A functional limit convergence towards brownian excursion.

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Abstract

We consider a random walk S in the domain of attraction of a standard normal law Z, *ie* there exists a positive sequence a_n such that S_n/a_n converges in law towards Z. The main result of this note is that the rescaled process $(S_{\lfloor nt \rfloor}/a_n, t \geqslant 0)$ conditioned to stay non-negative, to start and to come back *near the origin* converges in law towards the normalized brownian excursion.

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1 Introduction and the main result

It is a classical result that if a random walk S is in the domain of attraction of the standard normal law with norming sequence a_n, the rescaled process $(S_{\lfloor nt \rfloor / a_n})_{t \geqslant 0}$ converges in law towards the brownian motion (see [Bil68]). Denoting by $S^*, \mathbf{P}_x$ the random walk starting from x and conditioned to stay always positive (one can make sense of such a definition by means of a so called h-transform), it has recently been shown in [BJD06] and in [CC08] that if x/a_n vanishes as $n \to \infty$, the corresponding rescaled process converges in law towards the brownian meander. A natural question related to these results is whether conditioning on a *late* return near the origin (ie on $\{S_n^* = y\}$ with $y/a_n \to 0$ as $n \to \infty$) implies the convergence of $(S^*, \mathbf{P}_x)$ towards the brownian excursion.

Extending previous results from [BJD06], we show in this paper that such a convergence holds. Before stating precisely our main results, we recall the essentials of the conditioning to stay positive for an oscillating random walk.

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1.1 Conditioning a random walk to stay positive

Let $S_n = X_1 + \ldots + X_n$ be an integer valued aperiodic random walk. We write $\mathbf{P}_x$ the law of S started at x and for convenience we put $\mathbf{P} = \mathbf{P}_0$.

Next we introduce the strict descending *ladder process* $(T_k^-, H_k^-)_{k \geqslant 0}$ by setting $(T_0^-, H_0^-) = (0, 0)$ and

$$T_{k+1}^- := \min\{j > T_k^- | S_j < S_{T_k^-}\}, \qquad H_{k+1}^- = -S_{T_{k+1}^-}. \tag{1.1}$$

Note that under $\mathbf{P}$, (T^-, H^-) is a bivariate renewal process, that is a random walk on $(\mathbb{Z}^+)^2$ with step law supported on the first quadrant. The sequence T^- is the sequence of the so called *(strictly) descending ladder epochs*, the sequence H^- the sequence of *descending ladder heights*.

We denote by $V(\cdot)$ the renewal function associated to H^-, that is the positive function defined by

$$V(x) := \sum_{k \geqslant 0} \mathbf{P}(H_k^- \leqslant x). \tag{1.2}$$

Note in particular that $V(y)$ is the expected number of ladder points in the stripe $[0, \infty) \times [0, y]$. It follows that it is a subadditive and increasing function.

The *killed* random walk $\widehat{S}$ is a Markov chain defined in the following way. Let $\tau_{(-\infty,0)}$ denote the first entrance time of S into the negative half plane. Introducing $\{\Delta\}$ a cimetery state, for every n,

$$\widehat{S}_n := S_n \mathbf{1}_{\tau_{(-\infty,0)} > n} + \Delta \mathbf{1}_{\tau_{(-\infty,0)} \leqslant n}. \tag{1.3}$$

Then we denote S conditioned to stay non negative by $S_n^* = \sum_{i=1}^n X_i^*$. In our integer valued oscillating case this is a Markov chain on $\mathbb{Z}^+$ whose law is defined for any $n \in \mathbb{N}$ and for any $B \in \sigma(S_1, \ldots, S_n)$ by:

$$\mathbf{P}_x^*[B \cap \{S_n = y\}] := \frac{V(y)}{V(x)} \mathbf{P}_x[B \cap \{S_n = y\} \cap \mathcal{C}_n] = \frac{V(y)}{V(x)} \mathbf{P}_x[B \cap \{\widehat{S}_n = y\}], \tag{1.4}$$

where $\mathcal{C}_n = \{S_1 \geqslant 0, \ldots, S_n \geqslant 0\}$. The terminology is justified by the following weak convergence result

$$\mathbf{P}_x^* = \lim_{n \to \infty} \mathbf{P}_x\left(\cdot | \mathcal{C}_n\right) \tag{1.5}$$

which is proved in [BD94], Theorem 1.

1.2 A convergence towards the brownian excursion

From now on, we will always assume that S lies in the domain of attraction of the standard normal law. This means that the sequence (X_k) is iid and that for a suitable norming sequence (a_n) one has the weak convergence

$$S_n / a_n \Rightarrow \phi(x) dx, \qquad \phi(x) := \frac{1}{\sqrt{2\pi}} e^{-x^2/2}. \tag{1.6}$$

In particular this is the case when $\mathbf{E}[X_1] = 0$ and $\mathbf{E}[X_1^2] =: \sigma^2 < \infty$ with $a_n = \sigma\sqrt{n}$ by the central limit theorem.

By standard theory of stability, (see [Fel71] IX.8 and XVII.5) for (1.6) to hold it is necessary and sufficient that $\mathbf{E}[X_1] = 0$, that the truncated variance $\Phi(t) := \mathbf{E}[X_1^2 \mathbf{1}_{|X_1| \leqslant t}]$ is slowly varying at infinity (that is $\frac{\Phi(ct)}{\Phi(t)} \to 1$ as $t \to \infty$ for any $c > 0$) and that the sequence a_n satisfies $a_n^2 \sim n\Phi(a_n)$ as $n \to \infty$.

We define Ω as being the space $D([0,1], \mathbb{R})$ the set of càdlàg functions on $[0,1]$ endowed with the standard Skorohod topology (see [Bil68]) and for $n \in \mathbb{Z}^+$, we define the application X^n by:

$$X^n : \begin{array}{ccc} \mathbb{Z}^n & \longrightarrow & \Omega \\ (u_1, \ldots, u_n) & \mapsto & \left(\frac{\sum_{i=1}^{[nt]} u_i}{a_n} \right)_{t \in [0,1]} \end{array} . \tag{1.7}$$

For x, y positive integers, we denote by $P_n^{*,x,y}$ the law of S^* conditionally on the event $\{S_0^* = x, S_n^* = y\}$, and we define the probability laws on Ω:

$$Q_n^{x,y} := P_n^{*,x,y} \circ (X^n)^{-1}. \tag{1.8}$$

We can now state our main result:

Theorem 1.1. *Let x_n and y_n be positive integer valued sequences such that $x_n/a_n \to 0$ and $y_n/a_n \to 0$. Then, as $n \to \infty$, the following convergence holds in Ω:*

$$Q_n^{x_n, y_n} \Rightarrow e \tag{1.9}$$

where e denotes the law of the normalized brownian excursion.

The proof of this result will follow the standard procedure of showing finite dimensional convergence and tightness.

1.3 Some motivations and a short overview of the literature

The study of invariance principles for random walks is a very classical topic in probability (classical references are [Sko57], [Bil68]). Extending these invariance principles to conditioned random walks is far from being straightforward. Sometimes a clever representation can considerably simplify the proofs (like in [Bol76], [Don85] for the convergence towards the meander), but generally speaking such an issue demands some technical efforts, see [Igl74] for a convergence towards the meander or [Lig68] for the brownian bridge.

The more particular case of convergence towards the brownian excursion for the conditioned simple random walk conditioned by a late return to zero has first been proved in [DIM77]. Their results have been extended to the case where S has finite variance in [Kai76].

