THE HOROFUNCTION COMPACTIFICATION OF TEICHMÜLLER METRIC

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Abstract. We show that the horofunction compactification of the Teichmüller space with the Teichmüller metric is homeomorphic to the Gardiner-Masur compactification.

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Contents

1. Introduction

Let $S = S_{g,n}$ be an oriented surface of genus g with n punctures, $3g-3+n>0$. The Teichmüller space $\mathcal{T}(S)$ of S is the space of complex structures (or complete hyperbolic structures) X on S up to equivalence. We say that X is equivalent to Y, denoted by $X \sim Y$, if there is a conformal map $f : X \to Y$ homotopic to the identity map on S.

The Teichmüller metric on $\mathcal{T}(S)$ is the metric defined by

$$d_T(X,Y) := \frac{1}{2} \inf_f \log K(f)$$

where $f : X \to Y$ is a quasi-conformal map homotopic to the identity map of S and

$$K(f) := \operatorname{ess\,sup}_{x \in X} K_x(f) \geq 1$$

is the quasi-conformal dilatation of f , where

$$K_x(f) := \frac{|f_z| + |f_{\bar{z}}|}{|f_z| - |f_{\bar{z}}|}$$

is the pointwise quasiconformal dilatation at the point $x \in X$ with local conformal coordinate z. Teichmüller's theorem states that, given any $X, Y \in \mathcal{T}(S)$, there exists a unique quasi-conformal map f, called the Teichmüller map, such that

$$d_T(X,Y) = \frac{1}{2} \log K(f).$$

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Moreover, $\mathcal{T}(S)$ is a complete geodesic metric space, each Teichmüller geodesic is given by $\{X_t\}$, where there is a holomorphic quadratic differential q and a t-family of Teichmüller maps $f_t : X \to X_t$, with Beltrami differentials $\mu(f_t) = \frac{e^{2t}-1}{e^{2t}+1} \frac{\bar{q}}{|q|}$.

The aim of this paper is to relate two different compactification of $\mathcal{T}(S)$: one is the horofunction compactification of $\mathcal{T}(S)$ with the Teichmüller metric, which is defined on a quite general classes of metric spaces, and the other is the Gardiner-Masur compactification of $\mathcal{T}(S)$, defined by extremal length functions.

Recall that for a proper geodesic metric space (M, d), the horofunction compactification is defined by Gromov [3] in the following way. Fix a base point $x_0 \in M$. For each $z \in M$ we assign the function $\Psi_z : M \to \mathbb{R}$,

$$\Psi_z(x) = d(x, z) - d(x_0, z). \tag{1}$$

Let $C(M)$ be the space of continuous functions on M endowed with the compact-open topology, the topology of uniformly convergence on bounded subsets of M. Then the map $\Psi : M \to C(M), z \to \Psi_z$ is an embedding from M to $C(M)$. The closure $\overline{\Psi(M)}$ of $\Psi(M) \subset C(M)$ is compact, which is called the horofunction compatification of (M, d). The horofunction boundary is defined to be

$$M(\infty) = \overline{\Psi(M)} - \Psi(M),$$

and its elements are called horofunctions.

For the Teichmüller space $(\mathcal{T}(S), d_T)$, where d_T is the Teichmüller metric, we will define its horofunction compactification and show that

Theorem 1.1. *The horofunction compactification of* $(\mathcal{T}(S), d_T)$ *is homeomorphic to the Gardiner-Masur compactificaiton of* $\mathcal{T}(S)$.

Let $\mathrm{Isom}(M, d)$ be the isometry group of the metric space (M, d). Then we can check that for any isometry $g \in \mathrm{Isom}(M, d)$, any point $x \in M$, any horofunction $\xi \in M(\infty)$, the action of g on $M(\infty)$ is given by

$$(g \cdot \xi)(x) = \xi(g^{-1}(x)) - \xi(g^{-1}(x_0)).$$

From this we know that the action of the isometry group $\mathrm{Isom}(M, d)$ of M extends continuously to a homeomorphism on the horofunction compactification. Note that the mapping class group acts isometrically on the Teichmüller space with the Teichmüller metric (In fact, a famous theorem of Royden shows that the isometry group of $\mathcal{T}(S)$ with the Teichmüller metric is equal to the mapping class group of S.) As a result, we have the following corollary, which was proved by Miyachi [10] in a different way.

Theorem 1.2. *The action of mapping class group on* $\mathcal{T}(S)$ *extends continuously to the Gardiner-Masur boundary.*

Recall that a geodesic ray in a metric space (M, d) is an embedding γ from the interval $[0, \infty)$ to M such that

$$d(\gamma(s), \gamma(t)) = t - s,$$

for all $s, t \in [0, \infty)$, with $s < t$.

A map $\gamma : [0, \infty) \to M$ is called an almost-geodesic ray if for each $\epsilon > 0$ there exists $M \geq 0$ such that

$$|d(\gamma(0), \gamma(s)) + d(\gamma(s), \gamma(t)) - t| < \epsilon$$

for all $s, t \in [0, \infty)$ with $M \leq s \leq t$.

