New Constructions of Complex Manifolds

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Abstract

For a generic anti-canonical hypersurface in each smooth toric Fano 4−fold with rank 2 Picard group , we prove there exist three isolated rational curves in it . Moreover , for all these 4−folds except one , the contractions of generic anti-canonical hypersurfaces along the three rational curves can be deformed to smooth threefolds diffeomorphic to connected sums of $S^3 \times S^3$. In this manner, we obtain complex structures with trivial canonical bundles on some connected sums of $S^3 \times S^3$. This construction is an analogue of that in Friedman [7] , Lu and Tian [12] which used only quintics in $\mathbb{P}^4$.

1 Introduction

This paper is resulted from an attempt towards the *Reid's fantasy* for some families of Calabi-Yau threefolds . Let's first recall some notions .

Definition 1 ([14]). *Let Y be a Calabi-Yau threefold and $\phi : Y \to \bar{Y}$ be a birational contraction onto a normal variety . If there exists a complex deformation (smoothing) of $\bar{Y}$ to a Calabi-Yau threefold $\tilde{Y}$, then the process of going from Y to $\tilde{Y}$ is called a geometric transition and denoted by $T(Y, \bar{Y}, \tilde{Y})$. A transition $T(Y, \bar{Y}, \tilde{Y})$ is called conifold if $\bar{Y}$ admits only ordinary double points as singularities and the resolution morphism ϕ is a small resolution (i.e. replacing each ordinary double point by a smooth rational curve).*

Note for a conifold transition $T(Y, \bar{Y}, \tilde{Y})$, the exceptional set of the morphism ϕ is several not intersecting smooth rational curves each with normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ in Y and conversely , given some finite not intersecting smooth rational curves each with normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ in Y , we can contract them to get $\bar{Y}$ admitting only ordinary double points as singularities . The smoothing of $\bar{Y}$ has been studied by several people . For example , we have the following theorem :

Theorem 1 (Y. Kawamata , G.Tian , see [15]). *Let $\bar{Y}$ be a singular threefold with l ordinary double points as the only singular points $p_1, \ldots, p_l$. Let Y be a small resolution of $\bar{Y}$ by replacing p_i by smooth rational curves C_i . Assume that Y is cohomologically Kähler and has trivial canonical line bundle . Furthermore , we assume that the fundamental classes $[C_i]$ in $H^2(Y;\Omega_Y^2)$ satisfy a relation $\Sigma_i \lambda_i [C_i] = 0$ with each λ_i nonzero . Then $\bar{Y}$ can be deformed into a smooth threefold $\tilde{Y}$.*

A special case of the above theorem was obtained by R . Friedman in [6] .

The conifold transition process was firstly (locally)observed by H .Clemens in [3] , where he explained that locally a conifold transition is described by a suitable $S^3 \times D_3$ to $S^2 \times D_4$ surgery . Roughly speaking , for the conifold transition $T(Y, \bar{Y}, \tilde{Y})$ from Y to $\tilde{Y}$, it kills $2-$cycles in Y and increases $3-$cycles in $\tilde{Y}$. For a precise relation between their Betti numbers , one can consult Theorem 3.2 in [14] . In Theorem 1 , if the fundamental classes $[C_i]$ generates $H^4(Y;\mathbb{C})$, then we would have $b_2(\tilde{Y}) = 0$. By results of C.T.C.Wall in [16] , $\tilde{Y}$ would be diffeomorphic to a connected sum of $S^3 \times S^3$, and the number of copies is $\frac{b_3(\tilde{Y})}{2} + l - b_2(Y)$. We have the following two problems (cfr. [14]):

1. Wether every projective Calabi-Yau threefold is birational to a Calabi-Yau threefold Y such that $H^2(Y;\mathbb{C})$ is generated by rational curves and these curves satisfy the conditions in Theorem 1 .

2. Wether the moduli space $\mathcal{N}_r$ of complex structures on the connected sum of r copies of $S^3 \times S^3$ is irreducible .

If we could have positive answers to both of the above problems , then we would verify the famous :

Conjecture 1 (the Reid's fantasy , see [13]). *Up to some kind of inductive limit over r , the birational classes of projective Calabi-Yau threefolds can be fitted together , by means of conifold transitions , into one irreducible family parameterized by the moduli space $\mathcal{N}$ of complex structures over suitable connected sum of copies of $S^3 \times S^3$.*

In P.S.Green and T.Hübsch [8] , they proved that the moduli spaces of some CalabiCYau threefolds , which are complete intersections in products of projective spaces , were connected each other by conifold transitions . As for other families of Calabi-Yau threefolds , for example that which are anti-canonical hypersurfaces in toric Fano $4-$folds , M. Kreuzer and H. Skarke [11] proved they can be connected by geometric transitions , but the transitions they used are not conifold transitions . So a natural question is about the Reid's fantasy for these families of Calabi-Yau threefolds . In this paper , we study the first problem above . Our main result is :

Theorem 2. *For each toric smooth Fano $4-$folds X with rank 2 Picard group(the only toric Fano $4-$fold with rank 1 Picard group is $\mathbb{P}^4$) , let Y be a generic anti-canonical hypersurface of X. Then there exist three smooth rational curves C_i in Y such that each one has normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ in Y , the fundamental classes $[C_i]$ generate $H^4(Y;\mathbb{Z}) \simeq H_2(Y;\mathbb{Z})$ and they satisfy a relation $\Sigma_i \lambda_i [C_i] = 0$ with each λ_i nonzero . Moreover , with the exception of the variety B_1 in Batyrev's classification for smooth toric Fano $4-$folds [1], the three curves do not intersect with each other .*

Note by Lefschetz's hyperplane theorem , $H_2(Y;\mathbb{Z}) \simeq H_2(X;\mathbb{Z})$, so $H_2(Y;\mathbb{Z})$ is a free abelian group with rank 2 . Since X is simply connected , $H^2(X;\mathbb{Z})$ is also a free abelian group with rank 2 . Hence the classes $[C_i]$ generating $H_2(Y;\mathbb{Z})$ is equivalent to generating $H_2(X;\mathbb{Z})$, and this is also equivalent to the (dual) cohomology classes represented by C_i in $H^2(X;\mathbb{Z})$ generate $H^2(X;\mathbb{Z})$. Using Theorem 1 and the discussions above , we can get connected sums of $S^3 \times S^3$ after contracting these rational curves and smoothing . The number of copies of $S^3 \times S^3$ is summarized in the following table , in which the name of toric $4-$folds is from Batyrev [1] . :

Toric Fano $4-$fold	B_2	B_3	B_4	B_5	C_1	C_2	C_3	C_4
Number of copies of $S^3 \times S^3$	104	92	88	88	97	88	88	85

So according to Theorem 1 , we get complex structures with trivial canonical bundles on these connected sums of $S^3 \times S^3$. In Lu and Tian [12] , they obtained complex structures with trivial canonical bundles for the connected sum of m copies of $S^3 \times S^3$ for each $m \geq 2$, using quintics in $\mathbb{P}^4$, combing a preceding result of R. Friedman [7] .

The next question is about the relations of the complex structures obtained using different $4-$folds . For example , in the above table , we observe that for the varieties B_4 , B_5 , C_2, C_3 , we can obtain the same topological connected sum of $S^3 \times S^3$, but we don't know whether the various complex structures on it are the same , or at least lie in the same deformation class . We also would like to compare the complex structures obtained using $\mathbb{P}^4$ in Friedman [7] , Lu and Tian [12] and that obtained using these toric $4-$ folds with rank 2 Picard groups . Obviously this question is closely related to the second problem preceding Conjecture 1 . Because these complex manifolds are not Kähler manifolds (they have vanishing b_2), very few techniques are available in dealing with them , and very little is known about complex structures over connected sums of $S^3 \times S^3$. Still and all , the results obtained in this paper can be viewed to be a first step for finding connections of these Calabi-Yau threefolds and provide examples of non-intersecting isolated rational curves in some Calabi-Yau threefolds other than quintics (compare T.Johnsen and A.L.Knutsen [10], T. Johnsen and S. L. Kleiman [9]).

The paper is organized as follows .

In Sec. 2 , we recall the homogenous coordinates on a toric variety of D.Cox [4], and give the homogenous coordinates representations for embeddings of $\mathbb{P}^1$ to complete nonsingular toric varieties .

In Sec. 3 , we extend an argument from Clemens [2] to conclude that , if we can find some anti-canonical hypersurface containing a smooth rational curve with normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ and with a fixed topological type , then generic anti-canonical hypersurfaces will contain a smooth rational curve with normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ and with the same topological type . At the end of this section , we give a direct method to get the non-intersecting property for the rational curves in an anti-canonical hypersurface .

In Sec. 4 , we analyze each toric smooth Fano $4-$folds X with rank 2 Picard groups , and for each case we construct three rational curves C_{10} , C_{20} , C_{30} and three anti-canonical hypersurfaces Y_{10} , Y_{20} , Y_{30} such that C_{i0} lies in Y_{i0} with normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ for $i = 1, 2, 3$. We obtain the non-intersecting property for the rational curves , with the exception of the variety B_1 . Then using the results in Sec. 3 we get our main theorem .

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2 Embeddings of $\mathbb{P}^1$ to complete nonsingular toric varieties

In this section, we will describe all the embeddings of $\mathbb{P}^1$ to complete nonsingular toric varieties using the homogeneous coordinates on a toric variety of D.Cox [4]. First recall the homogeneous coordinates on a toric variety.

Let X be the toric variety determined by a fan Δ in $N \simeq \mathbb{Z}^n$. As usual , M will denote the $\mathbb{Z}$ dual of N , and cones in Δ will be denoted by σ . The one dimensional cones of Δ form the set $\Delta(1)$. And given $\rho \in \Delta(1)$, let n_ρ denote the generator of $\rho \cap N$. If σ is any cone in Δ , then $\sigma(1) = \{\rho \in \Delta(1) : \rho \subset \sigma\}$ is the set of one dimensional faces of the cone σ . Throughout of this section , we assume X to be a complete nonsingular toric variety . So for each maximal cone σ , $\{n_\rho : \rho \in \sigma(1)\}$ form a base of N .

Each $\rho \in \Delta(1)$ corresponds to a $T-$invariant irreducible Weil divisor D_ρ in X .

Where T is the torus in X . The free abelian group of $T-$invariant irreducible Weil divisor on X will be denoted by $\mathbb{Z}^{\Delta(1)}$.

