A twisted generalization of Lie-Yamaguti algebras

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Abstract

A twisted generalization of Lie-Yamaguti algebras, called Hom-Lie-Yamaguti algebras, is defined. Hom-Lie-Yamaguti algebras generalize Hom-Lie triple systems (and subsequently ternary Hom-Nambu algebras) and Hom-Lie algebras in the same way as Lie-Yamaguti algebras generalize Lie triple systems and Lie algebras. It is shown that the category of Hom-Lie-Yamaguti algebras is closed under twisting by self-morphisms. Constructions of Hom-Lie-Yamaguti algebras from classical Lie-Yamaguti algebras and Malcev algebras are given. It is observed that, when the ternary operation of a Hom-Lie-Yamaguti algebra expresses through its binary one in a specific way, then such a Hom-Lie-Yamaguti algebra is a Hom-Malcev algebra.

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1 Introduction

Using the Bianchi identities, K. Nomizu [14] characterized, by some identities involving the torsion and the curvature, reductive homogeneous spaces with canonical connection. K. Yamaguti [17] gave an algebraic interpretation of these identities by considering the torsion and curvature tensors of Nomizu's canonical connection as a bilinear and a trilinear algebraic operations satisfying some axioms, and thus defined what he called a "general

Lie triple system". M. Kikkawa [7] used the term "Lie triple algebra" for such an algebraic object. More recently, M.K. Kinyon and A. Weinstein [8] introduced the term "Lie-Yamaguti algebra" for this object.

A *Lie-Yamaguti algebra* $(V, *, \{,,\})$ is a vector space V together with a binary operation $*: V \times V \to V$ and a ternary operation $\{,,\}: V \times V \times V \to V$ such that

(A1) $x * y = -y * x$,
(A2) $\{x, y, z\} = -\{y, x, z\}$,
(A3) $\sigma[(x * y) * z + \{x, y, z\}] = 0$,
(A4) $\sigma[\{x * y, z, u\} = 0$,
(A5) $\{x, y, u * v\} = \{x, y, u\} * v + u * \{x, y, v\}$,
(A6) $\{x, y, \{u, v, w\}\} = \{\{x, y, u\}, v, w\} + \{u, \{x, y, v\}, w\}$
$+\{u, v, \{x, y, w\}\}$,

for all u, v, w, x, y, z in V and σ denotes the sum over cyclic permutation of x, y, z.

In [2] the notation "LY-algebra" is used for "Lie-Yamaguti algebra". So, likewise, we will write "Hom-LY algebra" for "Hom-Lie-Yamaguti algebra".

Observe that if $x * y = 0$, for all x, y in V, then $(V, *, \{,,\})$ reduces to a *Lie triple system* $(V, \{,,\})$ as defined in [16]. Originally, N. Jacobson [6] defined a Lie triple system as a submodule of an associative algebra that is closed under the iterated commutator bracket.

In this paper we consider a Hom-type generalization of LY algebras that we call Hom-LY algebras. Roughly, a Hom-type generalization of a given type of algebras is defined by twisting the defining identities of that type of algebras by a self-map in such a way that, when the twisting map is the identity map, one recovers the original type of algebras. The systematic study of Hom-algebras was initiated by A. Makhlouf and S.D. Silvestrov [11], while D. Yau [20] gave a general construction method of Hom-type algebras starting from classical algebras and a twisting self-map. For information on various types of Hom-algebras, one may refer to [1], [4], [5], [9]-[11], [19]-[22].

A Hom-type generalization of n-ary Lie algebras, n-ary Nambu algebras and n-ary Nambu-Lie algebras (i.e. Fillipov n-ary algebras) called n-ary Hom-Lie algebras, n-ary Hom-Nambu algebras and n-ary Hom-Nambu-Lie algebras respectively, is considered in [1]. Such a generalization is extended to the one of Hom-Lie triple systems and Hom-Jordan triple systems in [22]. We point out that the class of Hom-LY algebras encompasses the ones of ternary Hom-Nambu algebras, Hom-Lie triple systems (hence Jordan and Lie triple systems), Hom-Lie algebras (hence Lie algebras) and LY algebras.