A related result to ours that will turn out to be quite useful in our proofs is the convergence towards the brownian meander of a random walk in the domain of attraction of the normal law starting from x_n where x_n is $o(a_n)$ conditioned on $\mathcal{C}_n$ (see [Shi83, Remark 4]). Combining tightness arguments and local limit estimates, this result has been extended to the case where S is conditioned to stay positive by [BJD06], and their results in turn have been extended by quite different and somewhat lighter techniques in [CC08] to the case where S is in the domain of attraction of a *stable law* with index $\alpha \in (0,2]$ and with positivity parameter $\rho \in (0,1)$. Lacking a suitable representation under the form of an h-transform for the brownian excursion, our methods follow the same path as in [BJD06].

Besides the interest they have in their own, invariance principles are important in view of their applications. Let us mention one of them which is actually the main motivation of this paper. Consider the following homogeneous polymer model (a by now classical reference for polymer models is [Gia07]): for $N \in \mathbb{N}, y \in \mathbb{R}^+, a > 0$ and $\varepsilon \in \mathbb{R}$, we set

$$\frac{d\mathbf{P}^c_{N,a,\varepsilon}}{d\mathbf{P}} := \frac{1}{Z_{N,a,\varepsilon}} \exp\left(\varepsilon \sum_{i=1}^{N} \mathbf{1}_{S_i \in [0,a]}\right) \mathbf{1}_{S_N \in [0,a]} \tag{1.10}$$

where $\mathbf{P}$ is an aperiodic $\mathbb{Z}$ valued random walk in the domain of attraction of the standard normal law. The law $\mathbf{P}^c_{N,a,\varepsilon}$ may be viewed as an effective model for a $(1+1)$ dimensional interface above a wall with homogeneous impurities which are concentrated in the stripe $[0,\infty) \times [0,a]$. These impurities are either attracting or repelling the interface (depending on the sign of ε).

One standard goal related to this kind of models is to find the asymptotic behavior of the typical paths in the limit $N \to \infty$ and to study their dependence on ε and a. These limits have been resolved in the thesis [Soh10].

A common feature shared by this model and the classical homogeneous one is that the measure $\mathbf{P}^c_{N,a,\varepsilon}$ exhibits a remarkable decoupling between the contact level set $\mathcal{I}_N := \{i \leqslant N, S_i \in [0,a]\}$ and the excursions of S between two consecutive contact points (see [DGZ05] for more details in the standard homogeneous pinning case). In fact, conditionally on $I_N = \{t_1, \ldots, t_k\}$ and on $(S_{t_1}, \ldots, S_{t_k})$, the *bulk* excursions $e_i = \{e_i(n)\}_n := \big\{\{S_{t_i+n}\}_{0 \leqslant n \leqslant t_{i+1}-t_i}\big\}$ are independent under $\mathbf{P}^c_{N,a,\varepsilon}$ and are distributed like the random walk $(S, \mathbf{P}_{S_{t_i}})$ conditioned on the event $\big\{S_{t_{i+1}-t_i} \in [0,a], S_{t_i+j} > a, j \in \{1, \ldots, t_{i+1}-t_i-1\}\big\}$. It is therefore clear that to extract scaling limits on $\mathbf{P}^c_{N,a,\varepsilon}$, one has to combine good control over the law of the contact set $\mathcal{I}_N$ and suitable asymptotics properties of the excursions, and for this the utility of Theorem 1.1 emerges (see chapter 3 of the thesis [Soh10] for details).

1.4 Outline of the paper

The exposition of this paper will be organized as follows:

- in Section 2, we collect some preliminary facts.
- in Section 3, we discuss finite dimensional convergence and state our main technical lemma.
- in Section 4, we prove Lemma 3.1, which implies the finite dimensional convergence in Theorem 1.1.
- in Section 5, we show the tightness of the sequence of measures $(Q_n^{x_n,y_n})_n$, thus proving Theorem 1.1.
- in Section 6, we give a uniform equivalence for the tails of the random variable $\tau_{(-\infty,0)}$ under the law $\mathbf{P}_{x_n}$. This estimate is widely used in sections 4 and 5.

2 Some preliminary facts

2.1 Regular varying sequences

Throughout this note, for positive sequences α_n and β_n, we use the notation $\alpha_n \sim \beta_n$ to indicate that $\alpha_n/\beta_n \to 1$ as $n \to \infty$. Following Doney's terminology, for positive measurable functions g, h on $\mathbb{R}^+$, we will often say that the equivalence

$$g(x_n) \sim h(x_n) \tag{2.1}$$

is true *uniformly on the sequences x_n such that $x_n/a_n \to 0$*. By this we mean that, given any positive sequence ε_n such that $\varepsilon_n \to 0$ as $n \to \infty$, the convergence

$$\frac{g(x_n)}{h(x_n)} \to 1 \tag{2.2}$$

holds uniformly for every sequence $x_n \in \Delta_{\varepsilon_n}$ where

$$\Delta_{\varepsilon_n} := \{y \in \mathbb{Z}^{\mathbb{N}}, \forall n \geqslant 0, y_n \in [0, \varepsilon_n a_n]\}. \tag{2.3}$$

A positive sequence d_n is said to be slowly varying with index $\alpha \in \mathbb{R}$ (which we denote by $d_n \in \mathbb{R}_\alpha$) if $d_n \sim L_n n^\alpha$ where L_n is slowly varying at infinity that is for every positive t, $\lim_{n\to\infty} \frac{L_{[nt]}}{L_n} = 1$. If $d_n \in \mathbb{R}_\alpha$, we can (and will always assume) that $d_n = d(n)$ where $d(\cdot)$ is a continuous strictly monotone function whose inverse will be denoted $d^{-1}(\cdot)$ (see [BGT89, Theorem 1.5.3]). Observe that if $d_n \in \mathcal{R}_\alpha, d^{-1}(n) \in \mathcal{R}_{1/\alpha}$ and $1/d_n \in \mathcal{R}_{-\alpha}$.

The following basic uniform convergence property ([BGT89, Theorem 1.2.1]) will be often used in the sequel; if $d_n \in \mathcal{R}_\alpha$, then for every fixed $\varepsilon > 0$

$$d_{[tn]} = t^\alpha d_n(1 + o(1)) \tag{2.4}$$

uniformly for $t \in [\varepsilon, 1/\varepsilon]$.

2.2 Fluctuation theory

In a similar way as for the descending ladder process, one can define the weak ascending bivariate renewal process $(T_k^+, H_k^+)_k$ as $T_0^+ := 0$, $T_{k+1} := \min\{j > T_k^+, S_j \geqslant S_{T_k^+}\}$, $H_k^+ := S_{T_k^+}$ and

$$U(x) := \sum_{k \geqslant 0} \mathbf{P}(H_k^+ \leqslant x). \tag{2.5}$$

It is known that S_1 is in the domain of attraction (without centering) of a stable law if and only if (T_1^-, H_1^-) lies in a bivariate domain of attraction (see for example [DG93]). We can specialize this fact to our setting. By hypothesis, S_1 lies in the domain of attraction of the standard normal law, so that by standard fluctuation theory, $a_n \in \mathcal{R}_{1/2}$. We then define two sequences

$$\log(\frac{n}{\sqrt{2}}) = \sum_{m=1}^{\infty} \frac{\mathbf{P}[S_m < 0]}{m} e^{-\frac{m}{b_n}}, \qquad c_n := a(b_n). \tag{2.6}$$

