A DIRECT PROOF OF THE FIVE ELEMENT BASIS THEOREM

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Abstract. We present a direct proof of the consistency of the existence of a five element basis for the uncountable linear orders. Our argument is based on the approach of [9] and simplifies the original proof of Moore [11].

Introduction

In [11] Moore showed that PFA implies that the class of the uncountable linear orders has a five element basis, i.e., that there is a list of five uncountable linear orders such that every uncountable linear order contains an isomorphic copy of one of them. This basis consists of X, ω_1, ω_1^*, C, and C^*, where X is any suborder of the reals of cardinality ω_1 and C is any Countryman line[1]. It was previously known from the work of Baumgartner [5] and Abraham-Shelah [1], that, assuming a rather weak forcing axiom, the existence of a five element linear basis for uncountable linear orderings is equivalent to the following statement, called the *Coloring Axiom for Trees* (CAT):

> There is an Aronszajn tree T such that for every $K \subseteq T$ there is an uncountable antichain $X \subseteq T$ such that $\wedge(X)$ is either contained in or disjoint from K.

Here $\wedge(X)$ denotes the set of all pairwise meets of elements of X.

One feature of the argument from [11] is that it relied crucially on the Mapping Reflection Principle (MRP), a strong combinatorial principle previously introduced by Moore in [12], in order to prove the properness of the appropriate forcing notion. It was shown in [12] that MRP implies the failure of $\square_\kappa$, for all $\kappa \geq \omega_1$, and therefore its consistency requires very large cardinal axioms. However, it was not clear if any large cardinals were

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[1]Recall that a *Countryman line* is an uncountable linear order whose square is the union of countably many non-decreasing relations. The existence of such a linear order was proved by Shelah in [13].

needed for the relative consistency of CAT. Progress on this question was made by König, Larson, Moore and Veličković in [9] who reduced considerably the large cardinal assumptions in Moore's proof. They considered a statement φ which is a form of saturation of Aronszajn trees and showed that it can be used instead of MRP in the proof of the Key Lemma (Lemma 5.29) from [11]. Moreover, they showed that for the consistency of BPFA together with φ it is sufficient to assume the existence of a reflecting Mahlo cardinal. If one is only interested in the consistency of the existence of a five element basis for the uncountable linear orderings then even an smaller large cardinal assumption is sufficient (see [9] for details).

The purpose of this note is to present a direct proof of CAT, and therefore the existence of a five element linear basis, assuming the conjunction of BPFA and φ. The argument is much simpler than the original proof from [11]. It is our hope that by further understanding this forcing one will be able to determine if any large cardinal assumptions are needed for the consistency of CAT.

The paper is organized as follows. In §1 we present the background material on Aronszajn trees and the combinatorial principles ψ and φ. In §2 we start with a coherent special Aronszajn tree T and a subset K of T, define a new coloring of finite subsets of T and prove some technical lemmas. In §3 we define the main forcing notion $\partial^*(K)$ and show that it is proper. In §4 we complete the proof that BPFA together with φ implies CAT.

1. Saturation of Aronszajn trees

By an *Aronszajn tree* or simply an *A-tree* we mean an uncountable tree in which all levels and chains are countable. A *subtree* of an A-tree T is an uncountable downward closed subset of T. All our trees will be subtrees of $2^{<\omega_1}$ or products of such trees. If T is such a tree, $t \in T$ and $\xi < \mathrm{ht}(t)$ then $t \restriction \xi$ is the predecessor of t on level ξ of T. We start by discussing the notion of saturation of an Aronszajn tree.

Definition 1.1. *An Aronszajn tree T is* saturated *if whenever $\mathscr{A}$ is a collection of subtrees T which have pairwise countable intersection, $\mathscr{A}$ has cardinality at most ω_1.*

This statement follows from the stronger assertion shown by Baumgartner in [4] to hold after Levy collapsing an inaccessible cardinal to ω_2.

> For every Aronszajn tree T, there is a collection $\mathscr{B}$ of subtrees of T such that $\mathscr{B}$ has cardinality ω_1 and every subtree of T contains an element of $\mathscr{B}$.

However, in Baumgartner's model CH holds and we need to have saturation of Aronszajn trees together with BPFA. It is for this reason that a different approach was taken in [9]. We now recall the relevant definitions from this paper.

If $\mathscr{F}$ is a collection of subtrees of T, then $\mathscr{F}^\perp$ is the collection of all subtrees B of T such that for every A in $\mathscr{F}$, $A \cap B$ is countable. If $\mathscr{F}^\perp$ is empty, then $\mathscr{F}$ is said to be *predense*. For $\mathscr{F}$ a collection of subtrees of an Aronszajn tree T, we consider the following statements:

$\psi_0(\mathscr{F})$: There is a closed unbounded set $E \subseteq \omega_1$ and a continuous chain $\langle N_\nu : \nu \in E\rangle$ of countable subsets of $\mathscr{F}$ such that for every ν in E and t in T_ν there is a $\nu_t < \nu$ such that if $\xi \in (\nu_t, \nu) \cap E$, then there is $A \in \mathscr{F} \cap N_\xi$ such that $t \restriction \xi$ is in A.

$\varphi_0(\mathscr{F})$: There is a closed unbounded set $E \subseteq \omega_1$ and a continuous chain $\langle N_\nu : \nu \in E\rangle$ of countable subsets of $\mathscr{F} \cup \mathscr{F}^\perp$ such that for every ν in E and t in T_ν either
(1) there is a $\nu_t < \nu$ such that if $\xi \in (\nu_t, \nu) \cap E$, then there is $A \in \mathscr{F} \cap N_\xi$ such that $t \restriction \xi$ is in A, or
(2) there is a B in $\mathscr{F}^\perp \cap N_\nu$ such that t is in B.