The rest of the paper is organized as follows. In section 2 some basic facts on Hom-algebras and n-ary Hom-algebras are recalled. The emphasis point here is that the definition of a Hom-triple system (Definition 2.3) is more restrictive than the D. Yau's in [22]. However, with this vision of a Hom-triple system, we point out that any non-Hom-associative algebra (i.e. nonassociative Hom-algebra or Hom-nonassociative algebra) has a natural structure of a Hom-triple system (this is the Hom-counterpart of a similar well-known result connecting nonassociative algebras and triple systems). Then we give the definition of a Hom-LY algebra and make some observations on its relationships with some types of ternary Hom-algebras and with LY algebras. In section 3 we show that the category of Hom-LY algebras is closed under twisting by self-morphisms (Theorem 3.1). Subsequently, we show a way to construct Hom-LY algebras from LY algebras (or Malcev algebras) by twisting along self-morphisms (Corollary 3.2 and Corollary 3.3); this is an extension to binary-ternary algebras of a result due to D. Yau ([20], Theorem 2.3. Such an extension is first mentioned in [4], Corollary 4.6). In section 4 some relationships between Hom-LY algebras and Hom-Malcev algebras are considered. We show that when the ternary operation of a Hom-LY algebra expresses through its binary one in a specific way, then such a Hom-LY algebra turns out to be a Hom-Malcev algebra (Proposition 4.1). Moreover, with this expression of the ternary operation, it is observed that, in a Hom-Malcev algebra, the Hom-Malcev identity can be written in terms of this ternary operation and the original binary operation of the given Hom-Malcev algebra (Proposition 4.2). These considerations constitute the Hom-version of similar relationships between Malcev algebras and LY algebras ([13], [18]).

All vector spaces and algebras throughout will be over a ground field $\mathbb{K}$ of characteristic 0.

2 Ternary Hom-algebras. Definitions

We recall some basic facts about Hom-algebras, including ternary Hom-Nambu algebras. We note that the definition of a Hom-triple system given here (see Definition 2.3) is slightly more restrictive than the one given by D. Yau [22]. Then we give the definition of the main object of this paper (see Definition 2.6) and show its relationships with known structures such as ternary Hom-Nambu algebras, Hom-Lie triple systems, Hom-Lie algebras or Lie-Yamaguti algebras.

For definitions of n-ary Hom-algebras (n-ary Hom-Nambu and Hom-

Nambu-Lie algebras, n-ary Hom-Lie algebras, etc.) we refer to [1], [22]. Here, for our purpose, we restrict our concern to ternary Hom-algebras. In fact, as we shall see below, a Hom-Lie-Yamaguti algebra is a ternary Hom-Nambu algebra with an additional binary anticommutative operation satisfying some compatibility conditions.

Definition 2.1. [22] A *ternary Hom-algebra* $(V, [,,], \alpha = (\alpha_1, \alpha_2))$ consists of a $\mathbb{K}$-module V, a trilinear map $[,,] : V \times V \times V \to V$, and linear maps $\alpha_i : V \to V$, $i = 1, 2$, called the *twisting maps.* The algebra $(V, [,,], \alpha = (\alpha_1, \alpha_2))$ is said *multiplicative* if $\alpha_1 = \alpha_2 = \alpha$ and $\alpha([x, y, z]) = [\alpha(x), \alpha(y), \alpha(z)]$ for all $x, y, z \in V$.

For convenience, we assume throughout this paper that all Hom-algebras are multiplicative.

Definition 2.2. [1] A *(multiplicative) ternary Hom-Nambu algebra* is a (multiplicative) ternary Hom-algebra $(V, [,,], \alpha)$ satisfying

$$\begin{aligned}[\alpha(x), \alpha(y), [u, v, w]] = & [[x, y, u], \alpha(v), \alpha(w)] + [\alpha(u), [x, y, v], \alpha(w)] \\ & +[\alpha(u), \alpha(v), [x, y, w]], \end{aligned} \tag{2.1}$$

for all $u, v, w, x, y \in V$.

The condition (2.1) is called the *ternary Hom-Nambu identity.*

Definition 2.3. A *(multiplicative) Hom-triple system* is a (multiplicative) ternary Hom-algebra $(V, [,,], \alpha)$ such that
(i) $[u, v, w] = -[v, u, w]$,
(i) $\sigma[u, v, w] = 0$,
for all $u, v, w \in V$, where $\sigma[u, v, w] := [u, v, w] + [v, w, u] + [w, u, v]$.