Rieffel [13] proved that every almost-geodesic ray of (M, d) converges to a limit in $M(\infty)$. From this we have

Theorem 1.3. *Every Teichmüller (almost-)geodesic ray converges in the forward direction to a point in the Gardiner-Masur boundary.*

It is shown by Masur [8] that if a Teichmüller geodesic ray is uniquely ergodic or Strebel, then it converges to a point in the Thurston boundary. And there exists geodesic ray which does not have a limit in Thurston boudary, see Lenzhen [6]. As a result, it seems that the Gardiner-Masur boundary is more compatible with the Teichmüller metric, although much of the properties of the Gardiner-Masur boundary (such as topological dimension, geometric structure, etc.) are unknown.

A horofunction is called a Busemann point if there exists an almost-geodesic ray converging to it. Since the set of horofunctions of $(\mathcal{T}(S), d_T)$ is identified with the Gardiner-Masur boundary by Theorem 1.1, it is natural to ask the question whether every horofunction of $(\mathcal{T}(S), d_T)$ is a Busemann point, or equivalently,

Question 1.4. Can every point in the Gardiner-Masur boundary be an accumulation point of a Teichmüller almost-geodesic ray?

There is a result of Miyachi [10] that a projective class of rational measured lamination whose support consists of at least two simple closed geodesics can not be an accumulation point of any Teichmüller geodesic ray under the Gardiner-Masur compactification.

We have felt for some years that Teichmüller metric is natural for Gardiner-Masur compactification in some sense; while Thurston's Lipschitz metric is natural for Thurston compactification. Now we know that the horofunction compactification defined by Gromov build a bridge for these.

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2. Preliminaries

In fact, our proof of Theorem 1.1 is inspired by a recent result of C.Walsh [15] that Thuston's compactification $\mathcal{T}(S)^{Th}$ of $\mathcal{T}(S)$ is homeomorphic to the horofunction compactificaiton of $\mathcal{T}(S)$ with Thurston's Lipschitz metric. Moreover, for each $\mu \in \mathcal{PMF} = \partial\mathcal{T}(S)^{Th}$ (the Thurston boundary), the corresponding horofunction, which we denote by Ψ_μ^{Th}, is given by

$$\Psi_\mu^{Th}(X) = \log \sup_{\nu \in \mathcal{MF}} \frac{i(\mu,\nu)}{\ell_X(\nu)} - \log \sup_{\nu \in \mathcal{MF}} \frac{i(\mu,\nu)}{\ell_{X_0}(\nu)}, \tag{2}$$

where $X_0 \in \mathcal{T}(S)$ is a fixed base point and $i(\mu, \nu)$ is the intersection number of μ and ν.

To prove Theorem 1.1, we will construct the horofunctions of $\mathcal{T}(S)$ with the Teichmüller metric by replacing the the hyperbolic length $\ell_X(\mu)$ with the square root of the extremal length $\mathrm{Ext}_X(\mu)^{1/2}$ and the intersection number $i(\mu, \cdot)$ with a function $\mathcal{E}_\mu(\cdot)$ defined by Miyachi [10], and then follow the steps in the proof of Walsh [15].

We first recall some necessary definitions and then we will explain the Thurston's compactification and Gardiner-Masur compactification.

A nontrivial essential simple closed curves on S is a simple closed curve on S which is neither homotopic to a point on S nor homotopic to a puncture of S. Let $\mathcal{S}$ be the set of homotopy classes of nontrivial essential simple closed curves on S. Given a Riemann surface X and $\alpha \in \mathcal{S}$, the extremal length of α on X is defined by

$$\mathrm{Ext}_X(\alpha) = \sup_\sigma \frac{L_\sigma^2(\alpha)}{A(\sigma)},$$

where the supremum is over all conformal metrics $\sigma(z)|dz|$ on X, where

$$L_\sigma(\alpha) = \inf_{\alpha' \sim \alpha} \int_{\alpha'} \sigma(z)|dz|,$$

$\alpha' \sim \alpha$ means that α' is homotopic to α, and

$$A(\sigma) = \int_X \sigma^2(z)|dz|^2.$$

The following important formula is due to Kerckhoff [5].

Theorem 2.1. *Let X, Y be any two points of $\mathcal{T}(S)$. Then*

$$d_T(X,Y) = \frac{1}{2} \log \sup_{\alpha \in \mathcal{S}} \frac{\mathrm{Ext}_X(\alpha)}{\mathrm{Ext}_Y(\alpha)}.$$

A measured foliation on S is a foliation (with a finite number of singularities) with an invariant transverse measure. The singularities which are allowed are topologically the same as those that occur at $z = 0$ in the line field $z^{p-2}dz^2$. Two measured laminations μ and μ' are equivalent if, for all simple closed curves γ, the geometric intersection number $i(\gamma, \mu) = i(\gamma, \mu')$. Denote $\mathcal{MF}$ to be the space of equivalent classes of measured foliations. There is a special class of measured foliations that have the property that the complement of the critical leaves is homeomorphic to a cylinder, the leaves of the foliation on the cylinder are all freely homotopic to a simple closed curves γ. Such a foliation is completely determined as a point in $\mathcal{MF}$ by the height r of the cylinder and the isotopy class of γ. Denote such a foliation by (γ, r). Thurston [14] showed that $\mathcal{MF}$ is homeomorphic to a $6g-6$ dimensional ball and there is an embedding $\mathcal{S} \times \mathrm{R}_+ \to \mathcal{MF}$ whose image is dense in $\mathcal{MF}$. The density of (weighted) simple closed curves in $\mathcal{MF}$ allows us to replace the right hand side of Kerckhoff's formula by the supremum taken over all $\mu \in \mathcal{MF}$.

