Hölder's inequality,

$$\||u|^2u\|_{L^1_{x_1}L^2_{x_2,t}} \leqslant \|u\|_{L^\infty_{x_1}L^2_{x_2,t}}\|u\|^2_{L^2_{x_1}L^\infty_{x_2,t}}. \tag{1.20}$$

So, one needs at least to estimate $\|u\|_{L^2_{x_1}L^\infty_{x_2,t}}$. According to the integral equation, we need to show that $\|S(t)u_0\|_{L^2_{x_1}L^\infty_{x_2,t}}$ is bounded. As in [49], we have obtained that

$$\|\Box_k S(t)u_0\|_{L^p_{x_1}L^\infty_{x_2,t}(\mathbb{R}^{1+2})} \lesssim \langle k\rangle^{1/p}\|\Box_k u_0\|_{L^2(\mathbb{R}^2)}, \quad \forall\, p > 2. \tag{1.21}$$

When $p = 2$, there is a logarithmic divergence and we can not obtain the time-global estimate. To overcome this difficulty, we will use the Gabor frame. Roughly speaking, any function u_0 in L^2 and in any modulation space can be expressed in the form $\sum_{k,l\in\mathbb{Z}^2} c_{kl}e^{\mathrm{i}kx}e^{-|x-l|^2/2}$, it follows that

$$S(t)u_0 = \sum_{k,l\in\mathbb{Z}^2} c_{kl}e^{\mathrm{i}kx}e^{\mathrm{i}t(k_1^2-k_2^2)}\prod_{j=1}^{2}\frac{e^{-\frac{|x_1-l_1+2tk_1|^2}{2(1-2\mathrm{i}t)}-\frac{|x_2-l_2-2tk_2|^2}{2(1+2\mathrm{i}t)}}}{(1+4t^2)^{1/2}}. \tag{1.22}$$

We can get the following time-global estimates

$$\|\Box_k S(t)u_0\|_{L^2_{x_1}L^\infty_{x_2,t}(\mathbb{R}^{1+2})} \lesssim \langle k\rangle^{1/2}\|\Box_k u_0\|_{L^1(\mathbb{R}^2)}, \tag{1.23}$$

$$\|\Box_k \int S(t-\tau)f(\tau)d\tau\|_{L^2_{x_1}L^\infty_{x_2,t}(\mathbb{R}^{1+2})} \lesssim \langle k\rangle^{1/2}\|\Box_k f\|_{L^1_{x,t}(\mathbb{R}^{1+2})}. \tag{1.24}$$

Noticing that $\|\Box_k u\|_2 \leqslant \|\Box_k u\|_1$ and comparing (1.23) with (1.21), we see that there is a loss of spatial index in (1.23) and (1.24). According to (1.24), one need to bound (after ignoring the frequency localization and spatial regularity index)

$$\|\partial_{x_1}(|u|^2u)\|_{L^1_{x,t}(\mathbb{R}^{1+2})} \leqslant \|u_{x_1}\|_{L^\infty_{x_1}L^2_{x_2,t}}\|u\|^2_{L^2_{x_1}L^4_{x_2,t}}. \tag{1.25}$$

In another way,

$$\|\partial_{x_1}(|u|^2u)\|_{L^1_{x,t}(\mathbb{R}^{1+2})} \leqslant \|u_{x_1}\|_{L^3_{x,t}}\|u\|^2_{L^3_{x,t}}. \tag{1.26}$$

(1.25) is beneficial to the higher frequency part and (1.26) is useful for the lower frequency part. In summary, $L^\infty_{x_1}L^2_{x_2,t}$ is used for absorbing the derivative in nonlinearity. For the lower frequency part, $L^\infty_{x_1}L^2_{x_2,t}$ is a bad space and we use the Strichartz space $L^4_{x,t}\cap L^\infty_t L^2_x$ as a substitution. $L^2_{x_1}L^\infty_{x_2,t}$ is a maximal function space arising from the nonlinear estimate (1.20). In order to get a time-global estimate for the Schrödinger semi-group in $L^2_{x_1}L^\infty_{x_2,t}$, an intermediate space $L^1_{x,t}$ is introduced. Finally, the anisotropic space $L^2_{x_1}L^4_{x_2,t}$ and the generalized Strichartz space $L^3_{x,t}$ is employed for the nonlinear estimates in $L^1_{x,t}$.

2 Gabor frame estimates

Gabor frame is of importance in the time-frequency theory, its discrete form enable us to get an exact expression for the solution of the free Schrödinger equation, see below (2.2). The advantage of the expression (2.2) is that it has no singularity at $t = 0$ and easier to calculate than the following form

$$e^{-\mathrm{i}t\Delta}u_0 = ct^{-n/2}\int e^{\mathrm{i}c|x-y|^2/4t}u_0(y)dy.$$

In this section, we always denote $|\xi|^2_\pm = \sum_{j=1}^n \varepsilon_j\xi_j^2$, where $\varepsilon_j \in \{1, -1\}$ is arbitrary. For any $x \in \mathbb{R}^n$, we write $\bar{x} = (x_2, ..., x_n)$.

$$S(t) = \mathscr{F}^{-1}e^{-\mathrm{i}t|\xi|^2_\pm}\mathscr{F}, \quad \mathscr{A}f(t,x) = \int_0^t S(t-\tau)f(\tau,x)d\tau. \tag{2.1}$$

Proposition 2.1 (Gabor frame expression) *Let $s \in \mathbb{R}$, $1 \leqslant p, q < \infty$, $u_0 \in M^s_{p,q}$ and*

$$u_0(x) = \sum_{k,l\in\mathbb{Z}^n} c_{kl}e^{\mathrm{i}kx}e^{-\frac{|x-l|^2}{2}}.$$

Then we have

$$S(t)u_0 = \sum_{k,l\in\mathbb{Z}^n} c_{kl}e^{\mathrm{i}kx}e^{\mathrm{i}t|k|^2_\pm}\prod_{j=1}^n \frac{e^{-\frac{|x_j-l_j+2t\varepsilon_jk_j|^2}{2(1-2\mathrm{i}\varepsilon_jt)}}}{(1-2\mathrm{i}\varepsilon_jt)^{1/2}}. \tag{2.2}$$

Proof. In view of

$$\widehat{e^{\mathrm{i}kx}f} = \widehat{f}(\cdot - k), \quad \widehat{f(\cdot - l)} = e^{-\mathrm{i}l\xi}\widehat{f}, \quad \widehat{e^{-|x|^2/2}} = e^{-|\cdot|^2/2},$$

we see that

$$\widehat{u}_0(\xi) = \sum_{k,l\in\mathbb{Z}^n} c_{kl}e^{\mathrm{i}lk}e^{-\mathrm{i}l\xi}e^{-\frac{|\xi-k|^2}{2}}.$$

It follows that

$$S(t)u_0 = \sum_{k,l\in\mathbb{Z}^n} c_{kl}e^{\mathrm{i}lk}\mathscr{F}^{-1}\left(e^{\mathrm{i}t|\xi|^2_\pm}e^{-\mathrm{i}l\xi}e^{-\frac{|\xi-k|^2}{2}}\right).$$

In view of

$$\mathscr{F}^{-1}(f(\cdot - k)) = e^{\mathrm{i}kx}\mathscr{F}^{-1}f, \quad \mathscr{F}^{-1}(e^{-\mathrm{i}l\xi}f) = (\mathscr{F}^{-1}f)(\cdot - l),$$

we see that

$$\begin{aligned} S(t)u_0 &= \sum_{k,l\in\mathbb{Z}^n} c_{kl}e^{\mathrm{i}xk}\left[\mathscr{F}^{-1}\left(e^{\mathrm{i}t|\xi+k|^2_\pm}e^{-\frac{|\xi|^2}{2}}\right)\right](x-l) \\ &= \sum_{k,l\in\mathbb{Z}^n} c_{kl}e^{\mathrm{i}xk}e^{\mathrm{i}t|k|^2_\pm}\prod_{j=1}^n\left[\mathscr{F}^{-1}_{\xi_j}\left(e^{\mathrm{i}t\varepsilon_j\xi_j^2}e^{-\frac{\xi_j^2}{2}}\right)\right](x_j - l_j + 2t\varepsilon_jk_j). \end{aligned} \tag{2.3}$$

Using the fact that $\mathscr{F}^{-1}e^{-c\xi_1^2/2} = c^{-1/2}e^{-x_1^2/2c}$, we immediately have the result, as desired. $\square$

Proposition 2.2 *Let $n \geqslant 1$ and $1 \leqslant r, p, \bar{p} \leqslant \infty$. Assume that one of the following alternative conditions holds:*

$$n\left(\frac{1}{r} - \frac{1}{2} - \frac{1}{\bar{p}}\right) > \frac{1}{p}, \quad r \leqslant p, \quad or \tag{2.4}$$

$$n\left(\frac{1}{r} - \frac{1}{2} - \frac{1}{\bar{p}}\right) = \frac{1}{p}, \quad r < p < \infty. \tag{2.5}$$

Then we have

$$\|S(t)u_0\|_{L^p_{x_1} L^{\bar{p}}_{\bar{x},t}(\mathbb{R}^{1+n})} \lesssim \|u_0\|_{M^{1/p+1-1/r}_{r,1}}. \tag{2.6}$$

$$\|\mathscr{A} f\|_{L^p_{x_1} L^{\bar{p}}_{\bar{x},t}(\mathbb{R}^{1+n})} \lesssim \|f\|_{L^1(\mathbb{R},\ M^{1/p+1-1/r}_{r,1}(\mathbb{R}^n))}. \tag{2.7}$$

Proof. Let $u_0 = \sum_{k,l\in\mathbb{Z}^n} c_{kl} e^{\mathrm{i}kx} e^{-|x-l|^2/2}$. By Proposition 2.1,

$$\|S(t)u_0\|_{L^p_{x_1} L^{\bar{p}}_{\bar{x},t}} \leqslant \left\| \sum_{k,l\in\mathbb{Z}^n} |c_{kl}|(1+|t|)^{-n/2} \prod_{j=1}^n e^{-\frac{|x_j - l_j + 2t\varepsilon_j k_j|^2}{2(1+4t^2)}} \right\|_{L^p_{x_1} L^{\bar{p}}_{\bar{x},t}}$$

$$\leqslant \sum_{k\in\mathbb{Z}^n} \left\| \sum_{l_1\in\mathbb{Z}} \langle t\rangle^{-n/2} e^{-\frac{|x_1 - l_1 + 2t\varepsilon_1 k_1|^2}{2(1+4t^2)}} \left\| \sum_{\bar{l}\in\mathbb{Z}^{n-1}} |c_{kl}| \prod_{j=2}^n e^{-\frac{|x_j - l_j + 2t\varepsilon_j k_j|^2}{2(1+4t^2)}} \right\|_{L^{\bar{p}}_{\bar{x}}} \right\|_{L^p_{x_1} L^{\bar{p}}_t}. \tag{2.8}$$

Applying the fact that $\sup_{x>0} \langle x\rangle^N / e^x < \infty$ for any $N > 0$, we have

$$\left\| \sum_{\bar{l}\in\mathbb{Z}^{n-1}} |c_{kl}| \prod_{j=2}^n e^{-\frac{|x_j - l_j + 2t\varepsilon_j k_j|^2}{2(1+4t^2)}} \right\|_{L^{\bar{p}}_{\bar{x}}} \lesssim \left\| \sum_{\bar{l}\in\mathbb{Z}^{n-1}} |c_{kl}| \prod_{j=2}^n \left\langle \frac{|x_j - l_j + 2t\varepsilon_j k_j|}{1+|t|} \right\rangle^{-2} \right\|_{L^{\bar{p}}_{\bar{x}}}. \tag{2.9}$$

In view of Lemma B.1 we have (see Appendix)

$$\left\| \sum_{\bar{l}\in\mathbb{Z}^{n-1}} |c_{kl}| \prod_{j=2}^n e^{-\frac{|x_j - l_j + 2t\varepsilon_j k_j|^2}{2(1+4t^2)}} \right\|_{L^{\bar{p}}_{\bar{x}}} \lesssim \langle t\rangle^{(n-1)/r' + (n-1)/\bar{p}} \left(\sum_{\bar{l}\in\mathbb{Z}^{n-1}} |c_{kl}|^r \right)^{1/r}. \tag{2.10}$$

It follows from (2.8) and (2.10) that

$$\begin{aligned}
&\|S(t)u_0\|_{L^p_{x_1} L^{\bar{p}}_{\bar{x},t}} \\
&\lesssim \sum_{k\in\mathbb{Z}^n} \left\| \sum_{l_1\in\mathbb{Z}} \langle t\rangle^{-\frac{n}{2} + \frac{n-1}{r'} + \frac{n-1}{\bar{p}}} \left(\sum_{\bar{l}\in\mathbb{Z}^{n-1}} |c_{kl}|^r \right)^{1/r} e^{-\frac{|x_1 - l_1 + 2t\varepsilon_1 k_1|^2}{2(1+4t^2)}} \right\|_{L^p_{x_1} L^{\bar{p}}_t} \\
&= \sum_{k\in\mathbb{Z}^n,\, k_1\neq 0} \left\| \sum_{l_1\in\mathbb{Z}} \langle t\rangle^{-\frac{n}{2} + \frac{n-1}{r'} + \frac{n-1}{\bar{p}}} \left(\sum_{\bar{l}\in\mathbb{Z}^{n-1}} |c_{kl}|^r \right)^{1/r} e^{-\frac{|x_1 - l_1 + 2t\varepsilon_1 k_1|^2}{2(1+4t^2)}} \right\|_{L^p_{x_1} L^{\bar{p}}_t}
\end{aligned}$$

$$
+ \sum_{k\in\mathbb{Z}^n,\, k_1=0} \left\| \sum_{l_1\in\mathbb{Z}} \langle t\rangle^{-\frac{n}{2}+\frac{n-1}{r'}+\frac{n-1}{\bar p}} \left(\sum_{\bar l\in\mathbb{Z}^{n-1}} |c_{kl}|^r \right)^{1/r} e^{-\frac{|x_1-l_1|^2}{2(1+4t^2)}} \right\|_{L^p_{x_1} L^{\bar p}_t}
$$

$$
:= A_{\rm hi} + A_{\rm lo} \tag{2.11}
$$

We consider the estimate of $A_{\rm hi}$.

