# Examples of groups which are not weakly amenable

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Abstract. We prove that weak amenability of a locally compact group imposes a strong condition on its amenable closed normal subgroups. This extends non weak amenability results of Haagerup (1988) and Ozawa–Popa (2010). A von Neumann algebra analogue is also obtained.

## 1. Introduction

Let $G$ be a group, which is always assumed to be a locally compact topological group. The group $G$ is said to be *weakly amenable* if the Fourier algebra $\mathcal{A}G$ of $G$ has an approximate identity $(\varphi_n)$ which is uniformly bounded as Herz–Schur multipliers. (If one requires $(\varphi_n)$ to be bounded as elements in $\mathcal{A}G$, it becomes one of the equivalent definitions of amenability.) See Section 2 for the precise definition. Weak amenability is strictly weaker than amenability and passes to closed subgroups. It is proved by De Cannière–Haagerup, Cowling and Cowling–Haagerup ([dCH, Co, CH]) that real simple Lie groups of real rank one are weakly amenable (see also [Oz]), and by Haagerup ([Ha]) that real simple Lie groups of real rank at least two are not weakly amenable. For the latter fact, Haagerup proves that $\mathrm{SL}(2,\mathbb{R}) \ltimes \mathbb{R}^2$ is not weakly amenable. (See also [Do].) More recently, it is proved by Ozawa–Popa ([OP]) that the wreath product $\Lambda \wr \Gamma$ of a non-trivial group $\Lambda$ by a non-amenable discrete group $\Gamma$ is not "weakly amenable with constant 1." In this paper, we generalize these non weak amenability results as follows.

**Theorem A.** *Let $G$ be an weakly amenable group and $N$ be an amenable closed normal subgroup of $G$. Then, there is a $G \ltimes N$-invariant state on $L^\infty(N)$, where the semidirect product $G \ltimes N$ acts on $N$ by $(g,a)\cdot x = gaxg^{-1}$.*

In particular, the wreath product by a non-amenable group is never weakly amenable. The theorem also gives a new proof of Haagerup's result that $\mathrm{SL}(2,\mathbb{Z}) \ltimes \mathbb{Z}^2$ is not weakly amenable, without appealing to the lattice embedding into $\mathrm{SL}(2,\mathbb{R}) \ltimes \mathbb{R}^2$. We note for the sake of completeness that there is an even weaker variant of weak amenability, called the *approximation property* ([HK]), and $\mathrm{SL}(2,\mathbb{R}) \ltimes \mathbb{R}^2$ has the approximation property, while $\mathrm{SL}(n \geq 3,\mathbb{R})$ does not ([LdS]).

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As Theorem 3.5 in [OP], there is an analogous result for von Neumann algebras. We refer to Section 3 in [OP] and Section 4 of this paper for the terminology used in the following theorem.

**Theorem B.** *Let $M$ be a finite von Neumann algebra with the weak$^*$ completely bounded approximation property. Then, every amenable von Neumann subalgebra $P$ is weakly compact in $M$.*

It follows that a type $\mathrm{II}_1$ factor having the weak$^*$ completely bounded approximation property and property (T) (e.g., the group von Neumann algebra of a torsion-free lattice in $\mathrm{Sp}(1,n)$) is not isomorphic to a group-measure-space von Neumann algebra.

## 2. Preliminary on Herz–Schur multipliers

Let $G$ be a group. We denote by $\lambda$ the left regular representation of $G$ on $L^2(G)$, by $C^*_\lambda G$ the reduced group C$^*$-algebra and by $\mathcal{L}G$ the group von Neumann algebra of $G$. The *Fourier algebra* $\mathcal{A}G$ of $G$ consists of all functions $\varphi$ on $G$ such that there are vectors $\xi,\eta\in L^2(G)$ satisfying $\varphi(x)=\langle\lambda(x)\xi,\eta\rangle$ for every $x\in G$. (In other words, $\mathcal{A}G=L^2(G)*L^2(G)$.) It is a Banach algebra with the norm $\|\varphi\|=\inf\{\|\xi\|\|\eta\|\}$, where the infimum is taken over all $\xi,\eta\in L^2(G)$ as above. The Fourier algebra $\mathcal{A}G$ is naturally identified with the predual of $\mathcal{L}G$ under the duality pairing $\langle\varphi,\lambda(f)\rangle=\int_G\varphi f$ for $\varphi\in\mathcal{A}G$ and $\lambda(f)\in\mathcal{L}G$. If $H$ is a closed subgroup of $G$, then $\varphi|_H\in\mathcal{A}H$ for every $\varphi\in\mathcal{A}G$. A continuous function $\varphi$ on $G$ is called a *Herz–Schur multiplier* if there are a Hilbert space $\mathcal{H}$ and bounded continuous functions $\xi,\eta\colon G\to\mathcal{H}$ such that $\varphi(y^{-1}x)=\langle\xi(x),\eta(y)\rangle$ for every $x,y\in G$. The Herz–Schur norm of $\varphi$ is defined by

$$\|\varphi\|_{\mathrm{cb}}=\inf\{\|\xi\|_\infty\|\eta\|_\infty\},$$

where the infimum is taken over all $\xi,\eta\in C(G,\mathcal{H})$ as above. The Banach space of Herz–Schur multipliers is denoted by $B_2(G)$. Clearly, one has a contractive embedding of $\mathcal{A}G$ into $B_2(G)$. The Herz–Schur norm $\|\varphi\|_{\mathrm{cb}}$ coincides with the cb-norm of the corresponding multipliers on $\mathcal{L}G$ or on $C^*_\lambda G$:

$$\|\varphi\|_{\mathrm{cb}}=\|m_\varphi\colon\mathcal{L}G\ni\lambda(f)\mapsto\lambda(\varphi f)\in\mathcal{L}G\|_{\mathrm{cb}}=\|m_\varphi|_{C^*_\lambda G}\|_{\mathrm{cb}}.$$