Thurston introduced a compactification $\overline{\mathcal{T}(S)}^{Th}$ of $\mathcal{T}(S)$ such that the action of mapping class group (isotopy classes of orientation preserving homeomorphisms) on $\mathcal{T}(S)$ extended continuously to the boundary $\partial\overline{\mathcal{T}(S)}^{Th}$. We recall some of the fundamental results of Thurston as described in [1]. Again denote $\mathcal{S}$ the homotopy classes of essential simple closed curves with the discrete topology. Let $P(\mathbb{R}_+^{\mathcal{S}})$ be the projective space of $\mathbb{R}_+^{\mathcal{S}}$ and $\pi : \mathbb{R}_+^{\mathcal{S}} \to P(\mathbb{R}_+^{\mathcal{S}})$ be the natural projection. We endow $P(\mathbb{R}_+^{\mathcal{S}})$ with the product topology. There is a mapping $\tilde{\psi}$ from $\mathcal{T}(S)$ into $\mathbb{R}_+^{\mathcal{S}}$ which sends X to the function $\tilde{\psi}(X)$ defined by

$$\tilde{\psi}(X)(\alpha) = \ell_X(\alpha)$$

for all $\alpha \in \mathcal{S}$, where $\ell_X(\alpha)$ is the hyperbolic length of α on X. Thurston showed that $\psi = \pi \circ \tilde{\psi} : \mathcal{T}(S) \to P(\mathbb{R}_+^{\mathcal{S}})$ is an embedding.

There is also an embedding of $\mathcal{PMF}$ (the space of projective measured foliations) into $P(\mathbb{R}_+^{\mathcal{S}})$ which sends each projective class of measured foliation $[\mu]$ to the projective class of function

$$\gamma \to i(\mu, \gamma)$$

where $i(\mu; \gamma)$ is the geometric intersection number of measured foliations with homotopy classes of simple closed curves. Thurston proved that with these embeddings $\mathcal{PMF}$ is the boundary of $\mathcal{T}(S)$ and the the closure $\overline{\psi(\mathcal{T}(S))}$ of the image $\psi(\mathcal{T}(S))$ in $P(\mathbb{R}_+^{\mathcal{S}})$ is homeomorphic to the real $(6g-6+2n)$ dimensional closed ball. We denote $\overline{\mathcal{T}(S)}^{Th} = \overline{\psi(\mathcal{T}(S))}$ and call it the Thurston compactification of $\mathcal{T}(S)$. The complement $\partial_{Th}\mathcal{T}(S) = \overline{\psi(\mathcal{T}(S))} - \psi(\mathcal{T}(S))$ is called the Thurston boundary of $\mathcal{T}(S)$. We always identify $\partial_{Th}\mathcal{T}(S)$ with $\mathcal{PMF}$ without referring to the embedding.

Replacing the hyperbolic length functions $\ell_X(\alpha)$ by the square root of extremal length functions, Gardiner and Masur [2] defined the Gardiner-Masur compactification of $\mathcal{T}(S)$ and the corresponding boundary is called Gardiner-Masur boundary. Now we give the details. Define a mapping $\tilde{\phi}$ from $\mathcal{T}(S)$ into $\mathbb{R}_+^{\mathcal{S}}$ by

$$\tilde{\phi}(X)(\alpha) = \mathrm{Ext}_X(\alpha)^{1/2}$$

for all $\alpha \in \mathcal{S}$. Let $P(\mathbb{R}_+^{\mathcal{S}})$ be the projective space of $\mathbb{R}_+^{\mathcal{S}}$ and $\pi : \mathbb{R}_+^{\mathcal{S}} \to P(\mathbb{R}_+^{\mathcal{S}})$ be the natural projection. Gardiner and Masur [2] showed that $\phi = \pi \circ \tilde{\phi} : \mathcal{T}(S) \to P(\mathbb{R}_+^{\mathcal{S}})$ is an embedding and the closure $\overline{\phi(\mathcal{T}(S))}$ of the image $\phi(\mathcal{T}(S))$ in $P(\mathbb{R}_+^{\mathcal{S}})$ is compact. Denote $\overline{\mathcal{T}(S)}^{GM} = \overline{\phi(\mathcal{T}(S))}$. It's the Gardiner-Masur compactification of $\mathcal{T}(S)$. The complement $\partial_{GM}\mathcal{T}(S) = \overline{\phi(\mathcal{T}(S))} - \phi(\mathcal{T}(S))$ is called the Gardiner-Masur boundary of $\mathcal{T}(S)$.

Gardiner and Masur [2] also proved that $\partial_{Th}\mathcal{T}(S)$ is a proper subset of $\partial_{GM}\mathcal{T}(S)$. For further investigations about Thurston boundary, Gardiner-Masur boudary and their relations with the Teichmüller geometry, one refers to Gardiner and Masur [8] and recent works of Miyachi [10], [11].

3. Proof of Theorem 1.1

In this section, when saying that a sequence $P_n \in \overline{\mathcal{T}(S)}^{GM}$ converges to $P \in \overline{\mathcal{T}(S)}^{GM}$, we always refer to the convergence in the sense of the Gardiner-Masur compactification.

