QUASI-FREE ACTIONS OF FINITE GROUPS ON THE CUNTZ ALGEBRA $\mathcal{O}_\infty$

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Abstract. We show that any faithful quasi-free actions of a finite group on the Cuntz algebra $\mathcal{O}_\infty$ are mutually conjugate, and that they are asymptotically representable.

1. Introduction

The Cuntz algebra $\mathcal{O}_n$, $n = 2, 3, \cdots, \infty$, is the universal C^*-algebra generated by isometries $\{s_i\}_{i=1}^n$ with mutually orthogonal ranges, satisfying $\sum_{i=1}^n s_i s_i^* = 1$ if n is finite. It is well known that the two algebras $\mathcal{O}_2$ and $\mathcal{O}_\infty$, among the others, play special roles in the celebrated classification theory of Kirchberg algebras (see [15], [18]).

An action α of a group G on $\mathcal{O}_n$ is said to be *quasi-free* if $\alpha_g(\mathcal{H}_n) = \mathcal{H}_n$ for all $g \in G$, where $\mathcal{H}_n$ is the closed linear span of the generators $\{s_i\}_{i=1}^n$. We restrict our attention to finite G throughout this note. To develop a G-equivariant version of the classification theory, it is expected that G-actions on $\mathcal{O}_2$ with the Rohlin property and the quasi-free G-actions on $\mathcal{O}_\infty$ would play similar roles as $\mathcal{O}_2$ and $\mathcal{O}_\infty$ do in the case without group actions. Since we have already had a good understanding of the former thanks to [4], our task in this note is to investigate the latter, the quasi-free G-actions on $\mathcal{O}_\infty$.

The space $\mathcal{H}_n$ has a Hilbert space structure with inner product $t^*s = \langle s, t\rangle 1$, and a quasi-free G-action α gives a unitary representation $(\pi_\alpha, \mathcal{H}_\alpha)$, where $\pi_\alpha(g)$ is the restriction of α_g to $\mathcal{H}_\alpha$. It is known that the association $\alpha \mapsto \pi_\alpha$ gives a one-to-one correspondence between the quasi-free G actions on $\mathcal{O}_n$ and the unitary representations of G in $\mathcal{H}_n$. The conjugacy class of α depends on the unitary equivalence class of $(\pi_\alpha, \mathcal{H}_n)$, at least a priori. Indeed, it really does when n is finite, and this can be seen by computing the K-groups of the crossed product (see, for example, [2], [4], [5], [11]). However, when $n = \infty$, the pair $(\mathcal{O}_\infty, \alpha)$ is KK_G-equivalent to the pair $(\mathbb{C}, \mathrm{id})$, and there is no way to differentiate the quasi-free actions as far as K-theory is concerned.

One of the purposes of this note is to show that any two faithful quasi-free G-actions on $\mathcal{O}_\infty$ are indeed mutually conjugate for every finite group G (Corollary 5.2). Our main technical result is Theorem 4.1, an equivariant version of Lin-Phillips's result [10, Theorem 3.3], and Corollary 5.2 follows from it via Theorem 5.1, an equivariant version of Kirchberg-Phillips's $\mathcal{O}_\infty$ theorem [7, Theorem 3.15].

Supported in part by the Grant-in-Aid for Scientific Research (B) 22340032, JSPS.

Using Theorem 4.1, we also show that the quasi-free actions are asymptotically representable for any finite group G, which is another purpose of this note. The notion of asymptotic representability for group actions was introduced by the second-named author, and it is found to be important in the recent development of the classification of group actions on C^*-algebras (see [6], [11]).

The reader is referred to [18] for the basic properties and classification results for Kirchberg algebras. We denote by $\mathbb{K}$ the set of compact operators on a separable infinite dimensional Hilbert space. For a C^*-algebra A, we denote by $\tilde{A}$ and $M(A)$ the unitization and the multiplier algebra of A respectively. When A is unital, we denote by $U(A)$ the unitary group of A. For a homomorphism $\rho : A \to B$ between C^*-algebras A, B, we denote by $K_*(\rho)$ the homomorphism from $K_*(A)$ to $K_*(B)$ induced by ρ. We denote by $A \otimes B$ the minimal tensor product of A and B.

This work originated from the first-named author's unpublished preprint [3], where the idea of developing an equivariant version of Lin-Phillips's argument was introduced. Some results in this note are also obtained by N. C. Phillips, and the authors would like to thank him for informing of it.

2. Preliminaries for G-C^*-algebras

We fix a finite group G. By a G-C^*-algebra (A, α), we mean a C^*-algebra A with a fixed G-action α. We denote by A^G the fixed point algebra

$$\{a \in A |\ \alpha_g(a) = a,\ \forall g \in G\}.$$

We denote by $\{\lambda^\alpha_g\}_{g\in G}$ the implementing unitary representation of G in the crossed product $A \rtimes_\alpha G$. For a finite dimensional (not necessarily irreducible) unitary representation (π, H_π) of G, we introduce a homomorphism

$$\hat{\alpha}_\pi : A \rtimes_\alpha G \to (A \rtimes_\alpha G) \otimes B(H_\pi),$$

which is a part of the dual coaction of α, by $\hat{\alpha}_\pi(a) = a \otimes 1$ for $a \in A$, and $\hat{\alpha}_\pi(\lambda^\alpha_g) = \lambda^\alpha_g \otimes \pi(g)$ for $g \in G$. We denote by $\hat{G}$ the unitary dual of G, and by $\mathbb{Z}\hat{G}$ the representation ring of G. Then identifying $K_*(A \rtimes_\alpha G)$ with $K_*((A \rtimes_\alpha G) \otimes B(H_\pi))$, we get a $\mathbb{Z}\hat{G}$-module structure of $K_*(A \rtimes_\alpha G)$ from $K_*(\hat{\alpha}_\pi)$.

Let

$$e_\alpha = \frac{1}{\#G} \sum_{g\in G} \lambda^\alpha_g,$$

which is a projection in $(A \rtimes_\alpha G) \cap A^{G\prime}$. We denote by j_α the homomorphism from A^G into $A \rtimes_\alpha G$ defined by $j_\alpha(x) = xe_\alpha$. When A is simple and α is outer, that is, α_g is outer for every $g \in G \setminus \{e\}$, then $K_*(j_\alpha)$ is an isomorphism from $K_*(A^G)$ onto $K_*(A \rtimes_\alpha G)$. When A is purely infinite and simple, and α is outer, then A^G and $A \rtimes_\alpha G$ are purely infinite and simple.

A G-homomorphism φ from a G-C^*-algebra (A, α) into another G-C^*-algebra (B, β) is a homomorphism from A into B intertwining the two G-actions α and β. Such φ gives rise to an element in the equivariant KK-group $KK_G(A, B)$, which is denoted by $KK_G(\varphi)$. We denote by $\mathrm{Hom}_G(A, B)$ the set of nonzero G-homomorphisms from (A, α) into (B, β). Two actions α and β are said to be conjugate if there exists an invertible element in $\mathrm{Hom}_G(A,B)$. Two G-homomorphisms $\varphi, \psi \in \mathrm{Hom}_G(A,B)$ are said to be G-unitarily equivalent if there exists a unitary $u \in M(B)^G$ satisfying $\varphi(x) = u\psi(x)u^*$ for all $x \in A$. They are said to be G-asymptotically unitarily equivalent if there exists a norm continuous family of unitaries $\{u(t)\}_{t\geq 0}$ in $M(B)^G$ satisfying

$$\lim_{t\to\infty} \|\varphi(x) - \mathrm{Ad}\, u(t) \circ \psi(x)\|, \quad \forall x \in A.$$

If they satisfy the same condition with a sequence of unitaries $\{u_n\}_{n=1}^\infty$ in $M(B)^G$ instead of the continuous family, they are said to be G-approximately unitarily equivalent

For a free ultrafilter $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$ and a G-C^*-algebra (A,α), we use the following notation:

$$c_\omega(A) = \{(x_n) \in \ell^\infty(\mathbb{N}, A) | \lim_{n\to\omega} \|x_n\| = 0\},$$

$$A^\omega = \ell^\infty(\mathbb{N}, A)/c_\omega(A).$$

As usual, we often omit the quotient map from $\ell^\infty(\mathbb{N}, A)$ onto A^ω. We regard A as a C^*-subalgebra of A^ω consisting of the constant sequences, and we set $A_\omega = A^\omega \cap A'$. We denote by α^ω and α_ω the G-actions on A^ω and A_ω induced by α respectively, and we regard $(A^\omega, \alpha^\omega)$ and $(A_\omega, \alpha_\omega)$ as G-C^*-algebras.

