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Homogeneous spaces, multi-moment maps and (2,3)-trivial algebras

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Abstract

For geometries with a closed three-form we briefly overview the notion of multi-moment maps. We then give concrete examples of multi-moment maps for homogeneous hypercomplex and nearly Kähler manifolds. A special role in the theory is played by Lie algebras with second and third Betti numbers equal to zero. These we call (2,3)-trivial. We provide a number of examples of such algebras including a complete list in dimensions up to and including five.

1 Introduction

Symplectic geometry is a geometrisation of the theory of mechanical systems. A symplectic structure on a manifold M is defined by a closed 2-form ω that may be expressed locally as $\omega = dq_1 \wedge dp_1 + \cdots + dq_n \wedge dp_n$, where $\dim M = 2n$. In this context the concepts of linear and angular momentum are captured by the notion of moment map corresponding to a group of symmetries G of M preserving ω. It is an equivariant map $\mu\colon M \to \mathfrak{g}^*$, to the dual of the Lie algebra $\mathfrak{g}$ of G, characterised by the equation $d\langle \mu, \mathsf{X}\rangle = X \lrcorner \omega$, for each $\mathsf{X} \in \mathfrak{g}$. Here X denotes the corresponding vector field on M generated by $\mathsf{X} \in \mathfrak{g}$.

Developments in string and other field theories with Wess-Zumino terms [11, 13, 6, 2] have highlighted the importance of geometries associated to closed 3-forms c. Geometric aspects of these theories are not so well developed. In [9], developing [8], we introduce a notion of multi-moment map adapted to such geometries. The important features of our definition are that the target space depends only on the symmetry group G and not the underlying manifold M, in contrast to [4], and that existence of such maps are guaranteed in many circumstances. After reviewing the definition and basic properties, the rest of the paper is devoted to giving examples from hypercomplex and nearly Kähler geometry and to classifications of a special class of Lie algebras that arises.

We say the 3-form $c \in \Omega^3(M)$ defines a *strong geometry* if c is closed, meaning $dc = 0$. The lack of simple canonical descriptions for 3-forms means that in general a non-degeneracy assumption is not appropriate. However, if $X \lrcorner c = 0$ occurs only for $X = 0 \in T_xM$, we will say that the structure is *2-plectic*, following [2].

Suppose G acts on a strong geometry (M, c) preserving c. Then for each $\mathsf{X} \in \mathfrak{g}$, we have from the Cartan formula that $X \lrcorner c$ is a closed 2-form. Similarly, if X and Y are two commuting elements of $\mathfrak{g}$, then $(X \wedge Y) \lrcorner c = c(X, Y, \cdot)$ is a closed 1-form. If this can be integrated to a function $\nu_{X\wedge Y}$, then we have a multi-moment map for the $\mathbb{R}^2$ action generated by X and Y.

In general, the space of commuting elements in $\mathfrak{g}$ forms a complicated variety. We therefore introduce the *Lie kernel* $\mathcal{P}_{\mathfrak{g}}$ as the kernel of the map $\Lambda^2 \mathfrak{g} \to \mathfrak{g}$ induced by the Lie bracket $[\cdot,\cdot]$. A calculation shows that for $\mathsf{p} = \sum_i X_i \wedge Y_i \in \mathcal{P}_{\mathfrak{g}}$, the one-form $p \lrcorner c$ is closed. This leads to

Definition 1.1 [9] A *multi-moment map* for the action of a group G of symmetries of a strong geometry (M, c) is an equivariant map $\nu \colon M \to \mathcal{P}_{\mathfrak{g}}^*$ satisfying $d\langle \nu, \mathsf{p}\rangle = p \lrcorner c$, for all $\mathsf{p} \in \mathcal{P}_{\mathfrak{g}}$.

The following result summarises the important existence results for multi-moment maps.

Theorem 1.2 [9] *Multi-moment maps exist for the action of G on (M, c) if either*

(a) M is compact with $b_1(M) = 0$,
(b) G is compact, $b_1(M) = 0$ and G preserves a volume form on M,
(c) $c = db$ with $b \in \Omega^2(M)$ preserved by G, or
(d) $b_2(\mathfrak{g}) = 0 = b_3(\mathfrak{g})$.

In this paper, we will give examples related to cases (a–c) in hypercomplex and nearly Kähler geometries. Underlying the nearly Kähler examples is an analogue of the Kostant-Kirillov-Souriau construction, which we prove in [9]. This construction is also relevant for the Lie algebras satisfying the conditions of case (d); these we have dubbed as *(2,3)-trivial*. This new class replaces that of semi-simple algebras in the theory of symplectic moment maps, since semi-simple algebras $\mathfrak{h}$ are characterised by $b_1(\mathfrak{h}) = 0 = b_2(\mathfrak{h})$. We have found the following structure theorem:

Theorem 1.3 [9] *A Lie algebra $\mathfrak{g}$ is $(2,3)$-trivial if and only if $\mathfrak{g}$ is solvable, with derived algebra $\mathfrak{k} = \mathfrak{g}'$ of codimension one and satisfying $H^1(\mathfrak{k})^{\mathfrak{g}} = 0 = H^2(\mathfrak{k})^{\mathfrak{g}} = H^3(\mathfrak{k})^{\mathfrak{g}}$.*

In this paper we use this result to give full lists of $(2,3)$-trivial algebras in dimensions at most five and to show that every nilpotent algebra $\mathfrak{k}$ of dimension at most six occurs in the above theorem. We also give some constructions of infinite families of $(2,3)$-trivial Lie algebras.

2 A hypercomplex example

An HKT *structure* on a manifold is given by three complex structures I, J, $K = IJ = -JI$ with common Hermitian metric such that $Id\omega_I = Jd\omega_J = Kd\omega_K$. By [10, Prop. 6.2] it is unnecessary to check integrability of I, J and K. An example of a homogeneous HKT manifold is the compact simple Lie group $SU(3)$. In fact it admits a left-invariant SHKT structure, meaning that $c = -Id\omega_I$ is closed. As a consequence of left-invariance, we may think of the HKT structure on $SU(3)$ in terms of a corresponding algebraic structure on its Lie algebra. Write E_{pq} for the elementary 3×3-matrix with a 1 at position (p, q). Then the following 8 complex matrices constitute a basis for $\mathfrak{su}(3)$: $A_1 = i(E_{11} - E_{22})$, $A_2 = i(E_{22} - E_{33})$, $B_{pq} = E_{pq} - E_{qp}$, $C_{pq} = i(E_{pq} + E_{qp})$, for $p = 1, 2 < q = 2, 3$. We write $a_1, \ldots, c_{23}$ for the dual basis. Using the formula $d\alpha(X,Y) = -\alpha([X,Y])$, one finds that

$$\begin{aligned} da_1 &= -2b_{12}c_{12} - 2b_{13}c_{13},\ da_2 = -2b_{13}c_{13} - 2b_{23}c_{23},\\ db_{12} &= (2a_1 - a_2)c_{12} + b_{13}b_{23} + c_{13}c_{23},\ db_{13} = (a_1 + a_2)c_{13} - b_{12}b_{23} + c_{12}c_{23},\\ db_{23} &= (-a_1 + 2a_2)c_{23} + b_{12}b_{13} + c_{12}c_{13},\ dc_{12} = (-2a_1 + a_2)b_{12} - b_{13}c_{23} - b_{23}c_{13},\\ dc_{13} &= (-a_1 - a_2)b_{13} - b_{12}c_{23} + b_{23}c_{12},\ dc_{23} = (a_1 - 2a_2)b_{23} + b_{12}c_{13} + b_{13}c_{12}, \end{aligned} \tag{2.1}$$

where $\wedge$ signs have been omitted.

A metric is provided by minus the Killing form on $\mathfrak{su}(3)$; here we may take g to be the map $(X,Y) \mapsto -\operatorname{tr}(XY)$: $g = 2a_1^2 - a_1a_2 + 2(a_2^2 + b_{12}^2 + b_{13}^2 + b_{23}^2 + c_{12}^2 + c_{13}^2 + c_{23}^2)$.