Fix a point $X_0 \in \mathcal{T}(S)$ as the base-point of the horofunction compactification. For any $X \in \mathcal{T}(S)$, denote by K_X be the dilatation of the Teichmüller map between X_0 and X. Note that $d_T(X_0, X) = \frac{1}{2}\log K_X$. Consider the following function defined on $\mathcal{MF}$.

$$\mathcal{E}_X(\mu) = \frac{\mathrm{Ext}_X(\mu)^{1/2}}{K_X^{1/2}}, \mu \in \mathcal{MF}. \tag{3}$$

Such functions can be continuously extended to the Gardiner-Masur boundary [10]. They play the role in analogue with the intersection numbers $i(\mu, \cdot)$ in Thurston's compactification. See the following lemma of Miyachi [10].

Lemma 3.1. *For any $P \in \partial_{GM}\mathcal{T}(S)$, there is a non-negative continuous function $\mathcal{E}_P(\mu)$ defined on $\mathcal{MF}$, such that*

(i) $\mathcal{E}_P(t\mu) = t\mathcal{E}_P(\mu)$ for $t > 0$ and $\mu \in \mathcal{MF}$, and
(ii) the assignment $\mathcal{S} \ni \alpha \mapsto \mathcal{E}_P(\alpha)$ represents P as a point of $\partial_{GM}\mathcal{T}(S)$.
(iii) Furthermore, the function $\mathcal{E}_P(\cdot)$ is unique up to multiplication by a positive constant in the following sense: for any sequence $\{X_n\} \subset \mathcal{T}(S)$ converging to $P \in \overline{\mathcal{T}(S)}^{GM}$, there exist a subsequence $\{X_{n_j}\}$ such that $\mathcal{E}_{X_{n_j}}(\cdot)$ converges to a positive multiple of $\mathcal{E}_P(\cdot)$ uniformly on any compact subsets of $\mathcal{MF}$. Especially,

$$\lim_{n\to\infty} \frac{\mathrm{Ext}_{X_n}(\mu)^{1/2}}{\mathrm{Ext}_{X_n}(\nu)^{1/2}} = \frac{\mathcal{E}_P(\mu)}{\mathcal{E}_P(\nu)}$$

for all $\mu, \nu \in \mathcal{MF}$ with $\mathcal{E}_P(\nu) \neq 0$.

For any $P \in \overline{\mathcal{T}(S)}^{GM}$, we define

$$\mathcal{Q}(P) = \sup_{\nu \in \mathcal{MF}} \frac{\mathcal{E}_P(\nu)}{\mathrm{Ext}_{X_0}(\nu)^{1/2}}$$

and

$$\mathcal{L}_P(\cdot) : \mathcal{MF} \to \mathbb{R}^+ : \mu \to \frac{\mathcal{E}_P(\mu)}{\mathcal{Q}(P)}.$$

$\mathcal{L}_P(\cdot)$ is well-defined (multiplying the function $\mathcal{E}_P(\cdot)$ by a positive constant does't change the value of $\mathcal{L}_P(\cdot)$). We may consider $\mathcal{L}_P(\mu)$ as a function of the product space $\overline{\mathcal{T}(S)}^{GM} \times \mathcal{MF}$.

Lemma 3.2. *A sequence $\{P_n\}$ in $\overline{\mathcal{T}(S)}^{GM}$ converges to a point $P \in \overline{\mathcal{T}(S)}^{GM}$ if and only if $\mathcal{L}_{P_n}$ converges to $\mathcal{L}_P$ uniformly on compact set of $\mathcal{MF}$.*

Proof. Note that if $P \in \mathcal{T}(S)$, then

$$\mathcal{Q}(P) = \sup_{\nu \in \mathcal{MF}} \frac{\mathrm{Ext}_P^{1/2}(\nu)}{K_P^{1/2} \mathrm{Ext}_{X_0}(\nu)^{1/2}} = 1.$$

As a result,

$$\mathcal{L}_P(\mu) = \frac{\mathrm{Ext}_P^{1/2}(\mu)}{K_P^{1/2}}. \tag{4}$$

Then the lemma follows directly from the continuity of extremal length function.

Now we assume that $P_n \in \mathcal{T}(S), n = 1, 2, \cdots, P \in \partial_{GM}\mathcal{T}(S)$ and P_n converges in the Gardiner-Masur compactification to P. Let $\{P_{n_j}\}$ be any subsequence of $\{P_n\}$, such that for some $t_0 > 0$, $\mathcal{E}_{P_{n_j}}(\cdot)$ converges to $t_0\mathcal{E}_P(\cdot)$ uniformly on any compact subsets of $\mathcal{MF}$. By (iii) of Lemma 3.1, $\mathcal{Q}(P_{n_j})$ converges to $t_0\mathcal{Q}(P)$. Therefore,

$$\mathcal{L}_{P_{n_j}}(\mu) = \frac{t_0 \mathcal{E}_{P_{n_j}}(\mu)}{t_0 \mathcal{Q}(P_{n_j})} = \frac{\mathcal{E}_{P_{n_j}}(\mu)}{\mathcal{Q}(P_{n_j})},$$

which converges to $\mathcal{L}_P$ uniformly on compact set of $\mathcal{MF}$. Since the limit is independent of the choice of subsequence $\{P_{n_j}\}$, we know that $\mathcal{L}_{P_n}$ converges to $\mathcal{L}_P$ uniformly on compact set of $\mathcal{MF}$.

