On a General Linear Nonlocal Curvature Flow of Convex Plane Curves*

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Abstract

Motivated by Pan-Yang [PY] and Ma-Cheng [MC], we study a general linear nonlocal curvature flow for convex closed plane curves and discuss the short time existence and asymptotic convergence behavior of the flow.

Due to the linear structure of the flow, this partial differential equation problem can be resolved using an ordinary differential equation method, together with the help of representation formula for solutions to a linear heat equation.

1 Introduction.

Recently there has been some interest in the *nonlocal* flow of *convex* closed plane curves[1]. See the papers by Gage [GA1], Jiang-Pan [JP], Pan-Yang [PY], Ma-Cheng [MC], Ma-Zhu [MZ] and Lin-Tsai [LT1]. All of the above papers deal with the evolution of a given convex[2] simple closed plane curve γ_0. The general form of the equation is given by

$$\begin{cases} \dfrac{\partial X}{\partial t}(\varphi, t) = \left[F\left(k\left(\varphi, t\right)\right) - \lambda\left(t\right)\right] \mathbf{N}_{in}\left(\varphi, t\right) \\ X\left(\varphi, 0\right) = X_0\left(\varphi\right), \quad \varphi \in S^1, \end{cases} \tag{1}$$

which is a parabolic initial value problem. Here $X_0(\varphi) : S^1 \to \gamma_0$ is a smooth parametrization of γ_0; $k(\varphi, t)$ is the curvature of the evolving curve $\gamma_t = \gamma(\cdot, t)$ (parametrized by $X(\varphi, t)$) at the point φ; and $\mathbf{N}_{in}(\varphi, t)$ is the inward normal of the curve γ_t at time t. As for the speed, $F(k)$ is a given function of curvature satisfying the *parabolic condition* $F'(z) > 0$ for all z in its domain (usually $(0, \infty)$) and $\lambda(t)$ is a function of time, which may depend on certain global quantities of γ_t, say its length $L(t)$ or enclosed area $A(t)$, or others (see (4) and (5) below). In such a case the flow has nonlocal character. Note that if $\lambda(t)$ depends on γ_t, then it is not known beforehand. We only know $\lambda(0)$.

For k-type nonlocal flow, the following

$$F(k) - \lambda(t) = k - \frac{2\pi}{L(t)} \quad \text{(area-preserving, gradient flow of the IPD } L^2 - 4\pi A\text{)} \tag{2}$$

(IPD means *isoperimetric deficit*) and

$$F(k) - \lambda(t) = k - \frac{L(t)}{2A(t)} \quad \text{(gradient flow of the IPR } \frac{L^2}{4\pi A}\text{)} \tag{3}$$

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[1]There are already a lot of nonlocal curvature flow of hypersurfaces, especially those related to the so-called mean curvature flow. We will not mention them here.

[2]Here "convex" always means "strictly convex". A convex closed plane curve has positive curvature everywhere.

(IPR means *isoperimetric ratio*) and

$$F(k) - \lambda(t) = k - \frac{1}{2\pi}\int_0^{L(t)} k^2 ds \quad \text{(length-preserving)} \tag{4}$$

have been studied by Gage [GA1], Jiang-Pan [JP], and Ma-Zhu [MZ] respectively.

On the other hand, for $1/k$-type nonlocal flow (here the initial curve γ_0 must be *convex*, otherwise $k = 0$ somewhere), the following

$$F(k) - \lambda(t) = \frac{1}{L(t)}\int_0^{L(t)} \frac{1}{k} ds - \frac{1}{k} \quad \text{(area-preserving)} \tag{5}$$

and

$$F(k) - \lambda(t) = \frac{L(t)}{2\pi} - \frac{1}{k} \quad \text{(length-preserving, dual flow of Gage)} \tag{6}$$

and

$$F(k) - \lambda(t) = \frac{2A(t)}{L(t)} - \frac{1}{k} \quad \text{(dual flow of Jiang-Pan)} \tag{7}$$

have been studied by Ma-Cheng [MC], Pan-Yang [PY], and Lin-Tsai [LT1] respectively. The good thing about these $1/k$-type flows is that they produce a **linear** equation for the radius of curvature $1/k$, hence it is easier to deal with it. See the discussions below.

When the initial cure γ_0 is convex, the results claimed in each of the above mentioned papers are roughly more or less the same: the flow preserve the convexity of a given initial curve γ_0 and evolve it (if no singularity forming in finite time) to a round circle in C^∞ sense as $t \to \infty$. However, unfortunately, the C^∞ convergence proof in most of the above papers are usually omitted due to its cumbersome details.

We know that (see Gage [GA1]) for a family of time-dependent smooth simple closed curves $X(\varphi, t) : S^1 \times [0, T) \to \mathbb{R}^2$ with time variation

$$\frac{\partial X}{\partial t}(\varphi, t) = W(\varphi, t) \in \mathbb{R}^2, \tag{8}$$

its length $L(t)$ and enclosed area $A(t)$ evolve according to the following:

$$\frac{dL}{dt}(t) = -\int_{\gamma_t} \langle W,\ k\mathbf{N}_{in}\rangle\, ds, \qquad \frac{dA}{dt}(t) = -\int_{\gamma_t} \langle W,\ \mathbf{N}_{in}\rangle\, ds, \tag{9}$$

where s is the arc length parameter of the curve $X(\varphi, t)$. As for the curvature $k(\varphi, t)$, following computations similar to those in Gage-Hamilton [GH], we can obtain the equation

$$\frac{\partial k}{\partial t}(\varphi, t) = \left\langle \frac{\partial^2 W}{\partial s^2},\ \mathbf{N}_{in} \right\rangle - 2k\left\langle \frac{\partial W}{\partial s},\ \mathbf{T} \right\rangle \tag{10}$$

where $\mathbf{T} = \mathbf{T}(\varphi, t)$ is the unit tangent vector of $X(\varphi, t)$.