We consider the map

$$M \to \mathbb{Z}^{\Delta(1)} \texttt{ defined by } m \to D_m = \sum_{\rho\in\Delta(1)} < m, n_\rho > D_\rho$$

Where $<,>$ means the pairing between elements in M and N . The map is injective since $\Delta(1)$ spans $N \otimes_{\mathbb{Z}} \mathbb{R}$. We have an exact sequence

$$0 \to M \to \mathbb{Z}^{\Delta(1)} \to A_{n-1}(X) \to 0 \qquad (1)$$

Where $A_{n-1}(X)$ denotes the divisor class group of X . Since X is complete and nonsingular , $A_{n-1}(X) \simeq Pic(X)$ is a free abelian group of rank $d-n$, where d is the number of one-dimensional faces of Δ , and n is the dimension of X.

For each $\rho \in \Delta(1)$, we introduce a variable X_ρ . And for each cone $\sigma \in \Delta$, let $X_{\hat{\sigma}} = \prod_{\rho\notin\sigma} X_\rho$, then we can think of the variety

$$Z = \{x \in \mathbb{C}^{\Delta(1)} : X_{\hat{\sigma}} = 0 \texttt{ for all } \sigma \in \Delta\} \subset \mathbb{C}^{\Delta(1)}$$

as the "exceptional" subset of $\mathbb{C}^{\Delta(1)}$.

If we apply $Hom(\texttt{--}\ ,\mathbb{C}^*)$ to the exact sequence (1) , then we get the exact sequence

$$1 \to G \to (\mathbb{C}^*)^{\Delta(1)} \to T \to 1$$

where $G = Hom(A_{n-1}(X), \mathbb{C}^*)$ and $T = Hom(M, \mathbb{C}^*)$, both are products of copies of $\mathbb{C}^*$.

Since $(\mathbb{C}^*)^{\Delta(1)}$ acts naturally on $\mathbb{C}^{\Delta(1)}$, its subgroup $G \subset (\mathbb{C}^*)^{\Delta(1)}$ acts on $\mathbb{C}^{\Delta(1)}$ also . Now X is the geometric quotient of $\mathbb{C}^{\Delta(1)} - Z$ by G : $X \simeq \mathbb{C}^{\Delta(1)} - Z/G$

For each $\rho \in \Delta(1)$, we will canonically associate to it a line bundle L_ρ and a section s_ρ of this line bundle as follows .

For each maximal cone σ of Δ , denote the corresponding affine piece of X by $U_\sigma \simeq$ `Spec` $\mathbb{C}[\check{\sigma} \cap M]$, then define $f_{\rho,\sigma} \in M$:

$$f_{\rho,\sigma}(n_\tau) = \begin{cases} 1 & \text{if } \tau = \rho \ , \\ 0 & \text{otherwise .} \end{cases} \qquad (2)$$

for $\tau \in \Delta(1) \cap \sigma$.

Since $\{n_\tau : \tau \in \Delta(1) \cap \sigma\}$ form a base of N , $f_{\rho,\sigma}$ is well defined , and $f_{\rho,\sigma} \in \check{\sigma} \cap M$. So $\chi^{f_{\rho,\sigma}}$ is a regular function on U_σ . One can check easily that for two maximal cones σ_1 and σ_2 , $\chi^{f_{\rho,\sigma_1}} = g_{\sigma_1\sigma_2}\chi^{f_{\rho,\sigma_2}}$ on $U_{\sigma_1} \cap U_{\sigma_2}$, where $g_{\sigma_1\sigma_2}$ is a nowhere vanishing regular function on $U_{\sigma_1} \cap U_{\sigma_2}$. Using these $g_{\sigma_1\sigma_2}$ as transition functions , we ge a line bundle L_ρ , and those $\chi^{f_{\rho,\sigma}}$ for maximal cones ρ determine a section s_ρ of this line bundle . It's easy to see that the zero divisor of s_ρ is just the Weil divisor D_ρ .

Generally , if we choose $\rho_1, \rho_2, \dots, \rho_m \in \Delta(1)$, and $d_1, d_2, \dots, d_m$ integers , we define $f_{\sum_{i=1}^m d_i\rho_i,\sigma} = \sum_{i=1}^m d_i f_{\rho_i,\sigma}$, and similarly , using $g_{\sum_{i=1}^m d_i\rho_i,\sigma_1\sigma_2} = \chi^{f_{\sum_{i=1}^m d_i\rho_i,\sigma_1}} \big/ \chi^{f_{\sum_{i=1}^m d_i\rho_i,\sigma_2}}$ on the intersection of $U_{\sigma_1} \cap U_{\sigma_2}$ for two maximal cones σ_1 and σ_2 as transition functions , we get a meromorphic section $s_{\sum_{i=1}^m d_i\rho_i}$ of a line bundle $L_{\sum_{i=1}^m d_i\rho_i}$, which is isomorphic to $L_{\rho_1}^{d_1} \otimes L_{\rho_2}^{d_2} \otimes \dots \otimes L_{\rho_m}^{d_m}$. Moreover, this section is regular if $d_1, d_2, \dots, d_m$ are all nonnegative integers . one can check that for integers $d_1, d_2, \dots, d_m$ and $c_1, c_2, \dots, c_m$, if $\sum_{i=1}^m d_i D_{\rho_i} = \sum_{i=1}^m c_i D_{\rho_i}$ in A_{n-1} , then $g_{\sum_{i=1}^m d_i\rho_i,\sigma_1\sigma_2} = g_{\sum_{i=1}^m c_i\rho_i,\sigma_1\sigma_2}$ for any two maximal cones σ_1 and σ_2 . That is , the transition functions of $L_{\sum_{i=1}^m d_i\rho_i}$ and $L_{\sum_{i=1}^m c_i\rho_i}$ coincide . So in this case , the quotient $s_{\sum_{i=1}^m d_i\rho_i} \big/ s_{\sum_{i=1}^m c_i\rho_i}$ is well defined at the points on which $s_{\sum_{i=1}^m c_i\rho_i}$ is not vanishing .

Using the discussion above , we see that the sections s_ρ for $\rho \in \Delta(1)$ can be used to determine the homogenous coordinates of $X(\Delta) \simeq \mathbb{C}^{\Delta(1)} - Z \big/ G$

As an application of the homogenous coordinate description for the complete nonsingular toric variety $X(\Delta)$, we will obtain all of the homogenous coordinate representations for the anti-canonical hypersurfaces on X . Recall the anti-canonical bundle of X is isomorphic to $L_{\sum_{\Delta(1)}\rho}$, and a base of regular sections is determined by points in $Q = \{m \in M :< m, n_\rho >\geq -1, \forall \rho \in \Delta(1)\}$. Since for any maximal cone σ , the section $s_{\sum_{\Delta(1)}\rho}|_{U_\sigma} = \chi^{\sum_{\rho\in\sigma(1)} f_{\rho,\sigma}}$, so when restricted on U_σ , an anti-canonical section represented by $m \in Q$ is equal to $\chi^{\sum_{\rho\in\sigma(1)} f_{\rho,\sigma}+m}$. Then using the homogenous coordinates $(X_\rho)_{\rho\in\Delta(1)}$, $\chi^{\sum_{\rho\in\sigma(1)} f_{\rho,\sigma}+m} = 0$ is equivalent to $\prod_{\rho\in\Delta(1)} X_\rho^{<m,n_\rho>+1} = 0$. So any anti-canonical hypersurface on X has the form $\sum_{m\in Q} a_m \prod_{\rho\in\Delta(1)} X_\rho^{<m,n_\rho>+1} = 0$, where $a_m \in \mathbb{C}$ are complex numbers .

Now we will describe all the embeddings of $\mathbb{P}^1$ to $X(\Delta)$. Let $i : \mathbb{P}^1 \to X$ be a morphism from $\mathbb{P}^1$ to the toric variety X . Then we have a homomorphism of their Picard groups : $i^* : Pic(X) = A_{n-1}(X) \to Pic(\mathbb{P}^1) = \mathbb{Z}$. Under this homomorphism , suppose $i^*(L_\rho) = d_\rho$ for $\rho \in \Delta(1)$, then the section s_ρ is pulled back to a d_ρ form $f_\rho(s,t)$ on $\mathbb{P}^1$. So under the homogenous coordinates on X , the morphism i has the following form :

$$\mathbb{P}^1 \to X$$
$$(s,t) \to (f_\rho(s,t))_{\rho\in\Delta(1)}$$

We call this rational curve in X has type $(d_\rho)_{\rho\in\Delta(1)}$. It is a generalization of the concept of degree for rational curves in projective spaces . Note the homomorphism from $\mathbb{Z}^{\Delta(1)}$ to $\mathbb{Z}$ determined by the integers $d_\rho(\rho \in \Delta(1))$ is the composition of the map $\mathbb{Z}^{\Delta(1)} \to A_{n-1}(X)$ in the exact sequence (1) and the homomorphism $i^* : A_{n-1}(X) \to \mathbb{Z}$. So a set of integers $d_\rho(\rho \in \Delta(1))$ is induced by a morphism of $\mathbb{P}^1$ to X if and only if $d_\rho(\rho \in \Delta(1))$ satisfy $\sum_{\rho\in\Delta(1)} d_\rho n_\rho = 0$, where recall that n_ρ is the generator of $\rho \cap N$.

Next we want to determine the homology class represented by an embedding of $\mathbb{P}^1$. Since X is complete and nonsingular , $H^2(X,\mathbb{Z}) \simeq H_{2n-2}(X,\mathbb{Z}) \simeq A_{n-1}(X)$ is a finitely generated free abelian group . So if an embedding of $\mathbb{P}^1$ has the form $f_\rho(s,t)(\rho \in \Delta(1))$, where $f_\rho(s,t)$ is a degree d_ρ homogenous form of s,t . Then the cohomology class in $H^2(X,\mathbb{Z}) \simeq H_{2n-2}(X,\mathbb{Z}) \simeq A_{n-1}(X)$ represented by this rational curve is $\sum_{\rho\in\Delta(1)} d_\rho[D_\rho]$, where $[D_\rho]$ is the class in $A_{n-1}(X)$ represented by the divisor D_ρ . Note the type and the cohomological class determines each other for a rational curve in X .

3 Rational curves in a general anti-canonical hypersurface

In this section , we fix a complete nonsingular toric Fano 4-fold $X = X(\Delta)$, and use the same notations as the last section . We will prove that if an anti-canonical hypersurface of X contains a smooth rational curve C with normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$, then a generic anti-canonical hypersurface of X will contain a smooth rational curve with the same type as C, and the normal bundle is also $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$. More precisely, we have the following theorem .