It is not difficult to show that $\psi_0(\mathscr{F})$ implies that $\mathscr{F}$ is predense. It is also clear that $\psi_0(\mathscr{F})$ is a Σ_1-formula in the parameters $\mathscr{F}$ and T. While $\varphi_0(\mathscr{F})$ and $\psi_0(\mathscr{F})$ are equivalent if $\mathscr{F}$ is predense, $\varphi_0(\mathscr{F})$ is in general not a Σ_1-formula in $\mathscr{F}$ and T. Let φ be the assertion that whenever T is an Aronszajn tree and $\mathscr{F}$ is a family of subtrees T, $\varphi_0(\mathscr{F})$ holds and let ψ be the analogous assertion but with quantification only over $\mathscr{F}$ which are predense. As noted, φ implies ψ.

The following was proved as Corollary 3.9 in [9].

Proposition 1.2. *For a given family $\mathscr{F}$ of subtrees of an Aronszajn tree T, there is a proper forcing extension which satisfies $\varphi_0(\mathscr{F})$.* □

Remark 1.3. If we want to force φ it is natural to start with an inaccessible cardinal κ and do a countable support iteration of proper forcing notions $\langle \mathcal{P}_\alpha, \dot{\mathcal{Q}}_\beta; \alpha \leq \kappa, \beta < \kappa\rangle$. At stage α we can use $\diamondsuit_\kappa$ to guess an Aronszajn tree $\dot{T}_\alpha$ and a family $\dot{\mathcal{F}}_\alpha$ of subtrees of $\dot{T}_\alpha$ in the model $V^{\mathcal{P}_\alpha}$ and let $\dot{\mathcal{Q}}_\alpha$ be a $\mathcal{P}_\alpha$-name for the proper poset which forces $\varphi_0(\dot{\mathcal{F}}_\alpha)$. Suppose in the final model $V^{\mathcal{P}_\kappa}$ we have an Aronszajn tree $\dot{T}$ and a family $\dot{\mathcal{F}} \in V^{\mathcal{P}_\kappa}$ of subtrees of $\dot{T}$. In order to ensure that $\varphi_0(\dot{\mathcal{F}})$ holds in $V^{\mathcal{P}_\kappa}$ we need to find a stage α of the iteration at which $\dot{T}$ and $\dot{\mathcal{F}}$ are guessed, i.e. $\dot{T}_\alpha = \dot{T}$ and $\dot{\mathcal{F}} \restriction V^{\mathcal{P}_\alpha} = \dot{\mathcal{F}}_\alpha$ *and* moreover such that

$$(\dot{\mathcal{F}}^\perp)^{V^{\mathcal{P}_\kappa}} \restriction V^{\mathcal{P}_\alpha} = (\dot{\mathcal{F}}_\alpha^\perp)^{V^{\mathcal{P}_\alpha}}.$$

This is the reason why a Mahlo cardinal is used in the following theorem from [9].

Theorem 1.4. *If there is a cardinal which is both reflecting and Mahlo, then there is a proper forcing extension of L which satisfies the conjunction of* BPFA *and φ. In particular the forcing extension satisfies that the uncountable linear orders have a five element basis.* □

If one is interested only in the consistency that the uncountable linear order have a five element basis it was observed in [9] then a somewhat smaller large cardinal is sufficient. Indeed, for the desired conclusion one does not need the full strength of BPFA and one only needs $\varphi_0(\mathcal{F})$ for certain families of subtrees of an Aronszajn tree T which are Σ_1-definable using a subset of ω_1 as a parameter. The precise large cardinal assumption is that there is an inaccessible cardinal κ such that for every $\kappa_0 < \kappa$, there is an inaccessible cardinal $\delta < \kappa$ such that κ_0 is in $H(\delta)$ and $H(\delta)$ satisfies there are two reflecting cardinals which are greater than κ_0.

2. Colorings of Aronszajn trees

For the remainder of the paper we fix an Aronszajn tree $T \subseteq 2^{<\omega_1}$ which is coherent, special, and closed under finite modifications. The tree $T(\varrho_3)$ from [17] is such an example. Recall that T is *coherent* if for every $s, t \in T$ of the same height, say α, the set

$$D(s,t) = \{\xi < \alpha : s(\xi) \neq t(\xi)\}$$

is finite. By T_α we denote the α-th level of T, i.e. the set of nodes of height α. For $A \subseteq \omega_1$ we set $T \upharpoonright A = \bigcup_{\alpha \in A} T_\alpha$. If s and t are incomparable nodes in T, i.e. if $D(s,t)$ is non empty, we let

$$\Delta(s,t) = \min D(s,t).$$

We also let $s \wedge t$ denote the largest common initial segment of s and t, i.e. $s \upharpoonright \Delta(s,t)$. Given a subset X of T we let

$$\wedge(X) = \{s \wedge t : s, t \in X, s \text{ and } t \text{ incomparable}\}.$$

Note that if R_X is the tree induced by X, i.e. the set of all initial segments of elements of X, then $\wedge(R_X) = \wedge(X)$. We also let

$$\pi(X) = \{t \upharpoonright \mathrm{ht}(s) : s, t \in X \text{ and } \mathrm{ht}(s) \leq \mathrm{ht}(t)\}.$$

We let $\mathrm{lev}(X) = \{\mathrm{ht}(t) : t \in X\}$. If $\alpha \in \mathrm{lev}(X)$ we let $\pi_\alpha(X) = \pi(X) \cap T_\alpha$.