Remark. A more general definition of a Hom-triple system is given by D. Yau [22] without the requirements (i), (ii) as in Definition 2.3 above. Our definition here is motivated by the concern of giving a Hom-type analogue of the relationships between nonassociative algebras and triple systems (see Remark below).

A Hom-algebra in which the Hom-associativity is not assumed is called a nonassociative Hom-algebra [10] or a Hom-nonassociative algebra [19] (the expression of "non-Hom-associative" Hom-algebra is used in [4] for that type of Hom-algebras). With the notion of a Hom-triple system as above, we have the following

Proposition 2.4. *Any non-Hom-associative Hom-algebra is a Hom-triple system.*

Proof. Let $(A, \cdot, \alpha)$ be a non-Hom-associative algebra. Then $(A, [,], as(,,), \alpha)$ is a Hom-Akivis algebra with respect to $[x, y] := x \cdot y - y \cdot x$ (commutator) and $as(x, y, z) := xy \cdot \alpha(z) - \alpha(x) \cdot yz$ (Hom-associator), i.e. the Hom-Akivis identity

$\sigma[[x, y], \alpha(z)] = \sigma as(x, y, z) - \sigma as(y, x, z)$

holds for all x, y, z in A ([4]). Now define

$[x, y, z] := [[x, y], \alpha(z)] - as(x, y, z) + as(y, x, z)$

for all x, y, z in A. Then $[x, y, z] = -[y, x, z]$ and the Hom-Akivis identity implies that $\sigma[x, y, z] = 0$. Thus $(A, [, ,], \alpha)$ is a Hom-triple system. □

Remark. For $\alpha = Id$ (the identity map), we recover the triple system with ternary operation $[[x, y], z] - (x, y, z) + (y, x, z)$ that is associated to each nonassociative algebra, since any nonassociative algebra has a natural Akivis algebra structure with respect to the commutator and associator operations $[x, y]$ and (x, y, z), for all x, y, z (see, e.g., remarks in [4]).

Definition 2.5. [22] A *Hom-Lie triple system* is a Hom-triple system $(V, [, ,], \alpha)$ satisfying the ternary Hom-Nambu identity (2.1).

When $\alpha = Id$, a Hom-Lie triple system reduces to a Lie triple system.

We now give the definition of the basic object of this paper.

Definition 2.6. A *Hom-Lie-Yamaguti algebra* (Hom-LY algebra for short) is a quadruple $(L, *, \{, , \}, \alpha)$ in which L is a $\mathbb{K}$-vector space, "$*$" a binary operation and "$\{, , \}$" a ternary operation on L, and $\alpha : L \to L$ a Linear map such that

(B1) $\alpha(x * y) = \alpha(x) * \alpha(y)$,
(B2) $\alpha(\{x, y, z\}) = \{\alpha(x), \alpha(y), \alpha(z)\}$,
(B3) $x * y = -y * x$,
(B4) $\{x, y, z\} = -\{y, x, z\}$,
(B5) $\sigma[(x * y) * \alpha(z) + \{x, y, z\}] = 0$,
(B6) $\sigma[\{x * y, \alpha(z), \alpha(u)\} + \{z * y, \alpha(x), \alpha(u)\}] = 0$,
(B7) $\{\alpha(x), \alpha(y), u * v\} = \{x, y, u\} * \alpha^2(v) + \alpha^2(u) * \{x, y, v\}$,
(B8) $\{\alpha^2(x), \alpha^2(y), \{\alpha^2(u), \alpha^2(v), w\}\} = \{\{x, y, \alpha^2(u)\}, \alpha^4(v), \alpha^2(w)\}$

$$+\{\alpha^4(u),\{x,y,\alpha^2(v)\},\alpha^2(w)\}$$
$$+\{\alpha^4(u),\alpha^4(v),\{x,y,w\}\},$$

for all u,v,w,x,y,z in L and σ denotes the sum over cyclic permutation of x,y,z.

Note that the conditions **(B1)** and **(B2)** mean the multiplicativity of $(L,*,\{,,\},\alpha)$.

Remark. (1) If $\alpha = Id$, then the Hom-LY algebra $(L,*,\{,,\},\alpha)$ reduces to a LY algebra $(L,*,\{,,\})$ (see **(A1)-(A6)**).