In general, assume that $P_n \in \overline{\mathcal{T}(S)}^{GM}$ and P_n converges in the Gardiner-Masur compactification to $P \in \partial_{GM}\overline{\mathcal{T}(S)}$. It suffices to show that for any $\epsilon > 0$, there exists an N, such that for any $n > N$, $|\mathcal{L}_{P_n}(\cdot) - \mathcal{L}_P(\cdot)| < \epsilon$ uniformly on compact set of $\mathcal{MF}$. By the above argument, for each P_n, there exists a point $P'_n \in \mathcal{T}(S)$ such that $|\mathcal{L}_{P_n}(\cdot) - \mathcal{L}_{P'_n}(\cdot)| < \frac{\epsilon}{2^n}$ uniformly on compact set of $\mathcal{MF}$. Moreover, the sequence $\{P'_n\}$ is also converges to P. Then there is an sufficiently large N, such that for any $n > N$, we have $|\mathcal{L}_{P'_n}(\cdot) - \mathcal{L}_P(\cdot)| < \frac{\epsilon}{2}$ uniformly on compact set of $\mathcal{MF}$. It follows that $|\mathcal{L}_{P_n}(\cdot) - \mathcal{L}_P(\cdot)| < \epsilon$ uniformly on compact set of $\mathcal{MF}$.

On the other hand, for $P_n \in \overline{\mathcal{T}(S)}^{GM}, n = 1, 2, \cdots, P \in \partial_{GM}\overline{\mathcal{T}(S)}$, if $\mathcal{L}_{P_n}$ converges to $\mathcal{L}_P$ uniformly on compact set of $\mathcal{MF}$, we want to show that P_n converges to P. Let $\{Y_n\}$ be a subsequence of $\{P_n\}$ converging in $\partial_{GM}\overline{\mathcal{T}(S)}$ to a point Y. As the above discussion, we have that $\mathcal{L}_{Y_n}$ converges to $\mathcal{L}_Y$ uniformly on any compact set of $\mathcal{MF}$. Combining this with our assumption, we get that $\mathcal{L}_Y = \mathcal{L}_P$; that is, for any $\mu \in \mathcal{MF}$,

$$\frac{\mathcal{E}_Y(\mu)}{\mathcal{Q}(Y)} = \frac{\mathcal{E}_P(\mu)}{\mathcal{Q}(P)},$$

or equivalently,

$$\mathcal{E}_Y(\mu) = \frac{\sup_{\nu \in \mathcal{MF}} \frac{\mathcal{E}_P(\nu)}{\mathrm{Ext}_{X_0}(\nu)^{1/2}}}{\sup_{\nu \in \mathcal{MF}} \frac{\mathcal{E}_Y(\nu)}{\mathrm{Ext}_{X_0}(\nu)^{1/2}}} \mathcal{E}_P(\mu).$$

Therefore, $\mathcal{E}_Y$ equals to $\mathcal{E}_P$ up to a positive constant. We know that they represent the same point in $\overline{\mathcal{T}(S)}^{GM}$, by (ii) of Lemma 3.1. As we have show that any convergent subsequence of $\{P_n\}$ converges to P, it follows that P_n converges to P. □

For each $P \in \overline{\mathcal{T}(S)}^{GM}$, we define the map

$$\begin{aligned}\Psi_P(X) &= \log \sup_{\mu\in\mathcal{MF}} \frac{\mathcal{E}_P(\mu)}{\mathrm{Ext}_X(\mu)^{1/2}} - \log \sup_{\mu\in\mathcal{MF}} \frac{\mathcal{E}_P(\mu)}{\mathrm{Ext}_{X_0}(\mu)^{1/2}} \\ &= \log \sup_{\mu\in\mathcal{MF}} \frac{\mathcal{L}_P(\mu)}{\mathrm{Ext}_X(\mu)^{1/2}}\end{aligned}$$

for all $X \in \mathcal{T}(S)$.

Note that if $P \in \mathcal{T}(S)$, by (3) and Kerckhoff's formula,

$$\begin{aligned}\Psi_P(X) &= \log \sup_{\mu\in\mathcal{MF}} \frac{\mathrm{Ext}_P(\mu)^{1/2}}{K_P^{1/2}\mathrm{Ext}_X(\mu)^{1/2}} - \log \sup_{\mu\in\mathcal{MF}} \frac{\mathrm{Ext}_P(\mu)^{1/2}}{K_P^{1/2}\mathrm{Ext}_{X_0}(\mu)^{1/2}} \\ &= \log \sup_{\mu\in\mathcal{MF}} \frac{\mathrm{Ext}_P(\mu)^{1/2}}{\mathrm{Ext}_X(\mu)^{1/2}} - \log \sup_{\mu\in\mathcal{MF}} \frac{\mathrm{Ext}_P(\mu)^{1/2}}{\mathrm{Ext}_X(\mu)^{1/2}} \\ &= d_T(X,P) - d_T(X_0,P).\end{aligned}$$

In this case, the function Ψ_P coincides with the equation (1) for the case the metric space is $(\mathcal{T}(S), d_T)$.