Indeed, $\|\varphi\|_{\mathrm{cb}}\geq\|m_\varphi\|_{\mathrm{cb}}$ is easy to see: Given a factorization $\varphi(x^{-1}y)=\langle\xi(x),\eta(y)\rangle$ with $\xi,\eta\in C(G,\mathcal{H})$, we define $V_\xi\colon L^2(G)\to L^2(G,\mathcal{H})$ by $(V_\xi f)(x)=f(x)\xi(x^{-1})$, and likewise for $V_\eta$. Then, $\lambda(\varphi f)=V_\eta^*(\lambda(f)\otimes 1_{\mathcal{H}})V_\xi$ and $\|m_\varphi\|_{\mathrm{cb}}\leq\|\xi\|_\infty\|\eta\|_\infty$. We will give a proof of the converse inequality in Lemma 1, but sketch it here in the case of amenable groups. Let $N$ be an amenable group and $\varphi\in B_2(N)$. Since the unit character $\tau_0$ is continuous on $C^*_\lambda N$, the linear functional $\omega_\varphi=\tau_0\circ m_\varphi$ is bounded on $C^*_\lambda N$ and satisfies $\|\omega_\varphi\|\leq\|\varphi\|_{\mathrm{cb}}$. Let $(\pi,\mathcal{H})$ be the GNS representation for $|\omega_\varphi|$ and view $\pi$ as a continuous unitary $N$-representation. Then, there are vectors $\xi,\eta\in\mathcal{H}$ such that $\|\xi\|\|\eta\|=\|\omega_\varphi\|$ and $\varphi(x)=\langle\pi(x)\xi,\eta\rangle$ for every $x\in N$. (Hence, $\|\omega_\varphi\|=\|\varphi\|_{\mathrm{cb}}$.)

**Definition.** Let $G$ be a group. By an *approximate identity* on $G$, we mean a net $(\varphi_n)$ in $\mathcal{A}G$ which converges to 1 uniformly on compacta. It is *completely bounded* if

$$\|(\varphi_n)\|_{\mathrm{cb}} := \sup_n \|\varphi_n\|_{\mathrm{cb}} < +\infty.$$

A group $G$ is said to be *weakly amenable* if there is a completely bounded approximate identity on $G$. The Cowling–Haagerup constant $\Lambda_{\mathrm{cb}}(G)$ is defined to be

$$\Lambda_{\mathrm{cb}}(G) = \inf\{\|(\varphi_n)\|_{\mathrm{cb}} : (\varphi_n) \text{ a c.b.a.i. on } G\}.$$

Note that the above infimum is attained. See [CH, BO] for more information.

It is easy to see that if $H \leq G$ is a closed subgroup, then $\Lambda_{\mathrm{cb}}(H) \leq \Lambda_{\mathrm{cb}}(G)$. On this occasion, we record that the same inequality holds also for a "random" or "ME" subgroup in the sense of [Mo, Sa] (cf. [CZ]). For this, we only consider countable discrete groups $\Lambda$ and $\Gamma$. Recall that $\Lambda$ is an ME subgroup of $\Gamma$ if there is a standard measure space $\Omega$ on which $\Lambda \times \Gamma$ acts by measure-preserving transformations in such a way that each of $\Lambda$- and $\Gamma$-actions admits a fundamental domain and the measure of $\Omega_\Gamma := \Omega/\Gamma$ is finite. The action $\Lambda \curvearrowright \Omega$ gives rise to a measure-preserving action $\Lambda \curvearrowright \Omega_\Gamma$ and a measurable cocycle $\alpha\colon \Lambda \times \Omega_\Gamma \to \Gamma$ such that the action $\Lambda \curvearrowright \Omega$ is isomorphic (up to null sets) to the twisted action $\Lambda \curvearrowright \Omega_\Gamma \times \Gamma$, given by $a(t, g) = (at, \alpha(a, t)g)$ for $a \in \Lambda$, $t \in \Omega_\Gamma$ and $g \in \Gamma$. The map $\alpha$ satisfies the cocycle identity: $\alpha(ab, t) = \alpha(a, bt)\alpha(b, t)$ for every $a, b \in \Lambda$ and a.e. $t \in \Omega_\Gamma$. For $\varphi \in B_2(\Gamma)$, we denote the "induced" function on $\Lambda$ by $\varphi_\alpha$:

$$\varphi_\alpha(a) = \int_{\Omega_\Gamma} \varphi(\alpha(a, t))\, dt.$$

Here, we normalized the measure so that $|\Omega_\Gamma| = 1$. Since

$$\varphi_\alpha(b^{-1}a) = \int_{\Omega_\Gamma} \varphi(\alpha(b, b^{-1}at)^{-1}\alpha(a, t))\, dt = \int_{\Omega_\Gamma} \varphi(\alpha(b, b^{-1}t)^{-1}\alpha(a, a^{-1}t))\, dt,$$

one has $\varphi_\alpha \in B_2(\Lambda)$ and $\|\varphi_\alpha\|_{\mathrm{cb}} \leq \|\varphi\|_{\mathrm{cb}}$. Suppose now that $\varphi \in \mathcal{A}\Gamma$. Then, $\varphi_\alpha$ is a coefficient of the unitary $\Lambda$-representation $\sigma$ on $L^2(\Omega)$ induced by the measure-preserving action $\Lambda \curvearrowright \Omega$, i.e., there are $\xi, \eta \in L^2(\Omega)$ such that $\varphi_\alpha(a) = \langle \sigma(a)\xi, \eta\rangle$. Since $\Omega$ admits a $\Lambda$-fundamental domain, $\sigma$ is a multiple of the regular representation and $\varphi_\alpha \in \mathcal{A}\Lambda$. By inducing an approximate identity on $\Gamma$, one sees that if $\Gamma$ is weakly amenable, then so is $\Lambda$ and $\Lambda_{\mathrm{cb}}(\Lambda) \leq \Lambda_{\mathrm{cb}}(\Gamma)$.