(2) If $x*y=0$, for all $x,y\in L$, then $(L,*,\{,,\},\alpha)$ becomes a Hom-Lie triple system $(L,\{,,\},\alpha^2)$ and, subsequently, a ternary Hom-Nambu algebra (since, by Definition 2.5, any Hom-Lie triple system is automatically a ternary Hom-Nambu algebra).

(3) If $\{x,y,z\}=0$ for all $x,y,z\in L$, then the Hom-LY algebra $(L,*,\{,,\},\alpha)$ becomes a Hom-Lie algebra $(L,*,\alpha)$.

3 Constructions of Hom-Lie-Yamaguti algebras

In this section we consider construction methods for Hom-LY algebras. These methods allow to find examples of Hom-LY algebras starting from classical LY algebras or even from Malcev algebras.

First, as the main tool, we show that the category of (multiplicative) Hom-LY algebras is closed under self-morphisms.

Theorem 3.1. *Let $A_\alpha := (A,*,\{,,\},\alpha)$ be a multiplicative Hom-LY algebra and let β be an endomorphism of the algebra $(A,*,\{,,\})$ such that $\beta\alpha=\alpha\beta$. Define on A the operations*

$x *_\beta y := \beta(x*y)$,

$\{x,y,z\}_\beta := \beta^2(\{x,y,z\})$

for all x,y,z in A. Then $A_\beta := (A,*_\beta,\{,,\}_\beta,\beta\alpha)$ is a multiplicative Hom-LY algebra.

Proof. We have

$(\beta\alpha)(x*_\beta y) = (\beta\alpha)(\beta(x)*\beta(y)) = \beta((\alpha\beta)(x)*(\alpha\beta)(y))$

$= (\alpha\beta)(x)*_\beta(\alpha\beta)(y) = (\beta\alpha)(x)*_\beta(\beta\alpha)(y)$ and we get **(B1)**. Likewise, the condition $\beta\alpha=\alpha\beta$ implies **(B2)**. The identities **(B3)** and **(B4)** for A_β follow from the skew-symmetry of "$*$" and "$\{,,\}$" respectively.

Consider now $\sigma((x *_\beta y) *_\beta (\beta\alpha)(z)) + \sigma\{x, y, z\}_\beta$. Then

$$\begin{aligned}
\sigma((x *_\beta y) \quad *_\beta \quad & (\beta\alpha)(z)) + \sigma\{x, y, z\}_\beta \\
= \quad & \sigma[\beta(\beta(x * y) * \beta(\alpha(z)))] + \sigma[\beta^2(\{x, y, z\})] \\
= \quad & \sigma[\beta((\beta(x) * \beta(y)) * \alpha(\beta(z))] + \sigma[\beta^2(\{x, y, z\})] \\
& (\text{since } \beta\alpha = \alpha\beta) \\
= \quad & \beta(\sigma[(\beta(x) * \beta(y)) * \alpha(\beta(z))] + \sigma\{\beta(x), \beta(y), \beta(z)\}) \\
= \quad & \beta(0) \text{ (by } \mathbf{(B6)} \text{ for } A_\alpha) \\
= \quad & 0
\end{aligned}$$

and thus we get **(B5)** for A_β. Next,
$\{x *_\beta y, (\beta\alpha)(z), (\beta\alpha)(u)\}_\beta = \{\beta^3(x * y), \beta^3(\alpha(z)), \beta^3(\alpha(u))\}$
$= \beta^3(\{x * y, \alpha(z), \alpha(u)\})$.
Likewise we find that $\{z *_\beta y, (\beta\alpha)(x), (\beta\alpha)(u)\}_\beta = \beta^3(\{z * y, \alpha(x), \alpha(u)\})$. Therefore

$$\begin{aligned}
& \{x *_\beta y, (\beta\alpha)(z), (\beta\alpha)(u)\}_\beta + \{z *_\beta y, (\beta\alpha)(x), (\beta\alpha)(u)\}_\beta \\
= \quad & \sigma[\beta^3(\{x * y, \alpha(z), \alpha(u)\}) + \beta^3(\{z * y, \alpha(x), \alpha(u)\})] \\
= \quad & \beta^3(\sigma\{x * y, \alpha(z), \alpha(u)\} + \sigma\{z * y, \alpha(x), \alpha(u)\}) \\
= \quad & \beta^3(0) \text{ (by } \mathbf{(B6)} \text{ for } A_\alpha) \\
= \quad & 0
\end{aligned}$$