$$
A_{\rm hi} \lesssim \sum_{s=1}^{3} \sum_{k\in\mathbb{Z}^n,\, k_1\neq 0} \left\| \sum_{l_1\in\mathbb{Z}} \langle t\rangle^{-\frac{n}{2}+\frac{n-1}{r'}+\frac{n-1}{\bar p}} \left(\sum_{\bar l\in\mathbb{Z}^{n-1}} |c_{kl}|^r \right)^{1/r} e^{-\frac{|x_1-l_1+2t\varepsilon_1 k_1|^2}{2(1+4t^2)}} \chi_{\mathbb{D}_s} \right\|_{L^p_{x_1} L^{\bar p}_t}
$$

$$
:= \Upsilon_1 + \Upsilon_2 + \Upsilon_3, \tag{2.12}
$$

where we denote

$$
\begin{aligned}
\mathbb{D}_1 &= \{(x_1,t):\ |x_1-l_1| > 4|tk_1|\},\\
\mathbb{D}_2 &= \{(x_1,t):\ |x_1-l_1| < |tk_1|\},\\
\mathbb{D}_3 &= \{(x_1,t):\ |tk_1| \leqslant |x_1-l_1| \leqslant 4|tk_1|\}.
\end{aligned}
$$

We now estimate Υ_1. Usingt $\sup_{x>0}\langle x\rangle^\theta/e^x < \infty$ for any $\theta>0$, we have

$$
\begin{aligned}
\Upsilon_1 &\lesssim \sum_{k\in\mathbb{Z}^n,\, k_1\neq 0} \left\| \sum_{l_1\in\mathbb{Z}} \langle t\rangle^{-\frac{n}{2}+\frac{n-1}{r'}+\frac{n-1}{\bar p}} \left(\sum_{\bar l\in\mathbb{Z}^{n-1}} |c_{kl}|^r \right)^{1/r} e^{-\frac{|x_1-l_1|^2}{8(1+4t^2)}} \chi_{\mathbb{D}_1} \right\|_{L^p_{x_1} L^{\bar p}_t}\\
&\lesssim \sum_{k\in\mathbb{Z}^n,\, k_1\neq 0} \left\| \sum_{l_1\in\mathbb{Z}} \langle x_1-l_1\rangle^{-\frac{n}{2}+\frac{n-1}{r'}+\frac{n-1}{\bar p}} \left(\sum_{\bar l\in\mathbb{Z}^{n-1}} |c_{kl}|^r \right)^{1/r} \chi_{\mathbb{D}_1} \right\|_{L^p_{x_1} L^{\bar p}_t}\\
&\lesssim \sum_{k\in\mathbb{Z}^n,\, k_1\neq 0} \left\| \sum_{l_1\in\mathbb{Z}} \langle x_1-l_1\rangle^{-\frac{n}{2}+\frac{n-1}{r'}+\frac{n}{\bar p}} \left(\sum_{\bar l\in\mathbb{Z}^{n-1}} |c_{kl}|^r \right)^{1/r} \right\|_{L^p_{x_1}}.
\end{aligned} \tag{2.13}
$$

By Lemma B.1 and (2.13), we have

$$
\Upsilon_1 \lesssim \sum_{k\in\mathbb{Z}^n} \left(\sum_{l\in\mathbb{Z}^n} |c_{kl}|^r \right)^{1/r}.
$$

Noticing that $|t|\sim |x_1-l_1|/|k_1|$ in $\mathbb{D}_3$, we have

$$
\Upsilon_3 \lesssim \sum_{k\in\mathbb{Z}^n,\, k_1\neq 0} \left\| \sum_{l_1\in\mathbb{Z}} \left\langle \frac{x_1-l_1}{|k_1|} \right\rangle^{-\frac{n}{2}+\frac{n-1}{r'}+\frac{n-1}{\bar p}} \left(\sum_{\bar l\in\mathbb{Z}^{n-1}} |c_{kl}|^r \right)^{1/r} \chi_{\mathbb{D}_3} \right\|_{L^p_{x_1} L^{\bar p}_t}
$$

$$\lesssim \sum_{k\in\mathbb{Z}^n,\, k_1\neq 0} \left\| \sum_{l_1\in\mathbb{Z}} \left\langle \frac{x_1-l_1}{|k_1|} \right\rangle^{-\frac{n}{2}+\frac{n-1}{r'}+\frac{n}{\bar{p}}} \left(\sum_{\bar{l}\in\mathbb{Z}^{n-1}} |c_{kl}|^r \right)^{1/r} \right\|_{L^p_{x_1}}. \tag{2.14}$$

By Lemma B.1, we have

$$\Upsilon_3 \lesssim \sum_{k\in\mathbb{Z}^n} \langle k_1\rangle^{\frac{1}{r'}+\frac{1}{p}} \left(\sum_{l\in\mathbb{Z}^n} |c_{kl}|^r \right)^{1/r}. \tag{2.15}$$

For the estimate of Υ_2, we have

$$\begin{aligned}\Upsilon_2 &\lesssim \sum_{k\in\mathbb{Z}^n,\, k_1\neq 0} \left\| \sum_{l_1\in\mathbb{Z}} \langle t\rangle^{-\frac{n}{2}+\frac{n-1}{r'}+\frac{n-1}{\bar{p}}} \left(\sum_{\bar{l}\in\mathbb{Z}^{n-1}} |c_{kl}|^r \right)^{1/r} \chi_{\mathbb{D}_2} \right\|_{L^p_{x_1} L^{\bar{p}}_t} \\ &\lesssim \sum_{k\in\mathbb{Z}^n,\, k_1\neq 0} \left\| \sum_{l_1\in\mathbb{Z}} \left\langle \frac{x_1-l_1}{|k_1|} \right\rangle^{-\frac{n}{2}+\frac{n-1}{r'}+\frac{n}{\bar{p}}} \left(\sum_{\bar{l}\in\mathbb{Z}^{n-1}} |c_{kl}|^r \right)^{1/r} \right\|_{L^p_{x_1}}. \end{aligned} \tag{2.16}$$

By Lemma B.1, we see that Υ_2 has the same upper bound as Υ_3 in (2.15). Collecting the estimates of Υ_1, Υ_2 and Υ_3, we have the desired estimate.

We consider the estimate of $A_{\rm lo}$. We have

$$\begin{aligned} A_{\rm lo} &\lesssim \sum_{s=1}^{2} \sum_{k\in\mathbb{Z}^n,\, k_1=0} \left\| \sum_{l_1\in\mathbb{Z}} \langle t\rangle^{-\frac{n}{2}+\frac{n-1}{r'}+\frac{n-1}{\bar{p}}} \left(\sum_{\bar{l}\in\mathbb{Z}^{n-1}} |c_{kl}|^r \right)^{1/r} e^{-\frac{|x_1-l_1|^2}{2(1+4t^2)}} \chi_{\mathbb{E}_s} \right\|_{L^p_{x_1} L^{\bar{p}}_t} \\ &:= \Xi_1 + \Xi_2, \end{aligned} \tag{2.17}$$

where we denote

$$\begin{aligned} \mathbb{E}_1 &= \{(x_1,t):\ |x_1-l_1| > |t|\}, \\ \mathbb{E}_2 &= \{(x_1,t):\ |x_1-l_1| \leqslant |t|\}. \end{aligned}$$

Applying the fact $\sup_{x>0}\langle x\rangle^\theta/e^x < \infty$, we have

$$\begin{aligned}\Xi_1 &\lesssim \sum_{k\in\mathbb{Z}^n,\, k_1=0} \left\| \sum_{l_1\in\mathbb{Z}} \langle x_1-l_1\rangle^{-\frac{n}{2}+\frac{n-1}{r'}+\frac{n-1}{\bar{p}}} \left(\sum_{\bar{l}\in\mathbb{Z}^{n-1}} |c_{kl}|^r \right)^{1/r} \chi_{\mathbb{E}_1} \right\|_{L^p_{x_1} L^{\bar{p}}_t} \\ &\lesssim \sum_{k\in\mathbb{Z}^n,\, k_1=0} \left\| \sum_{l_1\in\mathbb{Z}} \langle x_1-l_1\rangle^{-\frac{n}{2}+\frac{n-1}{r'}+\frac{n}{\bar{p}}} \left(\sum_{\bar{l}\in\mathbb{Z}^{n-1}} |c_{kl}|^r \right)^{1/r} \right\|_{L^p_{x_1}}. \end{aligned} \tag{2.18}$$

By Lemma B.1, we have

$$\Xi_1 \lesssim \sum_{\bar{k}\in\mathbb{Z}^{n-1}} \left(\sum_{l\in\mathbb{Z}^n} |c_{(0,\bar{k})l}|^r \right)^{1/r}.$$

Analogous to the estimate of Υ_2, we can show that Ξ_2 has the same upper bound as Ξ_1. $\square$

Let us observe an endpoint case $r = 1$. We have

Corollary 2.3 *Let $n \geqslant 1$, $1 \leqslant p,\ \bar{p} \leqslant \infty$. Assume one of the following alternative conditions holds:*

$$n\left(\frac{1}{2} - \frac{1}{\bar{p}}\right) > \frac{1}{p}; \quad or$$

$$n\left(\frac{1}{2} - \frac{1}{\bar{p}}\right) = \frac{1}{p}, \quad 1 < p < \infty.$$

Then we have

$$\|S(t)u_0\|_{L^p_{x_1} L^{\bar{p}}_{\bar{x},t}(\mathbb{R}^{1+n})} \lesssim \|u_0\|_{M^{1/p}_{1,1}}. \tag{2.19}$$

$$\|\mathscr{A} f\|_{L^p_{x_1} L^{\bar{p}}_{\bar{x},t}(\mathbb{R}^{1+n})} \lesssim \|f\|_{L^1(\mathbb{R},\ M^{1/p}_{1,1}(\mathbb{R}^n))}. \tag{2.20}$$

3 Linear estimates with $\square_k$-decomposition

Corollary 3.1 (L^1-anisotropic estimates) *Let $n \geqslant 1$, $1 \leqslant p, \bar{p} \leqslant \infty$. Assume one of the following alternative conditions holds:*

$$n\left(\frac{1}{2} - \frac{1}{\bar{p}}\right) > \frac{1}{p}; \quad or$$

$$n\left(\frac{1}{2} - \frac{1}{\bar{p}}\right) = \frac{1}{p}, \quad 1 < p < \infty.$$

Then we have

$$\|\square_k S(t)u_0\|_{L^p_{x_1} L^{\bar{p}}_{\bar{x},t}(\mathbb{R}^{1+n})} \lesssim \langle k\rangle^{1/p} \|\square_k u_0\|_{L^1(\mathbb{R}^n)}, \tag{3.1}$$

$$\|\square_k \mathscr{A} f\|_{L^p_{x_1} L^{\bar{p}}_{\bar{x},t}(\mathbb{R}^{1+n})} \lesssim \langle k\rangle^{1/p} \|\square_k f\|_{L^1_{x,t}(\mathbb{R}^{n+1})}. \tag{3.2}$$

In particular, we have for $n = 2$,

$$\max_{q=4,\infty} \|\square_k S(t)u_0\|_{L^2_{x_1} L^q_{x_2,t}(\mathbb{R}^{1+2})} \lesssim \langle k\rangle^{1/2} \|\square_k u_0\|_{L^1(\mathbb{R}^2)}, \tag{3.3}$$

$$\max_{q=4,\infty} \|\square_k \mathscr{A} f\|_{L^2_{x_1} L^q_{x_2,t}(\mathbb{R}^{1+2})} \lesssim \langle k\rangle^{1/2} \|\square_k f\|_{L^1_{x,t}(\mathbb{R}^{n+1})}. \tag{3.4}$$

$$\|\square_k S(t)u_0\|_{L^3_{x,t}(\mathbb{R}^{1+2})} \lesssim \langle k\rangle^{1/3} \|\square_k u_0\|_{L^1(\mathbb{R}^2)}, \tag{3.5}$$

$$\|\square_k \mathscr{A} f\|_{L^3_{x,t}(\mathbb{R}^{1+2})} \lesssim \langle k\rangle^{1/2} \|\square_k f\|_{L^1_{x,t}(\mathbb{R}^{n+1})}. \tag{3.6}$$

Proof. By Proposition 2.3, we have

$$\|\Box_k S(t)u_0\|_{L^p_{x_1}L^{\bar{p}}_{\bar{x},t}(\mathbb{R}^{1+n})} \lesssim \|\Box_k u_0\|_{M^{1/p}_{1,1}}. \tag{3.7}$$

By definition and $\Box_k : L^r \to L^r$[2],

$$\|\Box_k u_0\|_{M^{1/p}_{1,1}} \leqslant \sum_{|l|_\infty \leqslant 1} \langle k+l\rangle^{1/p} \|\Box_{k+l}\Box_k u_0\|_1 \lesssim \langle k\rangle^{1/p}\|\Box_k u_0\|_1,$$

which implies the result, as desired. □

Proposition 3.2 (Smooth effects, [32, 49]) *For any $k = (k_1, \dots, k_n) \in \mathbb{Z}^n$, we have*

$$\|D^{1/2}_{x_1}\Box_k S(t)u_0\|_{L^\infty_{x_1}L^2_{\bar{x},t}} \lesssim \|\Box_k u_0\|_2, \tag{3.8}$$

$$\|\partial_{x_1}\Box_k \mathscr{A} f\|_{L^\infty_{x_1}L^2_{\bar{x},t}} \lesssim \|\Box_k f\|_{L^1_{x_1}L^2_{\bar{x},t}}. \tag{3.9}$$

Proposition 3.3 ($\Box_k$-Strichartz estimates, [47, 49]) *Let $4/n \leqslant p < \infty$. We have*

$$\|\Box_k S(t)u_0\|_{L^{2+p}_{t,x}\cap L^\infty_t L^2_x(\mathbb{R}^{1+n})} \lesssim \|\Box_k u_0\|_{L^2(\mathbb{R}^n)}, \tag{3.10}$$

$$\|\Box_k \mathscr{A} f\|_{L^\infty_t L^2_x \cap L^{2+p}_{t,x}(\mathbb{R}^{1+n})} \lesssim \|\Box_k f\|_{L^{(2+p)/(1+p)}_{t,x}(\mathbb{R}^{1+n})}. \tag{3.11}$$

Proposition 3.4 ($\Box_k$-interaction estimates, [49]) *Let $4/n \leqslant p < \infty$. We have*

$$\|\Box_k \partial_{x_1}\mathscr{A} f\|_{L^\infty_t L^2_x \cap L^{2+p}_{t,x}(\mathbb{R}^{1+n})} \lesssim \langle k_1\rangle^{1/2}\|\Box_k f\|_{L^1_{x_1}L^2_{\bar{x},t}(\mathbb{R}^{1+n})}, \tag{3.12}$$

$$\|\Box_k \mathscr{A}\partial_{x_1} f\|_{L^\infty_{x_1}L^2_{\bar{x},t}(\mathbb{R}^{1+n})} \lesssim \langle k_1\rangle^{1/2}\|\Box_k f\|_{L^{(2+p)/(1+p)}_{t,x}(\mathbb{R}^{1+n})}, \tag{3.13}$$

Proposition 3.5 (L^2-anisotropic estimates) *Let $n \geqslant 1$, $2 \leqslant q, \ \bar{q} \leqslant \infty$. Assume one of the following alternative conditions holds:*

$$n\left(\frac{1}{2} - \frac{2}{\bar{q}}\right) > \frac{2}{q}; \quad \textit{or}$$

$$n\left(\frac{1}{2} - \frac{2}{\bar{q}}\right) = \frac{2}{q}, \quad 2 < q < \infty.$$

Then we have

$$\|\Box_k S(t)u_0\|_{L^q_{x_1}L^{\bar{q}}_{\bar{x},t}(\mathbb{R}^{1+n})} \lesssim \langle k_1\rangle^{1/q}\|\Box_k u_0\|_2, \tag{3.14}$$

$$\|\Box_k \mathscr{A} f\|_{L^q_{x_1}L^{\bar{q}}_{\bar{x},t}(\mathbb{R}^{1+n})} \lesssim \langle k\rangle^{1/q}\|\Box_k f\|_{L^1_t L^2_x((\mathbb{R}^{n+1})}. \tag{3.15}$$

[2] $|l|_\infty = \max(|l_1|, |l_2|)$ for $l = (l_1, l_2) \in \mathbb{Z}^2$.