## 3. Proof of Theorem A

**Lemma 1.** *Let $N$ be an amenable closed normal subgroup of $G$ and $\varphi \in B_2(G)$. Then, there are a Hilbert space $\mathcal{H}$, functions $\xi, \eta \in C(G, \mathcal{H})$ and a continuous unitary representation $\pi$ of $N$ on $\mathcal{H}$ such that*

- $\|\xi\|_\infty = \|\eta\|_\infty = \|\varphi\|_{\mathrm{cb}}^{1/2}$*;*
- $\varphi(y^{-1}x) = \langle \xi(x), \eta(y)\rangle$ *for every $x, y \in G$;*
- $\pi(a)\xi(x) = \xi(ax)$ *and* $\pi(a)\eta(y) = \eta(ay)$ *for every $a \in N$ and $x, y \in G$.*

*Proof.* We follow Jolissaint's simple proof ([Jo]) of the inequality $\|\varphi\|_{\rm cb} \leq \|m_\varphi\|_{\rm cb}$. Since $N$ is amenable, the quotient map $q\colon G \to G/N$ extends to a $*$-homomorphism $q\colon C^*_\lambda G \to C^*_\lambda(G/N)$ between the reduced group C$^*$-algebras. Since $q\circ m_\varphi$ is completely bounded on $C^*_\lambda G$, a Stinespring type factorization theorem (Theorem B.7 in [BO]) yields a $*$-representation $\pi\colon C^*_\lambda G \to \mathbb{B}(\mathcal{H})$ and operators $V, W \in \mathbb{B}(L^2(G/N),\mathcal{H})$ such that $\|V\| = \|W\| \leq \|q\circ m_\varphi\|_{\rm cb}^{1/2}$ and $(q\circ m_\varphi)(X) = W^*\pi(X)V$ for $X \in C^*_\lambda G$. We view $\pi$ as a continuous unitary representation of $G$. Then, for a fixed unit vector $\zeta \in L^2(G/N)$, the maps $\xi(x) = \pi(x)V\lambda_{G/N}(q(x^{-1}))\zeta$ and $\eta(y) = \pi(y)W\lambda_{G/N}(q(y^{-1}))\zeta$ are continuous, $\|\xi\|_\infty, \|\eta\|_\infty \leq \|m_\varphi\|_{\rm cb}^{1/2}$ and $\varphi(y^{-1}x) = \langle \xi(x), \eta(y)\rangle$ for every $x, y \in G$. Moreover, $\pi(a)\xi(x) = \xi(ax)$ for $a \in N$, because $\lambda_{G/N}(a) = 1$. □

We denote by $\varphi^g$ the right translation of a function $\varphi$ by $g \in G$, i.e., $\varphi^g(x) = \varphi(xg^{-1})$.

**Lemma 2.** *Let $N$ be an amenable group, $\varphi \in B_2(N)$ and $a \in N$. Then,*

$$\|\frac{1}{2}\big(\varphi + \varphi^a\big)\|_{\rm cb}^2 + \|\frac{1}{2}\big(\varphi - \varphi^a\big)\|_{\rm cb}^2 \leq \|\varphi\|_{\rm cb}^2.$$

*Proof.* There are a continuous unitary representation $\pi$ of $N$ on a Hilbert space $\mathcal{H}$ and vectors $\xi, \eta \in \mathcal{H}$ such that $\|\xi\| = \|\eta\| = \|\varphi\|_{\rm cb}^{1/2}$ and $\varphi(x) = \langle \pi(x)\xi, \eta\rangle$ for every $x \in N$. Since $\big(\varphi \pm \varphi^a\big)(x) = \langle \pi(x)(\xi \pm \pi(a^{-1})\xi), \eta\rangle$, one has

$$\|\varphi + \varphi^a\|_{\rm cb}^2 + \|\varphi - \varphi^a\|_{\rm cb}^2 \leq \|\xi + \pi(a^{-1})\xi\|^2\|\eta\|^2 + \|\xi - \pi(a^{-1})\xi\|^2\|\eta\|^2 = 4\|\varphi\|_{\rm cb}^2.$$

□

For $\varphi \in B_2(G)$, we define $\varphi^*(x) := \overline{\varphi(x^{-1})}$, and say $\varphi$ is *self-adjoint* if $\varphi^* = \varphi$. For any $\varphi \in B_2(G)$, the function $(\varphi + \varphi^*)/2$ is self-adjoint and $\|(\varphi + \varphi^*)/2\|_{\rm cb} \leq \|\varphi\|_{\rm cb}$. Thus every approximate identity can be made self-adjoint without increasing norm. We fix a closed subgroup $N$ of $G$. A completely bounded approximate identity $(\varphi_n)$ on $G$ is said to be *$N$-optimal* if all $\varphi_n$ are self-adjoint, $\|(\varphi_n)\|_{\rm cb} = \Lambda_{\rm cb}(G)$ and

$$\|(\varphi_n|_N)\|_{\rm cb} = \inf\{\|(\psi_n|_N)\|_{\rm cb} : (\psi_n) \text{ a c.b.a.i. such that } \|(\psi_n)\|_{\rm cb} = \Lambda_{\rm cb}(G)\}.$$

Note that an $N$-optimal approximate identity exists (if $G$ is weakly amenable).