Lemma 2.1. *Let G be a finite group, and let (A,α) be a G-C^*-algebra. We assume that A is unital, purely infinite, and simple, and α is outer. Let $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$.*

1. *A^ω is purely infinite and simple, and α^ω is outer.*
2. *If A is a Kirchberg algebra, A_ω is purely infinite and simple, and α_ω is outer.*

Proof. (1) It is easy to show that A^ω is purely infinite and simple, and so it suffices to show that if $\theta \in \mathrm{Aut}(A)$ is outer, so is $\theta^\omega \in \mathrm{Aut}(A^\omega)$ induced by θ. Assume that θ is outer and θ^ω is inner. Then there exists $u = (u_n) \in U(A^\omega)$ satisfying $\mathrm{Ad}\, u = \theta^\omega$. We my assume that u_n is a unitary for all $n \in \mathbb{N}$. Since A is purely infinite, there exist a sequence of nonzero projections $\{p_n\}_{n=1}^\infty$ in A and a sequence of complex numbers $\{c_n\}_{n=1}^\infty$ with $|c_n| = 1$ such that $\{p_n u_n p_n - c_n p_n\}_{n=1}^\infty$ converges to 0. By replacing u_n with $\overline{c_n} u_n$ if necessary. we may assume $c_n = 1$. Since θ is outer, Kishimoto's result [8, Lemma 1.1] shows that there exists a sequence of positive elements $a_n \in p_n A p_n$ with $\|a_n\| = 1$ such that $\{a_n \theta(a_n)\}_{n=1}^\infty$ converging to 0. This is contradiction. Indeed, let $a = (a_n) \in A^\omega$, $p = (p_n) \in A^\omega$. On one hand we have $a\theta^\omega(a) = 0$, and on the other hand we have the following

$$a\theta^\omega(a) = auau^* = apupau^* = apau^* = a^2u^* \neq 0.$$

This shows that θ^ω is outer.

(2) The statement follows from [7, Proposition 3.4] and [13, Lemma 2]. ☐

Now we state two results, which are equivariant versions of well-known results in the classification theory of nuclear C^*-algebras. We omit their proofs, which are verbatim modifications of the original ones. The first one is an equivariant version of [18, Corollary 2.3.4].

Theorem 2.2. *Let G be a finite group, and let (A,α) and (B,β) be unital separable G-C^*-algebras. If there exist $\varphi \in \mathrm{Hom}_G(A,B)$ and $\psi \in \mathrm{Hom}_G(B,A)$ such that $\psi\circ\varphi$ is G-approximately unitarily equivalent to $\mathrm{id}_{(A,\alpha)}$ and $\varphi\circ\psi$ is G-approximately unitarily equivalent to $\mathrm{id}_{(B,\beta)}$, then the two actions α and β are conjugate.*

The following result is an equivariant version of [7, Proposition 3.13].

Theorem 2.3. *Let G be a finite group, and let (A,α), (B,β) be unital separable G-C^* algebras. We regard the minimal tensor product $B\otimes B$ as a G-C^*-algebra with the diagonal action $\alpha\otimes\alpha$, and define $\rho_l,\rho_r \in \mathrm{Hom}_G(B, B\otimes B)$ by $\rho_l(x) = x\otimes 1$ and $\rho_r(x) = 1\otimes x$ for $x\in B$. We assume that ρ_l and ρ_r are G-approximately unitarily equivalent. Then if there exists a unital homomorphism in $\mathrm{Hom}_G(B, A_\omega)$ with $\omega \in \beta\mathbb{N}\setminus\mathbb{N}$, the two G-actions α on A and $\alpha\otimes\beta$ on $A\otimes B$ are conjugate.*

3. Equivariant Rørdam's theorem

The purpose of this section is to show the following theorem, which is an equivariant version of Rørdam's theorem [17, Theorem 3.6],[18, Theorem 5.1.2].

Theorem 3.1. *Let G be a finite group, let α be a quasi-free action of G on $\mathcal{O}_n$ with finite n, and let (B,β) be a G-C^*-algebra. We assume that B is unital, purely infinite, and simple, and β is outer. For two unital G-homomorphisms $\varphi,\psi \in \mathrm{Hom}_G(\mathcal{O}_n, B)$, we set*

$$u_{\psi,\varphi} = \sum_{i=1}^{n} \psi(s_i)\varphi(s_i)^* \in U(B^G).$$

We introduce an endomorphism $\Lambda_\varphi \in \mathrm{End}(B^G)$ by

$$\Lambda_\varphi(x) = \sum_{i=1}^{n} \varphi(s_i)x\varphi(s_i^*), \quad x\in B^G.$$

Then the following conditions are equivalent.

1. *The G-homomorphisms φ and ψ are G-approximately unitarily equivalent.*
2. *The unitary $u_{\psi,\varphi}$ belongs to the closure of $\{v\Lambda_\varphi(v^*)\in U(B^G) \mid v\in B^G\}$.*
3. *The K_1-class $[u_{\psi,\varphi}] \in K_1(B^G)$ is in the image of $1-K_1(\Lambda_\varphi)$.*
4. *The K_1-class $K_1(j_\beta)([u_{\psi,\varphi}]) \in K_1(B\rtimes_\beta G)$ is in the image of $1-K_1(\hat{\beta}_{\pi_\alpha})$.*
5. *The equality $KK_G(\varphi) = KK_G(\psi)$ holds in $KK_G(\mathcal{O}_n, B)$.*

Proof. The equivalence of (1) and (2) follows from $\psi(s_i) = u_{\psi,\varphi}\varphi(s_i)$ and $v\varphi(s_i)v^* = v\Lambda_\varphi(v^*)\varphi(s_i)$.

The implication from (2) to (3) is trivial. In view of the proof of [17, Theorem 3.6], the implication from (3) to (2) is reduced to the Rohlin property of the shift automorphism of $(\bigotimes_{\mathbb{Z}} M_n(\mathbb{C}))^G$, where the G-action of the UHF algebra $\bigotimes_{\mathbb{Z}} M_n(\mathbb{C})$ is the product action $\bigotimes_{\mathbb{Z}} \mathrm{Ad}\,\pi_\alpha(g)$. This follows from Kishimoto's result [9, Theorem 2.1] (see [4, Lemma 5.5] for details).

The equivalence of (3) and (4) follows from Lemma 3.3 below.

We will show the equivalence of (4) and (5) in Appendix as it follows from a rather lengthy computation, and we do not really require it in the rest of this note. □

To show the equivalence of (3) and (4), we first recall the following well-known fact.

Lemma 3.2. *Let A be a C^*-algebra, and let $\{t_i\}_{i=1}^n \subset M(A)$ be isometries with mutually orthogonal ranges. Let $\{e_{ij}\}_{i,j=1}^n$ be the system of matrix units of the matrix algebra $M_n(\mathbb{C})$. We define two homomorphisms $\rho_1 : A \to A \otimes M_n(\mathbb{C})$ and $\rho_2 : A \otimes M_n(\mathbb{C}) \to A$ by $\rho_1(a) = a \otimes e_{11}$, and $\rho_2(a \otimes e_{ij}) = t_i a t_j^*$. Then $K_*(\rho_2)$ is the inverse of $K_*(\rho_1)$.*

Proof. Since $K_*(\rho_1)$ is an isomorphism, it suffices to show that the homomorphism $\rho_2 \circ \rho_1(x) = t_1 x t_1^*$ induces the identity on $K_*(A)$. This follows from a standard argument. □

Recall that we regard $K_*(\hat{\beta}_{\pi_\alpha})$ as an element of $\mathrm{End}(K_*(B \rtimes_\beta G))$ by identifying $K_*(B \rtimes_\beta G)$ with $K_*((B \rtimes_\beta G) \otimes B(\mathcal{H}_n))$.

Lemma 3.3. *With the above notation, we have the equality $K_*(j_\beta) \circ K_*(\Lambda_\varphi) = K_*(\hat{\beta}_{\pi_\alpha}) \circ K_*(j_\beta)$.*

Proof. Identifying $B(\mathcal{H}_n)$ with the linear span of $\{s_i s_j^*\}_{i,j=1}^n$ acting on $\mathcal{H}_n$ by left multiplication, we have

$$\pi_\alpha(g) = \sum_{i=1}^n \alpha_g(s_i)s_i^*.$$

We define a homomorphism $\rho : (B \rtimes_\beta G) \otimes B(\mathcal{H}_n) \to B \rtimes_\beta G$ by $\rho(x \otimes s_i s_j^*) = \varphi(s_i)x\varphi(s_j)^*$, which plays the role of ρ_2 in Lemma 3.2 with $A = B \rtimes_\beta G$ and $t_i = \varphi(s_i)$. Then for $x \in B^G$, we have

$$\begin{aligned} \rho \circ \hat{\beta}_{\pi_\alpha} \circ j_\beta(x) &= \frac{1}{\#G}\sum_{g\in G} \rho \circ \hat{\beta}_{\pi_\alpha}(\lambda_g^\beta x) = \frac{1}{\#G}\sum_{g\in G} \rho(\lambda_g^\beta x \otimes \pi_\alpha(g)) \\ &= \frac{1}{\#G}\sum_{g\in G}\sum_{i=1}^n \rho(\lambda_g^\beta x \otimes \alpha_g(s_i)s_i^*) = \frac{1}{\#G}\sum_{g\in G}\sum_{i=1}^n \varphi(\alpha_g(s_i))\lambda_g^\beta x \varphi(s_i)^* \\ &= \frac{1}{\#G}\sum_{g\in G}\sum_{i=1}^n \lambda_g^\beta \varphi(s_i)x\varphi(s_i)^* = j_\beta \circ \Lambda_\varphi(x), \end{aligned}$$

which proves the statement thanks to Lemma 3.2. □

4. Equivariant Lin-Phillips's theorem

The purpose of this section is to show the following theorem, which is an equivariant version of Lin-Phillips's theorem [10, Theorem 3.3], [18, Proposition 7.2.5].