We will also need to consider finite powers of our tree T. Given an integer n and a level T_α of T we let

$$T_\alpha^{[n]} = \{\tau \in T_\alpha^n : i < j \to \tau(i) \leq_{lex} \tau(j)\}$$

where $\leq_{lex}$ denotes the lexicographic ordering of T. We let $T^{[n]} = \bigcup_\alpha T^{[n]}_\alpha$. Morally, elements of $T^{[n]}$ are n-element subsets of T of the same height. In order to ensure that $T^{[n]}$ is closed under taking restrictions, it is necessary to allow for n-element sets with repetition and the above definition is a formal means to accommodate this. We will abuse notation and identify elements of $T^{[n]}$ which have distinct coordinates with the set of their coordinates. In our arguments, only the range of these sequences will be relevant.

If $\sigma \in T^{[n]}_\alpha$ and $\tau \in T^{[m]}_\alpha$, for some α, then, by abusing notation, we will write $\sigma \cup \tau$ is the sequence of length $n+m$ which enumerates the coordinates of σ and τ in $\leq_{lex}$-increasing order counting repetitions. We will also write $\sigma \subseteq \tau$ if σ is a subsequence of τ. $T^{[n]}$ will be considered as a tree with the coordinate-wise partial order induced by T. If $\sigma \in T^{[n]}$ and $\alpha < \mathrm{ht}(\sigma)$ we write $\sigma \restriction \alpha$ for the sequence $\langle \sigma(i) \restriction \alpha : i < n \rangle$. If $\sigma, \tau \in T^{[n]}$ are incomparable we will let

$$\Delta(\sigma, \tau) = \min\{\alpha : \sigma(i)(\alpha) \neq \tau(i)(\alpha), \text{ for some } i < n\}$$

and we will write $\sigma \wedge \tau$ for $\sigma \restriction \Delta(\sigma, \tau)$.

For $\sigma \in T^{[n]}$ let

$$D_\sigma = \langle D(\sigma(i), \sigma(0)) : i < n \rangle)$$

Suppose $\sigma, \tau \in T^{[n]}$ and $\mathrm{ht}(\sigma) \leq \mathrm{ht}(\tau)$. We say that the pair $\{\sigma, \tau\}$ is *regular* if $D_\tau \restriction \mathrm{ht}(\sigma) = D_\sigma$, i.e. for all $i < n$,

$$D(\tau(i), \tau(0)) \cap \mathrm{ht}(\sigma) = D(\sigma(i), \sigma(0)).$$

Note that in this case, for all $i, j < n$,

$$\Delta(\tau(i), \sigma(i)) = \Delta(\tau(j), \sigma(j)).$$

We say that a subset X of $T^{[n]}$ is *regular* if every pair of elements of X is regular. Note that if X is regular then so is the tree R_X generated by X. A *level sequence* of $T^{[n]}$ is a sequence $\{\sigma_\alpha : \alpha \in A\}$ where A is a subset of ω_1 and $\sigma_\alpha \in T^{[n]}_\alpha$, for all $\alpha \in A$. The following is a simple application of the Δ-system lemma and the Pressing Down Lemma.

Fact 2.1. *Let $\mathcal{A} = \{\sigma_\alpha : \alpha \in A\}$ be a level sequence. If A is uncountable (stationary) then there is an uncountable (stationary) subset B of A such that $\mathcal{B} = \{\sigma_\alpha : \alpha \in B\}$ is regular.* □

From now on we assume the conjunction of BPFA and φ. We are given a subset K of T and we want to find an uncountable antichain X in T such that $\wedge(X) \subseteq K$ or $\wedge(X) \cap K = \emptyset$. We will refer to K as a *coloring* of T. We first note that, for every integer n, K induces a coloring $K^{[n]}$ of $T^{[n]}$ defined by

$$K^{[n]} = K^n \cap T^{[n]}.$$

We let $\mathcal{F}_n$ be the collection of regular subtrees R of $T^{[n]}$ such that $\wedge(R) \cap K^{[n]} = \emptyset$. The following fact is immediate by using Fact 2.1.

Fact 2.2. *If $R \in \mathscr{F}_n^{\perp}$ then for every uncountable antichain $X \subseteq R$ there are $\sigma, \tau \in X$ such that $\sigma \wedge \tau \in K^{[n]}$.* $\square$

By $\varphi_0(\mathcal{F}_n)$ we can find a club C_n in ω_1 and a continuous increasing chain $\langle N_\xi^n : \xi \in C_n \rangle$ of countable subsets of $\mathcal{F}_n \cup (\mathcal{F}_n)^{\perp}$ witnessing $\varphi_0(\mathcal{F}_n)$. By replacing each of the C_n by their intersection we may assume that the C_n are all the same and equal to say C. We now define a new coloring of $T^{[n]} \restriction C$ as follows.

Definition 2.3. *A node $\sigma \in T^{[n]} \restriction C$ is in $K_\varphi^{[n]}$ if, letting α be the height of σ, there exists $R \in (\mathcal{F}_n)^{\perp} \cap N_\alpha^n$ such that $\sigma \in R$, i.e. if σ is in case (2) of the dichotomy for $\varphi_0(\mathcal{F}_n)$. We denote $(T^{[n]} \restriction C) \setminus K_\varphi^{[n]}$ by $L_\varphi^{[n]}$. We let $K_\varphi = \bigcup_n K_\varphi^{[n]}$ and $L_\varphi = \bigcup_n L_\varphi^{[n]}$.*

Remark 2.4. The induced coloring $T^{[n]} = K_\varphi^{[n]} \cup L_\varphi^{[n]}$, for $n < \omega$, is our analog of the notions of *acceptance* and *rejection* from [11]. The main difference is that these notions are defined in [11] relative to a given countable elementary submodel of $H(\omega_2)$ whereas our colorings do not make reference to any such model. This simplifies considerably the proof of properness of the main forcing notion we define in §3.

We now note some useful facts about these induced colorings.