Proof. By duality, it suffices to show that

$$\int_{\mathbb{R}} (\Box_k S(t)u_0, \phi(t))dt \lesssim \langle k_1\rangle^{1/q} \|\Box_k u_0\|_2 \|\phi\|_{L^{q'}_{x_1} L^{\bar{q}'}_{\bar{x},t}}. \tag{3.16}$$

It is easy to see that

$$\int_{\mathbb{R}} (\Box_k S(t)u_0, \phi(t))dt \lesssim \langle k_1\rangle^{1/q} \|\Box_k u_0\|_2 \sum_{|l|_\infty \leqslant 1} \left\| \Box_{k+l} \int S(-t)\phi(t)dt \right\|_2. \tag{3.17}$$

We have

$$\left\| \Box_k \int S(-t)\phi(t)dt \right\|_2^2 \leqslant \|\phi\|_{L^{q'}_{x_1} L^{\bar{q}'}_{\bar{x},t}} \left\| \Box_k \int S(t-s)\phi(s)ds \right\|_{L^{q}_{x_1} L^{\bar{q}}_{\bar{x},t}}. \tag{3.18}$$

Let us observe that

$$\Box_k \int S(t-s)\phi(s)ds = (\mathscr{F}^{-1} e^{\mathrm{i}t|\xi|^2_\pm} \eta_k(\xi)) * \phi,$$

where $*$ denotes the convolution on x and t. Applying Young's inequality, we obtain that

$$\left\| \Box_k \int S(t-s)\phi(s)ds \right\|_{L^{q}_{x_1} L^{\bar{q}}_{\bar{x},t}} \lesssim \|\mathscr{F}^{-1} e^{\mathrm{i}t|\xi|^2_\pm} \eta_k(\xi)|_{L^{q/2}_{x_1} L^{\bar{q}/2}_{\bar{x},t}} \|\phi\|_{L^{q'}_{x_1} L^{\bar{q}'}_{\bar{x},t}}. \tag{3.19}$$

Hence, if we can show that

$$\|\mathscr{F}^{-1} e^{\mathrm{i}t|\xi|^2_\pm} \eta_k(\xi)|_{L^{q/2}_{x_1} L^{\bar{q}/2}_{\bar{x},t}} \lesssim \langle k_1\rangle^{2/q}, \tag{3.20}$$

then the result follows. Indeed, in view of Corollary 2.3,

$$\|\mathscr{F}^{-1} e^{\mathrm{i}t|\xi|^2_\pm} \eta_k(\xi)|_{L^{q/2}_{x_1} L^{\bar{q}/2}_{\bar{x},t}} \lesssim \langle k_1\rangle^{2/q} \|\mathscr{F}^{-1}\eta_k\|_1 \lesssim \langle k_1\rangle^{2/q}. \qquad \Box$$

Proposition 3.6 *Let $1 \leqslant p \leqslant \infty$. Then $\Box_k S(t) : L^p \to L^p$ is uniformly bounded. More precisely,*

$$\|\Box_k S(t)u_0\|_{L^p} \lesssim (1+|t|^{n/2}) \|\Box_k u_0\|_{L^p} \tag{3.21}$$

uniformly holds for all $k \in \mathbb{Z}^n$.

Proof. See [5, 46].

4 Derivative NLS (1.3) and (1.4)

Now we introduce the following (semi-)norms

$$\|u\|_1^{\mathrm{sm}} = \sum_{k\in\mathbb{Z}^2,\, |k_1|\geqslant |k_2|\vee 10} \langle k_1\rangle^{3/2} \|\Box_k u\|_{L^\infty_{x_1} L^2_{x_2,t}}, \tag{4.1}$$

$$\|u\|_2^{\rm sm} = \sum_{k\in\mathbb{Z}^2,\, |k_2|\geqslant |k_1|\vee 10} \langle k_2\rangle^{3/2} \|\Box_k u\|_{L^\infty_{x_2} L^2_{x_1,t}}; \tag{4.2}$$

$$\|u\|_1^{\max} = \sum_{k\in\mathbb{Z}^2} \|\Box_k u\|_{L^2_{x_1} L^\infty_{x_2,t}}, \quad \|u\|_2^{\max} = \sum_{k\in\mathbb{Z}^2} \|\Box_k u\|_{L^2_{x_2} L^\infty_{x_1,t}}; \tag{4.3}$$

$$\|u\|_1^{\rm ant} = \sum_{k\in\mathbb{Z}^2} \|\Box_k u\|_{L^2_{x_1} L^4_{x_2,t}}, \quad \|u\|_2^{\rm ant} = \sum_{k\in\mathbb{Z}^2} \|\Box_k u\|_{L^2_{x_2} L^4_{x_1,t}}; \tag{4.4}$$

$$\|u\|^{\rm str} = \sum_{k\in\mathbb{Z}^2} \langle k\rangle \|\Box_k u\|_{L^\infty_t L^2_x \cap L^4_{x,t}}; \quad \|u\|^{\rm gstr} = \sum_{k\in\mathbb{Z}^2} \|\Box_k u\|_{L^3_{x,t}}. \tag{4.5}$$

We see that

$$\|u\|^{\rm sm} = \|u\|_1^{\rm sm} + \|u\|_2^{\rm sm}, \quad \|u\|^{\max} = \|u\|_1^{\max} + \|u\|_2^{\max}, \quad \|u\|^{\rm ant} = \|u\|_1^{\rm ant} + \|u\|_2^{\rm ant}.$$

We define the following

$$\mathcal{D} = \left\{ u : \sum_{\alpha=0,1} \sum_{i=1,2} \|\partial^\alpha_{x_i} u\|^{\rm sm\,\cap\,max\,\cap\,ant\,\cap\,str\,\cap\,gstr} \leqslant \delta \right\}, \tag{4.6}$$

and for any $u, v \in \mathcal{D}$,

$$d(u,v) = \sum_{\alpha=0,1} \sum_{i=1,2} \|\partial^\alpha_{x_i} (u-v)\|^{\rm sm\,\cap\,max\,\cap\,ant\,\cap\,str\,\cap\,gstr}. \tag{4.7}$$

We consider the following mapping $\mathscr{T}$ in $(\mathcal{D}, d)$,

$$\mathscr{T} : u \to S(t)u_0 - {\rm i}\int_0^t S(t-\tau) F(u(\tau))d\tau, \tag{4.8}$$

where

$$\begin{aligned} F(u) = \frac{(u^2_{x_1} - u^2_{x_2})\bar{u}}{1+|u|^2} &= \sum_{j=0}^\infty (-1)^j |u|^{2j} \bar{u} u^2_{x_1} - \sum_{j=0}^\infty (-1)^j |u|^{2j} \bar{u} u^2_{x_2} \\ &:= F_1(u) - F_2(u). \end{aligned} \tag{4.9}$$

Lemma 4.1 *Let $m \in H^{[n/2]+1}(\mathbb{R}^n)$, $1 \leqslant p_1, p_2 \leqslant \infty$. We have*

$$\|\mathscr{F}^{-1} m \mathscr{F} f\|_{L^{p_1}_{x_1} L^{p_2}_{\bar{x},t}} \lesssim \|m\|_{H^{[n/2]+1}} \|f\|_{L^{p_1}_{x_1} L^{p_2}_{\bar{x},t}}. \tag{4.10}$$

Proof. We have

$$\mathscr{F}^{-1} m \mathscr{F} f = \int_{\mathbb{R}^n} (\mathscr{F}^{-1}_\xi m)(y) f(t, x-y) dy.$$

It follows that

$$\|\mathscr{F}^{-1}m\mathscr{F}f\|_{L^{p_1}_{x_1}L^{p_2}_{\bar{x},t}} \leqslant \|\mathscr{F}^{-1}m\|_1 \sup_y \|f(t,x-y)\|_{L^{p_1}_{x_1}L^{p_2}_{\bar{x},t}}.$$

Applying the multiplier estimate

$$\|\mathscr{F}^{-1}m\|_1 \lesssim \|m\|_{H^{[n/2]+1}},$$

we have the result, as desired. □

Lemma 4.2 *We have*

$$\sum_{i=1,2}\sum_{\alpha=0,1} \|\partial^\alpha_{x_i} u\|^{\rm sm} \lesssim \|\partial_{x_1} u\|^{\rm sm}_1 + \|\partial_{x_2} u\|^{\rm sm}_2. \tag{4.11}$$

Proof. By Lemma 4.1, we see that $\|u\|^{\rm sm}_1 \lesssim \|\partial_{x_1} u\|^{\rm sm}_1$. By Lemma 4.1,

$$\begin{aligned}
\|\partial_{x_2} u\|^{\rm sm}_1 &= \sum_{k\in\mathbb{Z}^2,\,|k_1|\geqslant|k_2|\vee 10} \langle k_1\rangle^{3/2} \|\partial_{x_2}\Box_k u\|_{L^\infty_{x_1}L^2_{x_2,t}} \\
&\lesssim \sum_{k\in\mathbb{Z}^2,\,|k_1|\geqslant|k_2|\vee 10} \langle k_1\rangle^{3/2} \sum_{|l|_\infty\leqslant 1} \left\| \left(\frac{\xi_2}{\xi_1}\sigma_{k+l}\right)\right\|_{H^2} \|\partial_{x_1}\Box_k u\|_{L^\infty_{x_1}L^2_{x_2,t}} \\
&\lesssim \sum_{k\in\mathbb{Z}^2,\,|k_1|\geqslant|k_2|\vee 10} \langle k_1\rangle^{3/2} \|\partial_{x_1}\Box_k u\|_{L^\infty_{x_1}L^2_{x_2,t}},
\end{aligned} \tag{4.12}$$

which implies that $\|\partial_{x_2} u\|^{\rm sm}_1 \lesssim \|\partial_{x_1} u\|^{\rm sm}_1$. □

Lemma 4.3 *We have for any $k, k^{(s)} \in \mathbb{Z}^2$, $k^{(s)} = (k^{(s)}_1, k^{(s)}_2)$,*

$$\Box_k \bar{u} = \overline{\Box_{-k} u},$$

$$\Box_k(\Box_{k^{(1)}}u_1 ... \Box_{k^{(r)}}u_r) = 0$$

if $|k_i - k^{(1)}_i - ... - k^{(r)}_i| > r+1$, $i=1$ or $i=2$. Moreover,

$$\|\Box_k u\|_{L^\infty_{x,t}} \lesssim \|\Box_k u\|_{L^\infty_t L^2_x}$$

uniformly holds for all $k\in\mathbb{Z}^2$.

Proof. See [47].