Theorem 4.1. *Let G be a finite group, let α be a quasi-free action of G on $\mathcal{O}_\infty$, and let (B, β) be a unital G-C^*-algebra. We assume that B is purely infinite and simple, and β is outer. Then any two unital G-homomorphisms in $\mathrm{Hom}_G(\mathcal{O}_\infty, B)$ are G-approximately unitarily equivalent.*

Until the end of this section, we assume that G, $(\mathcal{O}_\infty, \alpha)$ and (B, β) are as in Theorem 4.1. To prove Theorem 4.1, we basically follow Lin-Phillips's strategy based on Theorem 3.1 in place of [17, Theorem 3.6], though we will take a short cut by using a ultraproduct technique.

Let n be a natural number larger than 2, and let $\mathcal{E}_n$ be the Cuntz-Toeplitz algebra, which is the universal C^*-algebra generated by isometries $\{t_i\}_{i=1}^n$ with mutually orthogonal ranges. Note that $p_n = 1 - \sum_{i=1}^n t_i t_i^*$ is a non-zero projection not as in the case of the Cuntz algebras. We denote by $\mathcal{K}_n$ the linear span of $\{t_i\}_{i=1}^n$. Quasi-free actions on $\mathcal{E}_n$ are defined as in the case of the Cuntz algebras. For a quasi-free action γ of G on $\mathcal{E}_n$, we denote by $(\pi_\gamma, \mathcal{K}_n)$ the corresponding unitary representation of G in $\mathcal{K}_n$.

Lemma 4.2. *Let γ be a quasi-free action of G on $\mathcal{E}_n$ with finite n, and let $\varphi, \psi \in \mathrm{Hom}_G(\mathcal{E}_n, B)$ be injective G-homomorphisms, either both unital or both nonunital. If $[\varphi(1)] = [\psi(1)] = 0$ in $K_0(B^G)$, then φ and ψ are G-approximately unitarily equivalent.*

Proof. In the same way as in the proof of Lemma 3.3, we can show

$$K_0(j_\beta)([\varphi(p_n)]) = K_0(j_\beta)([\varphi(1)]) - K_0(\hat{\beta}_{\pi_\gamma}) \circ K_0(j_\beta)([\varphi(1)]) = 0,$$

in $K_0(B \rtimes_\beta G)$. This implies $[\varphi(p_n)] = 0$ in $K_0(B^G)$, and for the same reason, $[\psi(p_n)] = 0$ in $K_0(B^G)$. Thus the statement follows from essentially the same argument as in the proof of [10, Proposition 1.7] by using Theorem 3.1 in place of [17, Theorem 3.6]. □

Since every quasi-free G-action on $\mathcal{O}_\infty$ is the inductive limit of a system of quasi-free actions of the form $\{(\mathcal{E}_{n_k}, \gamma^{(k)})\}_{k=1}^\infty$, we get

Corollary 4.3. *Let $\varphi, \psi \in \mathrm{Hom}_G(\mathcal{O}_\infty, B)$ be either both unital or both nonunital. If $[\varphi(1)] = [\psi(1)] = 0$ in $K_0(B^G)$, then φ and ψ are G-approximately unitarily equivalent.*

Let $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$ be a free ultrafilter, and let $\iota_\omega : \mathcal{O}_\infty \to {\mathcal{O}_\infty}^\omega$ the inclusion map. For $\varphi \in \mathrm{Hom}_G(\mathcal{O}_\infty, B)$, we denote by φ^ω the G-homomorphism in $\mathrm{Hom}_G({\mathcal{O}_\infty}^\omega, B^\omega)$ induced by φ. Then it is easy to show the following three conditions for $\varphi, \psi \in \mathrm{Hom}_G(\mathcal{O}_\infty, B)$ are equivalent:

(1) φ and ψ are G-approximately unitarily equivalent,
(2) $\varphi^\omega \circ \iota_\omega$ and $\psi^\omega \circ \iota_\omega$ are G-approximately unitarily equivalent,
(3) $\varphi^\omega \circ \iota_\omega$ and $\psi^\omega \circ \iota_\omega$ are G-unitarily equivalent.

Note that since G is a finite group, we have $(\mathcal{O}_{\infty\omega})^G = (\mathcal{O}_\infty^G)^\omega \cap \mathcal{O}_\infty'$ and $(B^\omega)^G = (B^G)^\omega$.

Proof of Theorem 4.1. Let $\varphi, \psi \in \mathrm{Hom}_G(\mathcal{O}_\infty, B)$ be unital. Since $\mathcal{O}_\infty$ is a Kirchberg algebra, the ω-central sequence algebra $\mathcal{O}_{\infty\omega}$ is purely infinite and simple. Let H be the kernel of $\alpha : G \to \mathrm{Aut}(\mathcal{O}_\infty)$. Since α is quasi-free, we may regard α as an outer action of G/H, and so α_ω is outer as an action of G/H. This implies that $(\mathcal{O}_{\infty\omega})^G$ is purely infinite and simple.

Choosing three nonzero projections $q_1, q_2, q_3 \in (\mathcal{O}_{\infty\omega})^G$ satisfying $q_1+q_2+q_3=1$ and $[1]=[q_1]=[q_2]=-[q_3]$ in $K_0((\mathcal{O}_{\infty\omega})^G)$, we introduce $\varphi_i, \psi_i \in \mathrm{Hom}_G(\mathcal{O}_\infty, B^\omega)$, $i=1,2,3$, by $\varphi_i(x)=\varphi^\omega(q_i x)$ and $\psi_i(x)=\psi^\omega(q_i x)$ for $x\in\mathcal{O}_\infty$. Then we have

$$\varphi(x)=\varphi_1(x)+\varphi_2(x)+\varphi_3(x), \quad x\in\mathcal{O}_\infty,$$
$$\psi(x)=\psi_1(x)+\psi_2(x)+\psi_3(x), \quad x\in\mathcal{O}_\infty,$$
$$[1]=[\varphi_1(1)]=[\varphi_2(1)]=-[\varphi_3(1)]=[\psi_1(1)]=[\psi_2(1)]=-[\psi_3(1)]\in K_0((B^\omega)^G).$$

Since $[(\varphi_2+\varphi_3)(1)]=[(\psi_2+\psi_3)(1)]=0$ in $K_0((B^\omega)^G)$, Corollary 4.3 implies that there exists a unitary $u\in U((B^\omega)^G)$ satisfying $u(\varphi_2+\varphi_3)(x)u^*=(\psi_2+\psi_3)(x)$ for $x\in\mathcal{O}_\infty$. We set $\varphi_1^u(x)=u\varphi_1(x)u^*$. Then φ_1^u is in $\mathrm{Hom}_G(\mathcal{O}_\infty, B^\omega)$ satisfying $\varphi_1^u(1)=\psi_1(1)$, and $\varphi^\omega\circ\iota_\omega$ and $\varphi_1^u+\psi_2+\psi_3$ are G-approximately unitarily equivalent. Since $(\varphi_1^u+\psi_3)(1)=(\psi_1+\psi_3)(1)$ whose class in $K_0((B^\omega)^G)$ is 0, Corollary 4.3 again implies that there exists a unitary $v\in U((B^\omega)^G)$ satisfying $v\psi_2(1)=\psi_2(1)$ and $v(\varphi_1^u+\psi_3)(x)v^*=(\psi_1+\psi_3)(x)$ for $x\in\mathcal{O}_\infty$. This shows that $vu\varphi(x)u^*v^*=\psi(x)$ for $x\in\mathcal{O}_\infty$, and so φ and ψ are G-approximately unitarily equivalent. □

5. Splitting theorem and Uniqueness theorem

Thanks to Theorem 4.1, we can obtain a G-equivariant version of Kirchberg-Phillips's $\mathcal{O}_\infty$ theorem [7, Theorem 3.15], [18, Theorem 7.2.6].

Theorem 5.1. *Let G be a finite group, and let (A,α) be a G-C^*-algebra. We assume that A is a unital Kirchberg algebra and α is outer. Let $\{\gamma^{(i)}\}_{i=1}^\infty$ be any sequence of quasi-free actions of G on $\mathcal{O}_\infty$. Then (A,α) is conjugate to*

$$(A\otimes\bigotimes_{i=1}^\infty\mathcal{O}_\infty, \alpha\otimes\bigotimes_{i=1}^\infty\gamma^{(i)}).$$

Proof. Let

$$(B,\beta)=(\bigotimes_{i=1}^\infty\mathcal{O}_\infty, \bigotimes_{i=1}^\infty\gamma^{(i)}),$$

and let $\rho_l, \rho_r\in\mathrm{Hom}_G(B, B\otimes B)$ be as in Theorem 2.3. Then Theorem 4.1 implies that ρ_r and ρ_l are G-approximately unitarily equivalent.