In [7, Thm. 4.2] Joyce proved the existence of hypercomplex structures on certain compact Lie groups. For $SU(3)$, Joyce's hypercomplex structure comes from taking a highest root $\mathfrak{su}(2)$, e.g., the span of A_1, B_{12}, C_{12} and the complement $\mathbb{H} + \mathbb{R}$, $\mathbb{H} \cong \langle B_{13}, C_{13}, B_{23}, C_{23}\rangle$ and $\mathbb{R} \cong \langle A_1 + 2A_2\rangle$. Concretely, let us write $\mathbf{I} = A_1$, $\mathbf{J} = B_{12}$ and $\mathbf{K} = C_{12}$. Then define I on $\mathbb{H}$ to be $\operatorname{ad}_{\mathbf{I}}$. Similarly J and K act on $\mathbb{H}$ by $\operatorname{ad}_{\mathbf{J}}$ and $\operatorname{ad}_{\mathbf{K}}$, respectively. On $\mathbb{R} + \mathfrak{su}(2)$ the actions of I, J and K are given by $IV = \mathbf{I}$, $JV = \mathbf{J}$ and $KV = \mathbf{K}$, where $V = (A_1 + 2A_2)/\sqrt{3}$. The complex structures I, J and K are now determined completely by the requirement that they square to -1. It is straightforward to check that $IJ = K = -JI$. Computations show that I, J and K are compatible with the metric, meaning $g(X,Y) = g(IX, IY)$, etc., for all $X, Y \in \mathfrak{su}(3)$. Defining $2a_1' = 2a_1 - a_2$ and $2a_2' = \sqrt{3}a_2$, we find that the non-degenerate 2-forms $\omega_I = g(I\cdot,\cdot)$, ω_J and ω_K are given by

$$\begin{gathered} \omega_I = -a_1'a_2' + b_{12}c_{12} + b_{13}c_{13} - b_{23}c_{23},\ \omega_J = a_2'b_{12} - a_1'c_{12} - b_{13}b_{23} - c_{13}c_{23},\\ \omega_K = a_2'c_{12} + a_1'b_{12} + b_{13}c_{23} + b_{23}c_{13}. \end{gathered}$$

Using (2.1) it then follows that

$$\begin{aligned} d\omega_I &= -\sqrt{3}a_1'(b_{13}c_{13} + b_{23}c_{23}) + a_2'(2b_{12}c_{12} + b_{13}c_{13} - b_{23}c_{23})\\ &\quad - b_{12}b_{13}c_{23} - b_{12}b_{23}c_{13} - b_{13}b_{23}c_{12} - c_{12}c_{13}c_{23},\\ d\omega_J &= 2a_1'a_2'c_{12} + a_1'(b_{13}c_{23} + b_{23}c_{13}) - a_2'(b_{13}b_{23} + c_{13}c_{23})\\ &\quad - \sqrt{3}b_{12}b_{13}c_{13} - \sqrt{3}b_{12}b_{23}c_{23} + b_{13}c_{12}c_{13} - b_{23}c_{12}c_{23},\\ d\omega_K &= -2a_1'a_2'b_{12} + a_1'(b_{13}b_{23} + b_{23}c_{13}) + a_2'(b_{13}c_{23} + b_{23}c_{13})\\ &\quad + \sqrt{3}b_{13}c_{12}c_{13} + \sqrt{3}b_{23}c_{12}c_{23} + b_{12}b_{13}c_{13} - b_{12}b_{23}c_{23}. \end{aligned}$$

From these formulae and the actions of I, J and K, we verify the HKT condition:

$$\begin{aligned} Id\omega_I &= a_1(2b_{12}c_{12} + b_{13}c_{13} - b_{23}c_{23}) - a_2(b_{12}c_{12} - b_{13}c_{13} - 2b_{23}c_{23})\\ &\quad - b_{23}c_{12}c_{13} - b_{13}c_{12}c_{23} - b_{12}c_{13}c_{23} - b_{12}b_{13}b_{23}\\ &= Jd\omega_J = Kd\omega_K. \end{aligned}$$

It is easy to check that $dc = 0$, and thus $(SU(3), g, I, J, K)$ is indeed an SHKT manifold as claimed.

Unfortunately the multi-moment map for the left action of $SU(3)$ on the strong geometry $(SU(3), c)$ is trivial. However, we may instead turn our attention towards the multi-moment maps ν_I, ν_J and ν_K associated with the 3-forms $d\omega_I$, $d\omega_J$ and $d\omega_K$ on $SU(3)$. As an example let us consider the multi-moment map $\nu_I\colon SU(3) \to \mathcal{P}^*_{\mathfrak{su}(3)}$, $\langle \nu, \mathsf{p} \rangle = \omega_I(p)$. The commutation relations for the chosen $\mathfrak{su}(3)$-basis may be used to establish a basis for $\mathcal{P}_{\mathfrak{su}(3)} \leqslant \Lambda^2\,\mathfrak{su}(3)$ whilst the exterior derivative, via equations (2.1), gives a basis for the submodule $\mathfrak{su}(3) \leqslant \Lambda^2\,\mathfrak{su}(3)$. We may now decompose ω_I at the identity: $\omega_I = \omega_I^{\mathfrak{su}(3)} + \omega_I^{\mathcal{P}} = 2\big(b_{12}c_{12} + b_{13}c_{13}\big) + \big(-\sqrt{3}a_1a_2/2 - (b_{12}c_{12} + b_{23}c_{23} - b_{13}c_{13})\big)$. It follows that

$$\mathrm{Ad}^*_{g^{-1}}\,\nu_I(g) = -\tfrac{\sqrt{3}}{2}a_1a_2 - (b_{12}c_{12} + b_{23}c_{23} - b_{13}c_{13}).$$

The image of ν_I is the orbit of $\beta_I = \nu_I(e)$ under $SU(3)$. This element is preserved by a maximal torus, invariant under I, and its orbit is a copy of $F_{1,2}(\mathbb{C}^3)$ inside $\mathcal{P}^*_{\mathfrak{su}(3)}$. At the algebraic level this is easily verified:

$$\begin{aligned}
\ker(\nu_I)_* = \ker d\nu_I &= \{\, A \in \mathfrak{su}(3) : d\nu_I(p, A) = 0 \text{ for all } \mathsf{p} \in \mathcal{P}_{\mathfrak{su}(3)} \,\} \\
&= \{\, A \in \mathfrak{su}(3) : c(Ip, IA) = 0 \text{ for all } \mathsf{p} \in \mathcal{P}_{\mathfrak{su}(3)} \,\} \\
&= I\{\, A \in \mathfrak{su}(3) : g([Ip], A) = 0 \text{ for all } \mathsf{p} \in \mathcal{P}_{\mathfrak{su}(3)} \,\} = [I\mathcal{P}_{\mathfrak{su}(3)}]^{\perp} = \langle A_1, V \rangle.
\end{aligned}$$

Similarly the images of ν_J and ν_K are full flags in $\mathbb{C}^3$. We find

$$\begin{aligned}
\mathrm{Ad}^*_{g^{-1}}\,\nu_J(g) &= \tfrac{\sqrt{3}}{2}\,(2(a_1 + a_2)b_{12} - (b_{23}c_{13} + b_{13}c_{23})) \\
&\quad + \tfrac{1}{14}\,(2(2a_1 - a_2)c_{12} - 5(b_{13}b_{23} + c_{13}c_{23}))\,, \qquad \ker(\nu_J)_* = \langle V, B_{12} \rangle, \\
\mathrm{Ad}^*_{g^{-1}}\,\nu_K(g) &= \tfrac{\sqrt{3}}{14}\,((3a_1 + 2a_2)c_{12} - 2(b_{13}b_{23} + c_{13}c_{23}))) \\
&\quad - \tfrac{1}{2}\,(2(8a_1 + 5a_2)b_{12} - 11(b_{13}c_{23} + b_{23}c_{13}))\,, \quad \ker(\nu_K)_* = \langle V, C_{12} \rangle.
\end{aligned}$$

Putting these together, we get an equivariant map $\underline{\nu} = (\nu_I, \nu_J, \nu_K)\colon SU(3) \to (\mathcal{P}^*_{\mathfrak{su}(3)})^3$. The image is the Aloff-Wallach space $A_{1,1} = SU(3)/T^1_{1,1}$. The relatively high dimension of this image indicates that multi-moment maps could be a useful tool to study homogeneous hyper-Hermitian structures.