2 A general linear nonlocal curvature flow; short time existence of a solution.

Regarding the short time existence of a solution to the nonlocal flow (1), for k-type nonlocal flow in (2), (3) or (4), one can use the same method as in Section 2 of Gage-Hamilton [GH] to show that for any smooth convex closed curve (or simple closed curve) γ_0, the flows in (2), (3), (4) all have a smooth solution defined on $S^1 \times [0, T)$ for short time $T > 0$. If γ_0 is convex, by continuity the convexity is also preserved for short time.

Let $H(p,q) : (0,\infty) \times (0,\infty) \to \mathbb{R}$ be a given (but can be arbitrary) smooth function of two variables. The goal of this paper is to first use an ODE method to explain that for a general $1/k$-type nonlocal flow of the form (with γ_0 convex, parametrized by X_0)

$$\begin{cases} \dfrac{\partial X}{\partial t}(\varphi,t) = \left(H(L(t), A(t)) - \dfrac{1}{k(\varphi,t)} \right) \mathbf{N}_{in}(\varphi,t) \\ X(\varphi,0) = X_0(\varphi), \quad \varphi \in S^1. \end{cases} \tag{11}$$

it has a smooth convex solution for short time $[0,T)$. After that, using the linear structure of $1/k$ (or the support function u) together with the help of the representation formula for a linear heat equation, we can solve $1/k$ (or the support function u) explicitly and give a precise description of the evolving curve γ_t and then discuss its possible convergence behavior.

If (11) has a convex solution defined on $S^1 \times [0,T)$ for some $T > 0$, then using normal (or tangent) angle $\theta \in S^1$ of the curve as parameter (as done in Gage-Hamilton [GH] and many others), the curvature evolution equation (10) becomes

$$\begin{aligned} \frac{\partial k}{\partial t}(\theta,t) &= \left\langle \frac{\partial^2 W}{\partial s^2}, \mathbf{N}_{in} \right\rangle - 2k \left\langle \frac{\partial W}{\partial s}, \mathbf{T} \right\rangle - \left\langle \frac{\partial W}{\partial s}, \mathbf{N}_{in} \right\rangle \frac{\partial k}{\partial \theta} \\ &= k^2(\theta,t) \left(-\frac{1}{k(\theta,t)} \right)_{\theta\theta} + k^2(\theta,t) \left(H(L(t), A(t)) - \frac{1}{k(\theta,t)} \right) \end{aligned} \tag{12}$$

where we have used $W = (H(L,A) - 1/k)\mathbf{N}_{in}$ and the relation $\partial/\partial s = k\partial/\partial\theta$ in (12). The attractive feature now is that the radius of curvature $1/k$ satisfies a **linear** equation on $S^1 \times [0,T)$

$$\frac{\partial}{\partial t} \frac{1}{k(\theta,t)} = \left(\frac{1}{k(\theta,t)} \right)_{\theta\theta} + \frac{1}{k(\theta,t)} - H(L(t), A(t)). \tag{13}$$

By (13), the evolution of $L(t)$ is given by

$$\frac{dL}{dt}(t) = L(t) - 2\pi H(L(t), A(t)), \quad \text{where } L(t) = \int_0^{2\pi} \frac{1}{k(\theta,t)} d\theta \tag{14}$$

(which is not self-contained because of the term $A(t)$) and then

$$\frac{\partial}{\partial t} \left(\frac{1}{k(\theta,t)} - \frac{L(t)}{2\pi} \right) = \left(\frac{1}{k(\theta,t)} - \frac{L(t)}{2\pi} \right)_{\theta\theta} + \left(\frac{1}{k(\theta,t)} - \frac{L(t)}{2\pi} \right), \tag{15}$$

where the term $H(L(t), A(t))$ has disappeared ! Thus the quantity $V(\theta,t) = [1/k(\theta,t) - L(t)/2\pi]\, e^{-t}$ satisfies a linear heat equation $V_t(\theta,t) = V_{\theta\theta}(\theta,t)$on $S^1 \times [0,T)$ with $V(\theta,0) = 1/k(\theta,0) - L(0)/2\pi$. By the representation formula for solution to a heat equation, the curvature $k(\theta,t)$ satisfies the simple-looking *integral equation*:

$$\frac{1}{k(\theta,t)} - \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{k(\theta,t)} d\theta = e^t \int_{-\infty}^{\infty} \frac{1}{2\sqrt{\pi t}} e^{-\frac{(\theta-\xi)^2}{4t}} \left(\frac{1}{k(\xi,0)} - \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{k(\sigma,0)} d\sigma \right) d\xi, \tag{16}$$

where the right hand side of the above is known. Note also that the function $1/k(\theta,t) - L(t)/2\pi$ is, if not identically equal to 0, somewhere positive and somewhere negative since its integral over S^1 is zero.