We always identify $\mathcal{PMF}$ with the cross-section $\{\mu \in \mathcal{MF} | \mathrm{Ext}_{X_0}(\mu) = 1\}$ and write Kerckhoff's formula as

$$d_T(X,Y) = \frac{1}{2}\log \sup_{\mu\in\mathcal{PMF}} \frac{\mathrm{Ext}_Y(\mu)}{\mathrm{Ext}_X(\mu)}.$$

Since $\mathcal{PMF}$ is compact, the supremum is attained by some $\mu \in \mathcal{PMF}$.

Lemma 3.3. *Let $\{X_n\}$ be a sequence of points in $\mathcal{T}(S)$ converges to a point P in the Gardiner-Masur boundary. Let Y be a point in $\mathcal{T}(S)$. Let $\{\mu_n\}$ be a sequence in $\mathcal{PMF}$ such that*

$$d_T(X_n,Y) = \frac{1}{2}\log \frac{\mathrm{Ext}_Y(\mu_n)}{\mathrm{Ext}_{X_n}(\mu_n)}.$$

Then any limit point $\mu_\infty \in \mathcal{PMF}$ of a convergent subsequence of the sequence $\{\mu_n\}$ satisfies $\mathcal{E}_P(\mu_\infty) = 0$.

Proof. Since $\{X_n\} \subset \mathcal{T}(S)$ converges to P, there exist a subsequence, still denoted by $\{X_n\}$, such that $\mathcal{E}_{X_n}(\cdot)$ converges to $t_0\mathcal{E}_P(\cdot)$ (with some constant $t_0 > 0$) uniformly on any compact subsets of $\mathcal{MF}$. It follows that for any limit point $\mu_\infty \in \mathcal{PMF}$ of convergent subsequence of the sequence $\{\mu_n\}$, the function $\mathrm{Ext}_Y(\mu_n)^{1/2}/\mathcal{E}_{X_n}(\mu_n)$ converges to $\mathrm{Ext}_Y(\mu_\infty)^{1/2}/t_0\mathcal{E}_P(\mu_\infty)$. On the other hand,

$$\begin{aligned}\frac{\mathrm{Ext}_Y(\mu_n)^{1/2}}{\mathcal{E}_{X_n}(\mu_n)} &= \frac{\mathrm{Ext}_Y(\mu_n)^{1/2}}{\mathrm{Ext}_{X_n}(\mu_n)^{1/2}} K_{X_n}^{1/2} \\ &= e^{d_T(X_n,Y)+d_T(X_n,X_0)}\end{aligned}$$

which tends to ∞ as X_n tends to the boundary. As a result, $\mathcal{E}_P(\mu_\infty) = 0$. □

Lemma 3.4. *Let $\{X_n\}$ be a sequence of points in $\mathcal{T}(S)$ converges to a point P in the Gardiner-Masur boundary. Let Y be a point in $\mathcal{T}(S)$. Let $\{\mu_n\}$ be a sequence in $\mathcal{PMF}$ such that*

$$d_T(X_n,Y) = \frac{1}{2}\log \frac{\mathrm{Ext}_Y(\mu_n)}{\mathrm{Ext}_{X_n}(\mu_n)}.$$

For any $\nu \in \mathcal{MF}$, if $\mathcal{E}_P(\nu) = 0$, then any limit point $\mu_\infty \in \mathcal{PMF}$ of a convergent subsequence of the sequence $\{\mu_n\}$ satisfies $i(\nu, \mu_\infty) = 0$.

Proof. Recall an inequality of Minsky [2],

$$i(\nu,\mu_n) \leq \mathrm{Ext}_{X_n}(\nu)^{1/2}\mathrm{Ext}_{X_n}(\mu_n)^{1/2}.$$

Note that

$$\begin{aligned}
\mathrm{Ext}_{X_n}(\nu)^{1/2}\mathrm{Ext}_{X_n}(\mu_n)^{1/2} &= K_{X_n}^{1/2}\frac{\mathrm{Ext}_{X_n}(\nu)^{1/2}}{K_{X_n}^{1/2}}\frac{\mathrm{Ext}_{X_n}(\mu_n)^{1/2}}{\mathrm{Ext}_Y(\mu_n)^{1/2}}\mathrm{Ext}_Y(\mu_n)^{1/2} \\
&= e^{d_T(X_0,X_n)}\mathcal{E}_{X_n}(\nu)\frac{1}{e^{d_T(Y,X_n)}}\mathrm{Ext}_Y(\mu_n)^{1/2} \\
&\leq \sup_{\mu\in\mathcal{PMF}}\{\mathrm{Ext}_Y(\mu)^{1/2}\}e^{d_T(X_0,X_n)-d_T(Y,X_n)}\mathcal{E}_{X_n}(\nu) \\
&\leq Ce^{d_T(X_0,Y)}\mathcal{E}_{X_n}(\nu),
\end{aligned}$$

where C is a constant depending on Y. By Lemma 3.1, we have $i(\nu,\mu_\infty) \leq Ct_0\mathcal{E}_P(\nu)$ for some $t_0 > 0$. This implies that if $\mathcal{E}_P(\nu) = 0$, then $i(\nu,\mu_\infty) = 0$. □

A measured foliation μ is minimal if no curves in S can be realized by leaves of μ. Equivalently, after Whitehead moves, the foliation has only dense leaves on S. Two measured foliation μ and ν are topologically equivalent if after Whitehead moves, the leaf structure are isotopic. μ is called uniquely ergodic if it is minimal and any topologically equivalent foliation is measured equivalent to a multiple of μ. The following lemma is proved in Masur [8].