Fact 2.5. *If there is a node t in $L_\varphi^{[1]}$ whose height is a limit point of C then there is an uncountable antichain X in T such that $\wedge(X) \cap K = \emptyset$, i.e. X is homogenous for $T \setminus K$.*

Proof. Assume t is such a node and let α be the height of t. By our assumption, case (1) of the dichotomy for $\varphi_0(\mathcal{F}_1)$ holds for t. Therefore, there exists $\eta < \alpha$ such that for every $\xi \in (\eta, \alpha) \cap C$ there is a $R \in \mathscr{F}_1 \cap N_\xi^1$ such that $t \restriction \xi \in R$. Since $(\eta, \alpha) \cap C$ is non empty if follows that $\mathcal{F}_1$ is non empty as well. Since members of $\mathcal{F}_1$ are precisely uncountable trees which are homogenous for $T \setminus K$, and every such tree contains an uncountable antichain, the conclusion follows. $\square$

Fact 2.6. *K_φ is closed for subsequences and L_φ is closed for supersequences which are in $\bigcup_n T^{[n]} \restriction C$.* $\square$

Fact 2.7. *If $\sigma \in K_\varphi$ and $\alpha = \mathrm{ht}(\sigma)$ is a limit of C then there is $\eta < \alpha$ such that $\sigma \restriction \xi \in K_\varphi$, for all $\xi \in (\eta, \alpha) \cap C$. Similarly for L_φ. We refer to this property as* continuity *of the induced coloring.* $\square$

Fact 2.8. *Suppose S is a stationary subset of C and $\mathcal{S} = \{\sigma_\xi : \xi \in S\}$ is a level sequence in $T^{[n]}$ consisting of elements of $K_\varphi^{[n]}$. Then there exist distinct $\xi, \eta \in S$ such that $\sigma_\xi \wedge \sigma_\eta \in K^{[n]}$.*

Proof. By using Fact 2.1 and shrinking S we may assume that that $\mathcal{S}$ is regular. For $\xi \in S$ since $\sigma_\xi \in K_\varphi^{[n]}$ there exists a tree $R_\xi \in N_\xi^n \cap (\mathcal{F}_n)^\perp$ such that $\sigma_\xi \in R_\xi$. By the Pressing Down Lemma and shrinking S again we may assume that the trees R_ξ are all the same and equal to say R. This simply means that $\mathcal{S} \subseteq R$. Since $R \in (\mathcal{F}_n)^\perp$, by Fact 2.2, there are $\xi \neq \eta \in S$ such that $\sigma_\eta \wedge \sigma_\xi \in K^{[n]}$, as required. □

Definition 2.9. *Let $\mathcal{S} = \{\sigma_\xi : \xi \in A\}$ be a regular level sequence in $T^{[n]}$, for some integer n. Then $\mathcal{P}_\mathcal{S}$ is the poset consisting of finite subsets p of $\mathcal{S}$ such that $\wedge(p) \cap K^{[n]} = \emptyset$, ordered by reverse inclusion.*

The following lemma is the main technical result of this section.

Lemma 2.10. *Let $\mathcal{S} = \{\sigma_\alpha : \alpha \in S\}$ and $\mathcal{Z} = \{\tau_\gamma : \gamma \in Z\}$ be two regular level sequences in $T^{[n]}$ and $T^{[m]}$ respectively such that S is a stationary subset of C and $\mathcal{S} \subseteq K_\varphi^{[n]}$. Assume that, for every $\alpha \in S$ and $\gamma \in Z$, if $\alpha < \gamma$ then*

$$\sigma_\alpha \cup \tau_\gamma \restriction \alpha \in L_\varphi^{[n+m]}.$$

Then $\mathcal{P}_\mathcal{Z}$ is c.c.c.

Proof. Before starting the proof, notice that by using the Pressing Down Lemma and shrinking S if necessary we may assume that there is a fixed tree $R_0 \in (\mathcal{F}_n)^\perp$ witnessing that σ_α is in $K_\varphi^{[n]}$, for all $\alpha \in S$. By shrinking S and Z if necessary we may moreover assume that for every $\alpha \in S, \gamma \in Z$, every $i < n$ and $j < m$, $\sigma_\alpha(i)$ and $\tau_\gamma(j)$ are incomparable in T.

Now, assume $\mathcal{A}$ is an uncountable subset of $\mathcal{P}_\mathcal{Z}$. We need to find distinct p and q in $\mathcal{A}$ which are compatible, i.e. such that $\wedge(p \cup q) \cap K^{[m]} = \emptyset$. By a standard Δ-system argument we can assume that all elements of $\mathcal{A}$ have a fixed size k and are mutually disjoint. For each $\alpha \in S$ we pick an element p_α of $\mathcal{A}$ such that $\mathrm{ht}(\tau) \geq \alpha$, for all $\tau \in p_\alpha$.

Fix for a moment one such α. Since by our assumption $\sigma_\alpha \cup \tau \restriction \alpha \in L_\varphi^{[n+m]}$, for all $\tau \in p_\alpha$, we can fix an ordinal $\eta_\alpha < \alpha$ such that for every $\xi \in (\eta_\alpha, \alpha) \cap C$ and every $\tau \in p_\alpha$ there is a tree $R \in \mathcal{F}_{n+m} \cap N_\xi^{n+m}$ such that $\sigma_\alpha \restriction \xi \cup \tau \restriction \xi \in R$. By applying the Pressing Down Lemma and shrinking S again we may assume that all the ordinals η_α are equal to some η_0.