so that we get **(B6)** for A_β. Further, using **(B7)** for A_α and condition $\alpha\beta = \beta\alpha$, we compute
$\{(\beta\alpha)(x), (\beta\alpha)(y), u \ *_\beta \ v\}_\beta = \beta^3(\{\alpha(x), \alpha(y), u * v\})$
$= \beta^3(\{x, y, u\} * \alpha^2(v) + \alpha^2(u) * \{x, y, v\}) = \beta(\beta^2(\{x, y, u\}) * (\beta^2\alpha^2)(v))$
$+\beta((\beta^2\alpha^2)(u) * \beta^2(\{x, y, v\})) = \{x, y, u\}_\beta \ *_\beta \ (\beta^2\alpha^2)(v)$
$+(\beta^2\alpha^2)(u) \ *_\beta \ \{x, y, v\}_\beta$
$= \{x, y, u\}_\beta \ *_\beta \ (\beta\alpha)^2(v) + (\beta\alpha)^2(u) \ *_\beta \ \{x, y, v\}_\beta$.
Thus **(B7)** holds for A_β. Using repeatedly the condition $\alpha\beta = \beta\alpha$ and the

identity **(B8)** for A_α, the verification of **(B8)** for A_β is as follows.

$$\begin{aligned}
&\{(\beta\alpha)^2(x),(\beta\alpha)^2(y),\{(\beta\alpha)^2(u),(\beta\alpha)^2(v),w\}_\beta\}_\beta\\
=\;&\{(\beta^2\alpha^2)(x),(\beta^2\alpha^2)(y),\{(\beta^2\alpha^2)(u),(\beta^2\alpha^2)(v),w\}_\beta\}_\beta\\
=\;&\beta^2(\{(\beta^2\alpha^2)(x),(\beta^2\alpha^2)(y),\beta^2(\{(\beta^2\alpha^2)(u),(\beta^2\alpha^2)(v),w\})\})\\
=\;&\beta^4(\{\alpha^2(x),\alpha^2(y),\{(\beta^2\alpha^2)(u),(\beta^2\alpha^2)(v),w\}\})\\
=\;&\beta^4(\{\alpha^2(x),\alpha^2(y),\{\alpha^2(\beta^2(u)),\alpha^2(\beta^2(v)),w\}\})\\
=\;&\beta^4(\{\alpha^4(\beta^2(u)),\alpha^4(\beta^2(v)),\{x,y,w\}\})\\
+\;&\beta^4(\{\{x,y,\alpha^2(\beta^2(u))\},\alpha^4(\beta^2(v)),\alpha^2(w)\}\\
+\;&\beta^4(\{\alpha^4(\beta^2(u)),\{x,y,\alpha^2(\beta^2(v))\},\alpha^2(w)\}\\
=\;&\beta^4(\{(\beta^2\alpha^4)(u),(\beta^2\alpha^4)(v),\{x,y,w\}\})\\
+\;&\beta^4(\{\{x,y,(\beta^2\alpha^2)(u)\},(\beta^2\alpha^4)(v),\alpha^2(w)\})\\
+\;&\beta^4(\{(\beta^2\alpha^4)(u),\{x,y,(\beta^2\alpha^2)(v)\},\alpha^2(w)\})\\
=\;&\beta^2(\{(\beta^4\alpha^4)(u),(\beta^4\alpha^4)(v),\beta^2(\{x,y,w\})\})\\
+\;&\beta^2(\{\beta^2(\{x,y,(\beta^2\alpha^2)(u)\}),(\beta^4\alpha^4)(v),(\beta^2\alpha^2)(w)\})\\
+\;&\beta^2(\{(\beta^4\alpha^4)(u),\beta^2(\{x,y,(\beta^2\alpha^2)(v)\}),(\beta^2\alpha^2)(w)\})\\
=\;&\{(\beta\alpha)^4(u),(\beta\alpha)^4(v),\{x,y,w\}_\beta\}_\beta\\
+\;&\{\{x,y,(\beta\alpha)^2(u)\}_\beta,(\beta\alpha)^4(v),(\beta\alpha)^2(w)\}_\beta\\
+\;&\{(\beta\alpha)^4(u),\{x,y,(\beta\alpha)^2(v)\}_\beta,(\beta\alpha)^2(w)\}_\beta.
\end{aligned}$$