**Proposition 3.** *Let $G$ be an weakly amenable group and $N$ be an amenable closed normal subgroup of $G$. Let $(\varphi_n)$ be an $N$-optimal approximate identity on $G$. Then, for every $g \in G$ and $a \in N$,*

$$\lim_n \|(\varphi_n - \varphi_n \circ {\rm Ad}_g)|_N\|_{\rm cb} = 0 \text{ and } \lim_n \|(\varphi_n - \varphi_n^a)|_N\|_{\rm cb} = 0.$$

*Proof.* We apply Lemma 1 for each $\varphi_n$ and find $(\pi_n, \mathcal{H}_n, \xi_n, \eta_n)$ satisfying the conditions stated there. In particular, $\|\xi\|_\infty = \|\eta\|_\infty \leq \Lambda_{\rm cb}(G)^{1/2}$ and $\varphi_n(y^{-1}x) = \langle \xi_n(x), \eta_n(y)\rangle$ for every $x, y \in G$. Let $g \in G$ be given and consider $\psi_n = (\varphi_n + \varphi_n^g)/2$. Since $(\psi_n)$ is a completely bounded approximate identity, one must have $\liminf_n \|\psi_n\|_{\rm cb} \geq \Lambda_{\rm cb}(G)$. Meanwhile, since $\varphi_n$ is self-adjoint,

$$\psi_n(y^{-1}x) = \frac{1}{4}\big(\langle \xi_n(x) + \xi_n(xg^{-1}), \eta_n(y)\rangle + \langle \eta_n(x) + \eta_n(xg^{-1}), \xi_n(y)\rangle\big)$$

and hence

$$\|\psi_n\|_{\mathrm{cb}} \le \Big\|\frac{1}{\sqrt{2}}\big(\frac{\xi_n+\xi_n^g}{2},\ \frac{\eta_n+\eta_n^g}{2}\big)\Big\|_{L^\infty(G,\mathcal{H}\oplus\mathcal{H})}\Big\|\frac{1}{\sqrt{2}}\big(\eta_n,\ \xi_n\big)\Big\|_{L^\infty(G,\mathcal{H}\oplus\mathcal{H})} \le \Lambda_{\mathrm{cb}}(G).$$

It follows that

$$\lim_n \Big\|\frac{1}{\sqrt{2}}\big(\frac{\xi_n+\xi_n^g}{2},\ \frac{\eta_n+\eta_n^g}{2}\big)\Big\|_{L^\infty(G,\mathcal{H}\oplus\mathcal{H})} = \Lambda_{\mathrm{cb}}(G)^{1/2},$$

which means that there is a net $z_n \in G$ such that

$$\lim_n \|\frac{\xi_n(z_n)+\xi_n(z_ng^{-1})}{2}\| = \Lambda_{\mathrm{cb}}(G)^{1/2} \text{ and } \lim_n \|\frac{\eta_n(z_n)+\eta_n(z_ng^{-1})}{2}\| = \Lambda_{\mathrm{cb}}(G)^{1/2}.$$

By the parallelogram identity, this implies that

$$\lim_n \|\xi_n(z_n)-\xi_n(z_ng^{-1})\| = 0 \text{ and } \lim_n \|\eta_n(z_n)-\eta_n(z_ng^{-1})\| = 0.$$

The unitary $N$-representation $\pi_n' = \pi_n \circ \mathrm{Ad}_{z_n}$ satisfies $\pi_n'(a)\xi_n(x) = \xi_n(z_naz_n^{-1}x)$,

$$\varphi_n(a) = \langle \pi_n'(a)\xi_n(z_n), \eta_n(z_n)\rangle \text{ and } (\varphi_n\circ\mathrm{Ad}_g)(a) = \langle \pi_n'(a)\xi_n(z_ng^{-1}), \eta_n(z_ng^{-1})\rangle$$

for $a \in N$. It follows that $\|(\varphi_n - \varphi_n\circ\mathrm{Ad}_g)|_N\|_{\mathrm{cb}} \to 0$. That $\|(\varphi_n - \varphi_n^a)|_N\|_{\mathrm{cb}} \to 0$ follows from $N$-optimality of $(\varphi_n)$ and Lemma 2. $\square$