Unfortunately, (16) is not enough for us to solve $k(\theta,t)$ uniquely (if $1/k(\theta,t)$ satisfies (16), so does $1/k(\theta,t)$ plus any function of time). This is not surprising because we have not used any information given by the nonlocal term $H(L,A)$ so far. Moreover, even if we find one $k(\theta,t)$ satisfying (16), we do not know whether it satisfies the evolution equation (13) or not. To overcome this, we need to explore other geometric quantities.

The *support function* $u(\theta,t)$ of a convex closed curve γ_t is, by definition, given by

$$u(\theta,t)=\langle X(\theta,t),\ (\cos\theta,\sin\theta)\rangle,\quad \theta\in S^1 \tag{17}$$

where $X(\theta,t)$ is the position vector of the unique point on γ_t with outward normal $\mathbf{N}_{out}$ equal to $(\cos\theta,\sin\theta)$. Using $u(\theta,t)$, one can express $L(t)$, $A(t)$ and curvature $k(\theta,t)$ of γ_t as (see the book by Schneider [S] for details)

$$k(\theta,t)=\frac{1}{u_{\theta\theta}(\theta,t)+u(\theta,t)} \tag{18}$$

and

$$L(t)=\int_0^{2\pi}u(\theta,t)\,d\theta,\quad A(t)=\frac{1}{2}\int_0^{2\pi}u(\theta,t)\left(u_{\theta\theta}(\theta,t)+u(\theta,t)\right)d\theta. \tag{19}$$

Under the nonlocal flow (11), the evolution of $u(\theta,t)$ on $S^1\times[0,T)$ is

$$\begin{aligned}\frac{\partial u}{\partial t}(\theta,t)&=u_{\theta\theta}(\theta,t)+u(\theta,t)-H(L(t),A(t))\\&=u_{\theta\theta}(\theta,t)+u(\theta,t)-H\left(\int_0^{2\pi}u(\theta,t)\,d\theta,\ \frac{1}{2}\int_0^{2\pi}u(\theta,t)\left(u_{\theta\theta}(\theta,t)+u(\theta,t)\right)d\theta\right).\end{aligned} \tag{20}$$

Again, we have

$$\frac{\partial}{\partial t}\left(u(\theta,t)-\frac{L(t)}{2\pi}\right)=\left(u(\theta,t)-\frac{L(t)}{2\pi}\right)_{\theta\theta}+\left(u(\theta,t)-\frac{L(t)}{2\pi}\right) \tag{21}$$

and similar to (16) we conclude

$$u(\theta,t)-\frac{L(t)}{2\pi}=e^t\int_{-\infty}^{\infty}\frac{1}{2\sqrt{\pi t}}e^{-\frac{(\theta-\xi)^2}{4t}}\left(u(\xi,0)-\frac{1}{2\pi}\int_0^{2\pi}u(\sigma,0)\,d\sigma\right)d\xi, \tag{22}$$

where the the right hand side of is known.

Denote the right hand side of (22) as $B(\theta,t)$. By (22) and (19), we can express the equation (14) for $L(t)$ as

$$\begin{aligned}\frac{dL}{dt}(t)&=L(t)-2\pi H\left(L(t),\ \frac{1}{2}\int_0^{2\pi}\left(B(\theta,t)+\frac{L(t)}{2\pi}\right)\left(B_{\theta\theta}(\theta,t)+B(\theta,t)+\frac{L(t)}{2\pi}\right)d\theta\right)\\&=L(t)-2\pi H\left(L(t),\ \frac{L^2(t)}{4\pi}+D(t)L(t)+E(t)\right),\end{aligned} \tag{23}$$

where

$$D(t):=\frac{1}{2\pi}\int_0^{2\pi}B(\theta,t)\,d\theta,\quad E(t):=\frac{1}{2}\int_0^{2\pi}B(\theta,t)\left(B_{\theta\theta}(\theta,t)+B(\theta,t)\right)d\theta \tag{24}$$

are both known functions of time, depending only on the initial data. One can simplify $D(t)$ and $E(t)$ further. Note that

$$B(\theta,t)=\frac{e^t}{2\sqrt{\pi t}}\int_{-\infty}^{\infty}e^{-\frac{\xi^2}{4t}}\left(u(\theta-\xi,0)-\frac{L(0)}{2\pi}\right)d\xi=\frac{e^t}{2\sqrt{\pi t}}\int_{-\infty}^{\infty}e^{-\frac{\xi^2}{4t}}u(\theta-\xi,0)\,d\xi-\frac{L(0)}{2\pi}e^t$$

and so

$$D(t)=\frac{1}{2\pi}\int_0^{2\pi}B(\theta,t)\,d\theta=\frac{1}{2\pi}\frac{e^t}{2\sqrt{\pi t}}\int_{-\infty}^{\infty}e^{-\frac{\xi^2}{4t}}\left(\int_0^{2\pi}\left(u(\theta-\xi,0)-\frac{L(0)}{2\pi}\right)d\theta\right)d\xi=0. \tag{25}$$

With (25), we find that

$$E(t)=\frac{1}{2}\int_0^{2\pi}B(\theta,t)\left(B_{\theta\theta}(\theta,t)+B(\theta,t)\right)d\theta=-\frac{1}{4\pi}\left(L^2(t)-4\pi A(t)\right)\leq 0. \tag{26}$$