Then $b_n \in \mathcal{R}_2$, $c_n \in \mathcal{R}_1$ and we have the weak convergence

$$\left(\frac{T_n^-}{b_n}, \frac{H_n^-}{a_n}\right) \Rightarrow Z, \qquad \mathbf{P}[Z \in (dx, dy)] = \frac{e^{-1/2x}}{\sqrt{2\pi}x^{3/2}} \mathbf{1}_{x \geqslant 0} \delta_1(dy), \tag{2.7}$$

where $\delta_1(dy)$ denotes the Dirac measure at $y = 1$. Note in particular that, like in the simple random walk case, T_1^- is attracted to Y, the stable law of index 1/2.

$$\frac{T_n^-}{b_n} \Rightarrow Y, \qquad \mathbf{P}[Y \in dx] = \frac{e^{-1/2x}}{\sqrt{2\pi}x^{3/2}} \mathbf{1}_{x \geqslant 0}. \tag{2.8}$$

We recall also that b_n is sharply linked to the tails of T_1^- by the relation

$$\mathbf{P}[T_1^- > b_n] \sim \sqrt{\frac{2}{\pi}} \frac{1}{n} \tag{2.9}$$

and it is known that this is a necessary and sufficient relation in order for b_n to be such that $T_n^-/b_n \Rightarrow Y$.

Equation (2.7) also implies that the process (H^-) follows a generalized law of large numbers, namely $\frac{H_n^-}{c_n} \Rightarrow 1$ (H_1^- is said to be *relatively stable*). Consequently the following equivalence holds (see [BGT89, Theorem 8.8.1])

$$V(x) \sim c^{-1}(x) =: \frac{x}{l^-(x)} \tag{2.10}$$

where $l^-(\cdot)$ is slowly varying at infinity. In a similar way, one can prove that the equivalence

$$U(x) \sim \frac{x}{l^+(x)} \tag{2.11}$$

is verified for some slowly varying function $l^+(\cdot)$.

2.3 The duality lemma and local limit estimates

Let $v(\cdot,\cdot)$ be the renewal mass function of the bivariate renewal process (H^-, T^-), that is

$$v(n,x) := \sum_k \mathbf{P}[T_k^- = n, H_k^- = x] \tag{2.12}$$

and $u(\cdot,\cdot)$ its counterpart for the process (H^+, T^+)

$$u(n,x) := \sum_k \mathbf{P}[T_k^+ = n, H_k^+ = x]. \tag{2.13}$$

The power of fluctuation theory for the study of random walks is linked to some fundamental identities, the most famous one being the so called "duality lemma" (see [Fel71, Chapter XII]):

$$\mathbf{P}[\mathcal{C}_n, S_n \in dx] = \mathbf{P}[n \text{ is a ladder epoch}, S_n \in dx] = u(n,x) \tag{2.14}$$

where by the event $\{n \text{ is a ladder epoch}\}$ we mean of course the disjoint union of the events $\cup_k \{T_k^+ = n\}$. The following equivalence about the asymptotics of $u(\cdot,\cdot)$ has been shown independently in [Car05] and in [BJD06]. Note that for the later, it is the chore of the proof of their main result.

Lemma 2.1. *Uniformly for $0 \leqslant y_n \leqslant K a_n$, one has the following equivalence:*

$$\mathbf{P}[\widehat{S}_n = y_n] = u(n, y_n) \sim \frac{U(y_n)}{n} \mathbf{P}[S_n = y_n]. \tag{2.15}$$

3 Finite dimensional convergence in Theorem 1.1

3.1 The law of the renormalized brownian excursion

For $x, y, t > 0$, we define $q_t(x,y)$ the transition function of the killed Brownian motion, that is

$$\begin{aligned} q_t(x,y) &:= \frac{1}{\sqrt{t}} r(\frac{x}{\sqrt{t}}, \frac{y}{\sqrt{t}}) \\ \text{where } r(u,v) &:= \sqrt{\frac{2}{\pi}} \sinh(uv) \exp(-\frac{u^2+v^2}{2}), \end{aligned} \tag{3.1}$$

and the following transition function :

$$\begin{aligned} l_t(y) &:= \frac{1}{t} r_0(\frac{y}{\sqrt{t}}) \\ \text{where } r_0(v) &:= \sqrt{\frac{1}{2\pi}} v \exp(-\frac{v^2}{2}). \end{aligned} \tag{3.2}$$

It is well known that (see [BS02]) for $k \in \mathbf{N}, 0 < t_1 < \ldots < t_k < 1$ and $f \in \mathcal{C}^b([0,1]^k, \mathbb{R})$, one has:

$$
\begin{aligned}
& e(f(\omega_{t_1}, \ldots, \omega_{t_1})) \\
& \qquad = 2\sqrt{2\pi} \int_{(\mathbb{R}^+)^k} f(x_1, \ldots, x_k) l_{t_1}(x_1) \ldots q_{t_k - t_{k-1}}(x_{k-1}, x_k) l_{1-t_k}(x_k) dx_1 \ldots dx_k.
\end{aligned}
\tag{3.3}
$$

To get Theorem 1.1, we have to show finite dimensional convergence, that is we show that for every positive integer k, $(t_1, \ldots, t_k) \in (0,1)^k$, $f \in \mathcal{C}^b((\mathbb{R}^+)^k, \mathbb{R})$:

$$
\begin{aligned}
& \frac{\mathbf{E}_{x_n}[f(\frac{S^*_{\lceil nt_1 \rceil}}{a_n}, \ldots, \frac{S^*_{\lceil nt_k \rceil}}{a_n}) \mathbf{1}_{S^*_n = y_n}]}{\mathbf{P}_{x_n}[S^*_n = y_n]} \\
& \qquad \to 2\sqrt{2\pi} \int_{\mathbb{R}^+} f(x_1, \ldots, x_k) l_{t_1}(x_1) q_{t_2 - t_1}(x_1, x_2) \ldots l_{1-t_k}(x_k) dx_1 \ldots dx_k
\end{aligned}
\tag{3.4}
$$

as $n \to \infty$.

3.2 Getting the convergence (3.4)

Our main tool to get this convergence is the following result which we prove in part 4:

Lemma 3.1. *For $K > 0$, uniformly in $x_n/a_n \to 0$ as $n \to \infty$ and in y_n such that $y_n/a_n \in [0, K]$, one has the following equivalence:*

$$
\mathbf{P}_{x_n}(\widehat{S}_n = y_n) \sim \frac{V(x_n)U(y_n)}{n} \mathbf{P}(S_n = y_n). \tag{3.5}
$$

The next result is a consequence of the Wiener Hopf factorization, it has been shown in [BJD06] and it will turn out to be useful numerous times in the sequel.