Lemma 3.5. *Assume that $\mu \in \mathcal{MF}$ is uniquely ergodic. If $\nu \in \mathcal{MF}$ satisfies $i(\mu,\nu) = 0$, then $\nu = c\mu$ for some constant $c \geq 0$.*

Moreover, it follows from Thurston theory that minimal uniquely ergodic measured foliations are dense in $\mathcal{MF}$ (see Masur [7] or Rees [12] for the proof).

Now we use the above lemmas to study the properties of the map Ψ. We will show that it is injective and continuous, then Theorem 1.1 follows from a topological argument.

Proposition 3.6. *The map $\Psi : \overline{\mathcal{T}(S)}^{GM} \to C(\mathcal{T}(S)) : P \to \Psi_P$ is injective.*

Proof. For any two distinct points $P, Q \in \overline{\mathcal{T}(S)}^{GM}$, it suffices to find a point $X \in \mathcal{T}(S)$ such that $\Psi_P(X) \neq \Psi_Q(X)$.

By Lemma 3.2 we know that $\mathcal{L}_P$ and $\mathcal{L}_Q$ are distinct. Without loss of generality, we assume that $\mathcal{L}_P(\mu) < \mathcal{L}_Q(\mu)$ for some $\mu \in \mathcal{PMF}$. By Lemma 3.2 again, there exist some $\epsilon > 0$ and a point $Y \in \mathcal{T}(S)$ which is sufficiently close to P in the Gardiner-Masur compactification, such that

$$\mathcal{L}_P(\mu) < (1+\epsilon)\mathcal{L}_Y(\mu) < \mathcal{L}_Q(\mu). \tag{5}$$

Since both $\mathcal{L}_P$ and $\mathcal{L}_Q$ are continuous, there exists a neighborhood $\mathcal{N}$ of μ in $\mathcal{PMF}$ such that equation (5) holds for all $\nu \in \mathcal{N}$. Moreover, we can also assume that

$$\mathcal{L}_P(\mu) < (1+\epsilon)\mathcal{L}_Y(\mu)$$

for any $\mu \in \mathcal{PMF}$. To obtain this, note that $\mathcal{PMF}$ is a compact set of $\mathcal{MF}$, thus we can choose the point Y to be sufficiently close to P such that $\mathcal{L}_Y(\cdot)$ is close to $\mathcal{L}_P(\cdot)$ uniformly on $\mathcal{PMF}$.

From [2] we know that the Gardiner-Masur boundary contains the Thurston boundary and the Thurston boundary can be identified to $\mathcal{PMF}$. From this any element of $\mathcal{PMF}$ can be viewed as an element of the Gardiner-Masur boundary. Since the set of uniquely ergodic measured laminations is dense in $\mathcal{PMF}$, we can choose a uniquely ergodic measured lamination $\mu_0 \in \mathcal{N}$. Consider μ_0 as a point in the Gardiner-Masur boundary and let $\{X_n\}$ be a sequences of points in $\mathcal{T}(S)$ converging to μ_0.

Let μ_n be the sequence in $\mathcal{PMF}$ such that

$$d_T(X_n, Y) = \frac{1}{2}\log\frac{\mathrm{Ext}_Y(\mu_n)}{\mathrm{Ext}_{X_n}(\mu_n)}.$$

Let μ_∞ be the limit point of a convergent subsequence of the sequence $\{\mu_n\}$. It follows from Lemma 3.3 that $\mathcal{E}_{\mu_0}(\mu_\infty) = 0$. By Lemma 3.4, $i(\mu_0, \mu_\infty) = 0$. Since μ_0 is uniquely ergodic, it follows from Lemma 3.5 that $\mu_\infty = \mu_0$. As a result, there is a sufficiently large number $N > 0$, such that for each $n > N$, the measured foliation μ_n lies in the neighborhood $\mathcal{N}$ of μ_0.

Let X be any of X_{n_0} with $n_0 > N$, then the supremum of $\mathcal{L}_Y(\cdot)/\mathrm{Ext}_X^{1/2}(\cdot)$ is attained in the set $\mathcal{N}$. To see this, from (4) we know that

$$\frac{\mathcal{L}_Y(\cdot)}{\mathrm{Ext}_X^{1/2}(\cdot)} = \frac{\mathrm{Ext}_Y^{1/2}(\cdot)}{\mathrm{Ext}_X^{1/2}(\cdot)K_Y^{1/2}}.$$

Therefore,

$$\begin{aligned}
\sup_{\mu\in\mathcal{N}}\frac{\mathcal{L}_Y(\mu)}{\mathrm{Ext}_X^{1/2}(\mu)} &\leq \sup_{\mu\in\mathcal{PMF}}\frac{\mathcal{L}_Y(\mu)}{\mathrm{Ext}_X^{1/2}(\mu)} \\
&= \frac{1}{K_Y^{1/2}}\sup_{\mu\in\mathcal{PMF}}\frac{\mathrm{Ext}_Y^{1/2}(\mu)}{\mathrm{Ext}_X^{1/2}(\mu)} \\
&= \frac{1}{K_Y^{1/2}}\frac{\mathrm{Ext}_Y^{1/2}(\mu_{n_0})}{\mathrm{Ext}_X^{1/2}(\mu_{n_0})} \\
&= \frac{\mathcal{L}_Y(\mu_{n_0})}{\mathrm{Ext}_X^{1/2}(\mu_{n_0})} \\
&\leq \sup_{\mu\in\mathcal{N}}\frac{\mathcal{L}_Y(\mu)}{\mathrm{Ext}_X^{1/2}(\mu)}
\end{aligned}$$

where μ_{n_0} is the measured lamination realizing $\sup_{\mu\in\mathcal{PMF}}\frac{\mathrm{Ext}_Y^{1/2}(\mu)}{\mathrm{Ext}_X^{1/2}(\mu)}$.