Theorem 3. *Suppose Y_0 is an anti-canonical hypersurface in X, C_0 is a smooth rational curve in X with type $(d_\rho)_{\rho\in\Delta(1)}$. Assume C_0 lies in the smooth part of Y_0, and the normal bundle satisfies $N_{C_0,Y_0} \simeq \mathcal{O}(-1) \oplus \mathcal{O}(-1)$ .Then*

1. *$d_\rho \geq -1$, for any $\rho \in \Delta(1)$.*

2. *For a generic anti-canonical hypersurface Y (so Y is smooth , according to Bertini's theorem), there is a smooth rational curve C embedded in Y , such that the type of C in X is the same as that of C_0 (so $[C] = [C_0]$ in $H_2(X,\mathbb{Z})$), and the normal bundle satisfies $N_{C,Y} \simeq \mathcal{O}(-1) \oplus \mathcal{O}(-1)$.*

Proof. We use an argument analogous to the one used in [2]. First of all we will construct two spaces parameterizing all the anti-canonical hypersurfaces in X and all the rational curves embedded in X with type $(d_\rho)_{\rho\in\Delta(1)}$ respectively . For the anti-canonical hypersurfaces , take $Q = \{w \in M :< w, n_\rho >\geq -1, \forall \rho \in \Delta(1)\}$. It is well known that Q is a finite set and we have shown in the last section that any anti-canonical hypersurface of X has the form $\sum_{w\in Q} a_w \prod_{\rho\in\Delta(1)} X_\rho^{<w,n_\rho>+1} = 0$, where $a_w \in \mathbb{C}$ are complex numbers , and obviously , not all the constants a_w are zero . Denote $d = \sharp|Q|$ as the number of elements in Q . Then we can take $\mathbb{P}^{d-1}$ as a parameter space for all the anti-canonical hypersurfaces in X .

For all the rational curves embedded in X with type $(d_\rho)_{\rho\in\Delta(1)}$, note under the homogenous coordinates of X , any such rational curve has the form $(f_{d_\rho}(s,t))_{\rho\in\Delta(1)}$, where s,t are the homogenous coordinates of $\mathbb{P}^1$, and $f_{d_\rho}(s,t)$ is a homogenous polynomial of s,t with degree d_ρ . By convention , $f_{d_\rho}(s,t) \equiv 0$ if $d_\rho < 0$. Let $\Delta_*(1) = \{\rho \in \Delta(1) : d_\rho < 0\}$, $\Delta^*(1) = \{\rho \in \Delta(1) : d_\rho \geq 0\}$. Suppose $f_{d_\rho} = \sum_{i=0}^{d_\rho} b_{\rho,i} s^i t^{d_\rho - i}$, for $\rho \in \Delta^*(1)$. Then it is natural to collect all the coefficients $b_{\rho,i}$ for $\rho \in \Delta^*(1)$, $i = 0, \ldots, d_\rho$ to construct a parameterizing space for all the rational curves with type $(d_\rho)_{\rho\in\Delta(1)}$ in X. Next we give the precise definition . Recall if $A_{n-1}(X) \simeq \mathbb{Z}^m$, then $X \simeq \mathbb{C}^{\Delta(1)} - Z/(\mathbb{C}^*)^m$, where $Z = \{x \in \mathbb{C}^{\Delta(1)} : X_{\hat\sigma} = \prod_{\rho\notin\sigma} X_\rho = 0 \texttt{ , for all } \sigma \in \Delta\} \subset \mathbb{C}^{\Delta(1)}$, and $(\mathbb{C}^*)^m$ acts on $\mathbb{C}^{\Delta(1)}$ in the form $(\lambda_1, \ldots, \lambda_m) \cdot (X_\rho)_{\rho\in\Delta(1)} = (\varphi_\rho(\lambda_1, \ldots, \lambda_m) X_\rho)_{\rho\in\Delta(1)}$, with $\varphi_\rho : (\mathbb{C}^*)^m \to \mathbb{C}^*$ a homomorphism for $\rho \in \Delta(1)$. Now define

$$\mathcal{M}' = \mathbb{C}^{\sharp(\Delta_*(1))} \times \mathbb{C}^{\sum_{\rho\in\Delta^*(1)}(d_\rho+1)} - Z'/(\mathbb{C}^*)^m$$

where

$$Z' = \{(b_{\rho,0})_{\rho\in\Delta_*(1)} \times (b_{\rho,0}, \ldots, b_{\rho,d_\rho})_{\rho\in\Delta^*(1)} : \prod_{\rho\notin\sigma} b_{\rho,i(\rho)} = 0, \forall\sigma \in \Delta, \forall i(\rho) \texttt{ such that}$$

$$0 \leq i(\rho) \leq d_\rho \texttt{ if } \rho \in \Delta^*(1) \texttt{ , and } i(\rho) = 0 \texttt{ if } \rho \in \Delta_*(1)\} \subseteq \mathbb{C}^{\sharp(\Delta_*(1))} \times \mathbb{C}^{\sum_{\rho\in\Delta^*(1)}(d_\rho+1)}$$

, and $(\mathbb{C}^*)^m$ acts on $\mathbb{C}^{\sharp(\Delta_*(1))} \times \mathbb{C}^{\sum_{\rho\in\Delta^*(1)}(d_\rho+1)}$ in the form

$$(\lambda_1, \ldots, \lambda_m) \cdot (b_{\rho,0})_{\rho\in\Delta_*(1)} \times (b_{\rho,0}, \ldots, b_{\rho,d_\rho})_{\rho\in\Delta^*(1)} =$$
$$(\varphi_\rho(\lambda_1, \ldots, \lambda_m) b_{\rho,0})_{\rho\in\Delta_*(1)} \times (\varphi_\rho(\lambda_1, \ldots, \lambda_m) b_{\rho,0}, \ldots, \varphi_\rho(\lambda_1, \ldots, \lambda_m) b_{\rho,d_\rho})_{\rho\in\Delta^*(1)}$$

Now define $\mathcal{M}$ to be the subvariety of $\mathcal{M}'$ with $b_{\rho,0} = 0$ for all $\rho \in \Delta_*(1)$. It's not hard to verify that $\mathcal{M}'$ is a nonsingular complete toric variety with dimension $4 + \sum_{\rho\in\Delta^*(1)} d_\rho$, and $\mathcal{M}$ is a nonsingular subvariety of $\mathcal{M}'$ with dimension $4 + \sum_{\rho\in\Delta^*(1)} d_\rho - \sharp(\Delta_*(1))$.

Consider the incidence variety

$$I = \{(a,b) \in \mathbb{P}^{d-1} \times \mathcal{M} : F_a(f^b_{d_\rho}(s,t))_{\rho\in\Delta(1)} \equiv 0\} \subseteq \mathbb{P}^{d-1} \times \mathcal{M}$$

where for $a = (a_w)_{w\in Q} \in \mathbb{P}^{d-1}$, $F_a(X_\rho)_{\rho\in\Delta(1)} = \sum_{w\in Q} a_w \prod_{\rho\in\Delta(1)} X_\rho^{<w,n_\rho>+1}$. And for $b = (0)_{\rho\in\Delta_*(1)} \times (b_{\rho,0}, \ldots, b_{\rho,d_\rho})_{\rho\in\Delta^*(1)} \in \mathcal{M}$, $f^b_{d_\rho}(s,t)) \equiv 0$ if $\rho \in \Delta_*(1)$, and $f^b_{d_\rho}(s,t)) = \sum_{i=0}^{d_\rho} b_{\rho,i} s^i t^{d_\rho-i}$, for $\rho \in \Delta^*(1)$.

Using the equality $\sum_{\rho\in\Delta(1)} d_\rho < w, n_\rho >= 0$ for any $w \in M$, we see that $F_a(f^b_{d_\rho}(s,t))_{\rho\in\Delta(1)}$ is a homogenous polynomial of s,t with degree $\sum_{\rho\in\Delta(1)} d_\rho$, if it is not 0 . Hence elementary dimension theory implies that every irreducible component of I has dimension not less than

$$dim\mathbb{P}^{d-1}+dim\mathcal{M}-1-\sum_{\rho\in\Delta(1)} d_\rho = dim\mathbb{P}^{d-1}+4+\sum_{\rho\in\Delta^*(1)} d_\rho-\sharp(\Delta_*(1))-1-\sum_{\rho\in\Delta(1)} d_\rho \quad (3)$$

On the other hand , the existence of C_0 and Y_0 in the hypothesis implies there is a point $(a_0, b_0) \in I$, where a_0 is the coefficients of the defining equation of Y_0, and b_0 denotes a parameterization for C_0 . Since the normal bundle of C_0 in Y_0 has no nonzero sections, C_0 is infinitesimally rigid in Y_0 . This implies that the fibre dimension of the projection $I \to \mathbb{P}^{d-1}$ at (a_0, b_0) is exactly 3 . In fact the fiber is parametrized by $PGL(2)$. So taking an irreducible component I_0 of I going through (a_0, b_0) , we have

$$dimI_0 \leq dim\mathbb{P}^{d-1} + 3 \quad (4)$$

Now (3)and (4) implies

$$dim\mathbb{P}^{d-1} + 4 + \sum_{\rho\in\Delta^*(1)} d_\rho - \sharp(\Delta_*(1)) - \sum_{\rho\in\Delta(1)} d_\rho - 1 \leq dimI_0 \leq dim\mathbb{P}^{d-1} + 3 \quad (5)$$

Since $\sum_{\rho\in\Delta^*(1)} d_\rho + \sum_{\rho\in\Delta_*(1)} d_\rho = \sum_{\rho\in\Delta(1)} d_\rho$, and $d_\rho \geq 0$ for $\rho \in \Delta^*(1)$, $d_\rho \leq -1$ for $\rho \in \Delta_*(1)$. We conclude that all the inequalities in (5) are in fact equalities , in particular $d_\rho \geq -1$, for any $\rho \in \Delta(1)$. That proves the first claim of the theorem .

Now $dimI_0 = dim\mathbb{P}^{d-1} + 3$ and that the fibre dimension of the composed morphism $I_0 \hookrightarrow I \to \mathbb{P}^{d-1}$ at (a_0, b_0) is exactly 3 will imply that this morphism $I_0 \to \mathbb{P}^{d-1}$ is surjective , and the generic fiber has dimension 3 . This will imply that for generic hypersurface Y , there is a rational curve C embedded in Y , such that the type of C in X is the same as that of C_0 , and the normal bundle satisfies $N_{C,Y} \simeq \mathcal{O}(-1) \oplus \mathcal{O}(-1)$. The smoothness of C comes from C_0 is smooth and that to be a regular embedding is an open condition on $\mathcal{M}$. So we have proven the second claim in the theorem .