Using Lemma 4.2, we have

$$\sum_{\alpha=0,1}\sum_{i=1,2} \|\partial^\alpha_{x_i}\mathscr{T}u\|^{\rm sm} \lesssim \|\partial_{x_1}\mathscr{T}u\|^{\rm sm}_1 + \|\partial_{x_2}\mathscr{T}u\|^{\rm sm}_2.$$

Lemma 4.4 *Let $u \in \mathcal{D}$. Then we have*

$$\|\partial_{x_1}\mathscr{T}u\|_1^{\rm sm} + \|\partial_{x_2}\mathscr{T}u\|_2^{\rm sm} \lesssim \|u_0\|_{M^2_{2,1}} + \delta^3\sum_{j=0}^{\infty} C^j\delta^{2j}.$$

Proof. First, we estimate $\|\partial_{x_1}\mathscr{T}u\|_1^{\rm sm}$.

$$\|\partial_{x_1}\mathscr{T}u\|_1^{\rm sm} \leqslant \|S(t)u_0\|_1^{\rm sm} + \|\partial_{x_1}\mathscr{A}F_1(u)\|_1^{\rm sm} + \|\partial_{x_1}\mathscr{A}F_2(u)\|_1^{\rm sm}.$$

By Proposition 3.2,

$$\begin{aligned}\|S(t)u_0\|_1^{\rm sm} &= \sum_{k\in\mathbb{Z}^2,\,|k_1|\geqslant|k_2|\vee 10} \langle k_1\rangle^{3/2}\|\partial_{x_1}\Box_k S(t)u_0\|_{L^\infty_{x_1}L^2_{x_2,t}} \\ &\lesssim \sum_{k\in\mathbb{Z}^2,\,|k_1|\geqslant|k_2|\vee 10} \langle k_1\rangle^{2}\|\Box_k u_0\|_2 \leqslant \|u_0\|_{M^2_{2,1}}.\end{aligned}$$

For convenience, we write

$$\begin{aligned}\mathbb{A}^{\lambda,i}_{\rm lo} &= \left\{(k^{(1)},...,k^{(\lambda)})\in(\mathbb{Z}^2)^\lambda:\ \max_{1\leqslant s\leqslant\lambda}|k_i^{(s)}|<10\right\},\\ \mathbb{A}^{\lambda,i}_{\rm hi} &= \left\{(k^{(1)},...,k^{(\lambda)})\in(\mathbb{Z}^2)^\lambda:\ \max_{1\leqslant s\leqslant\lambda}|k_i^{(s)}|\geqslant 10\right\},\end{aligned}\tag{4.13}$$

where $k^{(s)} = (k_1^{(s)}, k_2^{(s)})$. We see that

$$\begin{aligned}&\|\partial_{x_1}\mathscr{A}F_1(u)\|_1^{\rm sm}\\ &\leqslant \sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^2,\,|k_1|\geqslant|k_2|\vee 10}\langle k_1\rangle^{3/2}\|\partial_{x_1}\Box_k\mathscr{A}(|u|^{2j}\bar{u}u_{x_1}^2)\|_{L^\infty_{x_1}L^2_{x_2,t}}\\ &\lesssim \sum_{j=0}^{\infty}\left(\sum_{k^{(1)},...,k^{(2j+3)}\in\mathbb{A}^{2j+3,1}_{\rm lo}} + \sum_{k^{(1)},...,k^{(2j+3)}\in\mathbb{A}^{2j+3,1}_{\rm hi}}\right)\sum_{k\in\mathbb{Z}^2,\,|k_1|\geqslant|k_2|\vee 10}\langle k_1\rangle^{3/2}\\ &\qquad\times\left\|\partial_{x_1}\mathscr{A}\Box_k\left(\prod_{s=1}^{j}\Box_{k^{(s)}}u\prod_{s=j+1}^{2j+1}\Box_{k^{(s)}}\bar{u}\Box_{k^{(2j+2)}}u_{x_1}\Box_{k^{(2j+3)}}u_{x_1}\right)\right\|_{L^\infty_{x_1}L^2_{x_2,t}}\\ &:= I + II.\end{aligned}\tag{4.14}$$

By Proposition 3.4, Hölder's inequality and Lemma 4.3,

$$\begin{aligned}I \lesssim& \sum_{j=0}^{\infty}\sum_{k^{(1)},...,k^{(2j+3)}\in\mathbb{A}^{2j+3,1}_{\rm lo}}\sum_{k\in\mathbb{Z}^2,\,|k_1|\geqslant|k_2|\vee 10}\langle k_1\rangle^{2}\\ &\times\left\|\Box_k\left(\prod_{s=1}^{j}\Box_{k^{(s)}}u\prod_{s=j+1}^{2j+1}\Box_{k^{(s)}}\bar{u}\Box_{k^{(2j+2)}}u_{x_1}\Box_{k^{(2j+3)}}u_{x_1}\right)\right\|_{L^{4/3}_{x,t}}\end{aligned}$$

$$\lesssim \sum_{j=0}^{\infty} \sum_{k^{(1)},...,k^{(2j+3)}\in \mathbb{A}^{2j+3,1}_{\rm lo}} \sum_{k\in \mathbb{Z}^2,\, |k_1|\geqslant |k_2|\vee 10} \langle k_1\rangle^2 \prod_{s=1}^{2j} \|\Box_{k^{(s)}} u\|_{L^\infty_{x,t}} \|\Box_{k^{(2j+1)}} u\|_{L^4_{x,t}}$$

$$\times \|\Box_{k^{(2j+2)}} u_{x_1}\|_{L^4_{x,t}} \|\Box_{k^{(2j+3)}} u_{x_1}\|_{L^4_{x,t}} \prod_{i=1,2} \chi_{\left(|k_i - k_i^{(1)} - ... - k_i^{(2j+3)}|\leqslant 2j+4\right)}. \tag{4.15}$$

Noticing that there exist at most $O(j)$ terms in the summation $\sum_{k\in \mathbb{Z}^2,\, |k_1|\geqslant |k_2|\vee 10}$ and $|k_1| \leqslant Cj$ in (4.15), we easily see that

$$I \lesssim \sum_{j=0}^{\infty} C^j \sum_{k^{(1)},...,k^{(2j+3)}\in \mathbb{A}^{2j+3,1}_{\rm lo}} \prod_{s=1}^{2j} \|\Box_{k^{(s)}} u\|_{L^\infty_{x,t}} \|\Box_{k^{(2j+1)}} u\|_{L^4_{x,t}} \prod_{s=2j+2}^{2j+3} \|\Box_{k^{(s)}} u_{x_1}\|_{L^4_{x,t}}$$

$$\lesssim \sum_{j=0}^{\infty} C^j (\|u\|^{\rm str})^{2j+1} (\|u_{x_1}\|^{\rm str})^2. \tag{4.16}$$

For convenience, we further write

$$\mathbb{A}^{\lambda,i}_{\ell,\,\rm hi} = \left\{ (k^{(1)}, ..., k^{(\lambda)}) \in \mathbb{A}^{\lambda,i}_{\rm hi} : \, k_i^{(\ell)} = \max_{1\leqslant s\leqslant \lambda} |k_i^{(s)}| \right\}, \quad \ell = 1, ..., \lambda. \tag{4.17}$$

By Proposition 3.2,

$$II \lesssim \sum_{j=0}^{\infty} \left(\sum_{k^{(1)},...,k^{(2j+3)}\in \mathbb{A}^{2j+3,1}_{1,\rm hi}} + ... + \sum_{k^{(1)},...,k^{(2j+3)}\in \mathbb{A}^{2j+3,1}_{2j+3,\rm hi}} \right) \sum_{k\in \mathbb{Z}^2,\, |k_1|\geqslant |k_2|\vee 10} \langle k_1\rangle^{3/2}$$

$$\times \left\| \Box_k \left(\prod_{s=1}^{j} \Box_{k^{(s)}} u \prod_{s=j+1}^{2j+1} \Box_{k^{(s)}} \bar{u} \Box_{k^{(2j+2)}} u_{x_1} \Box_{k^{(2j+3)}} u_{x_1} \right) \right\|_{L^1_{x_1} L^2_{x_2,t}}$$

$$:= \sum_{j=0}^{\infty} (II_1 + ... + II_{2j+3}). \tag{4.18}$$

The estimates for $\sum_{j=0}^\infty II_1, ..., \sum_{j=0}^\infty II_{2j+1}$ are similar and it suffices to estimate $\sum_{j=0}^\infty II_{2j+1}$. If $|k_1^{(2j+1)}| = \max_{1\leqslant s\leqslant 2j+3} |k_1^{(s)}|$, by Hölder's inequality and Lemma 4.3, we have

$$\left\| \Box_k \left(\prod_{s=1}^{j} \Box_{k^{(s)}} u \prod_{s=j+1}^{2j+1} \Box_{k^{(s)}} \bar{u} \Box_{k^{(2j+2)}} u_{x_1} \Box_{k^{(2j+3)}} u_{x_1} \right) \right\|_{L^1_{x_1} L^2_{x_2,t}}$$

$$\leqslant \prod_{s=1}^{2j} \max_{v=u,\bar{u}} \|\Box_{k^{(s)}} v\|_{L^\infty_{x,t}} \|\Box_{k^{(2j+2)}} u_{x_1}\|_{L^2_{x_1} L^\infty_{x_2,t}} \|\Box_{k^{(2j+1)}} \bar{u} \Box_{k^{(2j+3)}} u_{x_1}\|_{L^2_{x,t}}$$

$$\times \prod_{i=1,2} \chi_{\left(|k_i - k_i^{(1)} - ... - k_i^{(2j+3)}|\leqslant 2j+4\right)}. \tag{4.19}$$

In (4.19), if $|k_1^{(2j+1)}| \geqslant |k_2^{(2j+1)}|$, by Hölder's inequality,

$$\|\Box_{k^{(2j+1)}} \bar{u} \Box_{k^{(2j+3)}} u_{x_1}\|_{L^2_{x,t}} \leqslant \|\Box_{k^{(2j+1)}} \bar{u}\|_{L^\infty_{x_1} L^2_{x_2,t}} \|\Box_{k^{(2j+3)}} u_{x_1}\|_{L^2_{x_1} L^\infty_{x_2,t}}; \tag{4.20}$$

and if $|k_2^{(2j+1)}| > |k_1^{(2j+1)}|$, by Hölder's inequality,

$$\|\Box_{k^{(2j+1)}}\bar{u}\Box_{k^{(2j+3)}}u_{x_1}\|_{L^2_{x,t}} \leqslant \|\Box_{k^{(2j+1)}}\bar{u}\|_{L^\infty_{x_2}L^2_{x_1,t}}\|\Box_{k^{(2j+3)}}u_{x_1}\|_{L^2_{x_2}L^\infty_{x_1,t}}. \tag{4.21}$$

We see that in II_{2j+1}, there exist at most Cj terms in the summation $\sum_{k\in\mathbb{Z}^2,\,|k_1|\geqslant|k_2|\vee 10}$ and $|k_1| \leqslant Cj(|k_1^{(2j+1)}| \vee |k_2^{(2j+1)}|)$ if $k^{(1)},...,k^{(2j+3)} \in \mathbb{A}^{2j+3,1}_{\rm hi}$, it follows that

$$\sum_{j=0}^\infty II_{2j+1} \lesssim \sum_{j=0}^\infty C^j(\|u\|^{\rm str})^{2j}\|u\|^{\rm sm}(\|\partial_{x_1}u\|^{\max})^2 \tag{4.22}$$

If $|k_1^{(2j+2)}| = \max_{1\leqslant s\leqslant 2j+3}|k_1^{(s)}|$, in view of Hölder's inequality, we have

$$\begin{aligned}
&\left\|\Box_k\left(\prod_{s=1}^{j}\Box_{k^{(s)}}u\prod_{s=j+1}^{2j+1}\Box_{k^{(s)}}\bar{u}\Box_{k^{(2j+2)}}u_{x_1}\Box_{k^{(2j+3)}}u_{x_1}\right)\right\|_{L^1_{x_1}L^2_{x_2,t}}\\
&\leqslant \prod_{s=1}^{2j}\max_{v=u,\bar{u}}\|\Box_{k^{(s)}}v\|_{L^\infty_{x,t}}\|\Box_{k^{(2j+1)}}\bar{u}\|_{L^2_{x_1}L^\infty_{x_2,t}}\|\Box_{k^{(2j+2)}}u_{x_1}\Box_{k^{(2j+3)}}u_{x_1}\|_{L^2_{x,t}}\\
&\quad\times\prod_{i=1,2}\chi_{\left(|k_i-k_i^{(1)}-...-k_i^{(2j+3)}|\leqslant 2j+4\right)}.
\end{aligned} \tag{4.23}$$

In (4.23), if $|k_1^{(2j+2)}| \geqslant |k_2^{(2j+2)}|$, by Hölder's inequality,

$$\|\Box_{k^{(2j+2)}}u_{x_1}\Box_{k^{(2j+3)}}u_{x_1}\|_{L^2_{x,t}} \leqslant \|\Box_{k^{(2j+2)}}u_{x_1}\|_{L^\infty_{x_1}L^2_{x_2,t}}\|\Box_{k^{(2j+3)}}u_{x_1}\|_{L^2_{x_1}L^\infty_{x_2,t}}; \tag{4.24}$$

and if $|k_2^{(2j+2)}| > |k_1^{(2j+2)}|$, by Hölder's inequality,

$$\|\Box_{k^{(2j+2)}}u_{x_1}\Box_{k^{(2j+3)}}u_{x_1}\|_{L^2_{x_1}L^2_{x_2,t}} \leqslant \|\Box_{k^{(2j+2)}}u_{x_1}\|_{L^\infty_{x_2}L^2_{x_1,t}}\|\Box_{k^{(2j+3)}}u_{x_1}\|_{L^2_{x_2}L^\infty_{x_1,t}}. \tag{4.25}$$

In any case, using an analogous way to (4.22), we have

$$\sum_{j=0}^\infty II_{2j+2} \lesssim \sum_{j=0}^\infty C^j(\|u\|^{\rm str})^{2j}\|u\|^{\max}\|u_{x_1}\|^{\max}\|u_{x_1}\|^{\rm sm}. \tag{4.26}$$

II_{2j+3} can be estimated by using a similar way as II_{2j+2}. So, we have shown that

$$\|\partial_{x_1}\mathscr{A}F_1(u)\|_1^{\rm sm} \lesssim \delta^3\sum_{j=0}^\infty C^j\delta^{2j} \tag{4.27}$$

holds for any $u\in\mathcal{D}$.