Lemma 3.2. *Let $K > 0$. Uniformly in the sequences $(x_n)_{n \geqslant 0}, (y_n)_{n \geqslant 0}$ such that $x_n/a_n \in [0, K], y_n/a_n \in [0, K]$, one has the following equivalence:*

$$
\frac{U(x_n)V(y_n)}{n} = 2\frac{x_n}{a_n}\frac{y_n}{a_n} + o(1) \quad as \quad n \to \infty \tag{3.6}
$$

Lemma 3.1 straightforwardly implies the equivalence:

$$
\mathbf{P}_{x_n}(S^*_n = y_n) \sim \frac{U(y_n)V(y_n)}{n} \mathbf{P}[S_n = y_n]. \tag{3.7}
$$

Of course, S^* is not reversible. Nevertheless, using time reversal, combining Lemma 3.2 and the equivalence (3.7) straightforwardly imply the following:

Lemma 3.3. *For $K > 0$, uniformly in $x_n/a_n \in [0, K]$ and in y_n such that $y_n/a_n \to 0$ as $n \to \infty$, one has the following equivalence:*

$$\mathbf{P}_{x_n}(S_n^* = y_n) \sim 2\frac{y_n^2}{a_n^2}\mathbf{P}(S_n = x_n) \sim 2\frac{y_n^2}{a_n^2}\frac{\phi(x_n/a_n)}{a_n}. \tag{3.8}$$

We finally recall the following proposition from [BJD06]:

Proposition 3.4. *Suppose x_n and y_n are integers such that*

$$x_n/a_n \to u > 0, \qquad y_n/a_n \to v > 0 \tag{3.9}$$

as $n \to \infty$. Then one has the convergence:

$$a_n\mathbf{P}[\widehat{S}_n = y_n] \to r(u, v) \tag{3.10}$$

It is then easy to check that combining the Lemmas 3.1, 3.2, 3.3 and the Proposition 3.4, one gets the convergence in (3.4), so that finite dimensional convergence in Theorem 1.1 holds.

4 Proof of Lemma 3.1

4.1 The case where y_n/a_n is bounded away from zero

We first assume that there exists $\varepsilon > 0$ such that for every $n, y_n/a_n \geqslant \varepsilon$.

We define $m_n := \inf\{S_j, j \leqslant n\}$ and $\mu_n := \inf\{j \leqslant n, S_j = m\}$ and their all time counterparts $m = \inf\{S_j, j \geqslant 0\}$ and $\mu := \inf\{j \geqslant 0, S_j = m\}$. Let $\eta > 0$ be fixed.

Alili and Doney have used the following equality in [AD01], it is an easy consequence of the duality lemma:

$$\begin{aligned}\mathbf{P}_{x_n}[\widehat{S}_n = y_n] &= \mathbf{P}_{x_n}[\widehat{S}_n = y_n; \mu_n < \eta n] + \mathbf{P}_{x_n}[\widehat{S}_n = y_n; \mu_n \geqslant \eta n] \\ &= \sum_{j=0}^{\eta n}\sum_{k=0}^{x_n \wedge y_n} \mathbf{P}_{x_n}[S_n = y_n, \mu_n = j, m_n = k] + \mathbf{P}_{x_n}[\widehat{S}_n = y_n; \mu_n \geqslant \eta n] \\ &= \sum_{j=0}^{\eta n}\sum_{k=0}^{x_n \wedge y_n} v(j, x_n - k)u(n - j, y_n - k) + \mathbf{P}_{x_n}[\widehat{S}_n = y_n; \mu_n \geqslant \eta n].\end{aligned} \tag{4.1}$$

We first treat the first term in the right hand side of the above equality. The assumptions on x_n, y_n imply that for large enough n, $x_n \wedge y_n = x_n$. Using Lemma 2.1, for large enough n, we get that:

$$\sum_{j=0}^{\eta n}\sum_{k=0}^{x_n \wedge y_n} v(j, x_n - k)u(n - j, y_n - k) \sim \sum_{j=0}^{\eta n}\sum_{k=0}^{x_n} v(j, x_n - k)\frac{U(y_n - k)\mathbf{P}_k[S_{n-j} = y_n]}{n - j} \tag{4.2}$$

as $n \to \infty$, so that:

$$g_n(\eta) \sum_{j=0}^{\eta n} \sum_{k=0}^{x_n} v(j,k) \leqslant \frac{n}{U(y_n)\mathbf{P}[S_n = y_n]} \sum_{j=0}^{\eta n} \sum_{k=0}^{x_n} v(j, x_n - k) \frac{U(y_n - k)\mathbf{P}_k[S_{n-j} = y_n]}{n-j}$$
$$\leqslant f_n(\eta) \sum_{j=0}^{\eta n} \sum_{k=0}^{x_n} v(j,k) \tag{4.3}$$

where we defined

$$f_n(\eta) := \sup_{j \leqslant \eta n, k \in [0, x_n]} \frac{U(y_n - k)\mathbf{P}_k[S_{n-j} = y_n]}{(1-\eta)U(y_n)\mathbf{P}[S_n = y_n]} \tag{4.4}$$

and

$$g_n(\eta) := \inf_{j \leqslant \eta n, k \in [0, x_n]} \frac{U(y_n - k)\mathbf{P}_k[S_{n-j} = y_n]}{U(y_n)\mathbf{P}[S_n = y_n]}. \tag{4.5}$$

Using the standard local limit theorem and equivalence (2.10), one gets easily that $\lim_{\eta \searrow 0} \limsup_{n\to\infty} f_n(\eta) = \lim_{\eta \searrow 0} \liminf_{n\to\infty} g_n(\eta) = 1$. Thus we are left with showing that

$$\sum_{j=0}^{\eta n} \sum_{k=0}^{x_n} v(j,k) \sim V(x_n). \tag{4.6}$$

Note that of course

$$\sum_{j=0}^{\infty} \sum_{k=0}^{x_n} v(j,k) = V(x_n), \tag{4.7}$$

so that we just have to show that

$$\frac{\sum_{j>\eta n} \sum_{k=0}^{x_n} v(j,k)}{V(x_n)} \to 0 \tag{4.8}$$

as $n \to \infty$ uniformly on x_n such that $x_n/a_n \to 0$. For this, we note that Lemma 2.1 implies

$$v(n,x) \sim \frac{V(x)\mathbf{P}[S_n = -x]}{n} \tag{4.9}$$

as $n \to \infty$ uniformly on $x \in [0, Ka_n]$ where $K > 0$, so that

$$\sum_{j>\eta n} \sum_{k=0}^{x_n} v(j,k) \sim \sum_{j>\eta n} \sum_{k=0}^{x_n} \frac{V(k)\mathbf{P}[S_j = -k]}{j}. \tag{4.10}$$

Using the fact that $V(\cdot)$ is increasing and the standard local limit theorem (here and later c is a positive constant which may vary from line to line):

$$\frac{\sum_{j>\eta n}\sum_{k=0}^{x_n} v(j,k)}{V(x_n)} \leqslant c \sum_{j>\eta n}\sum_{k=0}^{x_n} \frac{\phi(k/a_j)}{ja_j} \leqslant c \sum_{j>\eta n} \frac{x_n}{ja_j}. \tag{4.11}$$

Finally, as $a_n \in \mathcal{R}_{1/2}$, using property (2.4) it is easy to see that

$$\sum_{j>\eta n} \frac{a_n}{ja_j} \sim \int_\eta^\infty x^{-3/2} dx \tag{4.12}$$

and this entails (4.6). To conclude the case where y_n/a_n is bounded away from zero, we are left with showing that for any $\eta > 0$, one has:

$$\limsup_{n\to\infty} \frac{n\mathbf{P}_{x_n}[\widehat{S}_n = y_n; \mu_n \geqslant \eta n]}{V(x_n)U(y_n)\mathbf{P}[S_n = y_n]} = 0 \tag{4.13}$$

as $n \to \infty$. By the standard local limit theorem, there exists $a, b > 0$ such that $a \leqslant a_n \mathbf{P}[S_n = y_n] \leqslant b$. Using Lemma 3.2, we get that:

$$\frac{n\mathbf{P}_{x_n}[\widehat{S}_n = y_n; \mu_n \geqslant \eta n]}{V(x_n)U(y_n)\mathbf{P}[S_n = y_n]} = \frac{n\mathbf{P}^*_{x_n}[S_n = y_n; \mu_n \geqslant \eta n]}{V(y_n)U(y_n)\mathbf{P}[S_n = y_n]} \leqslant \frac{ca_n\mathbf{P}^*_{x_n}[S_n = y_n; \mu_n \geqslant \eta n]}{\varepsilon^2} \tag{4.14}$$

so that we have to show that

$$\limsup_{n\to\infty} a_n \mathbf{P}^*_{x_n}[S_n = y_n; \mu_n \geqslant \eta n] = 0. \tag{4.15}$$