*Proof of Theorem A.* Let $(\varphi_n)$ be an $N$-optimal approximate identity on $G$ and consider linear functionals $\omega_n = \tau_0 \circ m_{\varphi_n}$ on $C^*_\lambda N$, where $\tau_0$ is the unit character on $N$ (see Section 2). Since $\varphi_n \in \mathcal{A}G$, the linear functionals $\omega_n$ extend to ultraweakly-continuous linear functionals on the group von Neumann algebra $\mathcal{L}N$. Indeed, they are nothing but $\varphi_n|_N \in \mathcal{A}N = (\mathcal{L}N)_*$. One has $\|\omega_n\| \le \Lambda_{\mathrm{cb}}(G)$, $\omega_n(1_{\mathcal{L}N}) = \varphi_n(1_N)$ and, by Proposition 3, $\|\omega_n - \omega_n\circ\mathrm{Ad}_g\| \to 0$ and $\|\omega_n - \omega_n^a\| \to 0$ for every $g \in G$ and $a \in N$. We consider $\zeta_n := |\omega_n|^{1/2} \in L^2(N)$ and $\zeta_n' := \omega_n|\omega_n|^{-1/2} \in L^2(N)$ so that $\omega_n(x) = \langle X\zeta_n, \zeta_n'\rangle$ for $X \in \mathcal{L}N$. Here the absolute value and the square root are taken in the sense of the standard representation $\mathcal{L}N \subset \mathbb{B}(L^2(N))$. (In case where $N$ is abelian, the Fourier transform $L^2(N) \cong L^2(\widehat{N})$ implements $\mathcal{L}N \cong L^\infty(\widehat{N})$ and $(\mathcal{L}N)_* \cong L^1(\widehat{N})$, and the absolute value and square root are computed as ordinary functions on the Pontrjagin dual $\widehat{N}$.) We note that $\varphi_n(1) \le \|\zeta_n\|_2^2 \le \Lambda_{\mathrm{cb}}(G)$. By continuity of the absolute value (Proposition III.4.10 in [Ta]) and the Powers–Størmer inequality, one has $\|\zeta_n - \mathrm{Ad}_g\,\zeta_n\|_2 \to 0$ for every $g \in G$. Moreover, since

$$\|\zeta_n\|_2\|\zeta_n'\|_2 - \|\frac{\zeta_n+\lambda(a^{-1})\zeta_n}{2}\|_2\|\zeta_n'\|_2 \le \|\omega_n\| - \|\frac{\omega_n+\omega_n^a}{2}\| \to 0,$$

one has $\|\zeta_n - \lambda(a^{-1})\zeta_n\|_2 \to 0$ for every $a \in N$. Thus, any limit point of $(\zeta_n^2)$ in $L^\infty(N)^*$ is a non-zero positive $G \ltimes N$-invariant linear functional on $L^\infty(N)$. $\square$

## 4. Proof of Theorem B

We first fix notations. Throughout this section, $M$ is a finite von Neumann algebra with a distinguished faithful normal tracial state $\tau$, and $P$ is an amenable von Neumann subalgebra of $M$. The *normalizer* $\mathcal{N}(P)$ of $P$ in $M$ is

$$\mathcal{N}(P) = \{u \in \mathcal{U}(M) : \mathrm{Ad}_u(P) = P\},$$

where $\mathcal{U}(M)$ is the group of the unitary elements of $M$ and $\mathrm{Ad}_u(x) = uxu^*$. The GNS Hilbert space with respect to the trace $\tau$ is denoted by $L^2(M)$ and the vector in $L^2(M)$ associated with $x \in M$ is denoted by $\hat{x}$, i.e., $\langle \hat{x}, \hat{y} \rangle = \tau(y^*x)$ for $x, y \in M$. The complex conjugate $\bar{M} = \{\bar{a} : a \in M\}$ of $M$ acts on $L^2(M)$ from the right. Thus there is a $*$-representation $\varsigma$ of the algebraic tensor product $M \otimes \bar{M}$ on $L^2(M)$ defined by $\varsigma(a \otimes \bar{b})\hat{x} = \widehat{axb^*}$ for $a, b, x \in M$. We also use the bimodule notation $a\hat{x}b^*$ for $\varsigma(a \otimes \bar{b})\hat{x}$. Since $P$ is amenable, the $*$-homomorphism $\varsigma|_{M \otimes \bar{P}}$ is continuous with respect to the minimal tensor norm.

**Definition.** A von Neumann algebra $M$ is said to have the *weak$^*$ completely bounded approximation property*, or W$^*$CBAP in short, if there is a net of ultraweakly-continuous finite-rank maps $(\varphi_n)$ on $M$ such that $\varphi_n \to \mathrm{id}_M$ in the point-ultraweak topology and $\sup \|\varphi_n\|_{\mathrm{cb}} < +\infty$.

Recall that a finite von Neumann algebra $P$ is amenable (a.k.a. hyperfinite, injective, AFD, etc.) if the trace $\tau$ on $P$ extends to a $P$-central state $\omega$ on $\mathbb{B}(L^2(P))$. Here, a state $\omega$ is said to be *$P$-central* if $\omega \circ \mathrm{Ad}_u = \omega$ for every $u \in \mathcal{U}(P)$, or equivalently $\omega(ax) = \omega(xa)$ for every $a \in P$ and $x \in \mathbb{B}(L^2(P))$.

**Definition.** Let $P$ be a finite von Neumann algebra and $\mathcal{G}$ be a group acting on $P$ by trace-preserving $*$-automorphisms. We denote by $\sigma$ the corresponding unitary representation of $\mathcal{G}$ on $L^2(P)$. The action $\mathcal{G} \curvearrowright P$ is said to be *weakly compact* if there is a state $\omega$ on $\mathbb{B}(L^2(P))$ such that $\omega|_P = \tau$ and $\omega \circ \mathrm{Ad}_u = \omega$ for every $u \in \sigma(\mathcal{G}) \cup \mathcal{U}(P)$. (This forces $P$ to be amenable.) A von Neumann subalgebra $P$ of a finite von Neumann algebra $M$ is said to be *weakly compact* in $M$ if the conjugate action by the normalizer $\mathcal{N}(P)$ is weakly compact. See [OP] for more information.

If $M$ admits a crossed product decomposition $M = P \rtimes \Lambda$ and the "core" $P$ is non-atomic and weakly compact in $M$, then $\mathcal{L}\Lambda$ is co-amenable in $M$ (Proposition 3.2 in [OP]), which implies $M$ does not have property (T).