Also by

$$B_{\theta\theta}(\theta,t)+B(\theta,t)=\frac{e^t}{2\sqrt{\pi t}}\int_{-\infty}^{\infty}e^{-\frac{\xi^2}{4t}}\left(u_{\theta\theta}(\theta-\xi,0)+u(\theta-\xi,0)\right)d\xi-\frac{L(0)}{2\pi}e^t \tag{27}$$

we get

$$E(t)=\frac{1}{2}\int_0^{2\pi}\left\{\begin{array}{l}\left(\frac{e^t}{2\sqrt{\pi t}}\int_{-\infty}^{\infty}e^{-\frac{\xi^2}{4t}}u(\theta-\xi,0)\,d\xi-\frac{L(0)}{2\pi}e^t\right)\\ \times\left(\frac{e^t}{2\sqrt{\pi t}}\int_{-\infty}^{\infty}e^{-\frac{\xi^2}{4t}}\left[u_{\theta\theta}(\theta-\xi,0)+u(\theta-\xi,0)\right]d\xi\right)\end{array}\right\}d\theta=E_1(t)-E_2(t),$$

where

$$E_1(t)=\frac{1}{2}\int_0^{2\pi}\left\{\begin{array}{l}\left(\frac{e^t}{2\sqrt{\pi t}}\int_{-\infty}^{\infty}e^{-\frac{\xi^2}{4t}}u(\theta-\xi,0)\,d\xi\right)\\ \left(\frac{e^t}{2\sqrt{\pi t}}\int_{-\infty}^{\infty}e^{-\frac{\xi^2}{4t}}\left[u_{\theta\theta}(\theta-\xi,0)+u(\theta-\xi,0)\right]d\xi\right)\end{array}\right\}d\theta$$

and

$$E_2(t)=-\frac{1}{2}\int_0^{2\pi}\frac{L(0)}{2\pi}e^t\left(\frac{e^t}{2\sqrt{\pi t}}\int_{-\infty}^{\infty}e^{-\frac{\xi^2}{4t}}\left[u_{\theta\theta}(\theta-\xi,0)+u(\theta-\xi,0)\right]d\xi\right)d\theta=-\frac{L^2(0)}{4\pi}e^{2t}.$$

Thus the ODE (23) becomes

$$\begin{cases}\dfrac{dL}{dt}(t)=L(t)-2\pi H\left(L(t),\ \dfrac{L^2(t)}{4\pi}+E_1(t)-\dfrac{L^2(0)}{4\pi}e^{2t}\right)\\ L(0)=\text{ length of }\gamma_0,\qquad H(p,q):(0,\infty)\times(0,\infty)\to\mathbb{R}.\end{cases} \tag{28}$$

The ODE (28) for $L(t)$ is now *self-contained* and by standard ODE theory we can conclude the following immediately:

Lemma 1 *For the ODE in (28) with initial condition $L(0)>0$, there is a unique solution $L(t)$ defined on interval $[0,T_*)$ for some $T_*>0$.*

Now if we want to solve the initial value problem (11), we can go in the reverse direction and first use the ODE (28) to obtain $L(t)$ on $[0,T_*)$. Then we use (22) to obtain the support function $u(\theta,t)$. As long as the inequality $0<u_{\theta\theta}(\theta,t)+u(\theta,t)<\infty$ is satisfied, we can use it to construct a convex closed smooth curve γ_t. Finally we claim that this family of curves γ_t is indeed a solution to the nonlocal curvature flow (11) by checking that its radius of curvature $1/k$ or support function u satisfies the right equation.

Let

$$\begin{aligned}\tilde{u}(\theta,t)&:=B(\theta,t)+\frac{L(t)}{2\pi}\\ &=\frac{e^t}{2\sqrt{\pi t}}\int_{-\infty}^{\infty}e^{-\frac{\xi^2}{4t}}u(\theta-\xi,0)\,d\xi+\frac{L(t)}{2\pi}-\frac{L(0)}{2\pi}e^t,\quad(\theta,t)\in S^1\times[0,T_*)\end{aligned} \tag{29}$$

where $L(t)$ is from the solution in ODE (28) with initial condition $L(0)=L(\gamma_0)$ and $u(\cdot,0)$ is the support function of γ_0. The above $\tilde{u}(\theta,t)$ satisfies $\tilde{u}(\theta,0)=u(\theta,0)$ for all θ and

$$\begin{aligned}&\tilde{u}_{\theta\theta}(\theta,t)+\tilde{u}(\theta,t)\\ &=\frac{e^t}{2\sqrt{\pi t}}\int_{-\infty}^{\infty}e^{-\frac{\xi^2}{4t}}\left(u_{\theta\theta}(\theta-\xi,0)+u(\theta-\xi,0)\right)d\xi+\frac{L(t)}{2\pi}-\frac{L(0)}{2\pi}e^t\end{aligned} \tag{30}$$

with

$$\tilde{u}_{\theta\theta}(\theta,0)+\tilde{u}(\theta,0)=u_{\theta\theta}(\theta,0)+u(\theta,0)>0\quad\text{for all}\quad\theta\in S^1.$$

As S^1 is compact and $\tilde{u}(\theta,t)$ is continuous on $S^1\times[0,T_*)$, by choosing T_* smaller if necessary, we have

$$\tilde{u}_{\theta\theta}(\theta,t)+\tilde{u}(\theta,t)>0 \quad \text{for all} \quad (\theta,t)\in S^1\times[0,T_*). \tag{31}$$