3 Six-dimensional nearly Kähler manifolds

A *nearly Kähler structure*, briefly an NK structure, on a six-dimensional manifold may be specified by a 2-form σ and a 3-form ψ^+ whose common pointwise stabiliser in $GL(6, \mathbb{R})$ is isomorphic to $SU(3)$. The NK condition is then $d\sigma = \psi^+$ and $d\psi^- = -\frac{2}{3}\sigma^2$, where $\psi^+ + i\psi^- \in \Lambda^{3,0}$. We will indicate how each homogeneous strict NK six-manifold $G/H = F_{1,2}(\mathbb{C}^3)$, $\mathbb{C}P(3)$, $S^3 \times S^3$ and S^6, as classified by Butruille [3], may be realised as a 2-plectic orbit $G \cdot \beta$ in $\mathcal{P}^*_{\mathfrak{g}}$. Let $d_{\mathcal{P}}\colon \mathcal{P}^*_{\mathfrak{g}} \to \Lambda^3\,\mathfrak{g}^*$ be the map induced by d. Then our realisation is such that $\Psi = d_{\mathcal{P}}\beta$ induces ψ^+ via $\langle \Psi, \mathsf{X} \wedge \mathsf{Y} \wedge \mathsf{Z} \rangle = \psi^+(X, Y, Z)$ and $\sigma(X, Y) = \beta(\mathsf{X}, \mathsf{Y})$.

In each case the element $\beta \in \mathcal{P}^*_{\mathfrak{g}}$ must be chosen with some care. For instance neither of the 3 copies $F_{1,2}(\mathbb{C}^3) \subset \mathcal{P}^*_{\mathfrak{su}(3)}$ from section 2 behaves in the manner described above. On the other hand $SU(3)$ acting on the element $\beta_1 = b_{12}c_{12} + c_{13}b_{13} + b_{23}c_{23} \in \mathcal{P}^*_{\mathfrak{su}(3)}$ gives a copy of the full flag with forms $d_{\mathcal{P}}\beta_1$ and β_1 inducing the NK structure. The associated almost complex structure J is given by $J(B_{12}) = C_{12}$, $J(C_{13}) = B_{13}$, $J(B_{23}) = C_{23}$.

To obtain $\mathbb{C}P(3)$, we let $Sp(2)$ act on $\beta_2 = a_1b_{11} + b_{12}r + c_{12}q$ in $\mathcal{P}^*_{\mathfrak{sp}(2)}$. The chosen basis for $\mathfrak{sp}(2)$ consists of the 10 complex matrices $A_1 = i(E_{11} - E_{33})$, $A_2 = i(E_{22} - E_{44})$, $Q = E_{12} - E_{21} + E_{34} - E_{43}$, $R = i(E_{12} + E_{21} - E_{34} - E_{43})$, $B_{k\ell} = E_{k,2+\ell} + E_{\ell,2+k} - E_{2+k,\ell} - E_{2+\ell,k}$, $C_{k\ell} = i(E_{k,2+\ell} + E_{\ell,2+k} + E_{2+k,\ell} + E_{2+\ell,k})$, $1 \leqslant k \leqslant \ell \leqslant 2$. One easily verifies that $\operatorname{stab}_{\mathfrak{sp}(2)} \beta_1 = \mathfrak{u}(1) \oplus \mathfrak{su}(2)$, so that, up to discrete coverings, the orbit of β_2 is $\mathbb{C}P(3)$. We have

$$\begin{aligned} da_1 &= -2(4b_{11}c_{11} + b_{12}c_{12} + qr),\ db_{11} = 2a_1c_{11} + b_{12}q - c_{12}r, \\ db_{12} &= (a_1 + a_2)c_{12} + 2(-b_{11} + b_{22})q - 2(c_{11} + c_{22})r, \\ dc_{12} &= -(a_1 + a_2)b_{12} + 2(b_{11} + b_{22})r + 2(-c_{11} + c_{22})q, \\ dq &= (a_1 - a_2)r + 2(b_{11} - b_{22})b_{12} + 2(c_{11} - c_{22})c_{12}, \\ dr &= (-a_1 + a_2)q + 2(c_{11} + c_{22})b_{12} - 2(b_{11} + b_{22})c_{12}. \end{aligned}$$

Computations now show that β_2 and $d_{\mathcal{P}}\beta_2$ determine a NK structure which has almost complex structure given by $J(A_1) = \frac{1}{2}B_{11}$, $J(B_{12}) = R$ and $J(C_{12}) = Q$.

The homogeneous NK structure on $S^3 \times S^3$ is obtained on the orbit of $\beta_3 = e_1f_1 + e_2f_2 + e_3f_3 \in \mathcal{P}^*_{\mathfrak{su}(2)\oplus\mathfrak{su}(2)}$. Here e_i, f_i denotes a cyclic basis for $\mathfrak{su}(2)^* \oplus \mathfrak{su}(2)^*$, meaning $de_i = e_{i+1}e_{i+2}$ and $df_i = f_{i+1}f_{i+2}$ for $i \in \mathbb{Z}/3$. One may verify that the associated almost complex structure is given by $J(E_i) = (E_i + 2F_i)/\sqrt{3}$.

Finally we obtain S^6 as the G_2-orbit of the element $\beta_4 = b_1c_1 + b_3c_3 + c_4b_4 \in \mathcal{P}^*_{\mathfrak{g}_2}$. The chosen basis for $\mathfrak{g}_2$ is $A_1 = iH_1$, $A_2 = iH_2$, $B_a = X_a - Y_a$, $C_a = i(X_a + Y_a)$, $(1 \leqslant a \leqslant 6)$, with the elements $\{H_1, \dots, Y_6\}$ defined in [5, Table 22.1]. Now we have

$$\begin{aligned} db_1 &= (2a_1 - a_2)c_1 + b_3b_2 + c_3c_2 + 2(b_4b_3 + c_4c_3) + b_4b_5 + c_4c_5, \\ dc_1 &= (-2a_1 + a_2)b_1 + c_3b_2 + c_2b_3 + 2(c_4b_3 + c_3b_4) + b_4c_5 + b_5c_4, \\ db_3 &= (-a_1 + a_2)c_3 + b_2b_1 + c_1c_2 + 2(b_1b_4 + c_1c_4) + b_4b_6 + c_4c_6, \\ dc_3 &= (a_1 - a_2)b_3 + c_2b_1 + b_2c_1 + b_4c_6 + 2(b_1c_4 + b_4c_1) + b_4c_6 + b_6c_4, \\ db_4 &= a_1c_4 + 2(b_3b_1 + c_1c_3) + b_5b_1 + c_5c_1 + c_6c_3 + b_6b_3, \\ dc_4 &= b_4a_1 + 2(b_1c_3 + b_3c_1) + c_5b_1 + c_1b_5 + c_6b_3 + c_3b_6, \end{aligned}$$

and it can be verified that β_4 and $d_{\mathcal{P}}\beta_4$ induce a NK structure on S^6. In this case one has $J(B_1) = C_1$, $J(B_3) = C_3$ and $J(C_4) = B_4$.

Motivated by these examples, it makes sense to study the multi-moment map formalism for NK manifolds with less symmetry.