□

Remark 1. *Using some basic deformation theory , one can prove a similar result replacing the toric variety X by any complete smooth Fano $4-$fold .*

Remark 2. *In the above theorem, suppose there is a Zariski open subset U of $\mathcal{M}$ such that C_0 lies in U(more precisely, there is a parametrization of C_0 in $\mathcal{M}$ which lies in U). Then the rational curve C in the above theorem can be chosen to lie in U , too . This can be easily seen in the proof.*

According to the above theorem , for a nonsingular complete Fano toric variety X , if we can find some smooth rational curves C_{10}, ..., C_{l0} in X such that their types are all different with each other , and for each curve C_{i0},$1 \leq i \leq l$, there is an anti-canonical hypersurface Y_{i0} of X going through C_{i0} such that they satisfy the hypothesis of the above theorem. Then for generic anti-canonical hypersurface Y . Y contains smooth rational curves C_1, ..., C_l such that they all have normal bundles $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ in Y , and the type of C_i is equal to that of C_{i0} (so the cohomological class represented by C_i is equal to that represented by C_{i0}), for $1 \leq i \leq l$. So if the cohomological classes $[C_{10}]$, ..., $[C_{l0}]$ satisfy the condition in Theorem 1 , then the same is true for the curves C_i .

By Remark 2 , if we choose Zariski open subset U_i ($1 \leq i \leq l$) of $\mathcal{M}$ such that C_{i0} lies in U_i for each $1 \leq i \leq l$, then the curves C_1, ..., C_l in Y can be also chosen to lie in U_1, ..., U_l respectively. In practice, we usually choose U_i to be the Zariski subset of $\mathcal{M}$ which represents all the regular embeddings of $\mathbb{P}^1$, or at the same time, some homogenous coordinates of X have no zero points on the embedded rational curve.

Using the same notation as the last paragraph , note in the hypothesis of Theorem 1 , we require the smooth rational curves C_1, ..., C_l lying in Y do not intersect each other. So next we want to analyze when can we guarantee that for generic anti-canonical hypersurface Y, the rational curves C_1, ..., C_l lying in Y do not intersect each other. Take any two of these curves , suppose they are C_1, C_2 without loss of generality. We fix two Zariski open set U_1, U_2 of the corresponding parametrizing space $\mathcal{M}_1$, $\mathcal{M}_2$ such that C_i lies in U_i and every point in U_i represents a regular embedding, for $i = 1, 2$. Consider the subvariety U_{12} of $\mathcal{M}_1 \times \mathcal{M}_2$:

$$U_{12} = \{(b_1, b_2) \in U_1 \times U_2 : C_1 \cap C_2 \neq \emptyset\}$$

Where C_1, C_2 denotes the rational curves represented by b_1, b_2 respectively .

Consider the following incident variety :

$$J = \{(a, b_1, b_2) \in \mathbb{P}^{d-1} \times \; U_{12} : F_a(f_{d_\rho}^{b_1}(s,t))_{\rho \in \Delta(1)} = F_a(f_{d_\rho}^{b_2}(s,t))_{\rho \in \Delta(1)} \equiv 0\} \subseteq \mathbb{P}^{d-1} \times \mathcal{M}_1 \times \mathcal{M}_2$$

Roughly speaking, J represents the configuration that two intersecting rational curves lying in an anti-canonical hypersurface.

Now all we want to do is to find conditions to guarantee the dimension of the image of Pr_1 is strictly less than $dim\ \mathbb{P}^{d-1}$ (where Pr_1 denotes the natural projection morphism form J to $\mathbb{P}^{d-1}$), for then the image of Pr_1 is a lower dimensional constructible set (i.e. finite union of locally closed set) of $\mathbb{P}^{d-1}$, so the closure of this image is a lower dimensional closed subvariety in $\mathbb{P}^{d-1}$. Note each fibre of the natural projection morphism form J to $\mathbb{P}^{d-1}$ has dimension not less than 6 , because of the free action of $PGL(2)$ on each of the two rational curves. So it suffices to prove $dim\ J < dim\ \mathbb{P}^{d-1} + 6$. Considering the natural projection morphism Pr_2 from J to U_{12} , we only have to prove that each fibre of Pr_2 has dimension strictly less than $dim\ \mathbb{P}^{d-1} + 6 - dim\ U_{12}$. By the definition of J, for each point $(b_1, b_2) \in U_{12}$, the fibre of Pr_2 at (b_1, b_2) is the linear subspace of $\mathbb{P}^{d-1}$ such that its points represent exactly the anti-canonical hypersurfaces containing both of the rational curves represented by b_1 and b_2. So if for any rational curve pair (C_1, C_2) represented by a point in U_{12}, we can find $dim\ U_{12} - 5$ anti-canonical sections of X , such that the hypersurface corresponding to any nonzero linear combination of these anti-canonical sections never contain C_1 and C_2 at the same time, then we would be done . That is what we will do for each concrete toric variety in the next section. Unfortunately , this method fails in the case of the variety B_1 . That's why in our main theorem we have an exceptional case .

4 Examinations for toric Fano $4-$folds with rank 2 Picard groups

In this section , we give homogenous coordinates representations for each toric Fano $4-$folds with Picard group rank 2 . Using these representations , we will find three smooth rational curves C_{10}, C_{20}, C_{30} and anti-canonical hypersurfaces Y_{10}, Y_{20}, Y_{30} such that $[C_{i0}]$ satisfy the conditions in Theorem 1 , and C_{i0} is embedded in the smooth part of Y_{i0} with normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$. Then Theorem 3 implies a generic hypersurface Y will contain three smooth rational curves C_1, C_2, C_3 such that $[C_i]$ satisfy the conditions in Theorem 1 , and C_i is embedded in Y with normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ for $i = 1, 2, 3$. Similar to the definition of U_{12} at the end in the last section , we can define U_{ij} parameterizing intersecting rational curves C_i, C_j and analyze the dimension of the space U_{ij} . Finally we prove that a generic anti-canonical hypersurface does not contain intersecting rational curves with our chosen topological types .

4.1 The toric variety B_1

In the classification of Batyrev [1] , the toric variety B_1 is defined by a fan Δ in $\mathbb{R}^4$ such that $\Delta(1) = \{v_1, \dots, v_6\}$, $\Delta(1)$ generates $\mathbb{Z}^4$, and elements in $\Delta(1)$ satisfy the following linear relations (cf. [1]) :

$$\begin{cases} v_1 + v_2 + v_3 + v_4 = 3v_6 \\ v_5 + v_6 = 0 \end{cases}$$

$\mathbb{C}^* \times \mathbb{C}^*$ acts on this toric variety as follows :

$$(X_1, \ldots, X_6) \to (\lambda_1 X_1, \ldots, \lambda_1 X_4, \lambda_1^3 \lambda_2 X_5, \lambda_2 X_6)$$

For $(\lambda_1, \lambda_2) \in \mathbb{C}^* \times \mathbb{C}^*$.

Under this action , $X \simeq \mathbb{C}^6 - Z / \mathbb{C}^* \times \mathbb{C}^*$, where $Z = \{X_1X_5 = X_2X_5 = X_3X_5 = X_4X_5 = X_1X_6 = X_2X_6 = X_3X_6 = X_4X_6 = 0\}$

The anti-canonical forms of X are linear combinations of the following forms :

$$X_5^2 f_1(X_1, \ldots, X_4), X_5X_6f_4(X_1, \ldots, X_4), X_6^2 f_7(X_1, \ldots, X_4).$$

where $f_i(X_1, \ldots, X_4)$ denotes a degree i homogenous form of X_1, X_2, X_3, X_4 , for $i \geq 1$.

Now $H^2(X, \mathbb{Z}) \simeq H_6(X, \mathbb{Z}) \simeq A_3(X)$ is a rank 2 free abelian group , and $[D_4], [D_5]$ form a base of this group .

Consider the rational curve $(0, 0, 0, 1, s, t)$ in X , which has the type $(0, 0, 0, 0, 1, 1)$. Its cohomology class is $[D_5]$, and this rational curve is embedded in the anti-canonical hypersurface $X_1X_5^2 + X_2X_4^3X_5X_6 + X_3X_4^6X_6^2 = 0$ with normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$.

We explain the computation of the normal bundle .

Denote Y as the anti-canonical hypersurface $X_1X_5^2 + X_2X_4^3X_5X_6 + X_3X_4^6X_6^2 = 0$, and $i : \mathbb{P}^1 \to Y$ the embedding of the rational curve $(0, 0, 0, 1, s, t)$. Then we have two exact sequences :

$$N_{Y,X}^* \to \Omega_{X/\mathbb{C}} \to \Omega_{Y/\mathbb{C}} \to 0 \tag{6}$$

Where $N_{Y,X}^*$ is the conormal bundle of the hypersurface Y in the toric $4-$fold X . Pull back this exact sequence to $\mathbb{P}^1$ using i , we get :

$$i^*(N_{Y,X}^*) \to i^*(\Omega_{X/\mathbb{C}}) \to i^*(\Omega_{Y/\mathbb{C}}) \to 0 \tag{7}$$

Since the rational curve lies in the smooth part of Y , we have the exact sequence :

$$0 \to N_{\mathbb{P}^1,Y}^* \to i^*(\Omega_{Y/\mathbb{C}}) \to \Omega_{\mathbb{P}^1/\mathbb{C}} \to 0 \tag{8}$$

Where $N_{\mathbb{P}^1,Y}^* \simeq Hom(N_{\mathbb{P}^1,Y}, \mathcal{O}_{\mathbb{P}^1})$ is the conormal bundle of the rational curve in Y .

Using the two exact sequence (7) and (8) , we can compute concretely the rank 2 locally free sheaf $N_{\mathbb{P}^1,Y} \simeq \mathcal{O}(-1) \oplus \mathcal{O}(-1)$

Similarly , the rational curve $(s, t, 0, 0, 0, 1)$ has the type $(1, 1, 1, 1, 3, 0)$. Its cohomology class is $[D_4] + 3[D_5]$, and this rational curve is embedded in the anti-canonical section $X_3X_1^6X_6^2 + X_4X_1^4X_2^2X_6^2 + X_5X_2^4X_6 = 0$ with normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$.