We now estimate

$$\begin{aligned}
&\|\partial_{x_2}\mathscr{A}F_1(u)\|_2^{\rm sm}\\
&\leqslant \sum_{j=0}^\infty\sum_{k\in\mathbb{Z}^2,\,|k_2|\geqslant|k_1|\vee 10}\langle k_2\rangle^{3/2}\|\partial_{x_2}\Box_k\mathscr{A}(|u|^{2j}\bar{u}u_{x_1}^2)\|_{L^\infty_{x_2}L^2_{x_1,t}}
\end{aligned}$$

$$\lesssim \sum_{j=0}^{\infty} \left(\sum_{k^{(1)},\ldots,k^{(2j+3)} \in \mathbb{A}_{\rm lo}^{2j+3,2}} + \sum_{k^{(1)},\ldots,k^{(2j+3)} \in \mathbb{A}_{\rm hi}^{2j+3,2}} \right) \sum_{k\in\mathbb{Z}^2,\, |k_2|\geqslant |k_1|\vee 10} \langle k_2\rangle^{3/2}$$
$$\times \left\| \partial_{x_2} \mathscr{A} \Box_k \left(\prod_{s=1}^{j} \Box_{k^{(s)}} u \prod_{s=j+1}^{2j+1} \Box_{k^{(s)}} \bar{u} \Box_{k^{(2j+2)}} u_{x_1} \Box_{k^{(2j+3)}} u_{x_1} \right) \right\|_{L^\infty_{x_2} L^2_{x_1,t}}. \tag{4.28}$$

So, one can repeat the procedures as in the estimate of $\|\partial_{x_1} \mathscr{A} F_1(u)\|_1^{\rm sm}$ to obtain that

$$\|\partial_{x_2} \mathscr{A} F_1(u)\|_2^{\rm sm} \lesssim \delta^3 \sum_{j=0}^{\infty} C^j \delta^{2j} \tag{4.29}$$

holds for any $u \in \mathcal{D}$. Analogous to the above estimates,

$$\|\partial_{x_1} \mathscr{A} F_2(u)\|_1^{\rm sm} + \|\partial_{x_2} \mathscr{A} F_2(u)\|_2^{\rm sm} \lesssim \delta^3 \sum_{j=0}^{\infty} C^j \delta^{2j} \tag{4.30}$$

holds for any $u \in \mathcal{D}$. $\Box$

Lemma 4.5 *Let $u \in \mathcal{D}$. Then we have*

$$\sum_{\alpha=0,1} \sum_{i=1,2} \|\partial_{x_i}^{\alpha} \mathscr{T} u\|^{\max\cap {\rm ant}} \lesssim \|u_0\|_{M^2_{1,1}} + \delta^3 \sum_{j=0}^{\infty} C^j \delta^{2j}.$$

Proof. We have

$$\begin{aligned} &\sum_{\alpha=0,1} \sum_{i=1,2} \|\partial_{x_i}^{\alpha} \mathscr{T} u\|^{\max\cap {\rm ant}} \\ &\leqslant \sum_{\alpha=0,1} \sum_{i=1,2} \|\partial_{x_i}^{\alpha} S(t) u_0\|^{\max\cap {\rm ant}} + \sum_{\alpha=0,1} \sum_{i=1,2} \|\partial_{x_i}^{\alpha} \mathscr{A} F_1(u)\|^{\max\cap {\rm ant}} \\ &\quad + \sum_{\alpha=0,1} \sum_{i=1,2} \|\partial_{x_i}^{\alpha} \mathscr{A} F_2(u)\|^{\max\cap {\rm ant}}. \end{aligned}$$

By Corollary 3.1, we have for $\alpha = 0, 1$ and $i = 1, 2$,

$$\begin{aligned} \|\partial_{x_i}^{\alpha} S(t) u_0\|_1^{\max\cap {\rm ant}} &\leqslant \sum_{k\in\mathbb{Z}^2} \max_{p=\infty,4} \|\partial_{x_i}^{\alpha} \Box_k S(t) u_0\|_{L^2_{x_1} L^p_{x_2,t}} \\ &\lesssim \sum_{k\in\mathbb{Z}^2} \langle k_1\rangle^{1/2} \langle k_i\rangle \|\Box_k u_0\|_1 \leqslant \|u_0\|_{M^2_{1,1}}. \end{aligned}$$

Again, in view of Corollary 3.1,

$$\begin{aligned} &\sum_{\alpha=0,1} \sum_{i=1,2} \|\partial_{x_i}^{\alpha} \mathscr{A} F_1(u)\|_1^{\max\cap {\rm ant}} \\ &\leqslant \sum_{j=0}^{\infty} \sum_{\alpha=0,1} \sum_{i=1,2} \sum_{k\in\mathbb{Z}^2} \|\partial_{x_i}^{\alpha} \Box_k \mathscr{A} (|u|^{2j} \bar{u} u_{x_1}^2)\|_{L^2_{x_1} L^4_{x_2,t} \cap L^2_{x_1} L^\infty_{x_2,t}} \end{aligned}$$

$$
\begin{aligned}
&\leqslant \sum_{j=0}^{\infty} \sum_{k\in\mathbb{Z}^2,|k_1|\geqslant|k_2|} \sum_{k^{(1)},\ldots,k^{(2j+3)}\in\mathbb{Z}^2} \langle k_1\rangle^{3/2} \\
&\qquad \times \left\| \Box_k \left(\prod_{s=1}^{j} \Box_{k^{(s)}} u \prod_{s=j+1}^{2j+1} \Box_{k^{(s)}} \bar{u} \Box_{k^{(2j+2)}} u_{x_1} \Box_{k^{(2j+3)}} u_{x_1} \right) \right\|_{L^1_{x,t}} \\
&\quad + \sum_{j=0}^{\infty} \sum_{k\in\mathbb{Z}^2,|k_2|\geqslant|k_1|} \sum_{k^{(1)},\ldots,k^{(2j+3)}\in\mathbb{Z}^2} \langle k_2\rangle^{3/2} \\
&\qquad \times \left\| \Box_k \left(\prod_{s=1}^{j} \Box_{k^{(s)}} u \prod_{s=j+1}^{2j+1} \Box_{k^{(s)}} \bar{u} \Box_{k^{(2j+2)}} u_{x_1} \Box_{k^{(2j+3)}} u_{x_1} \right) \right\|_{L^1_{x,t}} \\
&:= \Upsilon_1 + \Upsilon_2. \qquad (4.31)
\end{aligned}
$$

We now estimate Υ_1. We have

$$
\begin{aligned}
\Upsilon_1 &\leqslant \sum_{j=0}^{\infty} \sum_{k\in\mathbb{Z}^2,|k_1|\geqslant|k_2|} \sum_{k^{(1)},\ldots,k^{(2j+3)}\in\mathbb{A}^{2j+3,1}_{\mathrm{lo}}} \langle k_1\rangle^{3/2} \\
&\qquad \times \left\| \Box_k \left(\prod_{s=1}^{j} \Box_{k^{(s)}} u \prod_{s=j+1}^{2j+1} \Box_{k^{(s)}} \bar{u} \Box_{k^{(2j+2)}} u_{x_1} \Box_{k^{(2j+3)}} u_{x_1} \right) \right\|_{L^1_{x,t}} \\
&\quad + \sum_{j=0}^{\infty} \sum_{k\in\mathbb{Z}^2,|k_1|\geqslant|k_2|} \sum_{k^{(1)},\ldots,k^{(2j+3)}\in\mathbb{A}^{2j+3,1}_{\mathrm{hi}}} \langle k_1\rangle^{3/2} \\
&\qquad \times \left\| \Box_k \left(\prod_{s=1}^{j} \Box_{k^{(s)}} u \prod_{s=j+1}^{2j+1} \Box_{k^{(s)}} \bar{u} \Box_{k^{(2j+2)}} u_{x_1} \Box_{k^{(2j+3)}} u_{x_1} \right) \right\|_{L^1_{x,t}} \\
&:= \Upsilon_{11} + \Upsilon_{12}. \qquad (4.32)
\end{aligned}
$$

By Hölder's inequality and Lemma 4.3,

$$
\begin{aligned}
\Upsilon_{11} &\leqslant \sum_{j=0}^{\infty} C^j \sum_{k^{(1)},\ldots,k^{(2j+3)}\in\mathbb{Z}^2} \prod_{s=1}^{2j} \|\Box_{k^{(s)}} u\|_{L^\infty_{x,t}} \prod_{s=2j+1}^{2j+3} \|\Box_{k^{(s)}} u\|_{L^3_{x,t}} \\
&\lesssim \sum_{j=0}^{\infty} C^j (\|u\|^{\mathrm{str}})^{2j} (\|u\|^{\mathrm{gstr}})^3. \qquad (4.33)
\end{aligned}
$$

In order to bound Υ_{12}, we further decompose $\mathbb{A}^{2j+3,1}_{\mathrm{hi}}$. We have

$$
\begin{aligned}
\Upsilon_{12} &\leqslant \sum_{j=0}^{\infty} \sum_{\ell=1}^{2j+3} \sum_{k\in\mathbb{Z}^2,|k_1|\geqslant|k_2|} \sum_{k^{(1)},\ldots,k^{(2j+3)}\in\mathbb{A}^{2j+3,1}_{\ell,\mathrm{hi}}} \langle k_1\rangle^{3/2} \\
&\qquad \times \left\| \Box_k \left(\prod_{s=1}^{j} \Box_{k^{(s)}} u \prod_{s=j+1}^{2j+1} \Box_{k^{(s)}} \bar{u} \Box_{k^{(2j+2)}} u_{x_1} \Box_{k^{(2j+3)}} u_{x_1} \right) \right\|_{L^1_{x,t}}
\end{aligned}
$$

$$:= \sum_{j=0}^{\infty} \sum_{\ell=1}^{2j+3} \Upsilon_{12,\ell}. \tag{4.34}$$

The estimates of $\Upsilon_{12,\ell}$ for $\ell = 1, ..., 2j$ are similar and we only need to estimate $\Upsilon_{12,1}$. By Hölder's inequality and Lemma 4.3,

$$\begin{aligned}
\sum_{j=0}^{\infty} \Upsilon_{12,1} \lesssim & \sum_{j=0}^{\infty} C^j \sum_{k^{(1)},...,k^{(2j+3)} \in \mathbb{A}^{2j+3,1}_{1,\mathrm{hi}}} \langle k_1^{(1)} \rangle^{3/2} \\
& \times \prod_{s=2}^{2j+1} \|\Box_{k^{(s)}} u\|_{L^\infty_{x,t}} \|\Box_{k^{(1)}} \bar{u} \Box_{k^{(2j+2)}} u_{x_1} \Box_{k^{(2j+3)}} u_{x_1}\|_{L^1_{x,t}} \\
\lesssim & \sum_{j=0}^{\infty} C^j \sum_{k^{(1)},...,k^{(2j+3)} \in \mathbb{A}^{2j+3,1}_{1,\mathrm{hi}},\, |k_1^{(1)}| \geqslant |k_2^{(1)}|} \langle k_1^{(1)} \rangle^{3/2} \prod_{s=2}^{2j+1} \|\Box_{k^{(s)}} u\|_{L^\infty_{x,t}} \\
& \times \|\Box_{k^{(1)}} u\|_{L^\infty_{x_1} L^2_{x_2,t}} \|\Box_{k^{(2j+2)}} u_{x_1}\|_{L^2_{x_1} L^4_{x_2,t}} \|\Box_{k^{(2j+3)}} u_{x_1}\|_{L^2_{x_1} L^4_{x_2,t}} \\
& + \sum_{j=0}^{\infty} C^j \sum_{k^{(1)},...,k^{(2j+3)} \in \mathbb{A}^{2j+3,1}_{1,\mathrm{hi}},\, |k_1^{(1)}| < |k_2^{(1)}|} \langle k_2^{(1)} \rangle^{3/2} \prod_{s=2}^{2j+1} \|\Box_{k^{(s)}} u\|_{L^\infty_{x,t}} \\
& \times \|\Box_{k^{(1)}} u\|_{L^\infty_{x_2} L^2_{x_1,t}} \|\Box_{k^{(2j+2)}} u_{x_1}\|_{L^2_{x_2} L^4_{x_1,t}} \|\Box_{k^{(2j+3)}} u_{x_1}\|_{L^2_{x_2} L^4_{x_1,t}} \\
\lesssim & \sum_{j=0}^{\infty} C^j (\|u\|^{\mathrm{str}})^{2j} \|u\|^{\mathrm{sm}} (\|u_{x_1}\|^{\mathrm{ant}})^2.
\end{aligned} \tag{4.35}$$

Analogous to $\Upsilon_{12,1}$, we have,

$$\sum_{j=0}^{\infty} (\Upsilon_{12,2j+2} + \Upsilon_{12,2j+3}) \lesssim \sum_{j=0}^{\infty} C^j (\|u\|^{\mathrm{str}})^{2j} \|u_{x_1}\|^{\mathrm{sm}} \|u\|^{\mathrm{ant}} \|u_{x_1}\|^{\mathrm{ant}}. \tag{4.36}$$

Hence, we have shown that

$$\Upsilon_1 \lesssim \delta^3 \sum_{j=0}^{\infty} C^j \delta^{2j}. \tag{4.37}$$

Using the same way as in the estimates of Υ_1, we can obtain that

$$\Upsilon_2 \lesssim \delta^3 \sum_{j=0}^{\infty} C^j \delta^{2j}. \tag{4.38}$$

The result follows. □

Lemma 4.6 *Let $u \in \mathcal{D}$. Then we have*

$$\sum_{\alpha=0,1} \sum_{i=1,2} \|\partial^\alpha_{x_i} \mathscr{T} u\|^{\mathrm{gstr}} \lesssim \|u_0\|_{M^2_{1,1}} + \delta^3 \sum_{j=0}^{\infty} C^j \delta^{2j}.$$

Proof. We have

$$\sum_{\alpha=0,1}\sum_{i=1,2}\|\partial^\alpha_{x_i}\mathscr{T}u\|^{\rm gstr}\leqslant\sum_{\alpha=0,1}\sum_{i=1,2}\|\partial^\alpha_{x_i}S(t)u_0\|^{\rm gstr}+\sum_{\alpha=0,1}\sum_{i=1,2}\|\partial^\alpha_{x_i}\mathscr{A}F_1(u)\|^{\rm gstr}$$
$$+\sum_{\alpha=0,1}\sum_{i=1,2}\|\partial^\alpha_{x_i}\mathscr{A}F_2(u)\|^{\rm gstr}.$$

By Corollary 3.1, we have for $\alpha=0,1$ and $i=1,2$,

$$\|\partial^\alpha_{x_i}S(t)u_0\|^{\rm gstr}=\sum_{k\in\mathbb{Z}^2}\|\partial^\alpha_{x_i}\Box_kS(t)u_0\|_{L^3_{x,t}}$$
$$\lesssim\sum_{k\in\mathbb{Z}^2}\langle k_1\rangle^{1/3}\langle k_i\rangle\|\Box_ku_0\|_1\leqslant\|u_0\|_{M^2_{1,1}}.$$

Again, in view of Corollary 3.1,

$$\begin{aligned}\sum_{\alpha=0,1}\sum_{i=1,2}\|\partial^\alpha_{x_i}\mathscr{A}F_1(u)\|^{\rm gstr}&\leqslant\sum_{j=0}^\infty\sum_{\alpha=0,1}\sum_{i=1,2}\sum_{k\in\mathbb{Z}^2}\|\partial^\alpha_{x_i}\Box_k\mathscr{A}(|u|^{2j}\bar u u^2_{x_1})\|_{L^3_{x,t}}\\&\leqslant\sum_{j=0}^\infty\sum_{k\in\mathbb{Z}^2,\,|k_1|\geqslant|k_2|}\langle k_1\rangle^{4/3}\|\Box_k(|u|^{2j}\bar u u^2_{x_1})\|_{L^1_{x,t}}\\&\quad+\sum_{j=0}^\infty\sum_{k\in\mathbb{Z}^2,\,|k_2|>|k_1|}\langle k_2\rangle^{4/3}\|\Box_k(|u|^{2j}\bar u u^2_{x_1})\|_{L^1_{x,t}}\\&\leqslant\Upsilon_1+\Upsilon_2,\end{aligned}\tag{4.39}$$

where Υ_1 and Υ_2 are the same as in (4.31). So,one can repeat the proof of Lemma 4.5 to obtain the result. □