To prove the statement applying Theorem 2.3, it suffices to construct a unital embedding of (B,β) in (A_ω,α_ω). For this, it suffices to construct a unital embedding of $(\mathcal{O}_\infty,\gamma^{(i)})$ into (A_ω,α_ω) for each i because the usual trick of taking subsequences can make the embeddings commute with each other. Let γ be the quasi-free action of G on $\mathcal{O}_\infty$ such that $(\pi_\gamma,\mathcal{H}_\infty)$ is unitarily equivalent to the infinite direct sum of the regular representation. Since there is a unital embedding of $(\mathcal{O}_\infty,\gamma^{(i)})$ into $(\mathcal{O}_\infty,\gamma)$, in order to prove the theorem, it only remains to construct a unital embedding of $(\mathcal{O}_\infty,\gamma)$ into (A_ω,α_ω).

Thanks to [13, Lemma 3], we can find a nonzero projection $e\in A_\omega$ satisfying $e\alpha_{\omega g}(e)=0$ for any $g\in G\setminus\{e\}$. We choose an isometry $v\in A_\omega$ satisfying $vv^*\leq e$, and set $s_{0,g}=\alpha_{\omega g}(v)$. Then $\{s_{0,g}\}_{g\in G}$ are isometries in A_ω with mutually orthogonal ranges satisfying $\alpha_{\omega g}(s_{0,h})=s_{0,gh}$. Let $p=\sum_{g\in G}s_{0,g}s_{0,g}^*$, which is a projection in $(A_\omega)^G$. Replacing v if necessary, we may assume that $p\neq 1$. Since $(A_\omega)^G$ is purely infinite and simple, we can find a sequence of partial isometries $\{w_i\}_{i=0}^{\infty}$ in $(A_\omega)^G$ with $w_0 = p$ such that $w_i^* w_i = p$ for all i, and $\{w_i w_i^*\}_{i=0}^{\infty}$ are mutually orthogonal. Let $s_{i,g} = w_i s_{0,g}$. Then $\{s_{i,g}\}_{(i,g)\in\mathbb{N}\times G}$ is a countable family of isometries in A_ω with mutually orthogonal ranges satisfying $\alpha_{\omega g}(s_{i,h}) = s_{i,gh}$. Thus we get the desirable embedding of $(\mathcal{O}_\infty, \gamma)$ into $(A_\omega, \alpha_\omega)$. □

Applying Theorem 5.1 to $A = \mathcal{O}_\infty$ with a faithful quasi-free action α, we obtain

Corollary 5.2. *Any two faithful quasi-free actions of a finite group on $\mathcal{O}_\infty$ are mutually conjugate.*

6. Asymptotic representability

Definition 6.1. An action α of a discrete group G on a unital C^*-algebra A is said to be *asymptotically representable* if there exists a continuous family of unitaries $\{u_g(t)\}_{t\geq 0}$ in $U(A)$ for each $g \in G$ satisfying

$$\lim_{t\to\infty} \|u_g(t) x u_g(t)^* - \alpha_g(x)\| = 0, \quad \forall x \in A,\ \forall g \in G,$$

$$\lim_{t\to\infty} \|u_g(t) u_h(t) - u_{gh}(t)\| = 0, \quad \forall g, h \in G,$$

$$\lim_{t\to\infty} \|\alpha_g(u_h(t)) - u_{ghg^{-1}}(t)\| = 0, \quad \forall g, h \in G.$$

An action α is said to approximately representable if α satisfies the above condition with a sequence $\{u_g(n)\}_{n\in\mathbb{N}}$ in place of the continuous family $\{u_g(t)\}_{t\geq 0}$.

Every asymptotically representable action is approximately representable, but the converse may not be true in general. When G is a finite abelian group, an action α is approximately representable if and only if its dual action has the Rohlin property. When G is a cyclic group of prime power order, approximately representable quasi-free actions on $\mathcal{O}_n$ with finite n are completely characterized in [5], and there exist quasi-free actions that are not approximately representable.

The purpose of this section is to show the following theorem:

Theorem 6.2. *Every quasi-free action of a finite group G on $\mathcal{O}_\infty$ is asymptotically representable.*

It is unlikely that one could show Theorem 6.2 directly from the definition of quasi-free actions. Our proof uses the intertwining argument between two model actions; one is obviously quasi-free, and the other is an infinite tensor product action, that can be shown to be asymptotically representable.

We first introduce the notion of K-trivial embeddings of the group C^*-algebra. We denote by $\{\lambda_g\}_{g\in G}$ the left regular representation of a finite group G. The group C^*-algebra $C^*(G)$ is the linear span of $\{\lambda_g\}_{g\in G}$.

Definition 6.3. Let G be a finite group, and let A be a unital C^*-algebra. An unital injective homomorphism $\rho : C^*(G) \to A$ is said to be a *K-trivial embedding* if $KK(\rho) = KK(C^*(G) \ni \lambda_g \mapsto 1 \in A)$.

For each irreducible representation (π, H_π) of G, we choose an orthonormal basis $\{\xi(\pi)_i\}_{i=1}^{n_\pi}$ of H_π, where $n_\pi = \dim \pi$. We set $\pi(g)_{ij} = \langle \pi(g)\xi(\pi)_i, \xi(\pi)_j\rangle$, and

$$e(\pi)_{ij} = \frac{n_\pi}{\#G} \sum_{g\in G} \overline{\pi(g)_{ij}} \lambda_g.$$

Then $\{e(\pi)_{ij}\}_{1\le i,j\le n_\pi}$ is a system of matrix units, and we have

$$\lambda_g = \sum_{\pi\in\hat{G}} \sum_{i,j=1}^{n_\pi} \pi(g)_{ij} e(\pi)_{ij}.$$

Let $C^*(G)_\pi$ be the linear span of $\{e(\pi)_{ij}\}_{i,j=1}^{\dim \pi}$. Then $C^*(G)_\pi$ is isomorphic to the matrix algebra $M_{n_\pi}(\mathbb{C})$, and $C^*(G)$ has the direct sum decomposition

$$C^*(G) = \bigoplus_{\pi\in\hat{G}} C^*(G)_\pi.$$

Let $\chi_\pi(g) = \mathrm{Tr}(\pi(g))$ be the character of π. Then

$$z(\pi) = \frac{n_\pi}{\#G} \sum_{g\in G} \overline{\chi_\pi(g)} \lambda_g = \sum_{i=1}^{n_\pi} e(\pi)_{ii}$$

is the unit of $C^*(G)_\pi$.

It is easy to show the following lemma:

Lemma 6.4. *Let G be a finite group, and let A, B be unital simple purely infinite C^*-algebras.*

1. *A unital injective homomorphism $\rho : C^*(G) \to A$ is a K-trivial embedding if and only if $[\rho(e(\pi)_{11})] = 0$ in $K_0(A)$ for any nontrivial irreducible representation π. When $K_0(A)$ is torsion free, it is further equivalent to the condition that $[\rho(z(\pi))] = 0$ in $K_0(A)$ for any nontrivial irreducible representation π.*
2. *Any two K-trivial unital embeddings of $C^*(G)$ into A are unitarily equivalent.*
3. *If $\rho : C^*(G) \to A$ and $\sigma : C^*(G) \to B$ are K-trivial embeddings, so is the tensor product embedding $C^*(G) \ni \lambda_g \mapsto \rho(\lambda_g) \otimes \sigma(\lambda_g) \in A \otimes B$.*

We now construct a K-trivial embedding of $C^*(G)$ into $\mathcal{O}_\infty$. We fix a nonzero projection $p \in \mathcal{O}_\infty$ with $[p] = 0$ in $K_0(\mathcal{O}_\infty)$, and fix unital embeddings

$$B(\ell^2(G)) \subset \mathcal{O}_2 \subset p\mathcal{O}_\infty p.$$

We denote by $\sigma_0 : C^*(G) \to p\mathcal{O}_\infty p$ the resulting embedding, and set $u_g = \sigma_0(\lambda_g) + 1 - p$. Then $\sigma : C^*(G) \ni \lambda_g \mapsto u_g \in \mathcal{O}_\infty$ is a K-trivial embedding of $C^*(G)$ into $\mathcal{O}_\infty$.

Using $\{u_g\}_{g\in G}$, we introduce a G-C^*-algebra (A, α) by

$$(A, \alpha_g) = \bigotimes_{k=1}^{\infty} (\mathcal{O}_\infty, \mathrm{Ad}\, u_g).$$

More precisely, we set

$$A_n = \bigotimes_{k=1}^{n} \mathcal{O}_\infty, \quad u_g^{(n)} = \bigotimes_{k=1}^{n} u_g,$$

and $\alpha_g^{(n)} = \mathrm{Ad}\, u_g^{(n)}$. Then (A, α) is the inductive limit of the system $\{(A_n, \alpha^{(n)})\}_{n=1}^\infty$ with the embedding $\iota_n : A_n \ni x \mapsto x \otimes 1 \in A_{n+1}$. The C^*-algebra A is isomorphic to $\mathcal{O}_\infty$, and the action α is outer.