4 Positive gradings of nilpotent algebras

A Lie algebra $\mathfrak{k}$ is *positively graded* if there is a vector space direct sum decomposition $\mathfrak{k} = \mathfrak{k}_1 + \dots + \mathfrak{k}_r$ with $[\mathfrak{k}_i, \mathfrak{k}_j] \subseteq \mathfrak{k}_{i+j}$ for all i, j. A grading of an n-dimensional Lie algebra $\mathfrak{k}$ may be specified in terms of an n positive integers, see Example 4.3. We have

G	β	$d_{\mathcal{P}}\beta$	$\mathcal{O} = G \cdot \beta$
$SU(3)$	$b_{12}c_{12}+c_{13}b_{13}+b_{23}c_{23}$	$3(b_{12}(b_{13}c_{23}+b_{23}c_{13})+c_{12}(b_{13}b_{23}+c_{13}c_{23}))$	$F_{1,2}(\mathbb{C}^3)$
$Sp(2)$	$a_1b_{11}+b_{12}r+c_{12}q$	$-3(a_1(b_{12}q-c_{12}r)+2b_{11}(b_{12}c_{12}+qr))$	$\mathbb{C}P(3)$
$SU(2)^2$	$e_1f_1+e_2f_2+e_3f_3$	$e_{12}f_3+e_{23}f_1+e_{31}f_2-e_1f_{23}-e_2f_{31}-e_3f_{12}$	$S^3\times S^3$
G_2	$b_1c_1+b_3c_3+c_4b_4$	$6(b_1(b_3c_4-c_3b_4)-c_1(b_3b_4+c_3c_4))$	S^6

Table 3.1. Realisations of the six-dimensional nearly Kähler manifolds as orbits in Lie kernels.

Proposition 4.1 *Any nilpotent Lie algebra of dimension at most six admits a positive grading. From dimension seven and above, there are nilpotent Lie algebras, which do not admit a positive grading.*

The nilpotent Lie algebras of dimension at most six and primitive positive gradings are listed in Table 4.1. Example 4.3 illustrates how gradings are found. In [9] we show that there are examples of nilpotent algebras $\mathfrak{k}$ of dimension seven that can not arise as the derived algebra of any $(2,3)$-trivial algebra $\mathfrak{g}$. It follows that such $\mathfrak{k}$ can not admit a positive grading.

Positive gradings can be used to generate $(2,3)$-trivial algebras:

Corollary 4.2 *Each of the* 50 *Lie algebras listed in Table 4.1 is the derived algebra of a completely solvable* $(2,3)$*-trivial Lie algebra.*

Proof Let $\mathfrak{g} = \langle A\rangle + \mathfrak{k}$, where $\mathfrak{k}$ is one of the algebras of Table 4.1 and ad_A acts as multiplication by i on $\mathfrak{k}_i$. Then $\mathfrak{g}$ is a solvable algebra. Moreover $(\Lambda^s\,\mathfrak{k})^{\mathfrak{g}} = \{0\}$ for $s \geqslant 1$, so that $H^1(\mathfrak{k})^{\mathfrak{g}} = \{0\} = H^2(\mathfrak{k})^{\mathfrak{g}} = H^3(\mathfrak{k})^{\mathfrak{g}}$. Thus, by Theorem 1.3, $\mathfrak{g}$ is $(2,3)$-trivial. Since ad_X has real eigenvalues for each $X \in \mathfrak{g}$ the Lie algebra is completely solvable. □

Example 4.3 A Lie algebra $\mathfrak{k}$ may be specified in terms of the action of $d\colon\ \mathfrak{k}^* \to \Lambda^2\,\mathfrak{k}^*$ on a basis for $\mathfrak{k}^*$. By $(0^2, 12, 13, 14{+}23, 24{+}15)$ we thus denote the nilpotent Lie algebra $\mathfrak{k}$ which has a basis $e^1,\dots,e^6$ for $\mathfrak{k}^*$ satisfying $de^1 = 0 = de^2$, $de^3 = e^1e^2$, ..., $de^6 = e^2e^4 + e^1e^5$. Assigning weights can now be formulated schematically as follows: $e^1 \to a$, $e^2 \to b$, $e^3 \to a+b$, $e^4 \to 2a+b$, $e^5 \to 3a+b = a+2b$, $e^6 \to 2(a+b) = 2(a+b)$. In particular we see that $2a = b$, so that a grading may be defined by $\mathfrak{k} = \mathfrak{k}_1 \oplus\cdots\oplus \mathfrak{k}_6$, where $\mathfrak{k}_i = \langle e_i\rangle$. Choosing $a = 1$, this weight decomposition is denoted by 123456. Following the proof of Corollary 4.2, we may now define a $(2,3)$-trivial extension of $\mathfrak{k}$: $(0, 12, 2.13, 3.14{+}23, 4.15{+}24, 5.16{+}25{+}34, 6.17{+}24{+}26)$.

5 Families of (2,3)-trivial algebras

While the method of positive gradings provides an effective tool for constructing examples of $(2,3)$-trivial algebras, it is inadequate if one aims for a general understanding of the $(2,3)$-trivial class. Therefore we now turn to give a complete list of such algebras in dimensions up to and including five.

In dimension one, the only Lie algebra is Abelian and is automatically $(2,3)$-trivial. In dimension two a Lie algebra is either Abelian or isomorphic to the $(2,3)$-trivial algebra $(0,21)$.

Structure	Grading
(0)	1
(0^2)	1^2
(0^3)	1^3
$(0^2,12)$	$1^2 2$
(0^4)	1^4
$(0^3,12)$	$1^3 2$
$(0^2,12,13)$	$1^2 23$
(0^5)	1^5
$(0^4,12), (0^4,12{+}34)$	$1^4 2$
$(0^3,12,13)$	$1^3 2^2$
$(0^3,12,14)$	$1^3 23$
$(0^3,12,13{+}24)$	$1^2 2^2 3$
$(0^2,12,13,23)$	$1^2 23^2$
$(0^2,12,13,14)$	$1^2 234$
$(0^2,12,13,14{+}23)$	12345
(0^6)	1^6
$(0^5,12), (0^5,12+34)$	$1^5 2$

Structure	Grading
$(0^4,12,13), (0^4,13{+}42,14{+}23), (0^4,12,34), (0^4,12,14{+}23)$	$1^4 2^2$
$(0^4,12,15)$	$1^4 23$
$(0^3,12,13,23)$	$1^3 2^3$
$(0^4,12,14{+}25), (0^4,12,15{+}34), (0^3,12,13,14), (0^3,12,23,14{\pm}35), (0^3,12,13,24), (0^3,12,13,14{+}35)$	$1^3 2^2 3$
$(0^3,12,14,24)$	$1^3 23^2$
$(0^3,12,14,15)$	$1^3 234$
$(0^3,12,13{+}14,24), (0^3,12,13{+}42,14{+}23), (0^3,12,13,14{+}23), (0^3,12,14,13{+}42)$	$1^2 2^2 3^2$
$(0^3,12,14{-}23,15{+}34)$	$1^2 2^2 34$
$(0^2,12,13,23,14{\pm}25), (0^2,12,13,23,14)$	$1^2 23^2 4$
$(0^2,12,13,14,15), (0^2,12,13,14,34{+}52)$	$1^2 2345$
$(0^3,12,14,15{+}23)$	$1^2 3234$
$(0^3,12,14,15{+}24)$	121345
$(0^3,12,14,15{+}23{+}24)$	$123^2 45$
$(0^2,12,13,14+23,24+15)$	123456
$(0^2,12,13,14{+}23,34{+}52)$	123457
$(0^2,12,13,14,23{+}15)$	134567

Table 4.1. Positive gradings of nilpotent Lie algebras of dimension $\leqslant 6$. Algebras are ordered according to their dimension and a primitive positive grading.

These first two examples are uninteresting from the point of view of multi-moment maps since they have $\mathcal{P}_{\mathfrak{g}} = \{0\}$. In next dimensions we have:

Proposition 5.1 *The $(2,3)$-trivial Lie algebras in dimensions three, four and five are listed in the Tables 5.1 and 5.2.*

We shall now give a proof of Proposition 5.1. Note that we do not discuss inequivalence of the algebras; imposing inequivalence would put further restrictions on the parameters, see for instance [1, Theorem 1.1, 1.5].

The starting point for our analysis is Theorem 1.3 which gives $\mathfrak{g} = \mathbb{R}A + \mathfrak{k}$, where $\mathfrak{k} = \mathfrak{g}'$ is nilpotent. The element A acts on $H^i(\mathfrak{k})$ as endomorphism with determinant a_i. Now $(2,3)$-triviality of $\mathfrak{g}$ may be rephrased as the non-vanishing of a_1, a_2 and a_3.