**Lemma 4.** *Every $P$-central state $\omega$ on $\mathbb{B}(L^2(P))$ decomposes uniquely as a sum $\omega = \omega_{\mathrm{n}} + \omega_{\mathrm{s}}$ of $P$-central positive linear functionals such that $\omega_{\mathrm{n}}|_P$ is normal and $\omega_{\mathrm{s}}|_P$ is singular. A trace-preserving action $\mathcal{G} \curvearrowright P$ is weakly compact if there is a positive linear functional $\omega$ on $\mathbb{B}(L^2(P))$ such that*

- *$\omega(p) > 0$ for every non-zero central projection $p$ in $P$,*
- *$\omega \circ \mathrm{Ad}_u = \omega$ for every $u \in \sigma(\mathcal{G}) \cup \mathcal{U}(P)$.*

*Proof.* We denote by $Z$ the center of $P$. Recall that every tracial state $\tau'$ on $P$ satisfies $\tau' = \tau'|_Z \circ E_Z$, where $E_Z \colon P \to Z$ is the center-valued trace. In particular, $\tau'$ is normal on $P$ if and only if it is normal on $Z$. Let $\omega$ be a $P$-central state and consider the normal/singular decomposition of the state $\omega|_Z$ (see Definition III.2.15 in [Ta]). There is an increasing sequence $(p_n)$ of projections in $Z$ such that $p_n \nearrow 1$ and $(\omega|_Z)_{\mathrm{s}}(p_n) = 0$ for all $n$ (see Theorem III.3.8 in [Ta]). We fix an ultralimit Lim on $\mathbb{N}$ and let $\omega_{\mathrm{n}}(x) = \mathrm{Lim}\,\omega(p_n x)$ and $\omega_{\mathrm{s}} = \omega - \omega_{\mathrm{n}}$. Since $\omega$ is $P$-central, these are $P$-central positive linear functionals on $\mathbb{B}(L^2(P))$, and $\omega|_Z = \omega_{\mathrm{n}}|_Z + \omega_{\mathrm{s}}|_Z$ is the normal/singular decomposition of $\omega|_Z$. Suppose that $\omega = \omega'_{\mathrm{n}} + \omega'_{\mathrm{s}}$ is another such decomposition. Then, since $\omega_{\mathrm{s}} + \omega'_{\mathrm{s}}$ is singular on $Z$, there is an increasing sequence $(q_n)$ of projections in $Z$ such that $q_n \nearrow 1$ and $(\omega_{\mathrm{s}} + \omega'_{\mathrm{s}})(q_n) = 0$ for all $n$. It follows that $\omega'_{\mathrm{n}}(x) = \lim \omega(q_n x) = \omega_{\mathrm{n}}(x)$ for every $x \in \mathbb{B}(L^2(P))$. This proves the first half of this lemma. For the second half, we first observe that we may assume $\omega$ is normal on $P$ by uniqueness of the normal/singular decomposition. Thus, there is $h \in L^1(Z)_+$ such that $\omega(z) = \tau(hz)$ for $z \in Z$. By assumption, $h$ has full support and is $\mathcal{G}$-invariant. Thus $\tilde{\omega}(x) := \mathrm{Lim}\,\omega((h + n^{-1})^{-1} x)$ defines a $\mathcal{G}$-invariant $P$-central state on $\mathbb{B}(L^2(P))$ such that $\tilde{\tau}|_Z = \tau|_Z$. □

**Lemma 5.** *Let $\varphi$ be a completely bounded map on $M$. Then, there are a $*$-representation of the minimal tensor product $M \otimes_{\min} \bar{P}$ on a Hilbert space $\mathcal{H}$ and operators $V, W \in \mathbb{B}(L^2(M), \mathcal{H})$ such that $\|V\| = \|W\| \le \|\varphi\|_{\mathrm{cb}}^{1/2}$ and*

$$\tau(y^* \varphi(a) x b^*) = \langle \varphi(a) \hat{x} b^*, \hat{y} \rangle = \langle \pi(a \otimes \bar{b}) V \hat{x}, W \hat{y} \rangle$$

*for every $a, x, y \in M$ and $b \in P$.*

*Proof.* Since the $*$-representation $\varsigma \colon M \otimes_{\min} \bar{P} \to \mathbb{B}(L^2(M))$ is continuous, a Stinespring type factorization theorem (Theorem B.7 in [BO]), applied to the completely bounded map $\varsigma \circ (\varphi \otimes \mathrm{id}_{\bar{P}})$, yields a $*$-representation $\pi \colon M \otimes_{\min} \bar{P} \to \mathbb{B}(\mathcal{H})$ and operators $V, W \in \mathbb{B}(L^2(M), \mathcal{H})$ such that $\|V\| \|W\| \le \|\varphi\|_{\mathrm{cb}}$ and

$$\varphi(a) \hat{x} b^* = \varsigma\big((\varphi \otimes \mathrm{id}_{\bar{P}})(a \otimes \bar{b})\big) \hat{x} = W^* \pi(a \otimes \bar{b}) V \hat{x}$$

for $a, x \in M$ and $b \in P$. □

Since W$^*$CBAP passes to a subalgebra (which is the range of a conditional expectation), we assume from now on that $P$ is *regular* in $M$, i.e., $\mathcal{N}(P)$ generates $M$ as a von Neumann algebra. We say a linear map $\varphi$ on $M$ is *$P$-cb* if there are a $*$-representation $\pi$ of $M \otimes_{\min} \bar{P}$ on a Hilbert space $\mathcal{H}$ and functions $V, W \in \ell_\infty(\mathcal{N}(P), \mathcal{H})$ such that

$$(*) \qquad \langle \varphi(a) \hat{x} b^*, \hat{y} \rangle = \langle \pi(a \otimes \bar{b}) V(x), W(y) \rangle$$

for every $a \in M$, $x, y \in \mathcal{N}(P)$ and $b \in P$. The $P$-cb norm of $\varphi$ is defined as

$$\|\varphi\|_P = \inf\{\|V\|_\infty \|W\|_\infty : (\pi, \mathcal{H}, V, W) \text{ satisfies } (*)\}.$$