Note that we always have $\tilde{u}_{\theta\theta}(\theta,t)+\tilde{u}(\theta,t)<\infty$ as long as time is finite. Therefore for each time $t\in[0,T_*)$ there exists a smooth convex closed curve $\tilde{\gamma}_t$ with support function equal to the above $\tilde{u}(\theta,t)$ (see Schneider [S] or Lemma 2 in [LT3]). In fact, the position vector $\tilde{P}(\theta,t)\in\mathbb{R}^2$ of $\tilde{\gamma}_t$ is given by

$$\tilde{P}(\theta,t)=\tilde{u}(\theta,t)(\cos\theta,\sin\theta)+\tilde{u}_\theta(\theta,t)(-\sin\theta,\cos\theta), \qquad \tilde{u}(\theta,t)=B(\theta,t)+\frac{L(t)}{2\pi}$$

$$=\begin{cases} e^t\int_{-\infty}^{\infty}\frac{1}{2\sqrt{\pi t}}e^{-\frac{\xi^2}{4t}}\left[u(\theta-\xi,0)(\cos\theta,\sin\theta)+u_\theta(\theta-\xi,0)(-\sin\theta,\cos\theta)\right]d\xi \\ +\left(\frac{L(t)}{2\pi}-\frac{L(0)}{2\pi}\right)(\cos\theta,\sin\theta)\end{cases} \tag{32}$$

By (29), the length $\tilde{L}(t)$ of $\tilde{\gamma}_t$ is given by

$$\tilde{L}(t)=\int_0^{2\pi}\tilde{u}(\theta,t)\,d\theta=\int_0^{2\pi}\left(B(\theta,t)+\frac{L(t)}{2\pi}\right)d\theta=L(t) \quad \text{for all} \quad t\in[0,T_*)$$

due to (25). Also by (24) the enclosed area $\tilde{A}(t)$ of $\tilde{\gamma}_t$ is given by

$$\begin{aligned}\tilde{A}(t)&=\frac{1}{2}\int_0^{2\pi}\tilde{u}(\theta,t)\left(\tilde{u}_{\theta\theta}(\theta,t)+\tilde{u}(\theta,t)\right)d\theta\\ &=\frac{1}{2}\int_0^{2\pi}\left(B(\theta,t)+\frac{L(t)}{2\pi}\right)\left(B_{\theta\theta}(\theta,t)+B(\theta,t)+\frac{L(t)}{2\pi}\right)d\theta\\ &=\frac{L^2(t)}{4\pi}+E_1(t)-\frac{L^2(0)}{4\pi}e^{2t} \quad \text{for all} \quad t\in[0,T_*).\end{aligned}$$

Now the support function of $\tilde{\gamma}_t$ has the evolution

$$\begin{aligned}\frac{\partial\tilde{u}}{\partial t}(\theta,t)&=\frac{\partial}{\partial t}\left(B(\theta,t)+\frac{L(t)}{2\pi}\right)=B_{\theta\theta}(\theta,t)+B(\theta,t)+\frac{\tilde{L}(t)}{2\pi}-H\left(\tilde{L}(t),\tilde{A}(t)\right)\\ &=\tilde{u}_{\theta\theta}(\theta,t)+\tilde{u}(\theta,t)-H\left(\tilde{L}(t),\tilde{A}(t)\right).\end{aligned}$$

Finally the curvature $\tilde{k}(\theta,t)>0$ of $\tilde{\gamma}_t$ is given by

$$\frac{1}{\tilde{k}(\theta,t)}=\tilde{u}_{\theta\theta}(\theta,t)+\tilde{u}(\theta,t) \tag{33}$$

and its evolution is

$$\begin{aligned}\frac{\partial}{\partial t}\frac{1}{\tilde{k}(\theta,t)}&=\left(\frac{\partial B}{\partial t}\right)_{\theta\theta}(\theta,t)+\left(\frac{\partial B}{\partial t}\right)(\theta,t)+\frac{\tilde{L}(t)}{2\pi}-H\left(\tilde{L}(t),\tilde{A}(t)\right)\\ &=\left(\frac{1}{\tilde{k}(\theta,t)}\right)_{\theta\theta}+\frac{1}{\tilde{k}(\theta,t)}-H\left(\tilde{L}(t),\tilde{A}(t)\right), \quad (\theta,t)\in S^1\times[0,T_*)\end{aligned} \tag{34}$$

which has the right form and is same as the original (13).

Now we can follow the same argument as in the proof of Theorem 4.1.4 (it says that the flow equation is equivalent to the curvature equation) of Gage-Hamilton [GH] to conclude that the nonlocal flow (11) has a solution for short time, defined on $S^1\times[0,T_*)$ for some small $T_*>0$.