Thus **(B8)** holds for A_β. Therefore, we get that A_β is a Hom-LY algebra. This finishes the proof. □

From Theorem 3.1 we have the following method of construction of Hom-LY algebras from LY algebras. This method is an extension to binary-ternary algebras of a result due to D. Yau ([20], Theorem 2.3), giving a general method of construction of Hom-algebras from their corresponding untwisted algebras. Such an extension to binary-ternary algebras is first mentioned in [4], Corollary 4.6.

Corollary 3.2. *Let $(A,*,[,,])$ be a LY algebra and β an endomorphism of $(A,*,[,,])$. If define on A a binary operation "$\tilde{*}$" and a ternary operation "$\{,,\}$" by*

$x\tilde{*}y := \beta(x*y)$,

$\{x,y,z\} := \beta^2([x,y,z])$,

then $(A,\tilde{*},\{,,\},\beta)$ is a Hom-LY algebra.

Proof. The proof follows if observe that Corollary 3.2 is Theorem 3.1 when $\alpha = Id$. □

Corollary 3.3. *Let* $(A, *)$ *be a Malcev algebra and* β *any endomorphism of* $(A, *)$*. Define on* A *the operations*
$x \tilde{*} y := \beta(x * y)$,
$\{x, y, z\} := \beta^2((x * y) * z - (y * z) * x - (z * x) * y)$.
Then $(A, \tilde{*}, \{,,\}, \beta)$ *is a Hom-LY algebra.*

Proof. If consider on A the ternary operation $[x, y, z] := (x * y) * z - (y * z) * x - (z * x) * y$, $\forall x, y, z \in A$, then $(A, *, [,,])$ is a LY algebra [18]. Moreover, since β is an endomorphism of $(A, *)$, we have $\beta([x, y, z]) = (\beta(x)*\beta(y))*\beta(z) - (\beta(y)*\beta(z))*\beta(x) - (\beta(z)*\beta(x))*\beta(y) = [\beta(x), \beta(y), \beta(z)]$ so that β is also an endomorphism of $(A, *, [,,])$. Then Corollary 3.2 implies that $(A, \tilde{*}, \{,,\}, \beta)$ is a Hom-LY algebra. □

4 Hom-Lie-Yamguti algebras and Hom-Malcev algebras

In this section we investigate conditions when a Hom-LY algebra reduces to a Hom-Malcev algebra. This consideration is based on the ternary operation $\{,,\}$ of a given Hom-LY algebra $(\mathfrak{m}, [,], \{,,\}, \alpha)$ that could be expressed through its binary one "$[,]$" as

$$\{x, y, z\} = -J_\alpha(x, y, z) + 2[[x, y], \alpha(z)], \tag{4.1}$$

for all x, y, z in $\mathfrak{m}$, where $J_\alpha(x, y, z) := \sigma[[x, y], \alpha(z)]$.

First we recall that a *Hom-Malcev algebra* [21] is a Hom-algebra $(A, [,], \alpha)$ such that "$[,]$" is skew-symmetric and that the *Hom-Malcev identity*

$$J_\alpha(\alpha(x), \alpha(y), [x, z]) = [J_\alpha(x, y, z), \alpha^2(x)] \tag{4.2}$$

holds for all x, y, z in A. It is observed [21] that when $\alpha = Id$, then (4.2) is the Malcev identity and thus a Hom-Malcev algebra reduces to a Malcev algebra. Other identities, equivalent to the identity (4.2), characterizing Hom-Malcev algebras are found ([21], Proposition 2.8). In [5] it is pointed out another defining identity of Hom-Malcev algebras. This latter identity is the most useful in the proof of the following

Proposition 4.1. *Let* $(\mathfrak{m}, [,], \{,,\}, \alpha)$ *be a Hom-LY algebra. If its ternary operation "*$\{,,\}$*" expresses through its binary one "*$[,]$*" as in (4.1) for all* x, y, z *in* $\mathfrak{m}$*, then* $(\mathfrak{m}, [,], \alpha)$ *is a Hom-Malcev algebra.*