Dimension three Let $\mathfrak{g}$ be a $(2,3)$-trivial algebra of dimension three. Then $\mathfrak{k}$ is nilpotent and two-dimensional, so $\mathfrak{k} \cong \mathbb{R}^2$. The element A acts on $\mathbb{R}^2$ invertibly and the induced action on $H^2(\mathbb{R}^2) \cong \Lambda^2\mathbb{R}^2 \cong \mathbb{R}$ is also invertible. So either A is diagonalisable over $\mathbb{C}$ with non-zero eigenvalues whose sum is non-zero, giving cases $\mathfrak{r}_{3,\lambda\neq-1,0}$ and $\mathfrak{r}'_{3,\lambda\neq0}$, or A acts with Jordan normal form $\left(\begin{smallmatrix}\lambda & 1\\ 0 & \lambda\end{smallmatrix}\right)$, $\lambda \neq 0$, giving case $\mathfrak{r}_3$.

$\mathfrak{r}_3$	$(0, 21{+}31, 31)$	
$\mathfrak{r}_{3,\lambda}$	$(0, 21, \lambda.31)$	$\lambda \neq -1, 0$
$\mathfrak{r}'_{3,\lambda}$	$(0, \lambda.21{+}31, -21{+}\lambda.31)$	$\lambda \neq 0$
$\mathfrak{r}_4$	$(0, 21{+}31, 31{+}41, 41)$	
$\mathfrak{r}_{4,\lambda}$	$(0, 21, \lambda.31{+}41, \lambda.41)$	$\lambda \neq -1, -\frac{1}{2}, 0$
$\mathfrak{r}_{4,\lambda(2)}$	$(0, 21, \lambda_1.31, \lambda_2.41)$	$\lambda_i, \lambda_1{+}\lambda_2 \neq -1, 0$
$\mathfrak{r}'_{4,\lambda(2)}$	$(0, \lambda_1.21, \lambda_2.31{+}41, -31{+}\lambda_2.41)$	$\lambda_1 \neq 0, \lambda_2 \neq -\frac{\lambda_1}{2}, 0$
$\mathfrak{d}_{4,\lambda}$	$(0, 21, \lambda.31, (1{+}\lambda).41{+}32)$	$\lambda \neq -2, -1, -\frac{1}{2}, 0$
$\mathfrak{d}'_{4,\lambda}$	$(0, \lambda.21{+}31, -21{+}\lambda.31, 2\lambda.41{+}32)$	$\lambda \neq 0$
$\mathfrak{h}_4$	$(0, 21{+}31, 31, 2.41{+}32)$	

Table 5.1. The three- and four-dimensional $(2,3)$-trivial Lie algebras. Our labelling of $\mathfrak{d}_\lambda$ differs from [1, Theorem 1.5].

$\mathfrak{r}_5$	$(0, 21{+}31, 31{+}41, 41{+}51, 51)$	
$\mathfrak{r}_{5(1),\lambda}$	$(0, 21, \lambda.31{+}41, \lambda.41{+}51, \lambda.51)$	$\lambda \neq -1, -\frac{1}{2}, 0$
$\mathfrak{r}_{5(2),\lambda}$	$(0, 21{+}31, 31, \lambda.41{+}51, \lambda.51)$	$\lambda \neq -2, -1, -\frac{1}{2}, 0$
$\mathfrak{r}_{5,\lambda(2)}$	$(0, 21, \lambda_1.31, \lambda_2.41{+}51, \lambda_2.51)$	$\lambda_i \neq -1, 0; \lambda_1{+}\lambda_2 \neq 0, -1;$ $1{+}2\lambda_2, \lambda_1{+}2\lambda_2 \neq 0$
$\mathfrak{r}_{5,\lambda(3)}$	$(0, 21, \lambda_1.31, \lambda_2.41, \lambda_3.51)$	$\lambda_i \neq -1, 0; \lambda_1{+}\lambda_2{+}\lambda_3 \neq 0;$ $\lambda_i{+}\lambda_j \neq -1, 0\ (i \neq j)$
$\mathfrak{r}'_{5,\lambda(2)}$	$(0, \lambda_1.21{+}31, \lambda_1.31, \lambda_2.41{+}51, -41{+}\lambda_2.51)$	$\lambda_i, \lambda_1{+}2\lambda_2 \neq 0$
$\mathfrak{r}'_{5,\lambda(3)}$	$(0, \lambda_1.21, \lambda_2.31, \lambda_3.41{+}51, -41{+}\lambda_3.51)$	$\lambda_i{\neq}0; \lambda_1{\neq}{-}\lambda_2; \lambda_1, \lambda_2{\neq}{-}2\lambda_3$
$\mathfrak{r}''_{5,\lambda}$	$(0, \lambda.21{+}31{+}41, -21{+}\lambda.31{+}51, \lambda.41{+}51, -41{+}\lambda.51)$	$\lambda \neq 0$
$\mathfrak{r}''_{5,\lambda(3)}$	$(0, \lambda_1.21{+}31, -21{+}\lambda_1.31, \lambda_2.41{+}\lambda_3.51, -\lambda_3.41{+}\lambda_2.51)$	$\lambda_i \neq 0$
$\mathfrak{d}_{5(1)}$	$(0, 21, 21{+}31, 31{+}41, 2.51{+}32)$	
$\mathfrak{d}^{\pm}_{5(2)}$	$(0, 21, 21{+}31, 2.41, 2.51 \pm 41{+}32)$	
$\mathfrak{d}_{5(1),\lambda}$	$(0, 21, \lambda.31, (1{+}\lambda).41, (1{+}\lambda).51{+}32{+}41)$	$\lambda \neq -2, -\frac{3}{2}, -1, -\frac{2}{3}, -\frac{1}{2}, 0$
$\mathfrak{d}_{5(2),\lambda}$	$(0, 21, 21{+}31, \lambda.41, 2.51{+}32)$	$\lambda \neq -3, -1, 0$
$\mathfrak{d}_{5,\lambda(2)}$	$(0, 21, \lambda_1.31, \lambda_2.41, (1{+}\lambda_1).51{+}32)$	$\lambda_1 \neq -2, -\frac{1}{2}, -1, 0; \lambda_2 \neq 0, -1;$ $\lambda_1{+}\lambda_2 \neq -2, 0; \lambda_2{+}2\lambda_1 \neq -1$
$\mathfrak{d}_{5(3),\lambda}$	$(0, \lambda.21, 31, 31{+}41, (1{+}\lambda).51{+}32)$	$\lambda \neq -3, -2, -1, -\frac{1}{2}, 0$
$\mathfrak{d}'^{\pm}_{5,\lambda}$	$(0, \lambda.21{+}31, -21{+}\lambda.31, 2\lambda.41, 2\lambda.51 \pm 41{+}32)$	$\lambda \neq 0$
$\mathfrak{d}'_{5,\lambda(2)}$	$(0, \lambda_1.21{+}31, -21{+}\lambda_1.31, \lambda_2.41, 2\lambda_1.51)$	$\lambda_1, \lambda_2 \neq 0$
$\mathfrak{p}_5$	$(0, 21, 21{+}31, 2.41{+}32, 3.51{+}42)$	
$\mathfrak{p}_{5,\lambda}$	$(0, 21, \lambda.31, (1{+}\lambda).41{+}32, (2{+}\lambda).51{+}42)$	$\lambda \neq -3, -2, -1, -\frac{1}{2}, 0$

Table 5.2. The five-dimensional $(2,3)$-trivial Lie algebras.