Then we fix $\theta \in (\eta, 1)$ and we have:

$$\begin{aligned} a_n\mathbf{P}^*_{x_n}[S_n = y_n; \mu_n \geqslant \eta n] = {} & \underbrace{a_n\mathbf{P}^*_{x_n}[\eta n \leqslant \mu_n \leqslant \theta n]}_{(1)} \\ & + \underbrace{a_n\mathbf{P}^*_{x_n}[\mu_n > \theta n]}_{(2)}. \end{aligned} \tag{4.16}$$

Making use of the Markov property, one gets:

$$\begin{aligned} (1) &= a_n \sum_{j=\eta n}^{\theta n}\sum_{k=0}^{x_n} \mathbf{P}^*_{x_n}\left[\mu_n = j, m_n = k\right] \mathbf{P}^*_k\left[S_{n-j} = y_n, \min_{l \leqslant n-j} S_l \geqslant k\right] \\ &\leqslant a_n \sum_{j=\eta n}^{\theta n}\sum_{k=0}^{x_n} \mathbf{P}^*_{x_n}[\mu_n = j, m_n = k] \frac{V(y_n)}{V(k)} \mathbf{P}_k\left[\widehat{S}_{n-j} = y_n, \min_{i \leqslant n-j} \widehat{S}_i \geqslant k\right]. \end{aligned} \tag{4.17}$$

Noting that one has the equality $\mathbf{P}_k\left[\widehat{S}_{n-j}=y_n, \min_{i\leqslant n-j}\widehat{S}_i\geqslant k\right]=\mathbf{P}\left[\widehat{S}_{n-j}=y_n-k\right]$, we get (note that $V(k)\geqslant 1$ for every k):

$$(1)\leqslant a_n\sum_{j=\eta n}^{\theta n}\sum_{k=0}^{x_n}\mathbf{P}^*_{x_n}\left[\mu_n=j, m_n=k\right]V(y_n)\mathbf{P}\left[\widehat{S}_{n-j}=y_n-k\right]. \tag{4.18}$$

Making use of Lemma 3.2, of Lemma 2.1 and of the fact that $x_n/a_n\to 0$ as $n\to\infty$, we get :

$$\begin{aligned}(1)&\leqslant ca_n\sum_{j=\eta n}^{\theta n}\sum_{k=0}^{x_n}\mathbf{P}^*_{x_n}[\mu_n=j, m_n=k]\frac{U(y_n)V(y_n)}{n-j}\mathbf{P}[S_{n-j}=y_n-k]\\&\leqslant cK^2\sum_{j=\eta n}^{\theta n}\mathbf{P}^*_{x_n}[\mu_n=j]\frac{n}{n-j}a_n\mathbf{P}[S_{n-j}=y_n].\end{aligned} \tag{4.19}$$

Making use of the standard local limit theorem, we have easily:

$$(1)\leqslant cK^2(1-\theta)^{-3/2}\mathbf{P}^*_{x_n}\left[\mu_n\geqslant\eta n\right]. \tag{4.20}$$

Evidently, for every n, one has $\mu_n\leqslant\mu$, so that

$$\mathbf{P}^*_{x_n}[\mu_n\geqslant\eta n]\leqslant\mathbf{P}^*_{x_n}[\mu\geqslant\eta n], \tag{4.21}$$

and it has been shown in [BJD06, Theorem 5.1] that for every $\eta>0$, uniformly in the sequences x_n such that $x_n/a_n\to 0$ as $n\to\infty$, the quantity $\mathbf{P}^*_{x_n}[\mu\geqslant\eta n]$ vanishes as $n\to\infty$.

For the second term in (4.16), we will need the following result which has been proved in [BJD06]:

Proposition 4.1. *For any $\kappa>0$, for $x_n/a_n\to 0$ as $n\to\infty$, one has the following convergence:*

$$\mathbf{P}^*_{x_n}\left[\max_{j\leqslant\mu}S_j\geqslant\kappa a_n\right]\to 0. \tag{4.22}$$

We give us $\kappa\in(0,\varepsilon)$ and for $n>0$, we note $\tau:=\inf\{j\geqslant 0, S_j\geqslant\kappa a_n\}$. Then we have:

$$\begin{aligned}(2)=&\underbrace{a_n\mathbf{P}^*_{x_n}[\mu_n\geqslant\eta n, S_n=y_n,\tau\geqslant\theta n]}_{(3)}\\&+\underbrace{a_n\mathbf{P}^*_{x_n}[\mu_n\geqslant\eta n, S_n=y_n,\tau<\theta n]}_{(4)}.\end{aligned} \tag{4.23}$$

Making use of the Markov property, we have:

$$
\begin{aligned}
(3) &\leqslant a_n \sum_{j=0}^{\kappa a_n} \mathbf{P}^*_{x_n}\left[\max_{i\leqslant \theta n} S_i \leqslant \kappa a_n, S_{\theta n} = j, S_n = y_n\right] \\
&\leqslant a_n \frac{V(y_n)}{V(x_n)} \sum_{j=0}^{\kappa a_n} \mathbf{P}_{x_n}\left[\max_{i\leqslant \theta n} \widehat{S}_i \leqslant \kappa a_n, \widehat{S}_{\theta n} = j\right] \mathbf{P}_j\left[\widehat{S}_{(1-\theta)n} = y_n\right] \\
&\leqslant a_n \frac{V(y_n)}{V(x_n)} \sum_{j=0}^{\kappa a_n} \mathbf{P}_{x_n}\left[\max_{i\leqslant \theta n} S_i \leqslant \kappa a_n, \tau_{(-\infty,0)} > \theta n, S_{\theta n} = j\right] \mathbf{P}\left[S_{(1-\theta)n} = y_n - j\right],
\end{aligned} \tag{4.24}
$$

where we recall that $\tau_{(-\infty,0)} = \inf\{j \geqslant 1, S_j \in (-\infty,0)\}$. Using the local limit theorem and the fact that $j \in [0, \kappa a_n]$, we get:

$$
\begin{aligned}
(3) \leqslant c\frac{V(y_n)\mathbf{P}_{x_n}\left[\tau_{(-\infty,0)} > \theta n\right]}{V(x_n)} \\
\times \mathbf{P}_{x_n}\left[\max_{i\leqslant \theta n} S_i \leqslant \kappa a_n \middle| \tau_{(-\infty,0)} > \theta n\right] \frac{1}{\sqrt{1-\theta}}\phi\left((\varepsilon-\kappa)(1-\theta)^{-1/2}\right).
\end{aligned} \tag{4.25}
$$

Using the remark 4 in [Shi83], we note that, as $n \to \infty$,

$$
\mathbf{P}_{x_n}\left[\max_{i\leqslant \theta n} S_i \leqslant \kappa a_n \middle| \tau_{(-\infty,0)} > \theta n\right] \to m\left[\sup_{[0,1]} \omega_t \leqslant \frac{\kappa}{\sqrt{\theta}}\right], \tag{4.26}
$$

where $m(\cdot)$ denotes the measure of the brownian meander.