Lemma 6.5. *Let the notation be as above.*

(1) *The action α is asymptotically representable.*
(2) *The embedding $\iota_\alpha : C^*(G) \ni \lambda_g \mapsto \lambda_g^\alpha \in A \rtimes_\alpha G$ gives KK-equivalence.*

Proof. (1) It suffices to construct a homotopy $\{v_g(t)\}_{t\in[0,1]}$ of unitary representations of G in A_3 satisfying $v_g(0) = u_g \otimes 1 \otimes 1$, $v_g(1) = u_g^{(2)} \otimes 1$, and $\alpha_g^{(3)}(v_h(t)) = v_{ghg^{-1}}(t)$. Since $\{u_g \otimes 1\}_{g\in G}$, $\{u_g^{(2)}\}_{g\in G}$, and $\{1 \otimes u_g\}_{g\in G}$ give K-trivial embeddings of $C^*(G)$ into A_2, there exist unitaries $w_1, w_2 \in U(A_2)$ satisfying $w_1(u_g \otimes 1)w_1^* = w_2(1 \otimes u_g)w_2^* = u_g^{(2)}$. Let $w = (w_1 \otimes 1)(1 \otimes w_2^*)$, which is a unitary in $A_3^G = A_3 \cap \{u_g^{(3)}\}'_{g\in G}$ satisfying $w(u_g \otimes 1 \otimes 1)w^* = u_g^{(2)} \otimes 1$. Since A_3^G is isomorphic to a finite direct sum of C^*-algebras Morita equivalent to $\mathcal{O}_\infty$, there exists a homotopy $\{w(t)\}_{t\in[0,1]}$ in $U(A_3^G)$ with $w(0) = 1$ and $w(1) = w$. Thus $v_g(t) = w(t)(u_g \otimes 1 \otimes 1)w(t)^*$ gives the desired homotopy.

(2) We identify $B_n = A_n \rtimes_{\alpha^{(n)}} G$ with the C^*-subalgebra of $A \rtimes_\alpha G$ generated by A_n and $\{\lambda_g^\alpha\}_{g\in G}$, and we denote by $\iota'_n : B_n \to B_{n+1}$ the embedding map. Then $A \rtimes_\alpha G$ is the inductive limit of the system $\{B_n\}_{n=1}^\infty$. Let $\iota_\alpha^{(n)} : C^*(G) \ni \lambda_g \mapsto \lambda_g^\alpha \in B_n$. Since we have $\iota'_n \circ \iota_\alpha^{(n)} = \iota_\alpha^{(n+1)}$, in order to prove the statement it suffices to show that $\iota_\alpha^{(n)}$ induces isomorphisms of the K-groups for every n.

Since $\alpha^{(n)}$ is inner, there exists an isomorphism $\theta_n : B_n \to A_n \otimes C^*(G)$ given by $\theta_n(a) = a \otimes 1$ for $a \in A_n$ and $\theta_n(\lambda_g^\alpha) = u_g^{(n)} \otimes \lambda_g$. Thus all we have to show is that the map $\theta_n \circ \iota_\alpha^{(n)} : C^*(G) \ni \lambda_g \mapsto u_g^{(n)} \otimes \lambda_g \in A_n \otimes C^*(G)$ induces isomorphisms of the K-groups. This follows from that fact that A_n is isomorphic to $\mathcal{O}_\infty$ and $\{u_g^{(n)}\}_{g\in G}$ gives an K-trivial embedding of $C^*(G)$ into A_n. □

Lemma 6.6. *For the G-C^*-algebra (A, α) as constructed above, any unital $\varphi \in$* $\mathrm{Hom}_G(A, A)$ *is G-asymptotically unitarily equivalent to* id.

Proof. Let $B = A \rtimes_\alpha G$, and let $\hat{\alpha} : B \to B \otimes C^*(G)$ be the dual coaction of α. Then φ extends to a unital endomorphism $\tilde{\varphi}$ in $\mathrm{End}(B)$ with $\tilde{\varphi}(\lambda_g^\alpha) = \lambda_g^\alpha$, which satisfies $\hat{\alpha} \circ \tilde{\varphi} = (\tilde{\varphi} \otimes \mathrm{id}_{C^*(G)}) \circ \hat{\alpha}$. By Lemma 6.5,(2), we have $KK(\tilde{\varphi}) = KK(\mathrm{id}_B)$. Thus Lemma 6.5,(1) and [6, Theorem 4.8] imply that there exists a continuous family of unitaries $\{u(t)\}_{t\geq 0}$ in A satisfying

$$\lim_{t\to\infty} \|u(t)xu(t)^* - \tilde{\varphi}(x)\| = 0, \quad \forall x \in B.$$

Setting $x = \lambda_g^\alpha$, we know that $\{\alpha_g(u(t)) - u(t)\}_{t\geq 0}$ converges to 1. Since G is a finite group, there exists a conditional expectation from A onto A^G, and we can construct a continuous family of unitaries $\{\tilde{u}(t)\}_{t\geq 0}$ in A^G such that $\{u(t) - \tilde{u}(t)\}_{t\geq 0}$ converges to 0 by a standard perturbation argument. Therefore φ and id are G-asymptotically unitarily equivalent. □

Proof of Theorem 6.2. Let γ be a faithful quasi-free G-action on $\mathcal{O}_\infty$. Thanks to Corollary 5.2, we may assume that $\mathcal{O}_\infty$ has the canonical generators $\{s_i\}_{i\in J}$ with $G\subset J$ satisfying $\gamma_g(s_h)=s_{gh}$. Since α is asymptotically representable, it suffices to show that α and γ are conjugate. Thanks to Theorem 5.1, the action α is conjugate to $\alpha\otimes\gamma$, and so there exists a unital embedding of $(\mathcal{O}_\infty,\gamma)$ into (A,α). Thus if there exists a unital embedding of (A,α) into $(\mathcal{O}_\infty,\gamma)$, Theorem 2.2, Theorem 4.1, and Lemma 6.6 imply that α and γ are conjugate. Since γ is conjugate to the infinite tensor product of its copies thanks to Theorem 5.1 again, all we have to show is that there exists a unital embedding of $(\mathcal{O}_\infty,\mathrm{Ad}\,u.)$ into $(\mathcal{O}_\infty,\gamma)$.

We denote by $\mathcal{O}_\infty^\gamma$ the fixed point subalgebra of $\mathcal{O}_\infty$ under the G-action γ. Since $\mathcal{O}_\infty^\gamma$ is purely infinite and simple, we can choose a nonzero projection $q_0\in\mathcal{O}_\infty^G$ with $[q_0]=0$ in $K_0(\mathcal{O}_\infty^\gamma)$. We set $q_1=\sum_{g\in G}s_gq_0s_g^*$. A similar argument as in the proof of Lemma 3.3 implies that $[q_1]=0$ in $K_0(\mathcal{O}_\infty^\gamma)$. We set

$$v_g=\sum_{h\in G}s_{gh}q_0s_h^*+1-q_1.$$

Then $\{v_g\}_{g\in G}$ is a unitary representation of G in $\mathcal{O}_\infty$ satisfying $\gamma_g(v_h)=v_{ghg^{-1}}$, and so $\{v_g^*\}_{g\in G}$ is a γ-cocycle. We show that this is a coboundary by using [4, Remark 2.6]. Indeed, we have

$$\begin{aligned}(6.1)\qquad \frac{1}{\#G}\sum_{g\in G}v_g^*\lambda_g^\gamma &= (1-q_1)e_\gamma+\frac{1}{\#G}\sum_{g\in G}\sum_{h\in G}s_hq_0s_{gh}^*\lambda_g^\gamma\\ &= (1-q_1)e_\gamma+\frac{1}{\#G}\sum_{g\in G}\sum_{h\in G}s_hq_0\lambda_g^\gamma s_h^*=(1-q_1)e_\gamma+\sum_{h\in G}s_hq_0e_\gamma s_h^*.\end{aligned}$$

This means that the class of this projection in $K_0(\mathcal{O}_\infty\rtimes_\gamma G)$ is

$$[(1-q_1)e_\gamma]+\#G[q_0e_\gamma]=[e_\gamma],$$

which implies that $\{v_g^*\}_{g\in G}$ is a coboundary. Thus there exists a unitary $v\in\mathcal{O}_\infty$ satisfying $v_g^*=v\gamma_g(v^*)$.