We summarize the results in the following table .

Table 1: B_1

rational curve C_{i0}	type	cohomology class	anti-canonical hypersurface Y_{i0}
(0,0,0,1,s,t)	(0,0,0,0,1,1)	$[D_5]$	$X_1X_5^2 + X_2X_4^3X_5X_6 + X_3X_4^6X_6^2 = 0$
(s,t,0,0,0,1)	(1,1,1,1,3,0)	$[D_4]+3[D_5]$	$X_3X_1^6X_6^2 + X_4X_1^4X_2^2X_6^2 + X_5X_2^4X_6 = 0$
(s,t,0,0,s^4,t)	(1,1,1,1,4,1)	$[D_4]+4[D_5]$	$X_3X_5^2 + X_4X_1^6X_6^2 + X_5X_6X_2^4 - X_6^2X_2^3X_1^4 = 0$

Where in each row , the computation shows C_{i0} is embedded in the smooth part of Y_{i0} with normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$.

Now the three curves $(0,0,0,1,s,t)$, $(s,t,0,0,0,1)$, $(s,t,0,0,s^4,t)$ are denoted by C_{10} ,C_{20}, C_{30} respectively , and the corresponding parameterizing spaces are denoted by $\mathcal{M}_1$,$\mathcal{M}_2$,$\mathcal{M}_3$. Take the Zariski open set U_1 of $\mathcal{M}_1$ such that points in U_1 represent regular embedded rational curves on which the homogenous X_4 does not have zero points . So U_1 parameterizes exactly smooth rational curves in X having the form $(c_1, c_2, c_3, c_4, \alpha_1(s,t), \beta_1(s,t))$, where $\alpha_1(s,t), \beta_1(s,t)$ denote linear forms of s,t , $c_i (1 \leq i \leq 4)$ are constants and c_4 is not zero. Similarly, take U_2 as the open set of $\mathcal{M}_2$ parameterizing regular embedded rational curves on which the homogenous coordinate X_6 does not have zero points . Take U_3 to be the open set of $\mathcal{M}_3$ parameterizing the regular embedded rational curves . Using the similar notation as the end of last section, we want to analyze $dim\ U_{ij} (i,j = 1,2,3, i \neq j)$ and find $dim\ U_{ij} - 5$ anti-canonical sections such that any nonzero linear combinations of these sections will not contain both of the rational curves represented by points in U_{ij}. This will suffice to prove that generic anti-canonical hypersurface will contain smooth rational curves in U_1,U_2,U_3 , and these curves do not intersect each other. We will use C_1,C_2,C_3 to denote curves represented by points in U_1,U_2,U_3 respectively.

The case of C_1 and C_2

By definition, C_1 has a representation with the form $(c_1, c_2, c_3, 1, \alpha_1(s,t), \beta_1(s,t))$, where $c_i (1 \leq i \leq 3)$ are constants , and $\alpha_1(s,t), \beta_1(s,t)$ are degree 1 homogenous forms of s,t. C_2 has a representation with the form $(\alpha_1(s,t), \beta_1(s,t), \gamma_1(s,t), \delta_1(s,t), \alpha_3(s,t), 1)$, where the subscript numbers denote the degrees of the corresponding homogenous forms of s,t. Now consider $dim\ U_{12}$. In the representation of C_2, there are 12 coefficients in the homogenous forms $\alpha_1(s,t), \beta_1(s,t), \gamma_1(s,t), \delta_1(s,t), \alpha_3(s,t)$, and modulo the action of $\mathbb{C}^* \times \mathbb{C}^*$, the appearance of C_2 will contribute 11 to $dim\ U_{12}$. When fixing C_2, since C_1 has to intersect with C_2, $(c_1, c_2, c_3, 1)$ has to lie in the rational curve $(\alpha_1(s,t), \beta_1(s,t), \gamma_1(s,t), \delta_1(s,t))$, this contributes 1 to $dim\ U_{12}$. At last, the 4 coefficients of $\alpha_1(s,t), \beta_1(s,t)$ in the representation of C_1 modulo the action of $\mathbb{C}^*$ will contribute 3 to $dim\ U_{12}$. So we get $dim\ U_{12} = 15$, and we have to find 10 anti-canonical sections satisfying the condition we just required.

Note that an invertible linear substitution of the homogenous coordinates X_1 , X_2 ,X_3 ,X_4 induces an automorphism of X, so for any point $(b_1, b_2) \in U_{12}$, the two rational curves C_1, C_2 represented by b_1, b_2 can be assumed to be $(0,0,0,1,s,t)$ and $(0,0,s,t,\alpha_3(s,t),1)$, after choosing appropriate homogenous coordinates on $\mathbb{P}^1$. Then its easy to verify the following 10 anti-canonical forms satisfy our requirement.

$$X_5^2X_4, X_5X_6X_4^4, X_6^2X_4^7, X_6^2X_3X_3^iX_4^{6-i}\ (i = 0,\dots,6).$$

The case of C_1 and C_3

Similar to the analysis in the last paragraph, $dim\ U_{13} = 17$, and without loss of generality, we can assume $C_1 = (0,0,0,1,s,t), C_3 = (0,0,s,t,\alpha_4(s,t),\alpha_1(s,t))$, since C_3 is a smooth rational curve , the homogenous forms $\alpha_4(s,t), \alpha_1(s,t)$ have no common factors in $\mathbb{C}[s,t]$. Pick a degree 3 homogenous form $\beta_3(s,t)$ such that it has no common factors with $\alpha_1(s,t)$, then it's easy to verify that the following 12 anti-canonical forms will suffice :

$$X_5^2X_4, X_5X_6X_4^4, X_6^2X_4^7, X_5^2X_3, X_5X_6X_3\beta_3(X_3,X_4), X_6^2X_3X_3^iX_4^{6-i}(i = 0,\dots,6).$$

The case of C_2 and C_3

In this case , our method fails . So we have an exceptional case in the main theorem .

4.2 The toric variety B_2

This variety is defined by a fan Δ such that elements in $\Delta(1)$ satisfy :

$$\begin{cases} v_1 + v_2 + v_3 + v_4 = 2v_6 \\ v_5 + v_6 = 0 \end{cases}$$

$\mathbb{C}^* \times \mathbb{C}^*$ acts on this toric variety as follows :

$$(X_1,\dots,X_6) \to (\lambda_1X_1,\dots,\lambda_1X_4,\lambda_1^2\lambda_2X_5,\lambda_2X_6)$$

Under this action , $X \simeq \mathbb{C}^6 - Z/\mathbb{C}^* \times \mathbb{C}^*$, where $Z = \{X_1X_5 = X_2X_5 = X_3X_5 = X_4X_5 = X_1X_6 = X_2X_6 = X_3X_6 = X_4X_6 = 0\}$

The anti-canonical forms of X are linear combinations of the following forms :

$$X_5^2f_2(X_1,\dots,X_4), X_5X_6f_4(X_1,\dots,X_4), X_6^2f_6(X_1,\dots,X_4).$$

where $f_i(X_1, \ldots, X_4)$ are degree i homogenous forms of X_1, X_2, X_3, X_4 , for $i \geq 1$.

Now $H^2(X, \mathbb{Z}) \simeq H_6(X, \mathbb{Z}) \simeq A_3(X)$ is a rank 2 free abelian group , and $[D_4], [D_5]$ form a base of this group .

Table 2: B_2

rational curve C_{i0}	type	cohomology class	anti-canonical section Y_{i0}
(0,0,0,1,s,t)	(0,0,0,0,1,1)	$[D_5]$	$X_1X_5^2X_4 + X_2X_4^3X_5X_6 + X_3X_4^5X_6^2 = 0$
(s,t,0,0,0,1)	(1,1,1,1,2,0)	$[D_4]+2[D_5]$	$X_3X_1^5X_6^2 + X_4X_1^3X_2^2X_6^2 + X_5X_2^4X_6 = 0$
(s,t,0,0,s^3,t)	(1,1,1,1,3,1)	$[D_4]+3[D_5]$	$X_3X_1X_5^2 + X_4X_2^5X_6^2 + X_5X_6X_1^2X_2^2 - X_6^2X_2X_1^5 = 0$

Denote the three rational curves in the first column by C_{10} , C_{20} , C_{30} respectively , and denote the corresponding parameterizing spaces by $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{M}_3$, then take the corresponding three Zariski open sets U_1 , U_2 , U_3 such that the rational curves parameterized by points in U_i are smooth . We will use the same notation as the last case and proceed in a similar way.

The case of C_1 and C_2

$dim\ U_{12} = 14$, C_1, C_2 can be assumed to be $(0,0,0,1,s,t), (0,0,s,t,\alpha_2(s,t),1)$, and the following 9 anti-canonical forms will suffice :

$$X_5^2X_4^2, X_5X_6X_4^4, X_6^2X_3^iX_4^{6-i}(i = 0, \ldots, 6).$$

The case of C_1 and C_3

$dim\ U_{13} = 16$, C_1, C_3 can be assumed to be $(0,0,0,1,s,t), (0,0,s,t,\alpha_3(s,t),\alpha_1(s,t))$, and the following 11 anti-canonical forms will suffice :

$$X_5^2X_4^2, X_5X_6X_4^4, X_6^2X_4^6, X_5^2X_3\hat{\beta}_1(X_3,X_4), X_5X_6X_3\hat{\beta}_3(X_3,X_4), X_6^2X_3X_3^iX_4^{5-i}(i = 0, \ldots, 5).$$

Where $\hat{\beta}_1(s,t), \hat{\beta}_3(s,t)$ are homogenous forms with degree $1, 3$ respectively and neither one has common factors with $\alpha_1(s,t)$. This can be used to guarantee that when restricting on the rational curve C_3 , the 8 forms

$$X_5^2X_3\hat{\beta}_1(X_3,X_4), X_5X_6X_3\hat{\beta}_3(X_3,X_4), X_6^2X_3X_3^iX_4^{5-i}(i = 0, \ldots, 5)$$

are linearly independent as degree 8 homogenous forms of s, t .