Dimension four For $\mathfrak{g}$ of dimension four we have $\mathfrak{k} \cong \mathbb{R}^3$ or the Heisenberg algebra $\mathfrak{h}_3 = (0^2, 21)$. The former gives the algebras from the $\mathfrak{r}$- and $\mathfrak{r}'$-series. Derivations of $\mathbb{R}^3$ are just linear endomorphisms; therefore the relevant list of extensions of $\mathbb{R}^3$ may be obtained from considerations of invertible 3×3 matrices in normal form: $A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix}$, $A_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$, $A_3 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ and $A_4 = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 1 \\ 0 & -1 & \lambda_2 \end{pmatrix}$. The element A_1 gives the family $\mathfrak{r}_{4,\lambda(2)}$, and the induced action on $H^1(\mathbb{R}^3) \cong \mathbb{R}^3$ is, up to sign, multiplication by the transpose of A_1. Using this observation one easily finds the induced actions on $H^2(\mathbb{R}^3) \cong \mathbb{R}^3$ and $H^3(\mathbb{R}^3) \cong \mathbb{R}$. We deduce that $(2,3)$-triviality holds if and only if the determinants $a_1 = \lambda_1\lambda_2$, $a_2 = (1+\lambda_1)(1+\lambda_2)(\lambda_1+\lambda_2)$ and $a_3 = 1+\lambda_1+\lambda_2$ do not vanish. The matrix A_2 gives us the algebra $\mathfrak{r}_{4,\lambda}$. In this case $a_1 = \lambda^2$, $a_2 = 2\lambda(1+\lambda)^2$, $a_3 = 1+2\lambda$, giving the restrictions on parameters in Table 5.1. The algebra $\mathfrak{r}_4$ corresponds to A_3. Finally, A_4 occurs when the action has 2 complex eigenvalues. The corresponding family is $\mathfrak{r}'_{4,\lambda(2)}$, where λ_1, λ_2 are restricted by $a_i \neq 0$ for $a_1 = \lambda_1(1+\lambda_2^2)$, $a_2 = 2\lambda_2(1+(\lambda_1+\lambda_2)^2)$ and $a_3 = \lambda_1 + 2\lambda_2$.

The Heisenberg algebra $\mathfrak{h}_3$ has $H^1(\mathfrak{h}_3) \cong \langle e^1, e^2 \rangle$, $H^2(\mathfrak{h}_3) \cong \langle e^{13}, e^{23} \rangle$, $H^3(\mathfrak{h}_3) \cong \langle e^{123} \rangle$. The action of A, being a derivation, is represented by a matrix of the form $\begin{pmatrix} B & 0 \\ \underline{b} & \operatorname{tr} B \end{pmatrix}$ with B a real 2×2-matrix and $\underline{b} = (b_1, b_2) \in \mathbb{R}^2$. To see this, write $\operatorname{ad}_A(e_i) = \sum_k b_i^k e_k$ and consider the relation $\operatorname{ad}_A(e_3) = \operatorname{ad}_A[e_1, e_2] = [\operatorname{ad}_A(e_1), e_2] + [e_1, \operatorname{ad}_A(e_2)]$. After the transformation $A \to A - b_2 e_1 + b_1 e_2$ we may assume $\underline{b} = 0$, so that the algebras are distinguished by the normal form of B. The family $\mathfrak{d}_{4,\lambda}$ arises when $B = \operatorname{diag}(1, \lambda)$. Restrictions on λ now follow from the determinants $a_1 = \lambda$, $a_2 = (2+\lambda)(1+2\lambda)$ and $a_3 = 2(1+\lambda)$ being non-zero. If $B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ one has the algebra $\mathfrak{h}_4$. Finally the action may have complex eigenvalues so that $B = \begin{pmatrix} \lambda & 1 \\ -1 & \lambda \end{pmatrix}$, corresponding to the family $\mathfrak{d}'_{4,\lambda}$. One finds $a_1 = 1+\lambda^2$, $a_2 = 1+9\lambda^2$ and $a_3 = 4\lambda$, so we must have $\lambda \neq 0$.

Dimension five A five-dimensional $(2,3)$-trivial Lie algebra has $\mathfrak{k} \cong \mathbb{R}^4$, $(0^3, 21)$ or $(0^2, 21, 31)$. In the Abelian case $H^1(\mathbb{R}^4) \cong \mathbb{R}^4$, $H^2(\mathbb{R}^4) \cong \mathbb{R}^6$, $H^3(\mathbb{R}^4) \cong \mathbb{R}^4$. The solvable extensions are found by taking invertible matrices in the normal forms $A_1, \dots, A_9$:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_3 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & \lambda_2 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$
$$\begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 1 \\ 0 & 0 & -1 & \lambda_3 \end{pmatrix}, \quad \begin{pmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & -1 & \lambda_2 \end{pmatrix}, \quad \begin{pmatrix} \lambda_1 & 1 & 0 & 0 \\ -1 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & \lambda_3 \\ 0 & 0 & -\lambda_3 & \lambda_2 \end{pmatrix}, \quad \begin{pmatrix} \lambda & 1 & 1 & 0 \\ -1 & \lambda & 0 & 1 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & -1 & \lambda \end{pmatrix}.$$

The matrix A_1 gives the family $\mathfrak{r}_{5,\lambda(3)}$ with restrictions on λ_i following from non-vanishing of $a_1 = \lambda_1\lambda_2\lambda_3$, $a_2 = \prod_i (1+\lambda_i) \prod_{i<j} (\lambda_i + \lambda_j)$, $a_3 = (\lambda_1+\lambda_2+\lambda_3) \prod_{i<j} (1+\lambda_j+\lambda_k)$. The form A_2 corresponds to the family $\mathfrak{r}_{5,\lambda(2)}$ and the determinants $a_1 = \lambda_1\lambda_2^2$, $a_2 = 2\lambda_2(1+\lambda_1)(1+\lambda_2)^2(\lambda_1+\lambda_2)^2$, and $a_3 = (1+\lambda_1+\lambda_2)^2(1+2\lambda_2)(\lambda_1+2\lambda_2)$ should be non-zero. From A_3 we obtain the family $\mathfrak{r}_{5(2),\lambda}$ with parameter value constrained by $a_1 = \lambda^2$, $a_2 = 4\lambda(1+\lambda)^4$, $a_3 = (1+2\lambda)^2(2+\lambda)^2$ being non-zero. The matrix A_4 corresponds to $\mathfrak{r}_{5(1),\lambda}$ with λ constrained by $a_i \neq 0$ for $a_1 = \lambda^3$, $a_2 = 8\lambda^3(1+\lambda)^3$, $a_3 = 3\lambda(1+2\lambda)^3$. The algebra $\mathfrak{r}_5$ is from A_5. Members

of the $\mathfrak{r}'$- and $\mathfrak{r}''$-series occur when ad_A has 2 or 4 complex eigenvalues. The algebra $\mathfrak{r}'_{5,\lambda(3)}$ corresponds to A_6; the conditions are that $a_1 = \lambda_1\lambda_2(1+\lambda_3^2)$, $a_2 = 2\lambda_3(\lambda_1+\lambda_2)(1+(\lambda_1+\lambda_3)^2)(1+(\lambda_2+\lambda_3)^2)$ and $a_3 = (\lambda_1+2\lambda_3)(\lambda_2+2\lambda_3)(1+(\lambda_1+\lambda_2+\lambda_3)^2)$ are non-zero. The form A_7 gives the family $\mathfrak{r}'_{5,\lambda(2)}$ with λ_1, λ_2 constrained by $a_1 = \lambda_1^2(1+\lambda_2^2)$, $a_2 = 4\lambda_1\lambda_2(1+(\lambda_1+\lambda_2)^2)^2$, $a_3 = (\lambda_1+2\lambda_2)^2(1+(2\lambda_1+\lambda_2)^2)$ being non-zero. The matrix A_8 has $\lambda_3 \neq 0$ and corresponds to the family $\mathfrak{r}''_{5,\lambda(3)}$; restrictions on parameter values follow from non-zero values for $a_1 = (1+\lambda_1^2)(\lambda_2^2+\lambda_3^2)$, $a_2 = 4\lambda_1\lambda_2((\lambda_1+\lambda_2)^2+(1+\lambda_3)^2)((\lambda_1+\lambda_2)^2+(1-\lambda_3)^2)$ and $a_3 = (\lambda_3^2+(2\lambda_1+\lambda_2)^2)(1+(\lambda_1+2\lambda_2)^2)$. Finally A_9 gives the algebra $\mathfrak{r}''_{5,\lambda}$. The determinants $a_1 = (1+\lambda^2)^2$, $a_2 = 64\lambda^4(1+\lambda^2)$ and $a_3 = (1+9\lambda^2)^2$ must be non-zero.