We set $w_g=v^*v_gv$, and claim that $\{w_g\}_{g\in G}$ gives a K-trivial embedding of $C^*(G)$ into $\mathcal{O}_\infty^\gamma$. Indeed,

$$\gamma_g(w_h)=\gamma_g(v^*)\gamma_h(v_h)\gamma_g(v)=v^*v_g^*v_{ghg^{-1}}v_gv=w_h,$$

which shows $w_g\in\mathcal{O}_\infty^\gamma$. Let $\rho:C^*(G)\ni\lambda_g\mapsto w_g\in\mathcal{O}_\infty^\gamma$. Thanks to Lemma 6.4,(1), in order to prove the claim it suffices to show that $[\rho(e(\pi)_{11})]=0$ in $K_0(\mathcal{O}_\infty^\gamma)$ for any nontrivial irreducible representation (π,H_π) of G. Indeed, we have

$$\begin{aligned}K_0(j_\gamma)([\rho(e(\pi)_{11})]) &= [\frac{n_\pi}{\#G^2}\sum_{g,h\in G}\overline{\pi(h)_{11}}\lambda_g^\gamma w_h]=[\frac{n_\pi}{\#G^2}\sum_{g,h\in G}\overline{\pi(h)_{11}}\lambda_g^\gamma v^*v_hv]\\ &= [\frac{n_\pi}{\#G^2}\sum_{g,h\in G}\overline{\pi(h)_{11}}\gamma_g(v^*)v_{ghg^{-1}}\lambda_g^\gamma v]=[\frac{n_\pi}{\#G^2}\sum_{g,h\in G}\overline{\pi(h)_{11}}v^*v_g^*v_{ghg^{-1}}\lambda_g^\gamma v]\\ &= [\frac{n_\pi}{\#G^2}\sum_{g,h\in G}\overline{\pi(h)_{11}}v_{hg^{-1}}\lambda_g^\gamma]=[\frac{n_\pi}{\#G^2}\sum_{g,h\in G}\overline{\pi(h)_{11}}v_hv_g^*\lambda_g^\gamma].\end{aligned}$$

Let $\rho_0 : C^*(G) \ni \lambda_g \mapsto v_g \in \mathcal{O}_\infty$. Equation (6.1) implies that this is equal to

$$[\rho_0(e(\pi)_{11})\sum_{k\in G} s_k q_0 e_\gamma s_k^*] = n_\pi[q_0 e_\gamma] = 0.$$

Thus the claim is shown.

We choose a unital embedding $\mu_0 : \mathcal{O}_\infty \to \mathcal{O}_\infty^\gamma$. Since both $\{\mu_0(u_g)\}_{g\in G}$ and $\{w_g\}_{g\in G}$ give K-trivial embeddings of $C^*(G)$ into $\mathcal{O}_\infty^\gamma$, Lemma 6.4,(2) shows that we may assume $\mu_0(u_g) = w_g$ by replacing μ_0 if necessary. Let $\mu(x) = v\mu_0(x)v^*$. Then

$$\begin{aligned}\gamma_g \circ \mu(x) &= \gamma_g(v)\mu_0(x)\gamma_g(v^*) = v_g v\mu_0(x)v^*v_g^* = vw_g\mu_0(x)w_g^*v^* \\ &= v\mu_0(u_g x u_g^*)v^* = \mu \circ \operatorname{Ad} u_g(x).\end{aligned}$$

Thus μ is the desired embedding of $(\mathcal{O}_\infty, \operatorname{Ad} u_\cdot)$ into $(\mathcal{O}_\infty, \gamma)$. □

From Theorem 6.2 and Lemma 6.6, we get

Corollary 6.7. *Let G be a finite group, and let γ be a quasi-free action of G on $\mathcal{O}_\infty$. Then any unital $\varphi \in \operatorname{Hom}_G(\mathcal{O}_\infty, \mathcal{O}_\infty)$ is G-asymptotically unitarily equivalent to* id.

7. Equivariant Rørdam group

Let A and B be simple C^*-algebras. For simplicity we assume that A and B are unital. Following Rørdam [18, p.40], we denote by $H(A, B)$ the set of the approximately unitary equivalence classes of nonzero homomorphisms from A into $B \otimes \mathbb{K}$. Choosing two isometries s_1 and s_2 satisfying the $\mathcal{O}_2$ relation in $M(B \otimes \mathbb{K})$, we can define the direct sum $[\varphi] \oplus [\psi]$ of two classes $[\varphi]$ and $[\psi]$ in $H(A, B)$ to be the class of the homomorphism

$$A \ni x \mapsto s_1\varphi(x)s_1^* + s_2\psi(x)s_2^* \in B \otimes \mathbb{K}.$$

This makes $H(A, B)$ a semigroup. When A is a separable simple nuclear C^*-algebra and B is a Kirchberg algebra, the Rørdam semigroup $H(A, B)$ is in fact a group. Moreover, if A satisfies the universal coefficient theorem, it is isomorphic to $KL(A, B)$, a certain quotient of $KK(A, B)$.

Let G be a finite group, and let α and β be outer G-actions on A and B respectively. We equip $B\otimes\mathbb{K}$ with a G-C^*-algebra structure by the diagonal action $\beta_g^s = \beta_g \otimes \operatorname{Ad} u_g$, where $\{u_g\}$ is a countable infinite direct sum of the regular representation of G. Then we can introduce an equivariant version $H_G(A, B)$ as the set of the G-approximately equivalence classes of nonzero G-homomorphisms in $\operatorname{Hom}_G(A, B \otimes \mathbb{K})$.

Theorem 7.1. *Let (A, α) and (B, β) be unital G-C^*-algebras with outer actions α and β. We assume that A is separable, simple, and nuclear, and B is a Kirchberg algebra. Then $H_G(A, B)$ is a group.*

Let (A, α) and (B, β) be as above. We say that $\varphi \in \operatorname{Hom}_G(A, B)$ is $\mathcal{O}_2$-*absorbing* if there exists $\varphi' \in \operatorname{Hom}_G(A \otimes \mathcal{O}_2, B)$ with $\varphi = \varphi' \circ \iota_A$, where $A \otimes \mathcal{O}_2$ is equipped with the G-action $\alpha \otimes \mathrm{id}_{\mathcal{O}_2}$, and $\iota_A : A \ni x \mapsto x \otimes 1 \in A \otimes \mathcal{O}_2$ is the inclusion map. We say that $\varphi \in \operatorname{Hom}_G(A, B)$ is $\mathcal{O}_\infty$-*absorbing* if there exists a unital embedding of $\mathcal{O}_\infty$ in $(\varphi(1)B^G\varphi(1)) \cap \varphi(A)'$.

The proof of Theorem 7.1 follows from essentially the same argument as in [18, Lemma 8.2.5] with the following lemma.

Lemma 7.2. *Let the notation be as above.*

(1) *Let $\varphi, \psi \in \mathrm{Hom}_G(A, B)$ be $\mathcal{O}_2$-absorbing G-homomorphisms, either both unital or both nonunital. Then φ and ψ are G-approximately unitarily equivalent.*
(2) *Any element in $\mathrm{Hom}_G(A, B)$ is G-approximately unitarily equivalent to a $\mathcal{O}_\infty$-absorbing one in $\mathrm{Hom}_G(A, B)$.*

Proof. (1) When φ and ψ are nonunital, the two projections $\varphi(1)$ and $\psi(1)$ are equivalent in B^G, and we may assume $\varphi(1) = \psi(1)$. Replacing B with $\varphi(1)B\varphi(1)$, we may assume that φ and ψ are unital.

Let γ be a faithful quasi-free action of G on $\mathcal{O}_\infty$. Since $(A \otimes \mathcal{O}_2, \alpha \otimes \mathrm{id}_{\mathcal{O}_2})$ is conjugate to $(\mathcal{O}_\infty \otimes \mathcal{O}_2, \gamma \otimes \mathrm{id}_{\mathcal{O}_2})$ thanks to [4, Corollary 4.3], it suffices to show that any unital $\varphi, \psi \in \mathrm{Hom}_G(\mathcal{O}_\infty \otimes \mathcal{O}_2, B)$ are G-approximately unitarily equivalent. Theorem 4.1 implies that there exists $u \in U((B^\omega)^G)$ satisfying $u\varphi(x\otimes 1)u^* = \psi(x\otimes 1)$ for any $x \in \mathcal{O}_\infty$, where $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$ is a free ultrafilter. Let $D = (B^\omega)^G \cap \psi(\mathcal{O}_\infty \otimes 1)'$. Then it suffices to show that the two unital homomorphisms $\rho, \sigma \in \mathrm{Hom}(\mathcal{O}_2, D)$ defined by $\rho(y) = u\varphi(1 \otimes y)u^*$, $\sigma(y) = \psi(1 \otimes y)$ for $y \in \mathcal{O}_2$, are approximately unitarily equivalent. Indeed, since $(B^\omega)^G \cap B' = (B_\omega)^G$ is purely infinite and simple, for any separable C^*-subalgebra C of D there exists a unital embedding of $\mathcal{O}_\infty$ in $D \cap C'$. Thus essentially the same proof of [15, Lemma 2.1.7] shows that $\mathrm{cel}(D)$ is finite (see [15, Lemma 2.1.1] for the definition). Therefore ρ and σ are approximately unitarily equivalent thanks to [17, Theorem 3.6].