We prove that the equivalence

$$
\mathbf{P}_{x_n}[\tau_{(-\infty,0)} > \theta n] \sim V(x_n)\mathbf{P}[T_1^- > \theta n] \tag{4.27}
$$

holds uniformly on the sequences x_n such that $x_n/a_n \to 0$ in Lemma 6.1, so that finally, using the convergence

$$
V(Ka_n)\mathbf{P}[T_1^- > \theta n] \to c\frac{K}{\sqrt{\theta}}, \tag{4.28}
$$

which one can deduce from part 2.2, one gets:

$$
\begin{aligned}
(3) &\leqslant cV(Ka_n)\mathbf{P}[T_1^- > \theta n] m\left[\sup_{[0,1]} \omega_t \leqslant \frac{\kappa}{\sqrt{\theta}}\right] \frac{1}{\sqrt{1-\theta}}\phi\left((\varepsilon-\kappa)(1-\theta)^{-1/2}\right) \\
&\leqslant cKm\left[\sup_{[0,1]} \omega_t \leqslant \frac{\kappa}{\sqrt{\theta}}\right] \frac{1}{\sqrt{\theta(1-\theta)}}\phi\left((\varepsilon-\kappa)(1-\theta)^{-1/2}\right)
\end{aligned} \tag{4.29}
$$

and for $\theta > 0$ fixed, the quantity in the right hand side above vanishes as $\kappa \searrow 0$.

We are left with the second term in equation (4.23). To get this, one notes that looking at the proof of Lemma 3.4 in [BJD06], it is not difficult to see that, with $c, c' > 0$ fixed, the convergence in (3.10) holds uniformly for (u, v) in the compact set $[c, c'] \times [\varepsilon, K]$. Note in particular the uniformity part in Lemma 3.1, the fact that the convergence in the local limit theorem is uniform on the sets $[ca_n, c'a_n]$ and finally the fact that the derivative of the function $(x, u) \mapsto \frac{x}{u^{3/2}}\phi(x/u^2)$ is uniformly bounded for $(x, u) \in [c, c'] \times (0, 1)$ (to get the uniform convergence of the Riemann's sums in the proof of Lemma 3.4 in [BJD06]).

Making use once again of the Markov property, this implies that:

$$
\begin{aligned}
(4) &\leqslant a_n \sum_{j \leqslant \theta n} \sum_{k \geqslant \kappa a_n} \mathbf{P}^*_{x_n}[\tau = j, S_j = k, \mu > \theta n] \mathbf{P}^*_k [S_{n-j} = y_n] \\
&\leqslant \sum_{j \leqslant \theta n} \sum_{k \geqslant \kappa a_n} \mathbf{P}^*_{x_n}[\tau = j, S_j = k, \mu > \theta n] \frac{V(y_n)}{V(k)} a_n \mathbf{P}_k \left[\widehat{S}_{n-j} = y_n\right].
\end{aligned}
\tag{4.30}
$$

Note that one can restrict the range of summation of k in the above expression over $[\kappa a_n, K' a_n]$ where $K' > 0$ is large enough and independent of n. Thus, using Proposition 3.4 and the fact that $r(\cdot, \cdot)$ is continuous, one obtains:

$$
\begin{aligned}
(4) &\leqslant c \frac{V(K)}{V(\kappa)\sqrt{1-\theta}} \frac{K'}{\varepsilon} \left[\sup_{u \in [\kappa, K'], v \in [\varepsilon, K]} r(u, v)\right] \sum_{j \leqslant \theta n} \sum_{k \geqslant \kappa a_n} \mathbf{P}^*_{x_n}[\tau = j, S_j = k, \mu_n > \theta n] \\
&\leqslant c \frac{V(K)}{V(\kappa)\sqrt{1-\theta}} \frac{K'}{\varepsilon} \left[\sup_{u \in [\kappa, K'], v \in [\varepsilon, K]} r(u, v)\right] \mathbf{P}^*_{x_n}[\max_{j \leqslant \mu_n} S_j \geqslant \kappa a_n]
\end{aligned}
\tag{4.31}
$$

and as evidently the inclusion of events $\{\max_{j \leqslant \mu_n} S_j \geqslant \kappa a_n\} \subset \{\max_{j \leqslant \mu} S_j \geqslant \kappa a_n\}$ holds, making use of Proposition 4.1, the last term in the equation above vanishes as $n \to \infty$ since $x_n / a_n \to 0$.

4.2 The case where y_n/a_n vanishes at infinity

This case relies heavily on the previous one. One has the equality:

$$
\begin{aligned}
\mathbf{P}_{x_n}\left[\widehat{S}_n = y_n\right] &= \underbrace{\sum_{z = \varepsilon a_n}^{K a_n} \mathbf{P}_{x_n}\left[\widehat{S}_{n/2} = z\right] \mathbf{P}_z \left[\widehat{S}_{n/2} = y_n\right]}_{(5)} \\
&\quad + \underbrace{\mathbf{P}_{x_n}\left[\widehat{S}_n = y_n, S_{n/2} \leqslant \varepsilon a_n, S_{n/2} \geqslant K a_n\right]}_{(6)}
\end{aligned}
\tag{4.32}
$$

We first show that the term in (5) yields the desired estimate, and then that the term in (6) is negligible with respect to the first one.

For the term in (5) , using time reversal and the case we just treated, one has of course:

$$\mathbf{P}_z[\widehat{S}_{n/2} = y_n] \sim \frac{U(y_n)V(z)}{n/2}\mathbf{P}[S_{n/2} = z] \tag{4.33}$$

so that, for $n \to \infty$, we have the equivalence:

$$\begin{aligned}\sum_{z=\varepsilon a_n}^{Ka_n} \mathbf{P}_{x_n}\left[\widehat{S}_{n/2} = z\right]\mathbf{P}_z\left[\widehat{S}_{n/2} = y_n\right] &\sim \sum_{z=\varepsilon a_n}^{Ka_n} \frac{V(x_n)U(z)}{n/2}\frac{U(y_n)V(z)}{n/2}\mathbf{P}[S_{n/2} = z]^2 \\ &\sim \frac{V(x_n)U(y_n)}{n}\sum_{z=\varepsilon a_n}^{Ka_n} 8\frac{z^2}{a_n^2}\frac{\phi(z/a_{n/2})^2}{a_{n/2}^2}\end{aligned} \tag{4.34}$$

where in the last equivalence we made use of the standard local limit theorem and of Lemma 3.3. Thus we are left with showing that

$$\lim_{\varepsilon\searrow 0, K\nearrow\infty}\lim_{n\to\infty} 8\sqrt{2\pi}a_n \sum_{z=\varepsilon a_n}^{Ka_n}\frac{z^2}{a_n^2}\frac{\phi(z/a_{n/2})^2}{a_{n/2}^2} = 1. \tag{4.35}$$

We use Riemann's sum and the fact that $(a_n) \in \mathcal{R}_{1/2}$ to get:

$$\begin{aligned}8\sum_{z=\varepsilon a_n}^{Ka_n}\frac{z^2}{a_n^2}\frac{\phi(z/a_{n/2})^2}{a_{n/2}^2} &\sim 16\sum_{z=\varepsilon a_n}^{Ka_n}\frac{z^2}{a_n^2}\frac{\phi(\sqrt{2}z/a_n)^2}{a_n^2} \\ &\sim \frac{16}{\sqrt{2\pi}}\sum_{z=\varepsilon a_n}^{Ka_n}\frac{z^2}{a_n^2}\frac{\phi(2z/a_n)}{a_n^2} \\ &\sim \frac{16}{\sqrt{2\pi}a_n}\int_\varepsilon^K u^2\phi(2u)du \\ &\sim \frac{2}{\sqrt{2\pi}a_n}\int_{\varepsilon/2}^{K/2} u^2\phi(u)du.\end{aligned} \tag{4.36}$$

and thus (4.35) is valid.