Lemma 4.7 *Let $u\in\mathcal{D}$. Then we have*

$$\sum_{\alpha=0,1}\sum_{i=1,2}\|\partial^\alpha_{x_i}\mathscr{T}u\|^{\rm str}\lesssim\|u_0\|_{M^2_{1,1}}+\delta^3\sum_{j=0}^\infty C^j\delta^{2j}.$$

Proof. We have

$$\sum_{\alpha=0,1}\sum_{i=1,2}\|\partial^\alpha_{x_i}\mathscr{T}u\|^{\rm str}\leqslant\sum_{\alpha=0,1}\sum_{i=1,2}\|\partial^\alpha_{x_i}S(t)u_0\|^{\rm str}+\sum_{\alpha=0,1}\sum_{i=1,2}\|\partial^\alpha_{x_i}\mathscr{A}F_1(u)\|^{\rm str}$$
$$+\sum_{\alpha=0,1}\sum_{i=1,2}\|\partial^\alpha_{x_i}\mathscr{A}F_2(u)\|^{\rm str}.$$

By Corollary 3.1, we have for $\alpha=0,1$ and $i=1,2$,

$$\|\partial^\alpha_{x_i}S(t)u_0\|^{\rm str}=\sum_{k\in\mathbb{Z}^2}\langle k\rangle\|\partial^\alpha_{x_i}\Box_kS(t)u_0\|_{L^\infty_tL^2_x\cap L^4_{x,t}}$$
$$\lesssim\sum_{k\in\mathbb{Z}^2}\langle k\rangle\langle k_i\rangle\|\Box_ku_0\|_2\leqslant\|u_0\|_{M^2_{2,1}}\leqslant\|u_0\|_{M^2_{1,1}}.$$

In view of Proposition 3.3, Lemma 4.3 and Hölder's inequality,

$$
\begin{aligned}
\|\mathscr{A}F_1(u)\|^{\rm str} &\leqslant \sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^2}\langle k\rangle\|\Box_k\mathscr{A}(|u|^{2j}\bar{u}u_{x_1}^2)\|_{L_t^\infty L_x^2\cap L_{x,t}^4}\\
&\leqslant \sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^2}\langle k\rangle\|\Box_k(|u|^{2j}\bar{u}u_{x_1}^2)\|_{L_{x,t}^{4/3}}\\
&\lesssim \sum_{j=0}^{\infty}C^j\sum_{k^{(1)},\dots,k^{(2j+3)}\in\mathbb{Z}^2}\sum_{k\in\mathbb{Z}^2}\langle|k^{(1)}|\vee\dots\vee|k^{(2j+3)|}\rangle\\
&\quad\times\left\|\Box_k\left(\prod_{s=1}^{j}\Box_{k^{(s)}}u\prod_{s=j+1}^{2j+1}\Box_{k^{(s)}}\bar{u}\Box_{k^{(2j+2)}}u_{x_1}\Box_{k^{(2j+3)}}u_{x_1}\right)\right\|_{L_{x,t}^{4/3}}\\
&\lesssim \sum_{j=0}^{\infty}C^j\sum_{k^{(1)},\dots,k^{(2j+3)}\in\mathbb{Z}^2}\langle|k^{(1)}|\vee\dots\vee|k^{(2j+3)|}\rangle\prod_{s=1}^{2j}\|\Box_{k^{(s)}}u\|_{L_{x,t}^\infty}\\
&\quad\times\|\Box_{k^{(2j+1)}}u\|_{L_{x,t}^4}\|\Box_{k^{(2j+2)}}u_{x_1}\|_{L_{x,t}^4}\|\Box_{k^{(2j+3)}}u_{x_1}\|_{L_{x,t}^4}\\
&\lesssim \sum_{j=0}^{\infty}C^j(\|u\|^{\rm str})^{2j+1}(\|u_{x_1}\|^{\rm str})^2.
\end{aligned}
\tag{4.40}
$$

Next, we estimate

$$
\begin{aligned}
\sum_{i=1,2}\|\partial_{x_i}\mathscr{A}F_1(u)\|^{\rm str} &\lesssim \sum_{j=0}^{\infty}\sum_{|k|\leqslant 20}\|\Box_k\mathscr{A}(|u|^{2j}\bar{u}u_{x_1}^2)\|_{L_t^\infty L_x^2\cap L_{x,t}^4}\\
&\quad+\sum_{j=0}^{\infty}\sum_{|k|>20}\langle k\rangle\sum_{i=1,2}\|\Box_k\partial_{x_i}\mathscr{A}(|u|^{2j}\bar{u}u_{x_1}^2)\|_{L_t^\infty L_x^2\cap L_{x,t}^4}\\
&:=\Gamma_1+\Gamma_2.
\end{aligned}
\tag{4.41}
$$

Using the same way as in (4.40), Γ_1 can be estimated by

$$
\Gamma_1\lesssim\sum_{j=0}^{\infty}C^j(\|u\|^{\rm str})^{2j+1}(\|\partial_{x_1}u\|^{\rm str})^2. \tag{4.42}
$$

Now we consider the estimate of Γ_2. We have for any $k\in\mathbb{Z}^2$, $|k_1|\geqslant|k_2|\vee 10$, similar to Lemma 4.1,

$$
\|\Box_k\partial_{x_2}f\|_{L_x^p}\lesssim\sum_{|l|_\infty\leqslant 1}\left\|\mathscr{F}^{-1}\left(\sigma_{k+l}\frac{\xi_2}{\xi_1}\right)\right\|_{L_x^1}\|\Box_k f\|_{L_x^p}\lesssim\|\Box_k f\|_{L_x^p}. \tag{4.43}
$$

It follows from (4.43) that

$$
\Gamma_2\lesssim\sum_{j=0}^{\infty}\sum_{|k|>20}\langle k\rangle\sum_{i=1,2}\|\Box_k\partial_{x_i}\mathscr{A}(|u|^{2j}\bar{u}u_{x_1}^2)\|_{L_t^\infty L_x^2\cap L_{x,t}^4}
$$

$$
\begin{aligned}
&\lesssim \sum_{j=0}^{\infty} \sum_{k\in\mathbb{Z}^2,\,|k_1|\geqslant|k_2|\vee 10} \langle k_1\rangle \|\Box_k \partial_{x_1} \mathscr{A}(|u|^{2j}\bar{u}u_{x_1}^2)\|_{L_t^\infty L_x^2\cap L_{x,t}^4} \\
&\quad + \sum_{j=0}^{\infty} \sum_{k\in\mathbb{Z}^2,\,|k_2|>|k_1|\vee 10} \langle k_2\rangle \|\Box_k \partial_{x_2} \mathscr{A}(|u|^{2j}\bar{u}u_{x_1}^2)\|_{L_t^\infty L_x^2\cap L_{x,t}^4} \\
&:= \Gamma_{21} + \Gamma_{22}.
\end{aligned} \tag{4.44}
$$

We have

$$
\begin{aligned}
\Gamma_{22} \lesssim \sum_{j=0}^{\infty} &\left(\sum_{k^{(1)},\dots,k^{(2j+3)}\in\mathbb{A}_{\rm lo}^{2j+3,2}} + \sum_{k^{(1)},\dots,k^{(2j+3)}\in\mathbb{A}_{\rm hi}^{2j+3,2}} \right) \sum_{k\in\mathbb{Z}^2,\,|k_2|\geqslant|k_1|\vee 10} \langle k_2\rangle \\
&\times \left\| \partial_{x_2}\mathscr{A}\Box_k \left(\prod_{s=1}^{j} \Box_{k^{(s)}} u \prod_{s=j+1}^{2j+1} \Box_{k^{(s)}} \bar{u} \Box_{k^{(2j+2)}} u_{x_2} \Box_{k^{(2j+3)}} u_{x_2} \right) \right\|_{L_t^\infty L_x^2 \cap L_{x,t}^4} \\
:= \Gamma_{22,1} + \Gamma_{22,2}.
\end{aligned} \tag{4.45}
$$

By Proposition 3.3,

$$
\begin{aligned}
\Gamma_{22,1} \lesssim \sum_{j=0}^{\infty} &\sum_{k^{(1)},\dots,k^{(2j+3)}\in\mathbb{A}_{\rm lo}^{2j+3,2}} \sum_{k\in\mathbb{Z}^2,\,|k_2|\geqslant|k_1|\vee 10} \langle k_2\rangle^2 \\
&\times \left\| \Box_k \left(\prod_{s=1}^{j} \Box_{k^{(s)}} \bar{u} \prod_{s=j+1}^{2j+1} \Box_{k^{(s)}} u \Box_{k^{(2j+2)}} u_{x_2} \Box_{k^{(2j+3)}} u_{x_2} \right) \right\|_{L_{x,t}^{4/3}}.
\end{aligned} \tag{4.46}
$$

Then we can use the same way as in (4.15)–(4.16) to obtain that

$$
\Gamma_{22,1} \lesssim \sum_{j=0}^{\infty} C^j (\|u\|^{\rm str})^{2j+1} (\|u_{x_2}\|^{\rm str})^2. \tag{4.47}
$$

By Proposition 3.4,

$$
\begin{aligned}
\Gamma_{22,2} \lesssim \sum_{j=0}^{\infty} &\sum_{k^{(1)},\dots,k^{(2j+3)}\in\mathbb{A}_{\rm hi}^{2j+3,2}} \sum_{k\in\mathbb{Z}^2,\,|k_2|>|k_1|\vee 10} \langle k_2\rangle^{3/2} \\
&\times \left\| \Box_k \left(\prod_{s=1}^{j} \Box_{k^{(s)}} u \prod_{s=j+1}^{2j+1} \Box_{k^{(s)}} \bar{u} \Box_{k^{(2j+2)}} u_{x_2} \Box_{k^{(2j+3)}} u_{x_2} \right) \right\|_{L_{x_2}^1 L_{x_1,t}^2},
\end{aligned} \tag{4.48}
$$

which reduces to the estimate of II as in (4.14) and we have

$$
\Gamma_{22,2} \lesssim \delta^3 \sum_{j=0}^{\infty} C^j \delta^{2j}. \tag{4.49}
$$

Analogous to $\Gamma_{22,2}$, $\Gamma_{22,1}$ can be bounded by the right hand side of (4.49). □

Proof of Theorem 1.1. By Lemmas 4.4, 4.5, 4.6 and 4.7, we immediately have for any $u \in \mathcal{D}$,

$$\sum_{\alpha=0,1} \sum_{i=1,2} \|\partial_{x_i}^{\alpha} \mathscr{T} u\|^{\rm sm\cap \max\cap ant\cap str\cap gstr} \lesssim \|u_0\|_{M^2_{1,1}} + \delta^3. \tag{4.50}$$

Similarly, for any $u, v \in \mathcal{D}$,

$$d(u,v) \lesssim \delta^2 d(u,v). \tag{4.51}$$

Following a standard contraction mapping argument, we obtain that (1.3) has a unique solution $u \in C(\mathbb{R}, M^2_{2,1}) \cap X$.

Finally, it suffices to show that $u \in C_{\rm loc}(\mathbb{R}, M^{3/2}_{1,1})$. Let $T > 0$ be arbitrary. By Proposition 3.6,

$$\begin{aligned} \|u\|_{C([-T,T];M^{3/2}_{1,1})} &\lesssim \langle T\rangle \|u_0\|_{M^{3/2}_{1,1}} + \langle T\rangle \int_{-T}^{T} \|F(u(\tau))\|_{M^{3/2}_{1,1}} d\tau \\ &\lesssim \langle T\rangle \|u_0\|_{M^{3/2}_{1,1}} + \langle T\rangle \sum_{k\in\mathbb{Z}^2} \langle k\rangle^{3/2} \|\Box_k F(u(\tau))\|_{L^1_{x,t}}, \end{aligned} \tag{4.52}$$

which reduces to the estimate as in Lemma 4.5 if we treat $T > 0$ as a fixed number. □

Proof of Theorem 1.3. In Lemmas 4.4, 4.5, 4.6 and 4.7, if we change the nonlinearity $|u|^{2j}\bar{u}u^2_{x_1}$ by $v_1v_2...v_{2j+3}$ (v_i can take any one of $u, \bar{u}, \partial_{x_j}u, \partial_{x_j}\bar{u}$), then the results and the proofs are essentially the same. So, we can use the same way as in Theorem 1.1 to obtain the result and the details are omitted. □

5 On hyperbolic Schrödinger map

In this section we prove our Corollary 1.2.