Proof. Observe that (4.1) and multiplicativity imply

$$\{\alpha(x), \alpha(y), z\} = -J_\alpha(\alpha(x), \alpha(y), z) + 2\alpha([[x, y], z]). \tag{4.3}$$

Then, setting (4.3) in **(B7)**, we get

$$\begin{aligned} -J_\alpha(\alpha(x), \alpha(y), [u, v]) = &[-J_\alpha(x, y, u), \alpha^2(v)] + [\alpha^2(u), -J_\alpha(x, y, v)] \\ &+[[2[x, y], \alpha(u)], \alpha^2(v)] + [\alpha^2(u), [2[x, y], \alpha(v)]] \\ &-2\alpha([[x, y], [u, v]]) \end{aligned}$$

and this last equality is written as

$$\begin{aligned} J_\alpha(\alpha(x), \alpha(y), [u, v]) = &[J_\alpha(x, y, u), \alpha^2(v)] + [\alpha^2(u), J_\alpha(x, y, v)] \\ &-2J_\alpha(\alpha(u), \alpha(v), [x, y]). \end{aligned} \tag{4.4}$$

The expression (4.4) is shown [5] to be equivalent to the Hom-Malcev identity (4.2). Therefore $(\mathfrak{m}, [,], \alpha)$ is a Hom-Malcev algebra. □

Remark. For $\alpha = Id$, the ternary operation (4.1) reduces to the ternary operation, defined by the identity (1.4) in [18], that is considered in Malcev algebras. Thus Proposition 4.1 is the Hom-analogue of the result of K. Yamaguti [18], which is the converse of a result of A.A. Sagle ([13], Proposition 8.3). The Hom-version of the sagle's result is the following

Proposition 4.2. *Let* $(\mathfrak{m}, [,], \alpha)$ *be a Hom-Malcev algebra and define on* $(\mathfrak{m}, [,], \alpha)$ *a ternary operation by (4.1). Then*

$$\{\alpha(x), \alpha(y), [u, v]\} = [\{x, y, u\}, \alpha^2(v)] + [\alpha^2(u), \{x, y, v\}] \tag{4.5}$$

for all u, v, x, y *in* $\mathfrak{m}$.

Proof. We write the identity (4.4) as

$$\begin{aligned} -J_\alpha(\alpha(x), \alpha(y), [u, v]) = &[-J_\alpha(x, y, u), \alpha^2(v)] + [\alpha^2(u), -J_\alpha(x, y, v)] \\ &+2J_\alpha(\alpha(u), \alpha(v), [x, y]) \end{aligned}$$

i.e.

$$\begin{aligned} -J_\alpha(\alpha(x), \alpha(y), [u, v]) = &[-J_\alpha(x, y, u), \alpha^2(v)] + [\alpha^2(u), -J_\alpha(x, y, v)] \\ &+2[\alpha([u, v]), \alpha([x, y])] + 2[[\alpha(v), [x, y]], \alpha^2(u)] \end{aligned}$$

$$+2[[[x,y],\alpha(u)],\alpha^2(v)]$$

or

$$-J_\alpha(\alpha(x),\alpha(y),[u,v]) + 2[\alpha([x,y]),\alpha([u,v])]$$
$$= [-J_\alpha(x,y,u) + 2[[x,y],\alpha(u)],\alpha^2(v)]$$
$$+[\alpha^2(u), -J_\alpha(x,y,v) + 2[[x,y],\alpha(v)]].$$

This last equality (according to (4.1) and using multiplicativity) means that

$$\{\alpha(x),\alpha(y),[u,v]\} = [\{x,y,u\},\alpha^2(v)] + [\alpha^2(u),\{x,y,v\}]$$

and therefore the proposition is proved. □

It could be expected that any Hom-Malcev algebra $(\mathfrak{m},[,],\alpha)$ with a ternary operation as in (4.1) has a Hom-LY structure. This is for further investigation. Combining Proposition 4.1 and Proposition 4.2, we get the following

Corollary 4.3. *In an anticommutative Hom-algebra $(A,[,],\alpha)$, the Hom-Malcev identity (4.2) is equivalent to (4.5), with $\{,,\}$ defined by (4.1).* □

The untwisted counterpart of Corollary 4.3 is Theorem 1.1 in [18].

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