Now, for each $\alpha \in S$, fix an enumeration $\{\upsilon_\alpha^0, \ldots, \upsilon_\alpha^{l_\alpha - 1}\}$ of distinct elements of $\{\tau \restriction \alpha : \tau \in p_\alpha\}$. We may assume that there is a fixed integer l such that $l_\alpha = l$, for all $\alpha \in S$. Moreover, by shrinking S further, we may assume that if $\alpha, \beta \in S$ are distinct then υ_α^i and υ_β^j are incomparable, for all $i, j < l$. For each $\alpha \in S$ let

$$F_\alpha = \{\sigma_\alpha(j) : j < n\} \cup \{\upsilon_\alpha^i(j) : i < l, j < m\}$$

and let

$$D_\alpha = \bigcup\{D(s,t) : s, t \in F_\alpha\}.$$

Then D_α is finite, so if α is a limit ordinal and we let $\xi_\alpha = \max(D_\alpha) + 1$ then $\xi_\alpha < \alpha$. By the Pressing Down Lemma and shrinking S yet again we may assume that there exists a fixed ordinal ξ, a sequence $\sigma \in T_\xi^{[n]}$, and sequences $\upsilon^i \in T_\xi^{[m]}$, for $i < l$, such that, for each $\alpha \in S$, we have:

(1) $\xi_\alpha = \xi$,
(2) $\sigma_\alpha \restriction \xi = \sigma$,
(3) $\upsilon_\alpha^i \restriction \xi = \upsilon^i$, for $i < l$.

Now, notice that if $\alpha, \beta \in S$ are distinct then, for every $i < l$,

$$\Delta(\upsilon_\alpha^i, \upsilon_\beta^i) = \Delta(\sigma_\alpha, \sigma_\beta).$$

Moreover, if i and j are distinct then

$$\upsilon_\alpha^i \wedge \upsilon_\beta^j = \upsilon_\alpha^i \wedge \upsilon_\alpha^j = \upsilon_\beta^i \wedge \upsilon_\beta^j \notin K^{[m]}.$$

Therefore, p_α and p_β are compatible in $\mathcal{P}_{\mathcal{Z}}$ provided $\upsilon_\alpha^i \wedge \upsilon_\beta^i \notin K^{[m]}$, for all $i < l$. We have finally set the stage for the proof of the lemma.

Fix a sufficiently large regular cardinal θ and a countable elementary submodel M of $H(\theta)$ containing all the relevant objects and such that $\delta = M \cap \omega_1$ belongs to S. Working in M fix a countable elementary submodel N of $H(\omega_2)$ containing all the relevant objects and let $\zeta = N \cap \omega_1$. Since, by our assumption $\eta_0, C \in N$, C is a club, and $N \in M$ we have that $\zeta \in (\eta_0, \delta) \cap C$. Therefore for each $i < l$ there exists a tree $A_i \in \mathcal{F}_{n+m} \cap N_\zeta^{n+m}$ such that $\sigma_\delta \restriction \zeta \cup \upsilon_\delta^i \restriction \zeta \in A_i$. Since $N_\zeta^{n+m} \subseteq N$ we know that $A_i \in N$, for all i. Let

$$H = \{\eta : \exists \alpha \in S[\alpha > \eta \wedge \forall i < l(\sigma_\alpha \restriction \eta \cup \upsilon_\alpha^i \restriction \eta \in A_i)]\}$$

Since all the parameters in the definition of H are in N, by elementarity of N it follows that $H \in N$. On the other hand $\zeta \in H \setminus N$, therefore H is uncountable. Fix a $1-1$ function $f : H \to S$ with $f \in N$ such that for every $\eta \in H$, $f(\eta)$ witnesses that $\eta \in H$. Then the set $X = \{\sigma_{f(\eta)} : \eta \in H\}$ belongs to N. We also know that X is an uncountable subset of R_0. Since $T^{[n]}$ is a special tree, by shrinking H we may assume that $Y = \{\sigma_{f(\eta)} \restriction \eta : \eta \in H\}$ is an antichain in $T^{[n]}$. Since $Y \subseteq R_0$ and $R_0 \in (\mathcal{F}_n)^\perp$, by Fact 2.2, there are distinct $\eta, \rho \in H$ such that:

$$\sigma_{f(\eta)} \wedge \sigma_{f(\rho)} = \sigma_{f(\eta)} \restriction \eta \wedge \sigma_{f(\rho)} \restriction \rho \in K^{[n]}.$$

Let $\alpha = f(\eta)$ and $\beta = f(\rho)$. We claim that p_α and p_β are compatible in $\mathcal{P}_{\mathcal{Z}}$. To see this, consider some $i < l$. We know that $\sigma_\alpha \restriction \eta \cup \upsilon^i_\alpha \restriction \eta$ and $\sigma_\beta \restriction \rho \cup \upsilon^i_\beta \restriction \rho$ belong to A_i. Therefore,

$$(\sigma_\alpha \cup \upsilon^i_\alpha) \wedge (\sigma_\beta \cup \upsilon^i_\beta) = (\sigma_\alpha \restriction \eta \cup \upsilon^i_\alpha \restriction \eta) \wedge (\sigma_\beta \restriction \rho \cup \upsilon^i_\beta \restriction \rho) \notin K^{[n+m]}.$$

Since $\sigma_\alpha \wedge \sigma_\beta \in K^{[n]}$ it follows that

$$\upsilon^i_\alpha \wedge \upsilon^i_\beta \notin K^{[m]}.$$

Since this is true for all i it follows that p_α and p_β are compatible. $\square$