Lemma 5.1 *Let $\kappa \geqslant 0$, $s_1, s_2 \in M^{\kappa}_{r,1}$ with $r \in [1,\infty]$ and $s = (s_1, s_2, s_3) \in \mathbb{S}^2$. Suppose that $\|s_i\|_{M^{\kappa}_{r,1}} \leqslant \eta \ll 1$ for $i = 1, 2$. Then we have $|s_3| - 1 \in M^{\kappa}_{r,1}$ and $\||s_3| - 1\|_{M^{\kappa}_{r,1}} \leqslant C_0\eta$.*

Proof. It is known that $M^{\kappa}_{r,1} \subset L^\infty$ is a Banach algebra, we see that $\|s_i\|_{L^\infty} \ll 1$. In view of Taylor's expansion we have

$$\begin{aligned} |s_3| - 1 &= \sqrt{1 - s_1^2 - s_2^2} - 1 \\ &= \sum_{n=1}^{\infty} \frac{(-1)^n}{n\,!} (s_1^2 \partial_x + s_2^2 \partial_y)^n \sqrt{1 + x + y}\,|_{(0,0)} \end{aligned}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \sum_{j=0}^{n} C_n^j s_1^{2j} \, s_2^{2(n-j)} \prod_{\lambda=0}^{n-1} (\frac{1}{2} - \lambda). \tag{5.1}$$

In view of the algebra property of $M^{\kappa}_{r,1}$,

$$\begin{aligned} \| |s_3| - 1 \|_{M^{\kappa}_{r,1}} &\leqslant \sum_{n=1}^{\infty} \sum_{j=0}^{n} C_n^j \| s_1^{2j} \, s_2^{2(n-j)} \|_{M^{\kappa}_{r,1}} \\ &\leqslant \sum_{n=1}^{\infty} \sum_{j=0}^{n} C_n^j C^{2n} \| s_1 \|^{2j}_{M^{\kappa}_{r,1}} \, \| s_2 \|^{2(n-j)}_{M^{\kappa}_{r,1}} \\ &\leqslant \sum_{n=1}^{\infty} \sum_{j=0}^{n} C_n^j C^{2n} \eta^{2n} := C_0 \eta, \end{aligned} \tag{5.2}$$

the result follows. □

Lemma 5.2 *Let $\kappa \geqslant 0$, $s_1, s_2 \in M^{\kappa}_{r,1}$ with $r \in [1, \infty]$ and $s = (s_1, s_2, s_3) \in \mathbb{S}^2$. Suppose that $\|s_i\|_{M^{\kappa}_{r,1}} \leqslant \eta \ll 1$ for $i = 1, 2$. Then we have $u_0 := (s_1 + is_2)/(1 + s_3) \in M^{\kappa}_{r,1}$ and $\|u_0\|_{M^{\kappa}_{r,1}} \leqslant C_1 \eta$.*

Proof. We may assume that $s_3 \geqslant 0$. Taking $\tilde{s}_3 = s_3 - 1$, we have $u_0 := (s_1 + is_2)/2(1 + \tilde{s}_3/2)$. Let us observe that

$$\frac{s_1 + is_2}{2(1 + \tilde{s}_3/2)} = (s_1 + is_2) \sum_{j=0}^{\infty} (-1)^j \left(\frac{\tilde{s}_3}{2} \right)^j. \tag{5.3}$$

Using the algebra property on $M^{\kappa}_{r,1}$, analogous to Lemma 5.1, we can obtain the result, as desired. □

By Lemmas 5.1 and 5.2, we see that $u_0 = (s_1(0) + is_2(0))/(1 + s_3(0)) \in M^2_{1,1}$ is small enough if $s_1(0), s_2(0) \in M^2_{1,1}$ with $s_0 = (s_1(0), s_2(0), s_3(0)) \in \mathbb{S}^2$ are sufficiently small. Hence, in view of Theorem 1.1, we obtain that (1.3) has a unique solution $u \in C(\mathbb{R}, M^2_{2,1}) \cap C(\mathbb{R}, M^{3/2}_{1,1}) \cap X$. Taking

$$s = \left(\frac{2Re\ u}{1 + |u|^2}, \ \frac{2Im\ u}{1 + |u|^2}, \ \frac{|u|^2 - 1}{1 + |u|^2} \right)$$

and applying the same way as in Lemmas 5.1 and 5.2, we have

$$s_1, s_2, |s_3| - 1 \in C(\mathbb{R}, M^2_{2,1}) \cap C(\mathbb{R}, M^{3/2}_{1,1}).$$

Finally, we show that $s_1, s_2, |s_3| - 1 \in X$ and we need the following

Lemma 5.3 *We have for $\bar{x} = (x_2, ..., x_n)$,*

$$\sum_{k \in \mathbb{Z}^n} \langle k \rangle^s \| \Box_k (u_1 ... u_N) \|_{L^p_{x_1} L^q_{\bar{x},t}}$$

$$\leqslant C^N \sum_{i=1}^{N} \left(\sum_{k^{(i)}\in\mathbb{Z}^n} \langle k^{(i)}\rangle^s \|\Box_k u_i\|_{L^p_{x_1} L^q_{\bar{x},t}} \right) \prod_{j\neq i,\, 1\leqslant j\leqslant N} \left(\sum_{k^{(j)}\in\mathbb{Z}^n} \langle k^{(j)}\rangle^s \|\Box_{k^{(j)}} u_j\|_{L^\infty_{x,t}} \right). \quad (5.4)$$

Proof. The result was essentially obtained in [47]. □

By Taylor's expansion

$$s_1 = \sum_{j=0}^{\infty} (-1)^j |u|^{2j} 2Re\, u,$$

$$\partial_{x_1} s_1 = 2Re\, u_{x_1} + \sum_{j=1}^{\infty} (-1)^j |u|^{2j} 2Re\, u_{x_1} + \sum_{j=1}^{\infty} (-1)^j j |u|^{2j-2} 2Re\, u \partial_{x_1} |u|^2.$$

By Lemmas 5.3 and 4.3, we have

$$\sum_{\alpha=0,1} \sum_{i=1,2} \|\partial^\alpha_{x_i} s_1\|^{\max\cap \mathrm{str}\cap \mathrm{gstr}\cap \mathrm{ant}} \lesssim \delta.$$

Finally, it suffices to estimate $\|\partial^\alpha_{x_i} s_1\|^{\mathrm{sm}}$, say, we bound $\|\partial_{x_1} s_1\|^{\mathrm{sm}}_1$. We have

$$\|\partial_{x_1} s_1\|^{\mathrm{sm}}_1 \leqslant 4\|u_{x_1}\|^{\mathrm{sm}}_1 + \sum_{j=1}^{\infty} \||u|^{2j} 2Re\, u_{x_1}\|^{\mathrm{sm}}_1 + \sum_{j=1}^{\infty} j \||u|^{2j-2} 2Re\, u \partial_{x_1} |u|^2\|^{\mathrm{sm}}_1. \quad (5.5)$$

We estimate

$$\begin{aligned} \||u|^{2j} 2Re\, u_{x_1}\|^{\mathrm{sm}}_1 \lesssim & \sum_{k\in\mathbb{Z}^2,\, |k_1|\geqslant |k_2|\vee 10} \left(\sum_{k^{(1)},\dots,k^{(2j+1)}\in\mathbb{A}^{2j+1,1}_{\mathrm{lo}}} + \sum_{k^{(1)},\dots,k^{(2j+3)}\in\mathbb{A}^{2j+1,1}_{\mathrm{hi}}} \right) \langle k_1\rangle^{3/2} \\ & \times \left\| \Box_k \left(\prod_{s=1}^{j} \Box_{k^{(s)}} u \prod_{s=j+1}^{2j} \Box_{k^{(s)}} \bar{u} \Box_{k^{(2j+1)}} Re\, u_{x_1} \right) \right\|_{L^\infty_{x_1} L^2_{x_2,t}} \\ := & \Lambda_{\mathrm{lo}} + \Lambda_{\mathrm{hi}}. \end{aligned} \quad (5.6)$$

Using the fact

$$\|\Box_k f\|_{L^\infty_{x_1} L^2_{x_2,t}} \leqslant \|\Box_k f\|_{L^2_{x_2,t} L^\infty_{x_1}} \leqslant \|\Box_k\|_{L^2_{x,t}},$$

and in view of Lemma 4.3 and Hölder's inequality, we have

$$\begin{aligned} \Lambda_{\mathrm{lo}} & \lesssim C^j \sum_{k^{(1)},\dots,k^{(2j+1)}\in\mathbb{Z}^2} \prod_{s=3}^{2j} \|\Box_{k^{(s)}} u\|_{L^\infty_{x,t}} \|\Box_{k^{(2j+1)}} u_{x_1}\|_{L^4_{x,t}} \|\Box_{k^{(1)}} u\|_{L^4_{x,t}} \|\Box_{k^{(2)}} u\|_{L^4_{x,t}} \\ & \lesssim C^j \|u_{x_1}\|^{\mathrm{str}} (\|u\|^{\mathrm{str}})^{2j}. \end{aligned} \quad (5.7)$$

Again, Lemma 4.3 and Hölder's inequality yield

$$\Lambda_{\mathrm{hi}} \lesssim C^j (\|u_{x_1}\|^{\mathrm{sm}} (\|u\|^{\mathrm{str}})^{2j} + \|u\|^{\mathrm{sm}} \|u_{x_1}\|^{\mathrm{str}} (\|u\|^{\mathrm{str}})^{2j-1}). \quad (5.8)$$

Collecting the estimates as in the above, we obtain that

$$\||u|^{2j} 2Re\, u_{x_1}\|_1^{\rm sm} \lesssim \sum_{j=1}^{\infty} C^j \delta^{2j+1} \lesssim \delta^3. \tag{5.9}$$

So, we have shown $\|\partial_{x_i}^{\alpha} s_1\|^{\rm sm} \lesssim \delta$. Similarly, we have the desired estimates for s_2 and $|s_3| - 1$. This finishes the proof of Corollary 1.2.

6 Initial data in weighted Sobolev spaces

If we can show that $H^{s+b,b}(\mathbb{R}^2) \subset M^s_{1,1}(\mathbb{R}^2)$ for any $b > 1$, then we get an exact proof of Corollaries 1.4 and 1.5.

Proposition 6.1 *Let $s \in \mathbb{R}$, $b > n/2$. We have $H^{s+b,b}(\mathbb{R}^n) \subset M^s_{1,1}(\mathbb{R}^n)$.*

Proof. It suffices to consider the case $s = 0$. We have

$$\|f\|_{M^0_{1,1}} \lesssim \left(\sum_{k \in \mathbb{Z}^n} \langle k \rangle^{-2b} \right)^{1/2} \|f\|_{M^b_{1,2}} \lesssim \|f\|_{M^b_{1,2}}.$$

For any $\tilde{b} > n/2$, using similar way as in Lemma 4.1,

$$\begin{aligned} \|\Box_k f\|_1 &\lesssim \sum_{|l|_\infty \leqslant 1} \|\mathscr{F}^{-1}(\sigma_{k+l} \langle \xi \rangle^{-b})\|_1 \|\mathscr{F}^{-1} \sigma_k \langle k \rangle^b \mathscr{F} f\|_1 \\ &\lesssim \langle k \rangle^{-b} \|\sigma_k \langle \xi \rangle^b \widehat{f}\|_{H^{\tilde{b}}}. \end{aligned} \tag{6.1}$$

It follows that

$$\|f\|_{M^b_{1,2}} \leqslant \left(\sum_{k \in \mathbb{Z}^n} \|\sigma_k \langle \xi \rangle^b \widehat{f}\|^2_{H^{\tilde{b}}} \right)^{1/2}. \tag{6.2}$$

If $\tilde{b} \in \mathbb{N}$, we see that

$$\|\sigma_k \widehat{g}\|_{H^{\tilde{b}}} \lesssim \sum_{|\alpha| \leqslant \tilde{b}} \|\partial^\alpha \widehat{g}\|_{L^2([k-1,k+1]^n)}.$$

Hence,

$$\left(\sum_{k \in \mathbb{Z}^n} \|\sigma_k \widehat{g}\|^2_{H^{\tilde{b}}} \right)^{1/2} \lesssim \sum_{|\alpha| \leqslant \tilde{b}} \|\partial^\alpha \widehat{g}\|_2 \sim \|\langle x \rangle^{\tilde{b}} g\|_2. \tag{6.3}$$

Considering the map $T : g \to \{\sigma_k \widehat{g} : \ k \in \mathbb{Z}^n\}$, (6.3) implies that $T : L^2(\mathbb{R}^n : \langle x \rangle^{\tilde{b}} dx) \to \ell^2(H^{\tilde{b}}(\mathbb{R}^n))$ is bounded for any $\tilde{b} \in \mathbb{N} \cup \{0\}$. For any $b > n/2$, we can choose $\tilde{b} > b$ with

$\tilde{b} \in \mathbb{N}$. In view of the real interpolation theory, we can interpolate $\ell^2(H^b(\mathbb{R}^n))$ between $\ell^2(L^2(\mathbb{R}^n))$ and $\ell^2(H^{\tilde{b}}(\mathbb{R}^n))$ and show that

$$\left(\sum_{k\in\mathbb{Z}^n} \|\sigma_k \widehat{g}\|^2_{H^b}\right)^{1/2} \lesssim \sum_{|\alpha|\leqslant \tilde{b}} \|\partial^\alpha \widehat{g}\|_2 \sim \|\langle x\rangle^b g\|_2. \tag{6.4}$$

Taking $g = \mathscr{F}^{-1}\langle\xi\rangle^b \mathscr{F} f$, we have from (6.2) and (6.4) that

$$\|f\|_{M^b_{1,2}} \lesssim \|\langle x\rangle^b \mathscr{F}^{-1}\langle\xi\rangle^b \mathscr{F} f\|_2.$$

Hence, we have $\|f\|_{M^0_{1,1}} \lesssim \|f\|_{H^{b,b}}$. □

A Gabor frame

We collect some results used in this paper for the Gabor frame, see for instance, Gröchenig [17]. Gabor frame is a fundamental tool in the theory of time-frequency analysis, which was first proposed by Gabor [15] in 1946. A system $\{e_j : \ j \in J\}$ in a Hilbert space $\mathcal{H}$ is said to be a frame if there exists two positive constant $A, B > 0$ such that for all $f \in \mathcal{H}$,

$$A\|f\| \leqslant \left(\sum_{j\in J} |\langle f, e_j\rangle|^2\right)^{1/2} \leqslant B\|f\|.$$

For convenience, we write $T_x f(y) = f(y-x)$, $M_\xi f(y) = e^{\mathrm{i}y\cdot\xi} f(y)$, T_x is a translation by x and M_ξ is a modulation by ξ. Let $g \in L^2(\mathbb{R}^n)$ and $\alpha, \beta > 0$. If

$$\mathcal{G}(g,\alpha,\beta) := \{T_{\alpha l} M_{\beta k} g : \quad k, l \in \mathbb{Z}^n\}$$

is a frame in L^2, then it is said to be a Gabor frame in L^2.

Proposition A.1 *Let $\mathcal{G}(g,\alpha,\beta)$ be a Gabor frame in L^2. Then any $f \in L^2$ has an expansion*

$$f = \sum_{k,l\in\mathbb{Z}^n} c_{kl} T_{\alpha l} M_{\beta k} g.$$

Moreover, $\|f\|_2 \sim \|(c_{kl})\|_{\ell^2}$.

Unfortunately, the generalization of Gabor frame in L^p with $p \neq 2$ is not available and the Gabor expansion only holds for the case $p = 2$. However, Proposition A.1 also holds for modulation spaces $M^s_{p,q}$ with $1 \leqslant p, q < \infty$:

Proposition A.2 *Let $\mathcal{G}(g,\alpha,\beta)$ be a Gabor frame in L^2. Then any $f \in M^s_{p,q}$ has an expansion*

$$f = \sum_{k,l\in\mathbb{Z}^n} c_{kl} T_{\alpha l} M_{\beta k} g.$$

Moreover, $\|f\|_{M^s_{p,q}} \sim \|(c_{kl})\|_{m^s_{p,q}}$, *where*

$$\|(c_{kl})\|_{m^s_{p,q}} = \left(\sum_k \langle k\rangle^{sq} \left(\sum_l |c_{kl}|^p\right)^{q/p}\right)^{1/q}.$$

A basic example of the Gabor frame is $\{e^{\mathrm{i}kx}e^{-|x-l|^2/2} : \ k,l \in \mathbb{Z}^n\}$.