As a result,

$$\begin{aligned}
\sup_{\mu\in\mathcal{PMF}}\frac{\mathcal{L}_P(\mu)}{\mathrm{Ext}_X^{1/2}(\mu)} &< \sup_{\mu\in\mathcal{PMF}}\frac{(1+\epsilon)\mathcal{L}_Y(\mu)}{\mathrm{Ext}_X^{1/2}(\mu)} \\
&= \sup_{\mu\in\mathcal{N}}\frac{(1+\epsilon)\mathcal{L}_Y(\mu)}{\mathrm{Ext}_X^{1/2}(\mu)} \\
&\leq \sup_{\mu\in\mathcal{PMF}}\frac{\mathcal{L}_Q(\mu)}{\mathrm{Ext}_X^{1/2}(\mu)}.
\end{aligned}$$

Thus $\Psi_P(X) < \Psi_Q(X)$.

□

Lemma 3.7. *The map* $\Psi : \overline{\mathcal{T}(S)}^{GM} \to C(\mathcal{T}(S)) : P \to \Psi_P$ *is continuous.*

Proof. Let P_n be a sequence of $\overline{\mathcal{T}(S)}^{GM}$ converging to a point P_0 in $\overline{\mathcal{T}(S)}^{GM}$. By Lemma 3.2, $\mathcal{L}_{P_n}$ converges to $\mathcal{L}_{P_0}$ uniformly on compact sets of $\mathcal{MF}$. For all $X \in \mathcal{T}(S)$, the square root of extremal function $\mathrm{Ext_X}^{1/2}$ is bounded away from zero on $\mathcal{PMF}$. We conclude that for any $X \in \mathcal{T}(S)$, $\frac{\mathcal{L}_{P_n}}{\mathrm{Ext}_X^{1/2}}$ converges uniformly on $\mathcal{PMF}$ to $\frac{\mathcal{L}_{P_0}}{\mathrm{Ext}_X^{1/2}}$. It follows that Ψ_{P_n} converges pointwise to Ψ_{P_0}. Note that the function $\Psi : \overline{\mathcal{T}(S)}^{GM} \times \mathcal{T}(S) \to \mathbb{R} : (P, X) \to \Psi_P(X)$ is continuous, and P_n converges to P_0

in $\overline{\mathcal{T}(S)}^{GM}$, then by a finite covering argument (see Lemma 3.1 of Walsh[15]), we know that $\Phi_{P_n}(\cdot)$ converges to $\Psi_{P_0}(\cdot)$ uniformly on compact set of $\mathcal{T}(S)$. By the definition of the topology of $C(\mathcal{T}(S))$, the map $\Psi : P \to \Psi_P(\cdot)$ is continuous. $\square$

Theorem 3.8. *The map Ψ is a homeomorphism between the horofunction compactification of $\mathcal{T}(S)$ with the Teichmüller metric and the Gardiner-Masur compactification of $\mathcal{T}(S)$.*

Proof. We have shown that $\Psi : \overline{\mathcal{T}(S)}^{GM} \to C(\mathcal{T}(S))$ is injective and continuous. Note that an embedding from a compact space to a Hausdorff space must be a homeomorphism to its image (see Kelley [4], Page 141 for the proof). As a result, $\Psi(\overline{\mathcal{T}(S)}^{GM})$ is a compact subset of $C(\mathcal{T}(S))$. Since the horofuction compactification is the closure of $\Psi(\mathcal{T}(S))$, we know that it is equal to $\Psi(\overline{\mathcal{T}(S)}^{GM})$. $\square$

Remark 3.9. Note that the Gardiner-Masur compactification $\overline{\mathcal{T}(S)}^{GM}$ is metrizable. There is a metrization of $\overline{\mathcal{T}(S)}^{GM}$ given by

$$\mathrm{dist}(P,Q) = \sup_{X\in\mathcal{T}(S)} |\Psi_P(X) - \Psi_Q(X)|$$

for any $P, Q \in \overline{\mathcal{T}(S)}^{GM}$. When $P, Q \in \mathcal{T}(S)$, we have

$$\mathrm{dist}(P,Q) = \sup_{X\in\mathcal{T}(S)} |d_T(X,P) - d_T(X,Q) + d_T(X_0,Q) - d_T(X_0,P)|.$$

It is interesting to study the geometric property of $\mathrm{dist}(\cdot,\cdot)$. For example, if we take any two Teichmüller rays $\gamma_1(t), \gamma_2(t)$ with initial point X_0, what is the meaning of the limit $\lim_{t\to\infty} \mathrm{dist}(\gamma_1(t), \gamma_2(t))$?

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