Lemma 2.11 ($\mathrm{MA}_{\aleph_1}$)**.** *Let $\mathcal{S} = \{\sigma_\alpha : \alpha \in S\}$ and $\mathcal{Z} = \{\tau_\gamma : \gamma \in Z\}$ be two regular level sequences in $T^{[n]}$ and $T^{[m]}$ respectively such that S and Z are stationary subsets of C. Assume $\mathcal{S} \subseteq K^{[n]}_\varphi$ and $\mathcal{Z} \subseteq K^{[m]}_\varphi$. Then there exist $\alpha \in S$ and $\gamma \in Z$ such that $\alpha < \gamma$ and*

$$\sigma_\alpha \cup \tau_\gamma \restriction \alpha \in K^{[n+m]}_\varphi.$$

Proof. For every $\gamma \in Z$ fix a tree $R_\gamma \in (\mathcal{F}_m)^\perp \cap N^m_\gamma$ such that $\tau_\gamma \in R_\gamma$, i.e. witnessing that $\tau_\gamma \in K^{[m]}$. Since Z is stationary by the Pressing Down Lemma and shrinking Z if necessary we may assume that all the R_γ are equal to some tree R. Assume towards contradiction that for every $\alpha \in S$ and $\gamma \in Z$, if $\alpha < \gamma$ then $\sigma_\alpha \cup \tau_\gamma \restriction \alpha \in L^{[n+m]}_\varphi$. By Lemma 2.10 $\mathcal{P}_{\mathcal{Z}}$ is c.c.c. By $\mathrm{MA}_{\aleph_1}$ we can find an uncountable subset Y of Z such that $\tau_\alpha \wedge \tau_\beta \notin K^{[m]}$, for every distinct $\alpha, \beta \in Y$. This means that the tree R^* generated by $\{\tau_\alpha : \alpha \in Y\}$ belongs to $\mathcal{F}_m$. However, $R^* \subseteq R$ and R is orthogonal to all trees in $\mathcal{F}_m$, a contradiction. $\square$

3. The forcing $\partial^*(K)$

In this section we define a notion of forcing $\partial^*(K)$ and prove that it is proper. We then show that either there is an uncountable subset Y of T such that $\wedge(Y) \cap K = \emptyset$ or forcing with $\partial^*(K)$ adds an uncountable subset X of $T \restriction C$ such that $\pi(X) = X$ and X is homogenous for K_φ. Then it will be easy to force again and obtain an uncountable subset Z of X such that $\wedge(Z) \subseteq K$. Before we start it will be convenient to define a certain club of countable elementary submodels of $H(\omega_2)$. Fix, for each $\delta < \omega_1$, a bijection $e_\delta : \omega \to T^{[n]}_\delta$.

Definition 3.1. *$\mathcal{E}$ is the collection of all countable elementary submodels M of $H(\omega_2)$ such that $T, C, K, \langle e_\delta : \delta < \omega_1\rangle$ as well as $\langle N^n_\xi : \xi \in C\rangle$, for $n < \omega$, all belong to M.*

We are now in the position to define the partial order $\partial^*(K)$.

Definition 3.2. *$\partial^*(K)$ consists of all pairs $(X_p, \mathcal{M}_p)$ such that:*

(1) *X_p is a finite subset of $T \restriction C$, $\pi(X_p) = X_p$, and $X_p \cap T_\alpha \in K_\varphi$, for all $\alpha \in \mathrm{lev}(X_p)$*[2].
(2) *$\mathcal{M}_p$ is a finite $\in$-chain of elements of $\mathcal{E}$ such that for every $x \in X_p$ there is $M \in \mathcal{M}_p$ such that $\mathrm{ht}(x) = M \cap \omega_1$.*

The order of $\partial^*(K)$ is the coordinatewise reverse inclusion, i.e. $q \leq p$ iff $X_p \subseteq X_q$ and $\mathcal{M}_p \subseteq \mathcal{M}_q$.

In what follows, for $p \in \partial^*(K)$, M_p^i denotes the i-th model in $\mathcal{M}_p$, in the enumeration induced by the heights of the models.

Theorem 3.3. *$\partial^*(K)$ is a proper forcing notion.*

Proof. Fix a countable $M \prec H(2^{|\partial^*(K)|^+})$ such that $\partial^*(K), \mathcal{E} \in M$. Given a condition $p = (X_p, \mathcal{M}_p) \in M$, we need to find $q \leq p$ that is $(\partial^*(K), M)$-generic. Set

$$q = (X_p, \mathcal{M}_p \cup \{M \cap H(\omega_2)\}).$$

We claim that q is as desired. To see this, fix a dense set $D \in M$ and a condition $r \leq q$. We need to find $s \in D \cap M$ which is compatible with r. By replacing r with a stronger condition we may assume that $r \in D$. Define

$$r' = (X_r \cap M, \mathcal{M}_r \cap M)$$

and

$$r^* = (X_r \setminus M, \mathcal{M}_r \setminus M)$$

and suppose that $|\mathcal{M}_r \setminus M| = l$. For every $i < l$ let $\delta_i = M_{r^*}^i \cap \omega_1$, let n_i be such that $e_{\delta_i}(n_i) = X_{r^*} \cap T_{\delta_i}$ and let $k_i = |X_{r^*} \cap T_{\delta_i}|$. We now define formulas θ_i, for $i < l$, by reverse induction on i.

$\theta_l(\xi_0, \dots, \xi_{l-1})$ holds if there is a condition $s = (X_s, \mathcal{M}_s) \in \partial^*(K)$ such that:

(1) $\mathcal{M}_s = \{M_s^0, \dots, M_s^{l-1}\}$,
(2) $M_s^i \cap \omega_1 = \xi_i$, for all $i < l$,
(3) $X_s \cap T_{\xi_i} = e_{\xi_i}(n_i)$ and $|X_s \cap T_{\xi_i}| = k_i$, for all $i < l$,
(4) $(X_{r'} \cup X_s, \mathcal{M}_{r'} \cup \mathcal{M}_s) \in D$.

Suppose θ_{i+1} has been defined for some $i < l$. Then

$$\theta_i(\xi_0, \dots, \xi_{i-1}) \text{ iff } Q\eta\, \theta_{i+1}(\xi_0, \dots, \xi_{i-1}, \eta).$$

Here $Q\eta\, \theta(\eta)$ means "there are stationary many η such that $\theta(\eta)$ holds".