We summarize the following:

Theorem 2 *(**short time existence**) For any smooth function $H(p,q) : (0,\infty)\times(0,\infty)\to\mathbb{R}$, the nonlocal flow (11) has a smooth convex solution defined on $S^1\times[0,T_*)$ for some time $T_*>0$. Moreover, during this time interval $[0,T_*)$, the parametrization of γ_t using its outward normal angle θ is given by*

$$P(\theta,t)=\begin{cases} e^t\int_{-\infty}^{\infty}\frac{1}{2\sqrt{\pi t}}e^{-\frac{\xi^2}{4t}}\left[u(\theta-\xi,0)(\cos\theta,\sin\theta)+u_\theta(\theta-\xi,0)(-\sin\theta,\cos\theta)\right]d\xi \\ +\left(\frac{L(t)}{2\pi}-\frac{L(0)}{2\pi}\right)(\cos\theta,\sin\theta). \end{cases} \tag{35}$$

To end this section we point out that under the flow (11) the isoperimetric deficit $L^2(t)-4\pi A(t)$ is always decreasing. If we compute

$$\begin{aligned} &\frac{d}{dt}\left(L^2(t)-4\pi A(t)\right) \\ &=2L(t)\left(L(t)-2\pi H(L(t),A(t))\right)-4\pi\left(\int_{\gamma_t}\frac{1}{k}ds-H(L(t),A(t))L(t)\right) \\ &=2L^2(t)-4\pi\int_{\gamma_t}\frac{1}{k}ds \end{aligned}$$

we see that there is **no** $H(L(t),A(t))$ term in the time derivative. Now by Green-Osher's inequality (also see Pan-Yang [PY])

$$\int_{\gamma_t}\frac{1}{k}ds\geq\frac{L^2(t)-2\pi A(t)}{\pi} \tag{36}$$

the above becomes

$$\frac{d}{dt}\left(L^2(t)-4\pi A(t)\right)\leq 2L^2(t)-4\left(L^2(t)-2\pi A(t)\right)=-2\left(L^2(t)-4\pi A(t)\right)\leq 0.$$

Hence

$$0\leq L^2(t)-4\pi A(t)\leq\left(L^2(0)-4\pi A(0)\right)e^{-2t} \tag{37}$$

as long as the solution exists.

On the other hand if we compute the time derivative of the isoperimetric ratio, we get

$$\begin{aligned} &\frac{d}{dt}\frac{L^2(t)}{4\pi A(t)} \\ &=\frac{4\pi A(t)\left(2L^2(t)-4\pi L(t)H(L(t),A(t))\right)-L^2(t)\,4\pi\left(\int_{\gamma_t}\frac{1}{k}ds-H(L(t),A(t))L(t)\right)}{(4\pi A(t))^2} \end{aligned}$$

and by the refined Green-Osher's inequality (see Lin-Tsai [LT1])

$$\int_{\gamma_t}\frac{1}{k}ds\geq\frac{2}{\pi}\left(L^2(t)-4\pi A(t)\right)+2A(t) \tag{38}$$

we obtain

$$\frac{d}{dt}\frac{L^2(t)}{4\pi A(t)}\leq\frac{L(t)}{4\pi(A(t))^2}\left[L^2(t)-4\pi A(t)\right]\left[H(L(t),A(t))-\frac{2}{\pi}L(t)\right]. \tag{39}$$

One can see that the three flows in (5), (6) and (7) are all isoperimetric ratio decreasing. The isoperimetric ratio is also decreasing as long as H is a negative function. See Remark 6 also.

Remark 3 *According to a private communication with S.-L. Pan, he also obtained the same refined Green-Osher's inequality (38) recently. Note that (38) is an improvement of (36) by* $(L^2 - 4\pi A)/\pi \geq 0$. *Unlike the situation in most of the isoperimetric inequalities where the equality cases occur only at circles, the equality case of (38) occurs if and only if the convex closed curve has support function* $u(\theta)$ *given by*

$$u(\theta) = a_0 + a_1 \cos\theta + b_1 \sin\theta + a_2 \cos 2\theta + b_2 \sin 2\theta, \quad \theta \in [0, 2\pi] \tag{40}$$

for some constants $a_0,\ a_1,\ b_1,\ a_2,\ b_2$ *satisfying*

$$u_{\theta\theta}(\theta) + u(\theta) = a_0 - 3a_2 \cos 2\theta - 3b_2 \sin 2\theta > 0 \quad \text{for all} \quad \theta \in [0, 2\pi]. \tag{41}$$

See Lin-Tsai [LT1] for details.

3 Convergence of the flow (11).

Assume that the solution $L(t)$ of the ODE (28) is defined on time interval $[0, \infty)$. Then the function

$$u(\theta, t) := B(\theta, t) + \frac{L(t)}{2\pi}$$

is defined on $S^1 \times [0, \infty)$ and satisfies equation (20) everywhere. This $u(\theta, t)$ clearly satisfies

$$u_{\theta\theta}(\theta, t) + u(\theta, t) < \infty \quad \text{for all} \quad (\theta, t) \in S^1 \times [0, \infty), \tag{42}$$

which implies that

$$k(\theta, t) = \frac{1}{u_{\theta\theta}(\theta, t) + u(\theta, t)} > 0 \quad \text{for all} \quad (\theta, t) \in S^1 \times [0, \infty). \tag{43}$$