We are left with showing that:

$$\begin{aligned}\limsup_{K\nearrow\infty}\lim_{n\to\infty}\frac{na_n}{V(x_n)U(y_n)}\mathbf{P}_{x_n}\left[\widehat{S}_n = y_n, \widehat{S}_{n/2} \geqslant Ka_n\right] = 0, \\ \limsup_{\varepsilon\searrow 0}\lim_{n\to\infty}\frac{na_n}{V(x_n)U(y_n)}\mathbf{P}_{x_n}\left[\widehat{S}_n = y_n, \widehat{S}_{n/2} \leqslant \varepsilon a_n\right] = 0.\end{aligned} \tag{4.37}$$

We define $\widetilde{S}$ as being the time reversed version of S, that is the random walk whose transitions are given by

$$\mathbf{P}[\widetilde{S}_1 = y] := \mathbf{P}[S_1 = -y], \qquad y \in \mathbb{Z}. \tag{4.38}$$

Note that

$$\begin{aligned} &\frac{na_n}{V(x_n)U(y_n)} \mathbf{P}_{x_n}\left[\widehat{S}_n = y_n, \widehat{S}_{n/2} \geqslant Ka_n\right] \\ &= \frac{na_n}{V(x_n)U(y_n)} \sum_{z \geqslant Ka_n} \mathbf{P}_{x_n}\left[\widehat{S}_{n/2} = z\right] \mathbf{P}_{y_n}\left[\widetilde{\widehat{S}}_{n/2} = z\right]. \end{aligned} \tag{4.39}$$

We recall that the following equivalences are shown in Lemma 6.1 below:

$$\begin{aligned} &\mathbf{P}_{x_n}[\tau_{(-\infty,0)} > n/2] \sim V(x_n)\mathbf{P}[T_1^- > n/2], \\ &\mathbf{P}_{y_n}[\tau_{(-\infty,0)} > n/2] \sim V(y_n)\mathbf{P}[\widetilde{T_1^-} > n/2] \end{aligned} \tag{4.40}$$

and that they hold uniformly for x_n, y_n which are $o(a_n)$.

Therefore, one deduces

$$\begin{aligned} &\frac{na_n}{V(x_n)U(y_n)} \mathbf{P}_{x_n}[\widehat{S}_n = y_n, \widehat{S}_{n/2} \geqslant Ka_n] \\ &\quad \sim n\mathbf{P}[T_1^- > n/2]\mathbf{P}[\widetilde{T_1^-} > n/2] \\ &\qquad \times \sum_{z \geqslant Ka_n} a_n \mathbf{P}_{x_n}[S_{n/2} = z | \tau_{(-\infty,0)} > n/2] \mathbf{P}_{y_n}[\widetilde{S}_{n/2} = z | \widetilde{\tau}_{(-\infty,0)} > n/2]. \end{aligned} \tag{4.41}$$

By the local limit theorem for the random walk conditioned to stay positive (see [Car05, Theorem 2]):

$$\sup_{z \in \mathbb{Z}} a_n \mathbf{P}_{x_n}[S_{n/2} = z | \tau_{(-\infty,0)} > n/2] =: C < \infty. \tag{4.42}$$

Recall that T_1^+ and T_1^- are attracted to stable laws of index $1/2$, so that by standard Tauberian theorems (see [Fel71, XIII 5.]):

$$\mathbf{P}[T_1^- > n] \sim \frac{1}{\sqrt{\pi}}\left(1 - \mathbf{E}\left[e^{-\frac{1}{n}T_1^-}\right]\right), \mathbf{P}[\widetilde{T}_1^- > n] \sim \frac{1}{\sqrt{\pi}}\left(1 - \mathbf{E}\left[e^{-\frac{1}{n}\widetilde{T}_1^-}\right]\right). \tag{4.43}$$

On the other hand, by the Wiener-Hopf factorization:

$$\begin{aligned} 1 - \mathbf{E}[e^{-\lambda T_1^-}] &= \exp\left(-\sum_{n=1}^{\infty} \frac{e^{-\lambda n}}{n} \mathbf{P}[S_n < 0]\right) \\ 1 - \mathbf{E}[e^{-\lambda T_1^+}] &= \exp\left(-\sum_{n=1}^{\infty} \frac{e^{-\lambda n}}{n} \mathbf{P}[S_n \geqslant 0]\right) \end{aligned} \tag{4.44}$$

hence, for $\lambda \searrow 0$,

$$\left(1-\mathbf{E}\left[e^{-\lambda T_1^-}\right]\right)\left(1-\mathbf{E}\left[e^{-\lambda T_1^+}\right]\right) = \exp\left(-\sum_{n=1}^{\infty}\frac{e^{-\lambda n}}{n}\right) = 1-e^{-\lambda} \sim \lambda \qquad (4.45)$$

therefore $\lim_{n\to\infty} n\mathbf{P}[T_1^- > n]\mathbf{P}[\widetilde{T}_1^- > n] = \frac{1}{\pi}$. Using finally the convergence towards the brownian meander, we get that

$$\begin{aligned}\frac{na_n}{V(x_n)U(y_n)}\mathbf{P}_{x_n}\left[\widehat{S}_n = y_n, \widehat{S}_{n/2} \geqslant Ka_n\right] &\leqslant \frac{C}{\pi}\mathbf{P}_{y_n}\left[\widetilde{S}_{n/2} \geqslant Ka_n \middle| \widetilde{T}_1^- > n\right] \\ &\leqslant cm\left[\omega_{1/2} > K\right]\end{aligned} \qquad (4.46)$$

and the last term vanishes as $K\to\infty$. Proceeding in the same way, it is easy to see that

$$\lim_{n\to\infty}\frac{na_n}{V(x_n)U(y_n)}\mathbf{P}_{x_n}\left[\widehat{S}_n = y_n, \widehat{S}_{n/2} \leqslant \varepsilon a_n\right] \leqslant cm\left[\omega_{1/2}\leqslant\varepsilon\right], \qquad (4.47)$$

and this last quantity also vanishes when $\varepsilon \searrow 0$, and this concludes the proof of Lemma 3.1.

5 Tightness of the measures $Q_n^{x_n,y_n}$

The proof of tightness is very similar to the one of [BJD06]. We first note that the process S under $\mathbf{P}^*_{x_n}[\cdot|S_n = y_n]$ is still a Markov chain, so that according to [Bil68, Theorem 8.4] , tightness will follow if we can show that for each positive ε and $K \in (0,1)$, there exists $\lambda > 0$ and an integer n_0 such that

$$\mathbf{P}^*_{x_n}\left[\max_{i\leqslant Kn} S_i \geqslant \lambda a_n \middle| S_n = y_n\right] \leqslant \frac{\varepsilon}{\lambda^2} \qquad (5.1)$$

for all $n \geqslant n_0$.