Remark 3.4. Notice that the parameters of each $\theta_i(\xi_0, \dots, \xi_{i-1})$ are in M, so if $\xi_0, \dots, \xi_{i-1} \in M$ then $\theta_i(\xi_0, \dots, \xi_{i-1})$ holds iff it holds in M. Thus, if W_i is the set of tuples $(\xi_0, \dots, \xi_{i-1})$ such that $\theta_i(\xi_0, \dots, \xi_{i-1})$ holds then $W_i \in M$, for all $i \leq l$. We set $W = \bigcup_{i \leq l} W_i$.

Notice also that if $\theta_i(\xi_0, \dots, \xi_{i-1})$ holds then $e_{\xi_j}(n_j) \in K_\varphi^{[k_j]}$, for all $j < i$.

[2] Here, of course, we identify $X_p \cap T_\alpha$ with its $\leq_{lex}$-increasing enumeration.

Claim 3.5. *$\theta_i(\delta_0, \ldots, \delta_{i-1})$ holds, for all $i \leq l$.*

Proof. We prove this by reverse induction on i. Notice that $\theta_l(\delta_0, \ldots, \delta_{l-1})$ holds as witnessed by the condition r^*. Suppose we have established $\theta_{i+1}(\delta_0, \ldots, \delta_i)$. Since $W_{i+1} \in M \cap H(\omega_2)$ and $M \cap H(\omega_2) = M^0_{r^*} \subseteq M^i_{r^*}$ it follows that the set

$$Z = \{\eta : (\delta_0, \ldots, \delta_{i-1}, \eta) \in W_{i+1}\}$$

also belongs to $M^i_{r^*}$. If Z were non stationary, by elementarity, there would be a club $E \in M^i_{r^*}$ disjoint from it, but $\delta_i \in Z$ and δ_i belongs to any club which is in $M^i_{r^*}$, a contradiction. $\square$

By Claim 3.5 we can pick a *stationary splitting tree* $U \subseteq W$. This means that $U \subseteq (\omega_1)^{\leq l}$ is a tree and for every node $t = (\xi_0, \ldots, \xi_{i-1}) \in U$ of height $i < l$ the set

$$S_t = \{\eta : (\xi_0, \ldots, \xi_{i-1}, \eta) \in U\}$$

is stationary. We can moreover assume that $U \in M$. We now build by induction an increasing sequence $(\xi_i : i < l)$ of ordinals in M such that:

(1) $(\xi_0, \ldots, \xi_i) \in U$, for all i,
(2) $e_{\xi_i}(n_i) \cup e_{\delta_0}(n_0) \restriction \xi_i \in K_\varphi^{[k_i+k_0]}$, for all i.

Suppose $j < l$ and we have picked ξ_i, for all $i < j$. Consider the set $S_j = \{\eta : (\xi_0, \ldots, \xi_{j-1}, \eta) \in U\}$.

Claim 3.6. *There is $\xi \in S_j \cap M$ such that $e_\xi(n_j) \cup e_{\delta_0}(n_0) \restriction \xi \in K_\varphi^{[k_j+k_0]}$.*

Proof. Assume otherwise. We know that S_j is stationary and that the level sequence $\mathcal{S}_j = \{e_\eta(n_j) : \eta \in S_j\}$ is contained in $K_\varphi^{[k_j]}$. By shrinking S_j we may also assume that $\mathcal{S}_j$ is regular. Let

$$Z = \{\eta : e_\eta(n_0) \in K_\varphi^{[k_0]} \wedge \forall \xi \in S_j \cap \eta [e_\xi(k_j) \cup e_\eta(k_0) \restriction \xi \notin K_\varphi^{[k_j+k_0]}]\}.$$

Then $Z \in M$ and since we assumed that $\delta_0 \in Z$ it follows that Z is stationary. By shrinking Z we may assume that the level sequence $\mathcal{Z} = \{e_\eta(k_0) : \eta \in Z\}$ is regular. Now, by Lemma 2.11 and $\mathrm{MA}_{\aleph_1}$ we obtain a contradiction. $\square$

Suppose $(\xi_0, \ldots, \xi_{l-1})$ has been constructed. Since $(\xi, \ldots, \xi_{l-1}) \in U \cap M$, by elementarity there is a condition $s \in \partial^*(K) \cap M$ witnessing this fact. Let

$$\bar{s} = (X_{r'} \cup X_s, \mathcal{M}_{r'} \cup \mathcal{M}_s).$$

Then by (4) in the statement of $\theta_l(\xi_0, \ldots, \xi_{l-1})$ we know that $\bar{s} \in D$. Since s and r' are both in M so is $\bar{s}$. We claim that $\bar{s}$ is compatible with r. To see this we define a condition u as follows. Let

$$X_u = \pi(X_r \cup X_s).$$

Note that $\mathrm{lev}(X_u) = \mathrm{lev}(X_{r'}) \cup \mathrm{lev}(X_s) \cup \mathrm{lev}(X_{r^*})$ and we have

$$X_u \cap T_\alpha = \begin{cases} e_{\delta_i}(n_i) & \text{if } \alpha = \delta_i, \text{ for some } i < l, \\ e_{\xi_i}(n_i) \cup e_{\delta_0}(n_0) \restriction \xi_i & \text{if } \alpha = \xi_i, \text{ for some } i < l, \\ X_{r'} \cap T_\alpha & \text{if } \alpha \in \mathrm{lev}(X_{r'}). \end{cases}$$

In all cases we have that $X_u \cap T_\alpha \in K_\varphi$. We let $\mathcal{M}_u = \mathcal{M}_{r'} \cup \mathcal{M}_s \cup M_{r^*}$. It follows that $u \leq \bar{s}, r$. This completes the proof of Theorem 3.3.