The case of C_2 and C_3

$dim\ U_{23} = 20$, the rational curve pair C_2, C_3 can be assumed to be :

$$C_2 = (\alpha_1(s,t), \beta_1(s,t), \gamma_1(s,t), \delta_1(s,t), \alpha_2(s,t), 1)$$

$$C_3 = (\tilde{\alpha}_1(s,t), \tilde{\beta}_1(s,t), \tilde{\gamma}_1(s,t), \tilde{\delta}_1(s,t), \alpha_3(s,t), \tau_1(s,t))$$

Note $(\alpha_1(s,t), \beta_1(s,t), \gamma_1(s,t), \delta_1(s,t))$, $(\tilde{\alpha}_1(s,t), \tilde{\beta}_1(s,t), \tilde{\gamma}_1(s,t), \tilde{\delta}_1(s,t))$ represent two rational curves in $\mathbb{P}^3$. Denote them by C_2' , C_3' respectively . Recall the notation at the end of Section 2 , if for the morphism form J to U_{23} , there is a rational curve pair C_2 , C_3 in the image of J such that C_2' does not coincide with C_3' as lines in $\mathbb{P}^3$, then modulo a re-parameterization and an automorphism of X , C_2 , C_3 can be assumed to be :

$$C_2 = (0, s, 0, t, \alpha_2(s,t), 1), C_3 = (0, 0, s, t, \alpha_3(s,t), \alpha_1(s,t)).$$

And the following 15 anti-canonical forms will suffice :

$$X_6^2 X_2^i X_4^{6-i} (i = 0, \ldots, 6), X_5^2 X_3 \hat{\beta}_1(X_3, X_4), X_5 X_6 X_3 \hat{\beta}_3(X_3, X_4), X_6^2 X_3 X_3^i X_4^{5-i} (i = 0, \ldots, 5).$$

Where the homogenous forms $\hat{\beta}_1$, $\hat{\beta}_3$ are the same as those in the case of C_1 and C_3 . If for any rational curve pair C_2, C_3 in the image of J , C_2' coincides with C_3' as lines in $\mathbb{P}^3$, then the dimension of this image $dim\ Im(J) \leq 18$, and modulo a re-parameterization and an automorphism of X , C_2 , C_3 can be assumed to be :

$$C_2 = (0, 0, s, t, \alpha_2(s,t), 1), C_3 = (0, 0, s, t, \alpha_3(s,t), \alpha_1(s,t)).$$

Since $\alpha_1(s,t)$ has no common factors with $\alpha_3(s,t)$, we can find degree 3 homogenous forms $\gamma_3(s,t), \delta_3(s,t)$ such that the 4 homogenous forms $\alpha_1\alpha_2$, α_3, γ_3, δ_3 are linearly independent as homogenous forms of s, t . Now consider the 7 anti-canonical forms :

$$\begin{aligned} &X_5^2 \alpha_1^2(X_3, X_4), X_5 X_6 \alpha_1(X_3, X_4) \alpha_3(X_3, X_4), X_5 X_6 \alpha_1(X_3, X_4) \gamma_3(X_3, X_4), \\ &\quad X_5 X_6 \alpha_1(X_3, X_4) \delta_3(X_3, X_4), X_6^2 \alpha_3(X_3, X_4) \alpha_3(X_3, X_4), \\ &\quad X_6^2 \alpha_3(X_3, X_4) \gamma_3(X_3, X_4), X_6^2 \alpha_3(X_3, X_4) \delta_3(X_3, X_4). \end{aligned} \tag{9}$$

When restricted to C_3, they reduce to 3 homogenous forms of s, t with degree 8 . Since the anti-canonical forms could generate all the degree 8 forms on C_3 , we can pick 6 anti-canonical forms $f_i (1 \leq i \leq 6)$ such that with the 7 forms in (9) , these 13 anti-canonical forms generate all the degree 8 forms on C_3 when restricted on it . Next it's direct to verify that when restricted on C_2 , the 7 forms in (11) generate a dimension 7, 6, or 5 linear subspace of degree 6 forms on C_2 , depending on $\alpha_2(s,t)$ and $\alpha_3(s,t)$ has a degree 0 , 1 , or 2 greatest common divisor respectively . According to the above analysis , we can always find $dim\ Im(J) - 5$ anti-canonical forms such that any nonzero linear combination of them does not contain C_2 and C_3 at the same time , for C_2 and C_3 represented by points in J .

4.3 The toric variety B_3

This variety is defined by a fan Δ such that elements in $\Delta(1)$ satisfy :

$$\begin{cases} v_1 + v_2 + v_3 + v_4 = v_6 \\ v_5 + v_6 = 0 \end{cases}$$

$\mathbb{C}^* \times \mathbb{C}^*$ acts on this toric variety as follows :

$$(X_1, \ldots, X_6) \to (\lambda_1 X_1, \lambda_1 X_2, \lambda_1 X_3, \lambda_1 X_4, \lambda_1\lambda_2 X_5, \lambda_2 X_6)$$

Under this action , $X \simeq \mathbb{C}^6 - Z/\mathbb{C}^* \times \mathbb{C}^*$, where $Z = \{X_1X_5 = X_2X_5 = X_3X_5 = X_4X_5 = X_1X_6 = X_2X_6 = X_3X_6 = X_4X_6 = 0\}$

The anti-canonical forms of X are linear combinations of the following forms :

$$X_5^2 f_3(X_1, \ldots, X_4), X_5X_6f_4(X_1, \ldots, X_4), \\ X_6^2 f_5(X_1, \ldots, X_4).$$

where $f_i(X_1, \ldots, X_4)$ are degree i homogenous forms , for $i \geq 1$.

Now $H^2(X, \mathbb{Z}) \simeq H_6(X, \mathbb{Z}) \simeq A_3(X)$ is a rank 2 free abelian group , and $[D_4], [D_5]$ form a base of this group .

Table 3: B_3

rational curve C_{i0}	type	cohomology class	anti-canonical section Y_{i0}
(s,t,0,0,1,0)	(1,1,1,1,0,-1)	$[D_4]$	$X_3X_1^2X_5^2 + X_4X_2^2X_5^2 + X_6X_5X_2^4$ =0
(s,t,0,0,0,1)	(1,1,1,1,1,0)	$[D_4] + [D_5]$	$X_3X_1^4X_6^2 + X_4X_1^2X_2^2X_6^2 + X_5X_6X_2^4$ =0
(1,0,0,0,s,t)	(0,0,0,0,1,1)	$[D_5]$	$X_2X_1^2X_5^2 + X_3X_1^3X_5X_6 + X_4X_1^4X_6^2$ =0

Denote the three rational curves in the first column by C_{10} , C_{20} , C_{30} respectively , and denote the corresponding parameterizing spaces by $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{M}_3$, then take the corresponding three Zariski open sets U_1 , U_2 , U_3 such that the rational curves parameterized by points in U_i are smooth , and the homogenous coordinates X_5 , X_6 , X_1 have no zero points on rational curves in U_1 , U_2 , U_3 respectively .

The case of C_1 and C_2

This case is trivial . Since the homogenous coordinate X_6 has no zero points on C_2 , and it's always zero on C_1 , so $U_{12} = \emptyset$.

The case of C_1 and C_3

$dim\ U_{13} = 11$, C_1 , C_3 can be assumed to be $(s,t,0,0,1,0)$, $(1,0,0,0,s,t)$, and the following 6 forms will suffice :

$$X_1^4X_5X_6, X_1^5X_6^2, X_5^2X_1^iX_2^{3-i}(0 \le i \le 3).$$

The case of C_2 and C_3

$dim\ U_{23} = 13$, C_2 , C_3 can be assumed to be $(s,t,0,0,\alpha_1(s,t),1)$, $(1,0,0,0,s,t)$, and the following 8 forms will suffice :

$$X_1^4X_5X_6, X_1^5X_6^2, X_5^2X_1^3, X_6^2X_2X_1^iX_2^{4-i}(0 \le i \le 4).$$

4.4 The toric variety B_4

This variety is defined by a fan Δ such that elements in $\Delta(1)$ satisfy :

$$\begin{cases} v_1 + v_2 + v_3 + v_4 = 0 \\ v_5 + v_6 = 0 \end{cases}$$

$\mathbb{C}^* \times \mathbb{C}^*$ acts on this toric variety as follows :

$$(X_1, \ldots, X_6) \to (\lambda_1X_1, \lambda_1X_2, \lambda_1X_3, \lambda_1X_4, \lambda_2X_5, \lambda_2X_6)$$

Under this action , $X \simeq \mathbb{C}^6 - Z \big/ \mathbb{C}^* \times \mathbb{C}^* \simeq \mathbb{P}^1 \times \mathbb{P}^3$, where $Z = \{X_1X_5 = X_2X_5 = X_3X_5 = X_4X_5 = X_1X_6 = X_2X_6 = X_3X_6 = X_4X_6 = 0\}$

The anti-canonical forms of X are linear combinations of the forms $f_4(X_1,X_2,X_3,X_4)g_2(X_5,X_6)$, where $f_i(X_1,X_2,X_3,X_4)$, $g_i(X_5,X_6)$ are degree i homogenous forms , for $i \ge 1$.

Now $H^2(X,\mathbb{Z}) \simeq H_6(X,\mathbb{Z}) \simeq A_3(X)$ is a rank 2 free abelian group , and $[D_4]$, $[D_5]$ form a base of this group .

Table 4: B_4

rational curve C_{i0}	type	cohomology class	anti-canonical section Y_{i0}
(1,0,0,0,s,t)	(0,0,0,0,1,1)	$[D_5]$	$X_2X_5^2X_1^3 + X_3X_5X_6X_1^3 + X_4X_6^2X_1^3$ =0
(s,t,0,0,1,0)	(1,1,1,1,0,0)	$[D_4]$	$X_3X_5^2X_1^3 + X_4X_5^2X_2^3 + X_6X_1^2X_5X_2^2$ =0
(s,t,0,0,s,t)	(1,1,1,1,1,1)	$[D_4]$+$[D_5]$	$X_3X_1^3X_5^2 + X_4X_1^3X_6^2 + X_2^4X_5X_6 - X_1X_2^3X_6^2$ =0

Denote the three rational curves in the first column by C_{10} , C_{20} , C_{30} respectively , and denote the corresponding parameterizing spaces by $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{M}_3$, then take the corresponding three Zariski open sets U_1 , U_2 , U_3 such that the rational curves parameterized by points in U_i are smooth .