(2) Since (B, β) is conjugate to $(B\otimes\mathcal{O}_\infty, \beta\otimes\mathrm{id}_{\mathcal{O}_\infty})$ thanks to [4, Corollary 2.10], the statement follows from the same argument as in the proof of [18, Lemma 8.2.5,(i)]. □

Remark 7.3. There are two natural homomorphisms

$$\mu : H_G(A, B) \to H(A, B),$$
$$\nu : H_G(A, B) \to H(A \rtimes_\alpha G, B \rtimes_\beta G).$$

The first one is the forgetful functor. Every $\varphi \in \mathrm{Hom}_G(A, B)$ extends to $\tilde{\varphi} \in \mathrm{Hom}(A \rtimes_\alpha G, B \rtimes_\beta G)$ by $\tilde{\varphi}(\lambda_g^\alpha) = \lambda_g^\beta$, and the second one is given by associating $[\tilde{\varphi}] \in H(A\rtimes_\alpha, G, B \rtimes_\beta G)$ with $[\varphi] \in H_G(A, B)$. The following hold for the two maps (see [6, Section 4] for more general treatment):

(1) If β has the Rohlin property, then μ is injective, and the image of μ is
$$\{[\rho] \in H(A, B)|\ [\beta_g^s \circ \rho] = [\rho \circ \alpha_g],\ \forall g \in G\}.$$
(2) If β is approximately representable, then ν is injective, and the image of ν is
$$\{[\rho] \in H(A \rtimes_\alpha G, B \rtimes_\beta G)|\ [\hat{\beta}^s \circ \rho] = [(\rho \otimes \mathrm{id}_{C^*(G)}) \circ \hat{\alpha}]\}.$$

Remark 7.4. Let $\hat{H}_G(A, B)$ be the set of the G-asymptotically equivalence classes of nonzero G-homomorphisms in $\mathrm{Hom}_G(A, B \otimes \mathbb{K})$. It is tempting to conjecture that the natural map from $\hat{H}_G(A, B)$ to the equivariant KK-group $KK_G(A, B)$ is an isomorphism, as it is the case for trivial G (see [15]).

8. Appendix

In this appendix, we show the equivalence of (4) and (5) in Theorem 3.1. Since our argument works for a compact group G, we assume that G is compact in what follows. Our proof is new even for trivial G. Let α be a quasi-free action of G on $\mathcal{O}_n$ with finite n, and let (B,β) be a unital G-C^*-algebra. Now the definition of the projection $e_\beta \in B \rtimes_\beta G$ should be modified to $e_\beta = \int_G \lambda_g^\beta dg$, where dg is the normalized Haar measure of G. For two unital $\varphi, \psi \in \mathrm{Hom}_G(\mathcal{O}_n, B)$, we define $u_{\varphi,\psi} \in U(B^G)$ as in Theorem 3.1.

Let $\mathcal{E}_n$ be the Cuntz-Toeplitz algebra with the canonical generators $\{t_i\}_{i=1}^n$. We denote by q_n the surjection $q_n : \mathcal{E}_n \to \mathcal{O}_n$ sending t_i to s_i for $i = 1, 2, \cdots, n$. Then the kernel J_n of q_n is the ideal generated by $p_n = 1 - \sum_{i=1}^n t_i t_i^*$, and is isomorphic to the compact operators $\mathbb{K}$. We denote by $i_n : J_n \to \mathcal{E}_n$ the inclusion map. Since $\mathcal{O}_n$ is nuclear, the exact sequence

$$0 \longrightarrow J_n \xrightarrow{i_n} \mathcal{E}_n \xrightarrow{q_n} \mathcal{O}_n \longrightarrow 0, \tag{8.1}$$

is semisplit, that is, there exists a unital completely positive lifting $l_n : \mathcal{O}_n \to \mathcal{E}_n$ of q_n. We denote by $\tilde{\alpha}$ the quasi-free action of G on $\mathcal{E}_n$ that is a lift of α. By replacing l_n with l_n^G given by

$$l_n^G(x) = \int_G \tilde{\alpha}_g \circ l_n \circ \alpha_{g^{-1}}(x) dg, \quad x \in \mathcal{O}_n,$$

we see that (8.1) is a semisplit exact sequence of G-C^*-algebras. Thus it induces the following 6-term exact sequence of KK_G-groups:

$$\begin{array}{ccccc} KK_G^0(J_n, B) & \xleftarrow{i_n^*} & KK_G^0(\mathcal{E}_n, B) & \xleftarrow{q_n^*} & KK_G^0(\mathcal{O}_n, B) \\ \delta \downarrow & & & & \uparrow \delta \\ KK_G^1(\mathcal{O}_n, B) & \xrightarrow[q_n^*]{} & KK_G^1(\mathcal{E}_n, B) & \xrightarrow[i_n^*]{} & KK_G^1(J_n, B) \end{array}.$$

Let H_n be the n-dimensional Hilbert space $\mathbb{C}^n$ with the canonical orthonormal basis $\{e_i\}_{i=1}^n$. We regard H_n as a $\mathbb{C} - \mathbb{C}$ bimodule with a G-action given by π_α. We denote by $\mathcal{F}_n$ the full Fock space

$$\mathcal{F}_n = \bigoplus_{m=0}^{\infty} H_n^{\otimes m},$$

with a unitary representation $\pi_{\mathcal{F}_n}$ of G coming from π_α. Identifying t_i with the creation operator of e_i acting on $\mathcal{F}_n$, we regard $\mathcal{E}_n$ as a C^*-subalgebra of $\mathbb{B}(\mathcal{F}_n)$. With this identification, we have $J_n = \mathbb{K}(\mathcal{F}_n)$, and p_n is the projection onto $H_n^{\otimes 0}$. We regard $\mathcal{F}_n$ as $J_n - \mathbb{C}$ bimodule, which gives the KK_G-equivalence of J_n and $\mathbb{C}$.

Pimsner's computation [16, Theorem 4.9] yields the following 6-term exact sequence:

$$\begin{array}{ccccc}
KK^0_G(\mathbb{C},B) & \xleftarrow{1-[H_n]\hat{\otimes}} & KK^0_G(\mathbb{C},B) & \longleftarrow & KK^0_G(\mathcal{O}_n,B) \\
{\scriptstyle \delta'}\downarrow & & & & \uparrow{\scriptstyle \delta'} \\
KK^1_G(\mathcal{O}_n,B) & \longrightarrow & KK^1_G(\mathbb{C},B) & \xrightarrow[1-[H_n]\hat{\otimes}]{} & KK^1_G(\mathbb{C},B)
\end{array},$$

where $[H_n]\hat{\otimes}$ denote the left multiplication of the class $[H_n]\in KK_G(\mathbb{C},\mathbb{C})$. Note that the identification of $KK^*_G(J_n,B)$ and $KK^*_G(\mathbb{C},B)$ is given by $[\mathcal{F}_n]\in KK_G(J_n,\mathbb{C})$, and so $\delta'=\delta\circ(\mathcal{F}_n\hat{\otimes})$.

With the Green-Julg isomorphism $h_*: KK^*_G(\mathbb{C},B)\to K_*(B\rtimes_\beta G)$ ([1, Theorem 11.7.1]), we have the commutative diagram

$$\begin{array}{ccc}
KK^*_G(\mathbb{C},B) & \xrightarrow{[\mathcal{H}_n]\hat{\otimes}} & KK^*_G(\mathbb{C},B) \\
{\scriptstyle h_*}\downarrow & & \downarrow{\scriptstyle h_*} \\
K_*(B\rtimes_\beta G) & \xrightarrow{K_*(\hat{\beta}_{\pi_\alpha})} & K_*(B\rtimes_\beta G)
\end{array},$$

and so we get the following 6-term exact sequence

$$\begin{array}{ccccc}
K_0(B\rtimes_\beta G) & \xleftarrow{1-K_0(\hat{\beta}_{\pi_\alpha})} & K_0(B\rtimes_\beta G) & \longleftarrow & KK^0_G(\mathcal{O}_n,B) \\
{\scriptstyle \delta''}\downarrow & & & & \uparrow{\scriptstyle \delta''} \\
KK^1_G(\mathcal{O}_n,B) & \longrightarrow & K_1(B\rtimes_\beta G) & \xrightarrow[1-K_1(\hat{\beta}_{\pi_\alpha})]{} & K_1(B\rtimes_\beta G)
\end{array},$$

with $\delta''=\delta\circ([\mathcal{F}_n]\hat{\otimes})\circ h_*^{-1}$. Now the proof of the equivalence of (4) and (5) in Theorem 3.1 follows from the next theorem.

Theorem 8.1. *With the above notation, we have*

$$\delta''(K_1(j_\beta)([u_{\psi,\varphi}]))=KK_G(\psi)-KK_G(\varphi).$$

The proof of Theorem 8.1 follows from a standard and rather tedious computation below. In what follows, we freely use the notation in Blackadar's book [1] for KK-theory. We regard $\mathbb{C}_1$, $C=C_0[0,1)$, and $S=C_0(0,1)$ as G-C^*-algebras with trivial G-actions.