$\square$

4. The main theorem

In this section we complete the proof of the main theorem saying that the conjunction of BPFA and φ implies CAT. Let G be V-generic over the poset $\partial^*(K)$ and define in $V[G]$:

$$X_G = \bigcup\{X_p : p \in G\}.$$

Note that $\pi(X_G) = X_G$, $\mathrm{lev}(X_G) \subseteq C$, and every finite subset of X_G contained in one level of T is in K_φ. Let $\dot{X}_G$ be a canonical $\partial^*(K)$-name for X_G. We first establish the following fact in the ground model V.

Lemma 4.1. *Suppose there is no uncountable antichain Y in T such that $\wedge(Y) \cap T = \emptyset$. Then there is a condition $p \in \partial^*(K)$ which forces that $\dot{X}_G$ is uncountable.*

Proof. Suppose the maximal condition forces that $\dot{X}_G$ is countable. Let θ be a sufficiently large regular cardinal and let M be a countable elementary submodel of $H(\theta)$ containing all the relevant objects. As shown in Theorem 3.3 $q = (\emptyset, \{M \cap H(\omega_2)\})$ is an $(M, \partial^*(K))$-generic condition. Therefore, $q \Vdash \dot{X}_G \subseteq M$. If there is a node t in $K_\varphi^{[1]} \cap T_\delta^{[1]}$ then $r = (\{t\}, \{M \cap H(\omega_2)\})$ is a condition stronger than q and $r \Vdash t \in \dot{X}_G$, a contradiction. Assume now that $T_\delta^{[1]} \subseteq L_\varphi^{[1]}$. Since δ is a limit point of C, by Fact 2.5 there is an uncountable antichain Y in T such that $\wedge(Y) \cap K = \emptyset$, as desired. $\square$

Now, assume there is no uncountable antichain $Y \subseteq T$ such that $\wedge(Y) \cap K = \emptyset$ and fix a V-generic G over $\partial^*(K)$ containing a condition as in Lemma 4.1. We work in $V[G]$. We can show that $\mathrm{lev}(X_G)$ is a club, but this is not necessary. Namely, let $\bar{X}_G$ be the closure of X_G in the tree topology. Then by Fact 2.7 all finite subsets of $\bar{X}_G$ contained in one level of T are in K_φ. Moreover, $\mathrm{lev}(\bar{X}_G)$ is equal to the closure of $\mathrm{lev}(X_G)$ in the order topology and is a club. Clearly, we also have $\pi(\bar{X}_G) = \bar{X}_G$.

Remark 4.2. Before continuing it is important to note a certain amount of absoluteness between V and $V[G]$. In V we defined $(\mathcal{F}_n)^V$ to be the collection of subtrees R of $T^{[n]}$ such that $\wedge(R) \cap K^{[n]} = \emptyset$. The same definition in $V[G]$ gives a larger collection $\mathcal{F}_n^{V[G]}$ of subtrees of $T^{[n]}$. Nevertheless, the definition of $\mathcal{F}_n$ is Σ_1 with parameters T and K. Since BPFA holds in V, if a certain tree $A \in V$ is in $(\mathcal{F}_n^V)^\perp$ then it is also in $(\mathcal{F}_n^{V[G]})^\perp$. It follows that the same sequences $\langle N_\xi^n : \xi \in C\rangle$ witness $\varphi(\mathcal{F}_n)$ in V and in $V[G]$, for all n. Therefore the definitions of the induced colorings $K_\varphi^{[n]}$, for $n < \omega$, are also absolute between V and $V[G]$.

Definition 4.3. *The poset $\mathcal{Q}$ consists of finite antichains p in $\bar{X}_G$ such that $\wedge(p) \subseteq K$, ordered by reverse inclusion.*

Claim 4.4. *$\mathcal{Q}$ is a c.c.c. poset.*

Proof. Suppose $\mathcal{A}$ is an uncountable subset of $\mathcal{Q}$. We need to find two elements of A which are compatible. By a standard Δ-system argument we may assume that the elements of $\mathcal{A}$ are disjoint and have the same size. For each $\alpha \in \operatorname{lev}(\bar{X}_G)$ choose $p_\alpha \in \mathcal{A}$ such that $\operatorname{ht}(t) \geq \alpha$, for all $t \in p$. We can assume that the p_α are distinct. Let σ_α be the enumeration in $\leq_{lex}$-increasing order of distinct elements of $\{t \restriction \alpha : t \in p_\alpha\}$. There is a stationary subset S of $\operatorname{lev}(\bar{X}_G)$ and an integer n such that σ_α has size n, for all $\alpha \in S$. Note that $\sigma_\alpha \in K_\varphi^{[n]}$, for all $\alpha \in S$. By shrinking S further we may assume that $\{\sigma_\alpha : \alpha \in S\}$ is a regular level sequence and that for every $\alpha, \beta \in S$ and every distinct $i, j < n$

$$\sigma_\alpha(i) \wedge \sigma_\beta(j) = \sigma_\alpha(i) \wedge \sigma_\alpha(j) \in K.$$

Now, by Fact 2.8 we can find distinct $\alpha, \beta \in S$ such that $\sigma_\alpha \wedge \sigma_\beta \in K^{[n]}$, i.e. $\sigma_\alpha(i) \wedge \sigma_\beta(i) \in K$, for all $i < n$. It follows that p_α and p_β are compatible in $\mathcal{Q}$. □

By a standard argument there is a condition $q \in \mathcal{Q}$ which forces the $\mathcal{Q}$-generic H to be uncountable. Therefore, by forcing with $\mathcal{Q}$ below q over $V[G]$ we obtain an uncountable antichain H of T such that $\wedge(H) \subseteq K$. Since $\partial^*(K) * \mathcal{Q}$ is proper, by BPFA, we have such an antichain in V. Thus, we have proved the main theorem which we now state.

Theorem 4.5. *Assume* BPFA *and φ. Then* CAT *holds and hence there is a five element basis for the uncountable linear orders.* □

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