B Function-sequence convolution

Considering the Gabor frame expression for the solutions of linear Schrödinger equations, we need to treat the convolution on variables $x \in \mathbb{R}^n$ and $l \in \mathbb{Z}^n$. Since x and l belong to different measure spaces, we can not directly use Young's and Hardy-Littlewood-Sobolev's inequalities. So, we need the following

Lemma B.1 *Let* $1 \leqslant p, r \leqslant \infty$. *Assume that* $\theta > 0$, $\theta > 1/r' + 1/p$ *with* $p \geqslant r$, *or* $\theta = 1/r' + 1/p \in (0,1)$ *and* $1 < p < \infty$. *Then we have for any* $b, c \in \mathbb{R}$ *with* $|c| \geqslant 1$,

$$\left\| \sum_{l\in\mathbb{Z}} |a_l| \left(1 + \frac{|x-l+b|}{|c|}\right)^{-\theta} \right\|_{L^p_x} \lesssim \langle c\rangle^{1/p+1/r'} \|(a_l)\|_{\ell^r}. \tag{B.1}$$

Proof. For convenience, we denote by $[x]$ the integer part of $x \in \mathbb{R}$. We have

$$\begin{aligned}
&\sum_{l\in\mathbb{Z}} |a_l| \left(1 + \frac{|x-l+b|}{|c|}\right)^{-\theta} \\
&= \sum_{l\in\mathbb{Z}} \int_l^{l+1} |a_l| \left(1 + \frac{|x-l+b|}{|c|}\right)^{-\theta} dy \\
&= \left(\sum_{l\geqslant[x+b]+2} + \sum_{l\leqslant[x+b]-1}\right) \int_l^{l+1} |a_l| \left(1 + \frac{|x-l+b|}{|c|}\right)^{-\theta} dy \\
&\quad + \sum_{l=[x+b]}^{[x+b]+1} \int_l^{l+1} |a_l| \left(1 + \frac{|x+b-l|}{|c|}\right)^{-\theta} dy.
\end{aligned} \tag{B.2}$$

Noticing that $|x-l+b| \geqslant |x-y+b+1|$ for any $y \in [l, l+1)$, $l \geqslant [x+b]+2$, we have

$$\begin{aligned}
&\sum_{l\geqslant[x+b]+2} \int_l^{l+1} |a_l| \left(1 + \frac{|x-l+b|}{|c|}\right)^{-\theta} dy \\
&\leqslant \sum_{l\geqslant[x+b]+2} \int_l^{l+1} |a_l| \left(1 + \frac{|x-y+b+1|}{|c|}\right)^{-\theta} dy \\
&\leqslant \int_{\mathbb{R}} \left(1 + \frac{|x-y+b+1|}{|c|}\right)^{-\theta} \sum_{l\in\mathbb{Z}} |a_l| \chi_{[l,l+1)}(y) dy.
\end{aligned} \tag{B.3}$$

Similarly,

$$\sum_{l\leqslant[x+b]-1}\int_l^{l+1}\left(1+\frac{|x-l+b|}{|c|}\right)^{-\theta}|a_l|dy$$
$$\leqslant\int_{\mathbb{R}}\left(1+\frac{|x-y+b|}{|c|}\right)^{-\theta}\sum_{l\in\mathbb{Z}}|a_l|\chi_{[l,l+1)}(y)dy, \tag{B.4}$$

and noticing that $|c|\geqslant 1$, for $l=[x+b],[x+b]+1$ one has that

$$\int_l^{l+1}\left(1+\frac{|x+b-l|}{|c|}\right)^{-\theta}|a_l|dy$$
$$\leqslant 4^\theta\int_{\mathbb{R}}\left(1+\frac{|x-y+b|}{|c|}\right)^{-\theta}|a_l|\chi_{[l,l+1)}(y)dy$$
$$\leqslant 4^\theta\int_{\mathbb{R}}\left(1+\frac{|x-y+b|}{|c|}\right)^{-\theta}\sum_{l\in\mathbb{Z}}|a_l|\chi_{[l,l+1)}(y)dy. \tag{B.5}$$

Applying Young's inequality in the case $\theta>1/r'+1/p$, we have

$$\left\|\int_{\mathbb{R}}\left(1+\frac{|x-y+b|}{|c|}\right)^{-\theta}\sum_{l\in\mathbb{Z}}|a_l|\chi_{[l,l+1)}(y)dy\right\|_{L^p_x}$$
$$\lesssim\|(1+|\cdot|/|c|)^{-\theta}\|_{L^{pr'/(p+r')}}\left\|\sum_{l\in\mathbb{Z}}|a_l|\chi_{[l,l+1)}\right\|_{L^r}$$
$$\lesssim\langle c\rangle^{1/r'+1/p}\left\|\sum_{l\in\mathbb{Z}}|a_l|\chi_{[l,l+1)}\right\|_{L^r}. \tag{B.6}$$

Using Hardy-Littlewood-Sobolev's inequality in the case $\theta=1/r'+1/p$, we also have the result, as desired. ☐

We note that the hidden constant in the right hand side of (B.1) is independent of $b,c\in\mathbb{R}$ with $|c|\geqslant 1$.

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References

[1] I. Bejenaru, On Schrödinger maps, Amer. J. Math., **130** (2008), 1033–1065.

[2] I. Bejenaru, Global results for Schrödinger maps in dimensions $n\geqslant 3$, Comm. Partial Differential Equations, **33** (2008), 451–477.

[3] I. Bejenaru, A. D. Ionescu, and C. E. Kenig, Global existence and uniqueness of Schrödinger maps in dimensions $d\geqslant 4$, Adv. Math. **215** (2007), 263–291.

[4] I. Bejenaru, A. D. Ionescu, C. E. Kenig and D. Tataru, Global Schrödinger maps in dimensions $d \geqslant 2$: small data in the critical Sobolev spaces, Ann. of Math., to appear.

[5] Á. Bényi; K. Gröchenig; K. A. Okoudjou; L. G. Rogers, Unimodular Fourier multipliers for modulation spaces. J. Funct. Anal. **246** (2007), 366C384.

[6] N. H. Chang, J. Shatah, and K. Uhlenbeck, Schrödinger maps, Comm. Pure Appl. Math. **53** (2000), 590–602.

[7] H. Chihara, The initial value problem for cubic semilinear Schrödinger equations with guage invariance, Publ. RIMS, **32** (1996), 445–471.

[8] P. Constantin and J. C. Saut, Local smoothing properties of dispersive equations, J. Amer. Math. Soc., **1** (1988), 413–446.

[9] P. A. Clarkson and J. A. Tuszyriski, Exact solutions of the multidimensional derivative nonlinear Schrödinger equation for many-body systems near criticality, J. Phys. A: Math. Gen., **23** (1990), 4269-4288.

[10] W. Y. Ding, Y. D. Wang, Schrödinger flow of maps into symplectic manifolds, Science in China Ser. A, **41** (1998), 746-755.

[11] W. Y. Ding and Y. D. Wang, Local Schrödinger flow into Kähler manifolds, Sci. China Ser. A, **44** (2001), 1446–1464.

[12] J. M. Dixon and J. A. Tuszynski, Coherent structures in strongly interacting many-body systems: II. Classical solutions and quantum fluctuations, J. Phys. A: Math. Gen., **22** (1989), 4895-4920.

[13] H. G. Feichtinger, On a new Segal algebra, Monatsh. Math., **92** (1981), 269–289.

[14] H. G. Feichtinger, F. Weisz, The Segal algebra $S_0(\mathbb{R}^d)$ and norm summability of Fourier series and Fourier transforms. Monatsh. Math., **148** (2006), no. 4, 333–349.

[15] D. Gabor, Theory of communication, J. IEE (London), **93** (III) (1946), 429–457.

[16] J. M. Ghidaglia, J. C. Saut, On the initial value problem for the Davey-Stewartson systems, Nonlinearity, **3** (1990), 475–506.

[17] K. Gröchenig, Foundations of Time-Frequency Analysis, Birkhäuser, Boston, MA, 2001.

[18] K. Gröchenig, Time-frequency analysis of Sjöstrand's class, Rev. Mat. Iberoam., **22** (2006), 703–724.

[19] K. Gröchenig, Zhang Hua, B. X. Wang, Global well posedness for the fourth order Schrödinger equation, Preprint.

[20] N. Hayashi, T. Ozawa, Remarks on nonlinear Schröinger equations in one space dimension, Differential Integral Equations, **7** (1994), 453–61.

[21] A. D. Ionescu and C. E. Kenig, Low-regularity Schrödinger maps, Differential Integral Equations **19** (2006), 1271–1300.

[22] A. D. Ionescu and C. E. Kenig, Low-regularity Schrödinger maps, II: global well-posedness in dimensions $d \geqslant 3$, Comm. Math. Phys., **271** (2007), 523–559.

[23] J. Kato, Existence and uniqueness of the solution to the modified Schrödinger map, Math. Res. Lett., **12** (2005), 171–186.

[24] J. Kato and H. Koch, Uniqueness of the modified Schrdinger map in $H^{3/4+}(\mathbb{R}^2)$, Comm. Partial Differential Equations, **32** (2007), 415–429.

[25] C. E. Kenig and A. Nahmod, The Cauchy problem for the hyperbolic-elliptic Ishimori system and Schrödinger maps, Nonlinearity, **18** (2005), 1987–2009.

[26] C. E. Kenig, G. Ponce, and L. Vega, Small solutions to nonlinear Schrödinger equations, Ann. Inst. H. Poincaré Anal. Non Linéaire, **10** (1993), 255–288.

[27] C. E. Kenig, G. Ponce, and L. Vega, Smoothing effects and local existence theory for the generalized nonlinear Schrödinger equations, Invent. Math., **134** (1998), 489–545.

[28] C. E. Kenig, G. Ponce, and L. Vega, The Cauchy problem for quasi-linear Schrödinger equations, Invent. Math., **158** (2004), 343–388.

[29] S. Klainerman, Long-time behavior of solutions to nonlinear evolution equations, Arch. Rational Mech. Anal., **78** (1982), 73–98.

[30] S. Klainerman, G. Ponce, Global small amplitude solutions to nonlinear evolution equations, Commun. Pure Appl. Math., **36** (1983), 133–141.

[31] L. D. Landau, Collected papers of L. D. Landau, Gordon and Breach Science Publishers, New York–London–Paris (1967).

[32] F. Linares and G. Ponce, On the Davey-Stewartson systems, Ann. Inst. H. Poincaré Anal. Non Linéaire, **10** (1993), 523–548.

[33] A. Nahmod, J. Shatah, L. Vega, and C. Zeng, Schrödinger maps and their associated frame systems, Int. Math. Res. Notices, **2007** (2007), article ID rnm088, 29 pages.

[34] A. Nahmod, A. Stefanov, and K. Uhlenbeck, On Schrödinger maps, Comm. Pure Appl. Math., **56** (2003), 114–151. Comm. Pure Appl. Math., **57** (2004), 833–839.

[35] T. Ozawa and J. Zhai, Global existence of small classical solutions to nonlinear Schrödinger equations, Ann. I. H. Poincaré, AN, **25** (2008), 303–311.

[36] N. Papanicolaou and T. N. Tomaras, Dynamics of magnetic vortices, Nuclear Phys. B, **360** (1991), 425–462.

[37] J. Shatah, Global existence of small classical solutions to nonlinear evolution equations, J. Differential Equations, **46** (1982), 409–423.

[38] J. Shatah and M. Struwe, The Cauchy problem for wave maps, Int. Math. Res. Notices, **2002** (2002), 555–571.

[39] P. Sjölin, Regularity of solutions to the Schrödinger equations, Duke Math. J., **55** (1987), 699–715.

[40] M. Sugimoto, M. Ruzhansky, A smoothing property of Schrödinger equations and a global existence result for derivative nonlinear equations, Advances in analysis, 315–320, World Sci. Publ., Hackensack, NJ, 2005.

[41] M. Sugimoto amd N. Tomita, The dilation property of modulation spaces and their inclusion relation with Besov spaces, J. Funct. Anal., **248** (2007), 79–106.

[42] A. Soyeur, The Cauchy problem for the Ishimori equations, J. Funct. Anal., **105** (1992), 233–255.

[43] T. Tao, Gauges for the Schrödinger map, http://www.math.ucla.edutao/preprints/Expository (unpublished).

[44] J. Toft, Continuity properties for modulation spaces, with applications to pseudo-differential calculus, I. J. Funct. Anal., **207** (2004), 399–429.

[45] J. A. Tuszynski and J. M. Dixon, Coherent structures in strongly interacting many-body systems: I. Derivation of dynamics, J. Phys. A: Math. Gen., **22** (1989), 4877-4894.

[46] B.X. Wang, L.F. Zhao and B.L. Guo, Isometric decomposition operators, function spaces $E^{\lambda}_{p,q}$ and applications to nonlinear evolution equations, J. Funct. Anal., **233** (2006), 1–39.

[47] B.X. Wang and H. Hudzik, The global Cauchy problem for the NLS and NLKG with small rough data, J. Differential Equations, **231** (2007), 36–73.

[48] B.X. Wang and C.Y. Huang, Frequency-uniform decomposition method for the generalized BO, KdV and NLS equations J. Differential Equations, **231** (2007), 36–73.

[49] B. X. Wang, L. J. Han, C. Y. Huang, Global well-Posedness and scattering for the derivative nonlinear Schrödinger equation with small rough data, Ann. I. H. Poincaré, AN, **26** (2009), 2253–2281.

[50] B. X. Wang and Y. Z. Wang, Global well posedness and scattering for the elliptic and non-elliptic derivative nonlinear Schrödinger equations with small data, Unpublished, arXiv:0803.2634.

[51] B. X. Wang and Y. Z. Wang, Global well-posedness and scattering for Derivative Schrödinger equation, Preprint.

[52] L. Vega, The Schrödinger equation: pointwise convergence to the initial data, Proc. Amer. Math. Soc., **102** (1988), 874–878.

[53] Y. L. Zhou, B. L. Guo, Weak solution of system of ferromagnetic chain with several variables. Sci. Sinica Ser. A, **30** (1987), 1251–1266.

[54] Y. Zhou, B. Guo and S. Tan, Existence and uniqueness of smooth solution for system of ferro-magnetic chain. Sci. China Ser. A, **34** (1991), 257–266.