To analyse the cases $(0^3,21)$ and $(0^2,21,31)$ we follow and modify arguments given in [12]. First consider $\mathfrak{k} \cong (0^3,21)$ which has $H^1(\mathfrak{k}) \cong \langle e^1,e^2,e^3\rangle$, $H^2(\mathfrak{k}) \cong \langle e^{13},e^{14},e^{23},e^{24}\rangle$ and $H^3 \cong \langle e^{124},e^{134},e^{234}\rangle$. Write $A(e_i) = a_i^k e_k$ for $i = 1,2,3,4$. From the relations $\mathrm{ad}_A(e_4) = [\mathrm{ad}_A(e_1),e_2] + [e_1,\mathrm{ad}_A(e_2)]$, $0 = \mathrm{ad}_A[e_i,e_3] = [\mathrm{ad}_A(e_i),e_3] + [e_i,\mathrm{ad}_A(e_3)]$, $i = 1,2$, we deduce that $a_4^4 = a_1^1 + a_2^2$ and $a_4^1 = 0 = a_4^2 = a_4^3 = a_3^2 = a_3^1$. After the transformation $A \to A - a_2^4 e_1 + a_1^4 e_2$ we may assume $a_1^4 = a_2^4 = 0$. The restriction $B = (b_i^k)$ of ad_A to the subspace $\langle e_1,e_2,e_3\rangle$ has $b_3^1 = 0 = b_3^2$. This may be transformed to Jordan form via $e_1 \to ae_1+be_2+ce_3$, $e_2 \to pe_1+qe_2+re_3$, $e_3 \to se_3$ with $aq-bp \neq 0$ and $s \neq 0$. Excluding degenerate matrices, we may therefore take $B = B_i$ to be one of: $\begin{pmatrix}1&0&0\\0&\lambda_1&0\\0&0&\lambda_2\end{pmatrix}$, $\begin{pmatrix}1&0&0\\1&1&0\\0&1&1\end{pmatrix}$, $\begin{pmatrix}1&0&0\\1&1&0\\0&0&\lambda\end{pmatrix}$, $\begin{pmatrix}\lambda_1&1&0\\-1&\lambda_1&0\\0&0&\lambda_2\end{pmatrix}$, $\begin{pmatrix}\lambda&0&0\\0&1&0\\0&1&1\end{pmatrix}$. Consider first the case $B_1 = \mathrm{diag}(1,\lambda_1,\lambda_2)$. If $\lambda_2 \neq 1+\lambda_1$ we may assume, making a change $e_3 \to e_3 + \alpha e_4$ if necessary, that $a_3^4 = 0$. This gives us the family $\mathfrak{d}_{5,\lambda(2)}$. The determinants $a_1 = \lambda_1\lambda_2$, $a_2 = (1+\lambda_2)(2+\lambda_1)(\lambda_1+\lambda_2)(1+2\lambda_1)$, and $a_3 = 2(1+\lambda_1)(2+\lambda_2+\lambda_1)(1+2\lambda_1+\lambda_2)$ must be non-zero. Turning next to the case $\lambda_2 = 1+\lambda_1$, let us assume $a_3^4 \neq 0$, otherwise we get a member of the family $\mathfrak{d}_{5,\lambda(2)}$. After rescaling $e_i \to |a_3^4|^{1/2} e_i$, for $i = 1,2$, $e_4 \to |a_3^4| e_4$, we obtain the families $\mathfrak{d}^{\pm}_{5(1),\lambda}$ given by $(0,21,\lambda.31,(1+\lambda).41,(1+\lambda).51+32\pm 41)$. Scaling (e_1,e_4,e_5) by factors $(\lambda,\lambda^{-1},-1)$ and interchanging e_2 and e_3 gives $\mathfrak{d}^{+}_{5(1),\lambda} \cong \mathfrak{d}^{-}_{5(1),1/\lambda}$, so there is only one family $\mathfrak{d}_{5(1),\lambda} := \mathfrak{d}^{+}_{5(1),\lambda}$. Restrictions on λ follow from non-vanishing of $a_1 = \lambda(1+\lambda)$, $a_2 = (2+\lambda)^2(1+2\lambda)^2$ and $a_3 = 2(1+\lambda)(3+2\lambda)(2+3\lambda)$. For the matrix B_2 we may assume that $a_3^4 = 0$, so that we have the (2,3)-trivial algebra $\mathfrak{d}_5$. The algebra $\mathfrak{d}_{5(2),\lambda}$ corresponds to the matrix B_3 with $a_3^4 = 0$. The following determinants $a_1 = \lambda$, $a_2 = 9(1+\lambda)^2$, and $a_3 = 4(3+\lambda)^2$ must be non-zero. For B_3 with $a_3^4 \neq 0$ we obtain the algebra $\mathfrak{d}^{\pm}_{5(2)}$; this requires a rescaling $e_i \to |a_3^4|^{1/2} e_i$, for $i = 1,2$, $e_4 \to |a_3^4| e_4$. From B_4 we obtain $\mathfrak{d}'_{5,\lambda(2)}$ when $a_3^4 = 0$. The requirement that $a_1 = \lambda_2(1+\lambda_1^2)$, $a_2 = (1+9\lambda_1^2)(1+(\lambda_1+\lambda_2)^2)$, $a_3 = 4\lambda_1(1+(3\lambda_1+\lambda_2)^2)$ are non-zero enforces restrictions on the λ_i's. When $a_3^4 \neq 0$ we find, after appropriate rescaling, that B_4 corresponds to the family $\mathfrak{d}'^{\pm}_{5,\lambda}$. The determinants $a_1 = 2\lambda(1+\lambda^2)$, $a_2 = (1+9\lambda^2)^2$ and $a_3 = 4\lambda(1+25\lambda^2)$ must be non-zero. For the matrix B_5 we must have $a_3^4 = 0$ and so we get the family $\mathfrak{d}_{5(3),\lambda}$. The allowed values for λ are deduced from the determinants $a_1 = \lambda$, $a_2 = 2(1+\lambda)(1+2\lambda)(2+\lambda)$, $a_3 = 4(1+\lambda)^2(3+\lambda)$ being non-zero.

Finally, for $\mathfrak{k} \cong (0^2,21,31)$ we have $H^1(\mathfrak{k}) = \langle e^1,e^2\rangle$, $H^2(\mathfrak{k}) \cong \langle e^{14},e^{23}\rangle$, $H^3(\mathfrak{k}) \cong \langle e^{134},e^{234}\rangle$. As above, write $A(e_i) = a_i^k e_k$. Considering the relations $0 = \mathrm{ad}_A[e_2,e_3] = [\mathrm{ad}_A(e_2),e_3] + [e_2,\mathrm{ad}_A(e_3)]$, $\mathrm{ad}_A(e_3) = [\mathrm{ad}_A(e_1),e_2] + [e_1,\mathrm{ad}_A(e_2)]$ and $\mathrm{ad}_A(e_4) = [\mathrm{ad}_A(e_1),e_3] + [e_1,\mathrm{ad}_A(e_3)]$, one finds $a_2^1 = a_3^1 = a_4^1 = a_3^2 = a_4^2 = a_4^3 = 0$, $a_2^3 = a_3^4$ and $a_4^4 = a_1^1 + a_3^3$, $a_3^3 = a_1^1 + a_2^2$. After the transformation $A \to A - a_2^3 e_1 + a_1^3 e_2 + a_1^4 e_3$ we may assume that ad_A takes the form $\mathrm{diag}(p, q, p+q, 2p+q) + A'$, where A' only has non-zero entries $a'^2_1 = a_1^2$ and $a'^4_2 = a_2^4$, below the diagonal. We then obtain $\mathfrak{p}_{5,\lambda}$ and $\mathfrak{p}_5$ as follows. As $\mathfrak{k} = \mathfrak{g}'$ one has $p \neq 0$ and we may to rescale ad_A by $1/p$. If $q \neq p$ we may consider the transformation $e_1 \to e_1 + a_1^2 e_2/(p-q)$. After appropriate transformations $e_1 \to e_1 + ae_4$, $e_2 \to e_2 + be_4$, we obtain the algebra $\mathfrak{p}_{5,\lambda}$ with $\lambda = q/p$. For this family we have determinants $a_1 = \lambda$, $a_2 = (1+2\lambda)(3+\lambda)$ and $a_3 = 6(1+\lambda)(2+\lambda)$, so that $a_i \neq 0$ enforces λ to be as specified in Table 5.2. Consider now $q = p$ and note that we may assume $a_1^2 \neq 0$, since otherwise we end with $\mathfrak{p}_{5,\lambda}$. Making a change $e_i \to a_1^2 e_i$ for $i = 2, 3, 4$ and then transforming $e_2 \to e_2 + ce_4$, we get the algebra $\mathfrak{p}_5$.