We also know that

$$u_{\theta\theta}(\theta, t) + u(\theta, t) > 0 \quad \text{for all} \quad (\theta, t) \in S^1 \times [0, T_*) \tag{44}$$

for some short time T_*. However, we can not exclude the possibility that $u_{\theta\theta}(\theta_0, t_0) + u(\theta_0, t_0) = 0$ at some $(\theta_0, t_0) \in S^1 \times [0, \infty)$. If this happens, then at time t_0 we can not use $u(\theta, t_0)$ to construct a smooth convex closed curve γ_{t_0}. In particular, this means that the flow (11) may develop a singularity at time t_0 with $k(\theta_0, t_0) = \infty$ for some θ_0. Note that

$$\begin{aligned} & u_{\theta\theta}(\theta, t) + u(\theta, t) \\ &= B_{\theta\theta}(\theta, t) + B(\theta, t) + \frac{L(t)}{2\pi} \\ &= \frac{e^t}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-\frac{\xi^2}{4t}} \left(u_{\theta\theta}(\theta - \xi, 0) + u(\theta - \xi, 0) - \frac{1}{2\pi} \int_0^{2\pi} u(\sigma, 0)\, d\sigma \right) d\xi + \frac{L(t)}{2\pi} \end{aligned} \tag{45}$$

and, unless γ_0 is a circle, the function $u_{\theta\theta}(\theta - \xi, 0) + u(\theta - \xi, 0) - \frac{1}{2\pi}\int_0^{2\pi} u(\sigma, 0)\, d\sigma$ is somewhere positive and somewhere negative, making it difficult to exclude the possibility.

We conclude the following:

Theorem 4 *Let* $H(p, q) : (0, \infty) \times (0, \infty) \to \mathbb{R}$ *be a smooth function and let* γ_0 *be a smooth convex closed curve. Assume that the ODE (28) for* $L(t)$ *is defined on* $[0, \infty)$. *Then the nonlocal flow (11) either develops a singularity (with* $k = \infty$ *somewhere) in finite time or the flow is defined on* $S^1 \times [0, \infty)$, *with each* γ_t *remaining smooth and convex, and its support function* $u(\theta, t)$ *satisfies the following* C^∞ *convergence on* S^1:

$$\begin{aligned} & \lim_{t\to\infty} \left(u(\theta, t) - \frac{L(t)}{2\pi} \right) \\ &= \left(\frac{1}{\pi} \int_0^{2\pi} u(\theta, 0) \cos\theta d\theta \right) \cos\theta + \left(\frac{1}{\pi} \int_0^{2\pi} u(\theta, 0) \sin\theta d\theta \right) \sin\theta, \quad \forall\ \theta \in S^1. \end{aligned} \tag{46}$$

Remark 5 *In the above theorem, the length $L(t)$ may go to infinity or approach a positive constant or tend to zero as $t \to \infty$. The same for the area $A(t)$ due to (37). The geometric meaning of the above theorem is that γ_t converges to a circle C_t with length $L(t)$ centered at the point*

$$P = \left(\frac{1}{\pi} \int_0^{2\pi} u(\theta, 0) \cos\theta d\theta, \ \frac{1}{\pi} \int_0^{2\pi} u(\theta, 0) \sin\theta d\theta \right) \in \mathbb{R}^2. \tag{47}$$

Remark 6 *If $H(p,q) : (0,\infty) \times (0,\infty) \to (-\infty, 0)$ is a negative function, then the ODE (28) implies that $L(t) \geq L(0) e^t$ for all $t \in [0,\infty)$. Now*

$$\begin{aligned} & u_{\theta\theta}(\theta,t) + u(\theta,t) \\ &= \frac{e^t}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-\frac{\xi^2}{4t}} \left(u_{\theta\theta}(\theta-\xi, 0) + u(\theta - \xi, 0) \right) d\xi + \left(\frac{L(t)}{2\pi} - \frac{L(0)}{2\pi} e^t \right) > 0 \end{aligned}$$

due to $u_{\theta\theta}(\theta,0) + u(\theta,0) > 0$ everywhere. Hence it is impossible for the flow to develop a singularity in finite time.

Proof. The proof is now straightforward. The convexity of γ_t is due to (43). For the convergence, the equation for $u(\theta,t) - L(t)/2\pi$ is linear with

$$\int_0^{2\pi} \left(u(\theta,t) - \frac{L(t)}{2\pi} \right) d\theta = 0 \quad \text{for all} \quad t \in [0,\infty).$$

Using Fourier series expansion for $u(\theta,t) - L(t)/2\pi$, the convergence result follows from standard theory. □

For the ODE (28), it is also possible that $L(t)$ is defined only on a finite time interval $[0, T_{\max})$ with either $\lim_{t \to T_{\max}} L(t) = \infty$ or $\lim_{t \to T_{\max}} L(t) = 0$. In the first case, we note that

$$\left| u(\theta,t) - \frac{L(t)}{2\pi} \right| \leq Ce^t \quad \text{for all} \quad t \in [0, T_{\max}), \tag{48}$$

where $C > 0$ is a constant depending only on the initial curve. Hence $u(\theta,t) \to \infty$ uniformly on S^1 as $t \to T_{\max}$. By

$$\frac{\partial}{\partial t}\left(\frac{\partial^m u}{\partial \theta^m} \right)(\theta,t) = \left(\frac{\partial^m u}{\partial \theta^m} \right)_{\theta\theta}(\theta,t) + \left(\frac{\partial^m u}{\partial \theta^m} \right)(\theta,t), \quad \int_0^{2\pi} \frac{\partial^m u}{\partial \theta^m}(\theta,t) d\theta = 0 \tag{49}$$

and using Fourier series expansion (or by the result in Chow-Gulliver [CG]) we know that for each $m \in \mathbb{N}$, $|(\partial^m u / \partial \theta^m)(\theta,t)|$ is uniformly bounded on $S^1 \times [0, T_{\max})$. Hence if we rescale the curve $\hat{\gamma}_t$ by considering $\hat{\gamma}_t = 2\pi\gamma_t / L(t)$, its support function $\hat{u}(\theta,t)$ will satisfy $\hat{u}(\theta,t) \to 1$ in $C^\infty(S^1)$ as $t \to T_{\max}$.