The case of C_1 and C_2

dim $U_{12} = 12$, C_1 , C_2 can be assumed to be $(1,0,0,0,s,t)$, $(s,t,0,0,1,0)$, and the following 7 forms will suffice :

$$X_5^i X_6^{4-i} X_1^2 (0 \leq i \leq 4), X_1 X_2 X_5^4, X_2^2 X_5^4.$$

The case of C_1 and C_3

dim $U_{13} = 14$, C_1 , C_3 can be assumed to be $(1,0,0,0,s,t)$, $(s,t,0,s,0,t)$ or $(1,0,0,0,s,t)$, $(s,t,0,0,s,t)$. In the first case , the following 10 forms will suffice :

$$X_5^i X_6^{4-i} X_1^2 (0 \leq i \leq 4), X_4^i X_6^{4-i} X_2^2 (0 \leq i \leq 4).$$

And in the second case , the following 10 forms will suffice :

$$X_5^i X_6^{4-i} X_1^2 (0 \leq i \leq 4), X_5^i X_6^{4-i} X_2^2 (0 \leq i \leq 4).$$

The case of C_2 and C_3

dim $U_{12} = 11$, C_2 , C_3 can be assumed to be $(s,t,0,0,1,0)$, $(s,t,0,0,s,t)$, and the following 6 forms will suffice :

$$X_1^i X_2^{2-i} X_5^4 (0 \leq i \leq 2), X_1^i X_2^{2-i} X_6^4 (0 \leq i \leq 2).$$

4.5 The toric variety B_5

This variety is defined by a fan Δ such that elements in $\Delta(1)$ satisfy :

$$\begin{cases} v_1 + v_2 + v_3 + v_4 = 0 \\ v_5 + v_6 = v_4 \end{cases}$$

$\mathbb{C}^* \times \mathbb{C}^*$ acts on this toric variety as follows :

$$(X_1, \ldots, X_6) \to (\lambda_1\lambda_2 X_1, \lambda_1\lambda_2 X_2, \lambda_1\lambda_2 X_3, \lambda_1 X_4, \lambda_2 X_5, \lambda_2 X_6)$$

Under this action , $X \simeq \mathbb{C}^6 - Z / \mathbb{C}^* \times \mathbb{C}^*$, where $Z = \{X_1X_5 = X_2X_5 = X_3X_5 = X_4X_5 = X_1X_6 = X_2X_6 = X_3X_6 = X_4X_6 = 0\}$

The anti-canonical forms of X are linear combinations of the following forms :

$$X_4^4 f_5(X_5,X_6), X_4^3 g_1(X_1,X_2,X_3) f_4(X_5,X_6), X_4^2 g_2(X_1,X_2,X_3) f_3(X_5,X_6),$$
$$X_4 g_3(X_1,X_2,X_3) f_2(X_5,X_6), g_4(X_1,X_2,X_3) f_1(X_5,X_6)$$

where $g_i(X_1,X_2,X_3)$, $f_i(X_5,X_6)$ are degree i homogenous forms , for $i \geq 1$.

Now $H^2(X,\mathbb{Z}) \simeq H_6(X,\mathbb{Z}) \simeq A_3(X)$ is a rank 2 free abelian group , and $[D_4]$, $[D_5]$ form a base of this group .

Table 5: B_5

rational curve C_{i0}	type	cohomology class	anti-canonical section Y_{i0}
(0,0,0,1,s,t)	(1,1,1,0,1,1)	$[D_5]$	$X_1X_4^3X_6^4 + X_2X_4^3X_5^4 + X_3X_4^3X_5^2X_6^2$ =0
(s,0,0,t,1,0)	(1,1,1,1,0,0)	$[D_4]$	$X_2X_1^3X_5 + X_3X_4^2X_1X_5^3 + X_6X_5^4X_4^4$ =0
(1,0,0,0,s,t)	(0,0,0,-1,1,1)	-$[D_4]$+$[D_5]$	$X_2X_1^3X_5 + X_3X_1^3X_6 + X_4X_6^2X_1^3$ =0

Denote the three rational curves in the first column by C_{10} , C_{20} , C_{30} respectively , and denote the corresponding parameterizing spaces by $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{M}_3$, then take the corresponding three Zariski open sets U_1 , U_2 , U_3 such that the rational curves parameterized by points in U_i are smooth , and the homogenous coordinates X_4 , X_5 , X_1 have no zero points on rational curves in U_1 , U_2 , U_3 respectively .

The case of C_1 and C_2

dim $U_{12} = 15$, C_1 , C_2 can be assumed to be $(\alpha_1(s,t), \beta_1(s,t), \gamma_1(s,t), 1, s, t)$, $(s, \tilde{\alpha}_1(s,t), \tilde{\beta}_1(s,t), t, 1, 0)$, and the following 10 forms will suffice :

$$X_4^4X_5^5, X_4^3X_1X_5^4, X_4^2X_1^2X_5^3, X_4X_1^3X_5^2, X_1^4X_5, X_4^4X_6X_5^iX_6^{4-i}(0 \le i \le 4).$$

The case of C_2 and C_3

dim $U_{23} = 11$, C_2 , C_3 can be assumed to be $(s, \tilde{\alpha}_1(s,t), \tilde{\beta}_1(s,t), t, 1, 0)$, $(1,0,0,0,s,t)$, and the following 6 forms will suffice :

$$X_4^4X_5^5, X_4^3X_1X_5^4, X_4^2X_1^2X_5^3, X_4X_1^3X_5^2, X_1^4X_5, X_1^4X_6.$$

The case of C_1 and C_3

This case is trivial . Since the homogenous coordinate X_4 has no zero points on C_1 , and it's always zero on C_3 , so $U_{13} = \emptyset$.

4.6 The toric variety C_1

This variety is defined by a fan Δ such that elements in $\Delta(1)$ satisfy :

$$\begin{cases} v_1 + v_2 + v_3 = 0 \\ v_4 + v_5 + v_6 = 2v_3 \end{cases}$$

$\mathbb{C}^* \times \mathbb{C}^*$ acts on this toric variety as follows :

$$(X_1, \ldots, X_6) \to (\lambda_1\lambda_2^2X_1, \lambda_1\lambda_2^2X_2, \lambda_1X_3, \lambda_2X_4, \lambda_2X_5, \lambda_2X_6)$$

Under this action , $X \simeq \mathbb{C}^6 - Z/\mathbb{C}^* \times \mathbb{C}^*$, where $Z = \{X_1X_4 = X_1X_5 = X_1X_6 = X_2X_4 = X_2X_5 = X_2X_6 = X_3X_4 = X_3X_5 = X_3X_6 = 0\}$

The anti-canonical forms of X are linear combinations of the following forms :

$$X_3^3 f_7(X_4,X_5,X_6), X_3^2 f_5(X_4,X_5,X_6) g_1(X_1,X_2), X_3 f_3(X_4,X_5,X_6) g_2(X_1,X_2),$$
$$f_1(X_4,X_5,X_6) g_3(X_1,X_2).$$

where $g_i(X_1,X_2)$, $f_i(X_4,X_5,X_6)$ are degree i homogenous forms , for $i \geq 1$.

Now $H^2(X,\mathbb{Z}) \simeq H_6(X,\mathbb{Z}) \simeq A_3(X)$ is a rank 2 free abelian group , and $[D_3]$, $[D_6]$ form a base of this group .

Table 6: C_1

rational curve C_{i0}	type	cohomology class	anti-canonical section Y_{i0}
(s,0,t,0,0,1)	(1,1,1,0,0,0)	$[D_3]$	$X_2X_1^2X_6 + X_4X_3^2X_6^4X_1 + X_5X_3^3X_6^6$ =0
(0,0,1,s,0,t)	(2,2,0,1,1,1)	$[D_6]$	$X_1X_3^2X_4^5 + X_2X_4^2X_3^2X_6^3 + X_5X_3^3X_6^6$ =0
(s,t,0,s,0,t)	(1,1,-1,1,1,1)	$[D_6]$-$[D_3]$	$X_3X_2^2X_6^3 + X_5X_1^3 + X_1X_6X_2^2 - X_2^3X_4$ =0

Denote the three rational curves in the first column by C_{10} , C_{20} , C_{30} respectively , and denote the corresponding parameterizing spaces by $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{M}_3$, then take the corresponding three Zariski open sets U_1 , U_2 , U_3 such that the rational curves parameterized by points in U_i are smooth , and the homogenous coordinates X_6 , X_3 have no zero points on rational curves in U_1 , U_2 respectively .

The case of C_1 and C_2

$dim\, U_{12} = 16$, C_1 , C_2 can be assumed to be $(s, \alpha_1(s,t), t, 0, 0, 1)$, $(\alpha_2(s,t), \beta_2(s,t), 1, s, 0, t)$, and the following 11 forms will suffice :

$$X_3^3X_6^7, X_3^2X_1X_6^5, X_1^2X_6^3X_3, X_6X_1^3, X_3^3X_4X_4^iX_6^{6-i} (0 \leq i \leq 6).$$

The case of C_1 and C_3

$dim\, U_{13} = 13$, C_1 , C_3 can be assumed to be $(s, \alpha_1(s,t), t, 0, 0, 1)$, $(\tilde{\alpha}_1(s,t), \tilde{\beta}_1(s,t), 0, s, 0, t)$, and the following 8 forms will suffice :

$$X_3^3X_6^7, X_3^2X_1X_6^5, X_1^2X_6^3X_3, X_6X_1^3, X_4X_1^3, X_4X_1^2X_2, X_4X_1X_2^2, X_4X_2^3.$$

The case of C_2 and C_3

This case is trivial . Since the homogenous coordinate X_3 has no zero points on C_2 , and it's always zero on C_3 , so $U_{23} = \emptyset$.

4.7 The toric variety C_2

This variety is defined by a fan Δ such that elements in $\Delta(1)$ satisfy :

$$\begin{cases} v_1 + v_2 + v_3 = 0 \\ v_4 + v_5 + v_6 = v_3 \end{cases}$$

$\mathbb{C}^* \times \mathbb{C}^*$ acts on this toric variety as follows :

$$(X_1, \ldots, X_6) \to (\lambda_1\lambda_2 X_1, \lambda_1\lambda_2 X_2, \lambda_1 X_3, \lambda_2 X_4, \lambda_2 X_5, \lambda_2 X_6)$$

Under this action , $X \simeq \mathbb{C}^6 - Z/\mathbb{C}^* \times \mathbb{C}^*$, where $Z = \{X_1X_4 = X_1X_5 = X_1X_6 = X_2X_4 = X_2X_5 = X_2X_6 = X_3X_4 = X_3X_5 = X_3X_6 = 0\}$

The anti-canonical forms of X are linear combinations of the following forms :

$$X_3^3 f_5(X_4, X_5, X_6), X_3^2 f_4(X_4, X_5, X_6) g_1(X_1, X_2), X_3 f_3(X_4, X_5, X_6) g_2(X_1, X_2), \\ f_2(X_4, X_5, X_6) g_3(X_1, X_2).$$

where $g_i(X_1, X_2)$, $f_i(X_4, X_5, X_6)$ are degree i homogenous forms , for $i \geq 1$.