We proceed quite similarly as in the last part of the proof of Lemma 3.1. We write:

$$\begin{aligned}\star &:= \mathbf{P}^*_{x_n}\left[\max_{i\leqslant Kn} S_i \geqslant \lambda a_n \middle| S_n = y_n\right] \\ &= \sum_{j\geqslant 0}\mathbf{P}_{x_n}\left[\max_{i\leqslant Kn}\widehat{S}_i \geqslant \lambda a_n, \widehat{S}_{Kn} = j \middle| \widehat{S}_n = y_n\right] \\ &\sim \frac{na_n}{V(x_n)U(y_n)}\sum_{j\geqslant 0}\mathbf{P}_{x_n}\left[\max_{i\leqslant Kn}\widehat{S}_i \geqslant \lambda a_n, \widehat{S}_{Kn} = j\right]\mathbf{P}_j\left[\widehat{S}_{n(1-K)} = y_n\right]\end{aligned} \qquad (5.2)$$

Using the same considerations as in the last part of the proof of Lemma 3.1 (by simply replacing $n/2$ by Kn or $(1-K)n$) , one gets that there exists a constant $C > 0$ such that:

$$\star \leqslant \frac{C}{\sqrt{1-K}} \sum_{j \geqslant 0} \mathbf{P}_{x_n} \left[\max_{i \leqslant Kn} S_i \geqslant \lambda a_n, S_{Kn} = j \middle| \tau_{(-\infty,0)} > Kn \right] \tag{5.3}$$

so that using the weak convergence towards the brownian meander, we get:

$$\star \leqslant \frac{C}{\sqrt{1-K}} m \left[\sup_{t \in [0,1]} \omega_t \geqslant \frac{\lambda}{\sqrt{K}} \right], \tag{5.4}$$

which for fixed K vanishes exponentially fast when λ becomes large, and in particular (5.1) holds. This concludes the proof of Theorem 1.1, and thus we are done.

6 Appendix

The following is the main result of this appendix:

Lemma 6.1. *Uniformly in x_n such that $x_n a_n^{-1} \to 0$ as $n \to \infty$, one has the following convergence:*

$$\frac{\mathbf{P}_{x_n}[\tau_{(-\infty,0)} > n]}{\mathbf{P}[T_1^- > n]} \sim V(x_n). \tag{6.1}$$

Note that it has been proved in [BD94] that

$$\liminf_{n \to \infty} \frac{\mathbf{P}_x[\tau_{(-\infty,0)} > n]}{\mathbf{P}[T_1^- > n]} \geqslant V(x) \tag{6.2}$$

in full generality (that is for every oscillating random walk S verifying $\mathbf{P}[S_1 > 0] \in (0,1)$). The convergence (6.1) has also been proved in [Kes63] in the lattice case for fixed x.

Proof. For $x > 0$, we denote by $\tau_x = \inf\{k \geqslant 1, S_k < -x\}$. One has the following identity:

$$\begin{aligned} \mathbf{P}_x[\tau_{(-\infty,0)} > n] &= \mathbf{P}[\tau_x > n] \\ &= \sum_{k=0}^{+\infty} \mathbf{P}[T_k^- \leqslant n < T_{k+1}^-, \tau_x > n] \\ &= \sum_{k=0}^{+\infty} \mathbf{P}[T_k^- \leqslant n < T_{k+1}^-, H_k^- < x] \\ &= \sum_{k=0}^{+\infty} \sum_{l=0}^{n} \mathbf{P}[T_k^- = l, H_k^- < x] \mathbf{P}[T_1^- > n - l] \end{aligned} \tag{6.3}$$

where in the last equality we made use of the Markov property. Thus:

$$\frac{\mathbf{P}_x[\tau_{(-\infty,0)} > n]}{V(x)} = \sum_{l=0}^{n} \frac{\mathbf{P}[\, l \text{ is a descending ladder epoch}, -S_l < x]}{V(x)} \mathbf{P}[T_1^- > n-l]. \tag{6.4}$$

We recall a strong version of Iglehart's lemma ([AD99, Lemma 5]):

Lemma 6.2. *Let $c_n, d_n(z)$be two sequences where z belongs to a subset Δ of $\mathbb{R}$. Define e_n on Δ by:*

$$e_n(z) := \sum_{j=0}^{n-1} d_j(z) c_{n-j}. \tag{6.5}$$

Assume that there exist $c > 0$ such that the following condition holds uniformly on $z \in \Delta$:

$$\sum_{j=1}^{n} d_j(z) \to d(z) < \infty \ \text{ and } \ n d_n(z) \leqslant c \tag{6.6}$$

Assume moreover that the sequence c_n is regularly varying with index $-\rho$ where $\rho \in (0,1)$. Then the equivalence $e_n(z) \sim d(z) c_n$ holds uniformly on $z \in \Delta$.

We already pointed out that:

$$\mathbf{P}[T_1^- > n] \sim \frac{b^{-1}(n)}{\sqrt{2\pi n}} \ \text{ as } \ n \to \infty. \tag{6.7}$$

Recalling that $b(\cdot) \in \mathcal{R}_2$, one has $b^{-1}(n)/n \in \mathcal{R}_{-1/2}$, which implies that the sequence $\left(\mathbf{P}[T_1^- > n]\right)_n$ verifies the hypothesis of the sequence c of Lemma 6.2 with $\rho = 1/2$.

On the other hand, we write

$$1 = \sum_{l \geqslant 0} \frac{\mathbf{P}\left[\, l \text{ is a descending ladder epoch}, -S_l < x\right]}{V(x)} = \sum_{l \geqslant 0} \frac{\sum_{j \in [0,x]} v(l,j)}{V(x)} \tag{6.8}$$

and thus we want to prove that the sequence $d_l(x) = \frac{\sum_{j \in [0,x]} v(l,j)}{V(x)}$ satisfies the second conditions of Lemma 6.2 with $\Delta_{(\varepsilon_n)} = \left\{(x_n) \in \mathbb{Z}^{\mathbb{N}}, \forall n, x_n \in [0, \varepsilon_n a_n]\right\}$ where ε_n is a given positive sequence which vanishes at infinity.

We first note that the uniform convergence of the series on $\Delta_{(\varepsilon_n)}$ has already been proved in the first part of the proof of Lemma 3.1.

For the second point, we consider a sequence $(x_n)_n \in \Delta_{(\varepsilon_n)}$. For $l > 0$ and making use of Lemma 2.1 (note in particular the uniformity part of it) and of the local limit theorem:

$$\begin{aligned}\sum_{j\in[0,x_n]} v(l,j) &\leqslant x_n \sup_{j\,\leqslant\, x_n} v(l,j)\\ &\leqslant c\frac{x_n}{na_n}V(x_n)\\ &\leqslant c\frac{\varepsilon_n}{n}V(x_n)\end{aligned} \tag{6.9}$$

and as $\varepsilon_n \to 0$, both conditions of the first part of (6.6) are fulfilled by the sequence $\left(\frac{\sum_{j\in[0,x]} v(l,j)}{V(x)}\right)_l$.

Thus we get that the following equivalence holds uniformly on $\Delta_{(\varepsilon_n)}$:

$$\frac{\mathbf{P}_{x_n}\left[\tau_{(-\infty,0)} > n\right]}{V(x_n)} \sim \mathbf{P}\left[T_1^- > n\right]. \tag{6.10}$$

This entails that the following equivalence holds uniformly on x_n such that $x_n a_n^{-1} \to 0$ as $n \to \infty$:

$$\lim_{n\to\infty} \frac{\mathbf{P}_{x_n}\left[\tau_{(-\infty,0)} > n\right]}{\mathbf{P}[T_1^- > n]} \sim V(x_n) \tag{6.11}$$

which is equation (6.1). □

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