It is indeed a norm and the infimum is attained (for the latter fact, use ultraproduct). By the above lemma, $\|\varphi\|_P \le \|\varphi\|_{\mathrm{cb}}$. By an *approximate identity*, we mean a net $(\varphi_n)$

of ultraweakly-continuous finite-rank maps such that $\varphi_n \to \mathrm{id}_M$ in the point-ultraweak topology and $\sup \|\varphi_n\|_P < +\infty$. It exists if $M$ has the W*CBAP. We define

$$\Lambda_P(M) = \inf\{\sup_n \|\varphi_n\|_P : (\varphi_n) \text{ an approximate identity}\}.$$

For a map $\varphi$ on $M$, we define $\varphi^*(a) = \varphi(a^*)^*$ and say $\varphi$ is *self-adjoint* if $\varphi = \varphi^*$. We note that if $(\pi, \mathcal{H}, V, W)$ satisfies $(*)$ for $\varphi$, then $(\pi, \mathcal{H}, W, V)$ satisfies $(*)$ for $\varphi^*$. In particular, $(\varphi+\varphi^*)/2$ is self-adjoint and $\|(\varphi+\varphi^*)/2\|_P \le \|\varphi\|_P$. Thus, any approximate identity can be made self-adjoint without increasing norm. For a $P$-cb map $\varphi$, we define a bounded linear functional $\mu_\varphi$ on $M \otimes_{\min} \bar{P}$ by

$$\mu_\varphi(a \otimes \bar{b}) := \tau(\varphi(a)b^*) = \langle \varphi(a)\hat{1}b^*, \hat{1}\rangle = \langle \pi(a \otimes \bar{b})V(1), W(1)\rangle.$$

Note that $\|\mu_\varphi\| \le \|\varphi\|_P$. If $\varphi$ is ultraweakly-continuous and finite-rank, then $\mu_\varphi$ extend to an ultraweakly-continuous linear functional on the von Neumann algebra $M \bar{\otimes} \bar{P}$.

**Proposition 6.** *Let $M$ be a finite von Neumann algebra having the W*CBAP and $(\varphi_n)$ be a self-adjoint approximate identity such that* $\sup_n \|\varphi_n\|_P = \Lambda_P(M)$. *Then, the net* $\mu_n := \mu_{\varphi_n}|_{P\bar{\otimes}\bar{P}}$ *satisfies the following properties:*

- $\mu_n$ *are self-adjoint and ultraweakly-continuous for all $n$;*
- $\sup \|\mu_n\| \le \Lambda_P(M)$ *and* $\mu_n(a \otimes \bar{1}) \to \tau(a)$ *for every* $a \in P$;
- $\|\mu_n - \mu_n^{v\otimes\bar{v}}\| \to 0$ *for every* $v \in \mathcal{U}(P)$, *where* $\mu_n^{v\otimes\bar{v}}(a \otimes \bar{b}) = \mu_n((a \otimes \bar{b})(v \otimes \bar{v})^*)$;
- $\|\mu_n - \mu_n \circ \mathrm{Ad}_{u\otimes\bar{u}}\| \to 0$ *for every* $u \in \mathcal{N}(P)$.

*Proof.* The first two conditions are easy to see. Let $u \in \mathcal{N}(P)$ be given, and define $\varphi_n^u$ by $\varphi_n^u(a) = \varphi_n(au^*)u$ for $a \in M$. We note that $\mu_{\varphi_n^u}|_{P\bar{\otimes}\bar{P}} = \mu_n^{u\otimes\bar{u}}$ if $u \in \mathcal{U}(P)$. Thus, it suffices to show

$$\lim_n \|\mu_{\varphi_n} - \mu_{\varphi_n^u}\| = 0 \text{ and } \lim_n \|\mu_{\varphi_n} - \mu_{\varphi_n} \circ \mathrm{Ad}_{u\otimes\bar{u}}\| = 0.$$

Take $(\pi_n, \mathcal{H}_n, V_n, W_n)$ satisfying $(*)$ and $\lim \|V_n\|_\infty = \lim \|W_n\|_\infty = \Lambda_P(M)^{1/2}$. It follows that

$$\langle \varphi_n^u(a)\hat{x}b^*, \hat{y}\rangle = \langle \varphi_n(au^*)\widehat{ux}b^*, \hat{y}\rangle = \langle \pi_n(a \otimes \bar{b})\pi_n(u^* \otimes \bar{1})V_n(ux), W_n(y)\rangle$$

for every $a \in M$, $b \in P$ and $x, y \in \mathcal{N}(P)$. Hence with $V_n^u(x) = \pi_n(u^* \otimes \bar{1})V_n(ux)$, the quadruplet $(\pi_n, \mathcal{H}_n, V_n^u, W_n)$ satisfies $(*)$ for $\varphi_n^u$. Note that $\|V_n^u\|_\infty = \|V_n\|_\infty$. We define $W_n^u$ similarly. Since $\varphi_n$ is self-adjoint, $(\pi_n, \mathcal{H}_n, W_n, V_n)$ (resp. $(\pi_n, \mathcal{H}_n, W_n^u, V_n)$) satisfies $(*)$ for $\varphi_n$ (resp. $\varphi_n^u$), too. Thus, for $\psi_n = (\varphi_n + \varphi_n^u)/2$, one has

$$\|\psi_n\|_P \le \left\| \frac{1}{\sqrt{2}}\Big(\frac{V_n + V_n^u}{2}, \frac{W_n + W_n^u}{2}\Big)\right\|_{\ell_\infty(\mathcal{N}(P),\mathcal{H}\oplus\mathcal{H})} \left\| \frac{1}{\sqrt{2}}\big(W_n, V_n\big)\right\|_{\ell_\infty(\mathcal{N}(P),\mathcal{H}\oplus\mathcal{H})}.$$