This concludes the proof of Proposition 5.1.

Unimodular The lists of $(2,3)$-trivial algebras in dimensions up to and including five reveal that algebraic properties of this class are not fully reflected in low-dimensional examples. A Lie algebra $\mathfrak{g}$ is called *unimodular* if the homomorphism $\chi\colon \mathfrak{g} \to \mathbb{R}$ given by $\chi(x) = \mathrm{tr}(\mathrm{ad}(x))$ has trivial image. By direct inspection we observe that the $(2,3)$-trivial Lie algebras of dimensions two, three and four are not unimodular. On the other hand there are infinitely many five-dimensional algebras with this property:

Corollary 5.2 *The unimodular $(2,3)$-trivial Lie algebras of dimension up to and including five are* $\mathbb{R}$, $\mathfrak{r}_{5(1),-1/3}$, $\mathfrak{r}_{5,\lambda,-(1+\lambda)/2}$, $\mathfrak{r}_{5,\lambda,\mu,-(1+\lambda+\mu)}$, $\mathfrak{r}'_{5,\lambda,-\lambda}$, $\mathfrak{r}''_{5,\lambda,-\lambda,\mu}$, $\mathfrak{r}'_{5,\lambda,\mu,-(\lambda+\mu)/2}$, $\mathfrak{d}_{5(2),-4}$, $\mathfrak{d}_{5,\lambda,-2(1+\lambda)}$, $\mathfrak{d}_{5(3),-3/2}$, $\mathfrak{d}'_{5,\lambda,-4\lambda}$ *and* $\mathfrak{p}_{5,-4/3}$, *where parameters satisfy the conditions in Table 5.2.*

Higher dimensions The quest for higher-dimensional examples is easily met. Indeed, one may construct infinite families of $(2,3)$-trivial Lie algebras following the methods invoked in the proof of Proposition 5.1. In fact all the families appearing in dimension five have higher dimensional generalisations. Let us show how to obtain the following examples:

- $\mathfrak{r}_n$: $(0, 21{+}31, 31{+}41, \dots, (n{-}1)1{+}n1, n1)$,
- $\mathfrak{r}_{n(k-1),\lambda}$: $(0, 21{+}31, \dots, (k{-}1)1 + k1, k1, \lambda.(k{+}1)1 + (k{+}2)1, \dots, \lambda.(n{-}1)1 + n1, \lambda.n1)$, with $k > 2$ and $\lambda \neq 0, -1, -2, -1/2$,
- $\mathfrak{r}_{n,\lambda(k)}$: $(0, 21, \lambda_1.31, \dots, \lambda_{k-1}.(k{+}1)1, \lambda_k.(k{+}2)1 + (k{+}3)1, \dots, \lambda_k.(n{-}1)1 + n1, \lambda_k.n1)$, with $n > k+2$ and non-zero λ_i, $1{+}\lambda_i$, $\lambda_i{+}\lambda_j$, $1{+}2\lambda_k$, $\lambda_i{+}2\lambda_k$, $1{+}\lambda_i{+}\lambda_j$ $(i < j)$ and $\lambda_i{+}\lambda_j{+}\lambda_\ell$ $(i < j < \ell)$,
- $\mathfrak{d}_{n,\lambda(n-3)}$: $(0, 21, \lambda_1.31, \dots, \lambda_{n-3}.(n{-}1)1, (1{+}\lambda_1).n1{+}32)$, with $\lambda_i \neq 0, -1$ for all i, $\lambda_1 \neq -2, -1/2, -\lambda_i, -1/2(1+\lambda_i), -2{-}\lambda_i, -\lambda_i{-}\lambda_j$ for $1 < i$, $1 < i < j$ and non-zero $\lambda_i + \lambda_j$, $1{+}\lambda_i + \lambda_j$ $(1 < i < j)$, $\lambda_i{+}\lambda_j{+}\lambda_k$ $(1 < i < j < k)$.

The members of the $\mathfrak{r}$-series have $\mathfrak{k} \cong \mathbb{R}^{n-1}$ and ad_A belongs to the list $J(n-1,1)$, $J(k-1,1) \oplus J(n-k,\lambda)$, $\mathrm{diag}(1, \lambda_1, \dots, \lambda_{k-1}) \oplus J(n-k-1, \lambda_k)$, where $J(m,a)$ is an $m \times m$-Jordan block with a on the diagonal and 1 immediately above the diagonal. The first matrix corresponds to $\mathfrak{r}_n$, the second corresponds to the families $\mathfrak{r}_{n(k-1),\lambda}$ and the third one gives $\mathfrak{r}_{n,\lambda(k)}$. For the latter two cases the requirement that A must act invertibly on cohomology enforces some restrictions on parameters. As A acts on $H^1(\mathbb{R}^{n-1}) \cong \mathbb{R}^{n-1}$ by a lower triangular matrix, these restrictions are easily determined: the sum of one, two or three diagonal elements must be non-zero.

The family $\mathfrak{d}_{n,\lambda(n-3)}$ has $\mathfrak{k} \cong (0^{n-2}, 21)$ and ad_A is $\mathrm{diag}(1, \lambda_1, \dots, \lambda_{n-3}, 1+\lambda_1)$. Now A acts diagonally on $\mathfrak{k}^*$, and restrictions on parameters may therefore be read off directly from the cohomology groups $H^1(\mathfrak{k}) \cong \mathfrak{k}^* \ominus \langle e^{n-1} \rangle$, $H^2(\mathfrak{k}) \cong \Lambda^2 \mathfrak{k}^* \ominus \langle e^{12}, e^{i(n-1)} \colon i > 2 \rangle$, $H^3(\mathfrak{k}) \cong \Lambda^3 \mathfrak{k}^* \ominus \langle e^{12i}, e^{jk(n-1)} \colon 2 < i < n-1, 2 < j < k \rangle$.

An alternative way of constructing infinite families of $(2,3)$-trivial algebras goes via positive gradings of infinite families:

- $\mathfrak{f}_n^1$: $(0, 21, 31, 2.41 + 32, 3.51 + 42, \dots, (n-2).n1 + (n-1)2)$,
- $\mathfrak{f}_n^2$: $(0, 21, 2.31, 3.41 + 32, 4.51 + 42, 5.61 + 52 + 43, \dots, (n-1).n1 + (n-1)2 + (n-2)3)$,
- $\mathfrak{f}_n^3$: $(0, 21, 31, 2.41 + 32, \dots, (n-3).(n-1)1 + (n-2)2, (n-2).n1 + (n-1)2 - (n-1)3 + (n-2)3 - \cdots)$.

Here $(\mathfrak{f}_n^1)' = (0^2, 21, \dots, (n-2)1)$ has positive grading $1^2 2 \cdots (n-2)$. The derived algebra $(\mathfrak{f}_n^2)' = (0^2, 21, 31, 41 + 32, \dots, (n-2)1 + (n-3)2)$ admits grading $12 \cdots (n-1)$ and $(\mathfrak{f}_n^3)' = (0^2, 21, 31, \dots, (n-3)1, (n-2)1 - (n-2)2 + (n-3)3 - \cdots - (-1)^k (k+1)k)$ with $n = 2k+1$ has positive grading $1^2 23 \cdots (n-2)$.

In conclusion, the above exposition shows that the class of $(2,3)$-trivial algebras is quite rich. Yet, because of Theorem 1.3, this class is much more accessible than the larger class of solvable algebras.

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