Now $H^2(X, \mathbb{Z}) \simeq H_6(X, \mathbb{Z}) \simeq A_3(X)$ is a rank 2 free abelian group , and $[D_3]$, $[D_6]$ form a base of this group .

Table 7: C_2

rational curve C_{i0}	type	cohomology class	anti-canonical section Y_{i0}
(s,0,t,0,0,1)	(1,1,1,0,0,0)	$[D_3]$	$X_2X_1^2X_6^2 + X_4X_3^2X_6^3X_1 + X_5X_3^3X_6^4 = 0$
(0,0,1,s,0,t)	(1,1,0,1,1,1)	$[D_6]$	$X_1X_3^2X_4^4 + X_2X_4^2X_3^2X_6^2 + X_5X_3^3X_6^4 = 0$
(1,0,0,s,0,t)	(0,0,-1,1,1,1)	$-[D_3]+[D_6]$	$X_2X_1^2X_4^2 + X_3X_2^2X_5^3 + X_5X_1^3X_6 = 0$

Denote the three rational curves in the first column by C_{10} , C_{20} , C_{30} respectively , and denote the corresponding parameterizing spaces by $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{M}_3$, then take the corresponding three Zariski open sets U_1 , U_2 , U_3 such that the rational curves parameterized by points in U_i are smooth , and the homogenous coordinates X_6 , X_3 have no zero points on rational curves in U_1 , U_2 respectively .

The case of C_1 and C_2

$dim\, U_{12} = 14$, C_1 , C_2 can be assumed to be $(s, \alpha_1(s,t), t, 0, 0, 1)$, $(\tilde{\alpha}_1(s,t), \tilde{\beta}_1(s,t), 1, s, 0, t)$, and the following 9 forms will suffice :

$$X_3^3X_6^5, X_3^2X_1X_6^4, X_1^2X_6^3X_3, X_6^2X_1^3, X_3^3X_4X_4^iX_6^{4-i}(0 \leq i \leq 4).$$

The case of C_1 and C_3

$dim\ U_{13} = 11$, C_1 , C_3 can be assumed to be $(s, \alpha_1(s,t), t, 0, 0, 1)$, $(1, 0, 0, s, 0, t)$, and the following 6 forms will suffice :

$$X_3^3 X_6^5, X_3^2 X_1 X_6^4, X_1^2 X_6^3 X_3, X_6^2 X_1^3, X_4 X_6 X_1^3, X_4^2 X_1^3.$$

The case of C_2 and C_3

This case is trivial . Since the homogenous coordinate X_3 has no zero points on C_2 , and it's always zero on C_3 , so $U_{23} = \emptyset$.

4.8 The toric variety C_3

This variety is defined by a fan Δ such that elements in $\Delta(1)$ satisfy :

$$\begin{cases} v_1 + v_2 + v_3 = 0 \\ v_4 + v_5 + v_6 = v_1 + v_2 = -v_3 \end{cases}$$

$\mathbb{C}^* \times \mathbb{C}^*$ acts on this toric variety as follows :

$$(X_1, \ldots, X_6) \to (\lambda_1 \lambda_2^{-1} X_1, \lambda_1 \lambda_2^{-1} X_2, \lambda_1 X_3, \lambda_2 X_4, \lambda_2 X_5, \lambda_2 X_6)$$

Under this action , $X \simeq \mathbb{C}^6 - Z \diagup \mathbb{C}^* \times \mathbb{C}^*$, where $Z = \{X_1X_4 = X_1X_5 = X_1X_6 = X_2X_4 = X_2X_5 = X_2X_6 = X_3X_4 = X_3X_5 = X_3X_6 = 0\}$

The anti-canonical forms of X are linear combinations of the following forms :

$$X_3^3 f_1(X_4, X_5, X_6), X_3^2 f_2(X_4, X_5, X_6) g_1(X_1, X_2), X_3 f_3(X_4, X_5, X_6) g_2(X_1, X_2),$$
$$f_4(X_4, X_5, X_6) g_3(X_1, X_2).$$

where $g_i(X_1, X_2)$, $f_i(X_4, X_5, X_6)$ are degree i homogenous forms , for $i \geq 1$.

Now $H^2(X, \mathbb{Z}) \simeq H_6(X, \mathbb{Z}) \simeq A_3(X)$ is a rank 2 free abelian group , and $[D_3]$, $[D_6]$ form a base of this group .

Table 8: C_3

rational curve C_{i0}	type	cohomology class	anti-canonical section Y_{i0}
(s,0,t,0,0,1)	(1,1,1,0,0,0)	$[D_3]$	$X_2X_1^2X_6^4 + X_4X_3^2X_6X_1 + X_5X_3^3$ =0
(0,0,1,s,0,t)	(-1,-1,0,1,1,1)	$[D_6]$	$X_1X_3^2X_4^2 + X_2X_3^2X_4^2 + X_5X_3^3$ =0
(1,0,0,s,0,t)	(0,0,1,1,1,1)	$[D_3]$+$[D_6]$	$X_2X_1^2X_4^4 + X_3X_1^2X_4^2X_6 + X_5X_1^3X_6^3$ =0

Denote the three rational curves in the first column by C_{10} , C_{20} , C_{30} respectively , and denote the corresponding parameterizing spaces by $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{M}_3$, then take the corresponding three Zariski open sets U_1 , U_2 , U_3 such that the rational curves parameterized by points in U_i are smooth , and the homogenous coordinates X_6 , X_3 , X_1 have no zero points on rational curves in U_1 , U_2 , U_3 respectively .

The case of C_1 and C_2

$dim\ U_{12} = 10$, C_1 , C_2 can be assumed to be $(s, \alpha_1(s,t), t, 0, 0, 1)$, $(0, 0, 1, s, 0, t)$, and the following 5 forms will suffice :

$$X_3^3X_6, X_3^2X_1X_6^2, X_1^2X_6^3X_3, X_6^4X_1^3, X_3^3X_4.$$

The case of C_1 and C_3

$dim\ U_{13} = 13$, C_1 , C_3 can be assumed to be $(s, \alpha_1(s,t), t, 0, 0, 1)$, $(1, 0, \tilde{\alpha}_1(s,t), t, 0, s)$, and the following 8 forms will suffice :

$$X_3^3X_6, X_3^2X_1X_6^2, X_1^2X_6^3X_3, X_6^4X_1^3, X_4X_1^3X_4^iX_6^{3-i}(0 \le i \le 3).$$

The case of C_2 and C_3

This case is trivial . Since the homogenous coordinate X_1 has no zero points on C_3 , and it's always zero on C_2 , so $U_{23} = \emptyset$.

4.9 The toric variety C_4

This variety is defined by a fan Δ such that elements in $\Delta(1)$ satisfy :

$$\begin{cases} v_1 + v_2 + v_3 = 0 \\ v_4 + v_5 + v_6 = 0 \end{cases}$$

$\mathbb{C}^* \times \mathbb{C}^*$ acts on this toric variety as follows :

$$(X_1, \ldots, X_6) \to (\lambda_1X_1, \lambda_1X_2, \lambda_1X_3, \lambda_2X_4, \lambda_2X_5, \lambda_2X_6)$$

Under this action , $X \simeq \mathbb{C}^6 - Z \diagup \mathbb{C}^* \times \mathbb{C}^* \simeq \mathbb{P}^2 \times \mathbb{P}^2$, where $Z = \{X_1X_4 = X_1X_5 = X_1X_6 = X_2X_4 = X_2X_5 = X_2X_6 = X_3X_4 = X_3X_5 = X_3X_6 = 0\}$

The anti-canonical forms of X are linear combinations of the forms $f_3(X_1, X_2, X_3)g_3(X_4, X_5, X_6)$, where $f_i(X_1, X_2, X_3)$, $g_i(X_4, X_5, X_6)$ are degree i homogenous forms , for $i \ge 1$.

Now $H^2(X, \mathbb{Z}) \simeq H_6(X, \mathbb{Z}) \simeq A_3(X)$ is a rank 2 free abelian group , and $[D_3]$, $[D_6]$ form a base of this group .

In the following Table 9 , denote the three rational curves in the first column by C_{10} , C_{20} , C_{30} respectively , and denote the corresponding parameterizing spaces by $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{M}_3$, then take the corresponding three Zariski open sets U_1 , U_2 , U_3 such that the rational curves parameterized by points in U_i are smooth .

Table 9: C_4

rational curve C_{i0}	type	cohomology class	anti-canonical section Y_{i0}
(s,t,0,1,0,0)	(1,1,1,0,0,0)	$[D_3]$	$X_5X_4^2X_2^3+X_6X_4^2X_1X_2^2+X_3X_4^3X_1^2=0$
(1,0,0,s,t,0)	(0,0,0,1,1,1)	$[D_6]$	$X_2X_1^2X_5^3+X_3X_1^2X_4X_5^2+X_6X_1^3X_4^2=0$
(s,t,0,s,t,0)	(1,1,1,1,1,1)	$[D_3]+[D_6]$	$X_3X_1^2X_4X_5^2+X_6X_1^3X_4^2+X_1X_2^2X_5^3-X_2^3X_4X_5^2=0$

The case of C_1 and C_2

dim $U_{12}=12$, C_1 , C_2 can be assumed to be $(s,t,0,1,0,0)$, $(1,0,0,s,t,0)$, and the following 7 forms will suffice :

$$X_1^iX_2^{3-i}X_4^3(0\le i\le 3), X_1^3X_5X_4^iX_5^{2-i}(0\le i\le 2).$$

The case of C_1 and C_3

dim $U_{13}=13$. C_1 , C_3 can be assumed to be $(s,t,0,1,0,0)$, $(s,0,t,s,t,0)$ or $(s,t,0,1,0,0)$, $(s,t,0,s,t,0)$. And in the first case , the following 8 forms will suffice :

$$X_1^iX_2^{3-i}X_4^3(0\le i\le 3), X_1^iX_3^{3-i}X_5^3(0\le i\le 3).$$

In the second case , the following 8 forms will suffice :

$$X_1^iX_2^{3-i}X_4^3(0\le i\le 3), X_1^iX_2^{3-i}X_5^3(0\le i\le 3).$$

The case of C_2 and C_3

This follows from a similar argument as we did in the last case .

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