Meanwhile, since $(\psi_n)$ is an approximate identity, one must have $\liminf \|\psi_n\|_P \ge \Lambda_P(M)$. It follows that

$$\lim_n \left\| \frac{1}{\sqrt{2}}\Big(\frac{V_n + V_n^u}{2}, \frac{W_n + W_n^u}{2}\Big)\right\|_{\ell_\infty(\mathcal{N}(P),\mathcal{H}\oplus\mathcal{H})} = \Lambda_P(M)^{1/2}$$

and hence there is a net $(z_n)$ in $\mathcal{N}(P)$ such that

$$\lim_n \Big\| \frac{1}{\sqrt{2}}\big(\frac{(V_n+V_n^u)(z_n)}{2}, \frac{(W_n+W_n^u)(z_n)}{2}\big)\Big\|_{\mathcal{H}\oplus\mathcal{H}} = \Lambda_P(M)^{1/2}.$$

By the parallelogram identity, this implies that

$$\lim_n \|V_n(z_n) - V_n^u(z_n)\| = 0 \text{ and } \lim_n \|W_n(z_n) - W_n^u(z_n)\| = 0.$$

Let $\pi_n' = \pi_n \circ (\mathrm{id}_M \otimes \mathrm{Ad}_{\bar{z}_n^{-1}})$. Since

$$\mu_{\varphi_n}(a\otimes\bar{b}) = \langle \varphi_n(a)\hat{z}_n \,\mathrm{Ad}_{z_n^{-1}}(b)^*, \hat{z}_n\rangle = \langle \pi_n'(a\otimes\bar{b})V_n(z_n), W_n(z_n)\rangle,$$
$$\mu_{\varphi_n^u}(a\otimes\bar{b}) = \langle \varphi_n(au^*)\widehat{uz_n} \,\mathrm{Ad}_{z_n^{-1}}(b)^*, \hat{z}_n\rangle = \langle \pi_n'(a\otimes\bar{b})V_n^u(z_n), W_n(z_n)\rangle,$$

and

$$\big(\mu_{\varphi_n}\circ \mathrm{Ad}_{u\otimes\bar{u}}\big)(a\otimes\bar{b}) = \langle \varphi_n(uau^*)\widehat{uz_n}\,\mathrm{Ad}_{z_n^{-1}}(b)^*, \widehat{uz_n}\rangle = \langle \pi_n'(a\otimes\bar{b})V_n^u(z_n), W_n^u(z_n)\rangle,$$

we conclude that $\|\mu_{\varphi_n} - \mu_{\varphi_n^u}\| \to 0$ and $\|\mu_{\varphi_n} - \mu_{\varphi_n}\circ \mathrm{Ad}_{u\otimes\bar{u}}\| \to 0$. □

*Proof of Theorem B.* Since $M$ has the W*CBAP, there is a net $(\mu_n)$ satisfying the conclusion of Proposition 6. We view $\mu_n$ as an element in $L^1(P\,\bar{\otimes}\,\bar{P})$ (see Section 2 in [OP]) and let $\zeta_n = |\mu_n|^{1/2} \in L^2(P\,\bar{\otimes}\,\bar{P})$ and $\zeta_n' = \mu_n|\mu_n|^{-1/2} \in L^2(P\,\bar{\otimes}\,\bar{P})$ so that $\mu_n(X) = \langle X\zeta_n, \zeta_n'\rangle$ for $X \in P\,\bar{\otimes}\,\bar{P}$. By continuity of the absolute value (Proposition III.4.10 in [Ta]) and the Powers–Størmer inequality, one has $\|\zeta_n - \mathrm{Ad}_{u\otimes\bar{u}}\,\zeta_n\|_2 \to 0$ for every $u \in \mathcal{N}(P)$. Since

$$2\|\mu_n\| \approx \|\mu_n + \mu_n^{v\otimes\bar{v}}\| \le \|\zeta_n + (v\otimes\bar{v})\zeta_n\|_2\|\zeta_n'\|_2 \le 2\|\zeta_n\|_2\|\zeta_n'\|_2 = 2\|\mu_n\|,$$

one also has $\|\zeta_n - (v\otimes\bar{v})\zeta_n\| \to 0$ for every $v \in \mathcal{U}(P)$. Now, fix an ultralimit Lim and define $\omega$ on $\mathbb{B}(L^2(P))$ by $\omega(x) = \mathrm{Lim}\langle (x\otimes\bar{1})\zeta_n, \zeta_n\rangle$. Then $\omega$ is an $\mathcal{N}(P)$-invariant $P$-central positive linear functional satisfying

$$\omega(p) = \mathrm{Lim}_n\,|\mu_n|(p\otimes\bar{1}) \ge \mathrm{Lim}_n\,|\mu_n(p\otimes\bar{1})| = \tau(p)$$

for every central projection $p$ in $P$. By Lemma 4, we are done. □

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