We conclude the following:

Theorem 7 *If the ODE (28) for $L(t)$ is defined on $[0, T_{\max})$ with $T_{\max} < \infty$ and $\lim_{t \to T_{\max}} L(t) = \infty$. Then the nonlocal flow (11) either develops a singularity before $T_{\max}$ or the flow is defined on $S^1 \times [0, T_{\max})$, with each γ_t remaining smooth and convex, and the support function $\hat{u}(\theta,t)$ of the rescaled curve $\hat{\gamma}_t = 2\pi\gamma_t / L(t)$ satisfies $\hat{u}(\theta,t) \to 1$ in $C^\infty(S^1)$ as $t \to T_{\max}$.*

For the case $\lim_{t \to T_{\max}} L(t) = 0$, by (16) we have

$$\frac{1}{k(\theta,t)} = Z(\theta,t) + \frac{L(t)}{2\pi}, \quad Z(\theta,t) = e^t \int_{-\infty}^{\infty} \frac{1}{2\sqrt{\pi t}} e^{-\frac{(\theta-\xi)^2}{4t}} \left(\frac{1}{k(\xi,0)} - \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{k(\sigma,0)} d\sigma \right) d\xi$$

where the function $Z(\theta,t)$ is somewhere positive and somewhere negative (unless γ_0 is a circle) for each $t \in [0,\infty)$ due to $\int_0^{2\pi} Z(\theta,t)\, d\theta = 0$. Therefore before time $T_{\max}$ we must have $k = \infty$ somewhere. Hence we conclude:

Theorem 8 *If the ODE (28) for $L(t)$ is defined on $[0, T_{\max})$ with $T_{\max} < \infty$ and $\lim_{t \to T_{\max}} L(t) = 0$. Then if γ_0 is not a circle, the nonlocal flow (11) must develop a singularity before time $T_{\max}$ with $k = \infty$ somewhere.*

If the ODE (28) for $L(t)$ is defined only on a maximal domain $[0, T_{\max})$ with $T_{\max} < \infty$, then $(L(t), A(t))$ must approach the boundary of the domain $(0,\infty)\times(0,\infty)$ of $H(p,q)$. If $\lim_{t \to T_{\max}} L(t) = \infty$ (or 0), then the same for $\lim_{t \to T_{\max}} A(t)$ due to (37) and the isoperimetric inequality. There remains the last case that $\lim_{t \to T_{\max}} L(t) = \ell \in (0,\infty)$ but $\lim_{t \to T_{\max}} A(t) = 0$. In such a case, by Gage's inequality for convex closed curves

$$\int_{\gamma_t} k^2 ds \geq \frac{\pi L(t)}{A(t)} \to \infty \quad \text{as} \quad t \to T_{\max}$$

we must have $\lim_{t \to T_{\max}} k_{\max}(t) = \infty$. We conclude:

Theorem 9 *Assume the ODE (28) for $L(t)$ is defined on $[0, T_{\max})$ with $T_{\max} < \infty$ and $\lim_{t \to T_{\max}} L(t) = \ell \in (0,\infty)$. If $\lim_{t \to T_{\max}} A(t) = 0$, then we must have $\lim_{t \to T_{\max}} k_{\max}(t) = \infty$.*

4 Conclusion.

We can allow the nonlocal term $H(L(t), A(t))$ to be more general. For example in Ma-Cheng [MC], they considered an interesting nonlocal flow of the form

$$\frac{\partial X}{\partial t}(\varphi, t) = \left(\frac{1}{L(t)} \int_0^{L(t)} \frac{1}{k(\varphi,t)} ds - \frac{1}{k(\varphi,t)} \right) \mathbf{N}_{in}(\varphi, t), \tag{50}$$

which is an *area-preserving* flow. For this flow the support function $u(\theta,t)$ and the length $L(t)$ still satisfies the same equation (21), and so $u(\theta,t) = L(t)/2\pi + B(\theta,t)$, where $B(\theta,t)$ is given by (22). The ODE for $L(t)$ in this case is

$$\begin{aligned} \frac{dL}{dt}(t) &= L(t) - \frac{2\pi}{L(t)} \int_0^{L(t)} \frac{1}{k(\varphi,t)} ds = L(t) - \frac{2\pi}{L(t)} \int_0^{2\theta} \left[u_{\theta\theta}(\theta,t) + u(\theta,t)\right]^2 d\theta \\ &= L(t) - \frac{2\pi}{L(t)} \int_0^{2\theta} \left(B_{\theta\theta}(\theta,t) + B(\theta,t) + \frac{L(t)}{2\pi} \right)^2 d\theta. \end{aligned} \tag{51}$$

Again (51) is a self-contained ODE and has short time existence. Thus Ma-Cheng's flow has solution for short time. In view of this, we see that the nonlocal flow (11) has short time existence as long as the quantities in the nonlocal term H can be expressed in terms of the support function $u(\theta,t)$. Since the position vector of a convex closed curve γ_t is uniquely determined by its support function via the formula (32), we can conclude that essentially *any* $1/k$-type nonlocal flow has short time existence and can be dealt with by the above ODE method.

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