[1, Theorem 19.5.7] shows that δ' is given by the left multiplication of the class δ_{q_n} of the extension (8.1) in $KK^1_G(\mathcal{O}_n,\mathbb{C})=KK_G(\mathcal{O}_n,\mathbb{C}_1)$, whose Kasparov module $(E_1,\phi_1,F_1)\in\mathbb{E}_G(\mathcal{O}_n,\mathbb{C}_1)$ is given as follows. By the Stinespring dilation of the G-equivariant lifting $l^G_n:\mathcal{O}_n\to\mathcal{E}_n\subset\mathbb{B}(\mathcal{F}_n)$, we get a Hilbert space H including $\mathcal{F}_n$, with a unitary representation π_H of G extending $\pi_{\mathcal{F}_n}$, satisfying the following condition: there is a unital G-homomorphism $\Phi:\mathcal{O}_n\to\mathbb{B}(H)$ such that if P is the projection from H onto $\mathcal{F}_n$, then $l^G_n(x)=P\Phi(x)P$ for any $x\in\mathcal{O}_n$. Now we have

$$(E_1,\phi_1,F_1)=(H\hat{\otimes}\mathbb{C}_1,\Phi\hat{\otimes}1,(2P-1)\hat{\otimes}\varepsilon),$$

where $\varepsilon=1\oplus-1$ is the generator of $\mathbb{C}_1\cong C^*(\mathbb{Z}_2)$.

Let $z(t) = e^{2\pi i t}$, and let θ be the element in $\mathrm{Hom}_G(C_0(0,1), B)$ determined by $\theta(z-1) = u_{\psi,\varphi} - 1$. Then $h_1^{-1} \circ K_1(j_\beta)([u_{\psi,\varphi}])$ is given by

$$KK_G(\theta) \in KK_G(C_0(0,1), B) \cong KK_G(\mathbb{C}_1, B).$$

In order to compute the Kasparov product of $\delta_{q_n} \in KK_G(\mathcal{O}_n, \mathbb{C}_1)$ and $KK_G(\theta) \in KK_G(S, B)$, we need to identify $KK_G(S, B)$ with $KK_G(\mathbb{C}_1, B)$ explicitly, and we need the invertible element $\boldsymbol{x} \in KK_G(\mathbb{C}_1, S)$ defined in [1, Section 19.2]. By the extension

$$0 \longrightarrow S \longrightarrow C \longrightarrow \mathbb{C} \longrightarrow 0,$$

we get an invertible element in $KK_G(\mathbb{C}, S\hat{\otimes}\mathbb{C}_1)$. Then $\boldsymbol{x}$ is the image of this element by the isomorphism

$$\begin{aligned} \tau_{\mathbb{C}_1} : KK_G(\mathbb{C}, S\hat{\otimes}\mathbb{C}_1) &\to KK_G(\mathbb{C}\hat{\otimes}\mathbb{C}_1, S\hat{\otimes}\mathbb{C}_1\hat{\otimes}\mathbb{C}_1) \\ &= KK_G(\mathbb{C}_1, S\hat{\otimes}M_2(\mathbb{C})) = KK_G(\mathbb{C}_1, S). \end{aligned}$$

For the identification of $\mathbb{C}_1\hat{\otimes}\mathbb{C}_1$ and $M_2(\mathbb{C})$ with standard even grading, we follow the convention in the proof of [1, Theorem 18.10.12] (our computation really depends on it). A direct computation shows that $\boldsymbol{x}$ is given by the Kasparov module $(E_2, \phi_2, F_2) \in \mathbb{E}_G(\mathbb{C}_1, S)$ with $E_2 = \mathbb{C}^2\hat{\otimes}(S \oplus S)$,

$$F_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\phi_2(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes Q, \quad \phi_2(\varepsilon) = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \otimes Q,$$

where the projection $Q \in M_2(M(S))$ is given by

$$Q(t) = \begin{pmatrix} 1-t & \sqrt{t(1-t)} \\ \sqrt{t(1-t)} & t \end{pmatrix},$$

and the grading of E_2 is given by

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

With this $\boldsymbol{x}$, we have

$$\delta''(K_1(j_\beta)([u_{\psi,\varphi}])) = \delta_{q_n}\hat{\otimes}_{\mathbb{C}_1}\boldsymbol{x}\hat{\otimes}_S KK_G(\theta) = \theta_*(\delta_{q_n}\hat{\otimes}_{\mathbb{C}_1}\boldsymbol{x}),$$

and so our task now is to compute $\delta_{q_n}\hat{\otimes}_{\mathbb{C}_1}\boldsymbol{x}$ explicitly.

Lemma 8.2. *The class $\delta_{q_n}\hat{\otimes}_{\mathbb{C}_1}\boldsymbol{x} \in KK_G(\mathcal{O}_n, S)$ is given by the quasi-homomorphism $\rho = (\rho^{(0)}, \rho^{(1)})$ from $\mathcal{O}_n$ to S such that $\rho^{(0)}$ and $\rho^{(1)}$ are unital homomorphisms from $\mathcal{O}_n$ to $\mathbb{B}(H\hat{\otimes}S)$ with $\rho^{(0)}(x) = \Phi(x)\hat{\otimes}1$ and*

$$\rho^{(1)}(x) = \big(P\hat{\otimes}1 + (1-P)\hat{\otimes}z\big)(\Phi(x)\hat{\otimes}1)\big(P\hat{\otimes}1 + (1-P)\hat{\otimes}z\big)^*.$$

Proof. We regard $H\hat{\otimes}S$ as a $\mathcal{O}_n$-S bimodule with trivial grading, and we set $E = (H\hat{\otimes}S)\oplus(H\hat{\otimes}S)^{op}$. We denote by $\Psi : S \to Q(S\oplus S)$ a Hilbert S-module isomorphism given by

$$\Phi(f)(t) = (\sqrt{1-t}f(t), \sqrt{t}f(t)).$$

Then $E_1\hat{\otimes}_{\mathbb{C}_1}E_2$ is identified with E via the identification of $(\xi_1\hat{\otimes}f_1, \xi_2\hat{\otimes}f_2) \in E$ and

$$\xi_1\hat{\otimes}1\hat{\otimes}_{\mathbb{C}_1}(1,0)\hat{\otimes}\Psi(f_1) + \xi_2\hat{\otimes}1\hat{\otimes}_{\mathbb{C}_1}(0,1)\hat{\otimes}\Psi(f_2) \in H\hat{\otimes}\mathbb{C}_1\hat{\otimes}_{\mathbb{C}_1}\mathbb{C}^2\hat{\otimes}(S\oplus S).$$

We claim that $\delta_{q_n}\hat{\otimes}_{\mathbb{C}_1}\boldsymbol{x}$ is given by the Kasparov module $(E,\phi,F) \in \mathbb{E}_G(\mathcal{O}_n, S)$ with

$$\phi(x) = \mathrm{diag}(\Phi(x)\otimes 1, \Phi(x)\otimes 1),$$

$$F = \begin{pmatrix} 0 & 1\hat{\otimes}c \\ 1\hat{\otimes}c & 0 \end{pmatrix} + \begin{pmatrix} 0 & -i(2P-1)\hat{\otimes}s \\ i(2P-1)\hat{\otimes}s & 0 \end{pmatrix},$$

where $c(t) = \cos(\pi t)$, $s(t) = \sin(\pi t)$. Indeed, it is easy to show that (E,ϕ,F) is a Kasparov module, and the graded commutator $[F_1\hat{\otimes}1_{E_2}, F]$ is positive. We show that F is a F_2-connection (see [1, Definition 18.3.1] for the definition). Let $\xi \in H$, $x = (x_1,x_2) \in \mathbb{C}^2$, and $f = (f_1,f_2) \in S\oplus S$. Then we have

$$T_{\xi\hat{\otimes}1}(x\hat{\otimes}f) = (x_1\xi\hat{\otimes}(\sqrt{1-t}f_1 + \sqrt{t}f_2), x_2\xi\hat{\otimes}(\sqrt{1-t}f_1+\sqrt{t}f_2)) \in E,$$

$$T_{\xi\hat{\otimes}\varepsilon}(x\hat{\otimes}f) = (-ix_2\xi\hat{\otimes}(\sqrt{1-t}f_1 + \sqrt{t}f_2), ix_1\xi\hat{\otimes}(\sqrt{1-t}f_1+\sqrt{t}f_2)) \in E.$$

A direct computation shows that $T_{\xi\hat{\otimes}1}\circ F_2 - F\circ T_{\xi\hat{\otimes}1}$ and $T_{\xi\hat{\otimes}\varepsilon}\circ F_2 + F\circ T_{\xi\hat{\otimes}\varepsilon}$ are in $\mathbb{K}(E_2,E)$. Since F_2 and F are self-adjoint, we see that F is a F_2-connection. Therefore (E,ϕ,F) gives the Kasparov product $\delta_{q_n}\hat{\otimes}_{\mathbb{C}_1}\boldsymbol{x}$.

Note that F satisfies $F = F^*$, $F^2 = 1$. Let

$$U = \begin{pmatrix} 1\otimes 1 & 0 \\ 0 & 1\hat{\otimes}c + i(2P-1)\hat{\otimes}s \end{pmatrix},$$

which is a unitary in $\mathbb{B}(E)$. Then we have

$$U^*FU = \begin{pmatrix} 0 & 1\otimes 1 \\ 1\otimes 1 & 0 \end{pmatrix},$$

$$U^*\phi(x)U = \begin{pmatrix} \rho^{(0)}(x) & 0 \\ 0 & \rho^{(1)}(x) \end{pmatrix},$$

which finish the proof. □

To continue the proof, we need more detailed information of the homomorphism Φ.

Lemma 8.3. *Let the notation be as above.*

(1) *We can choose Φ so that it has the following form with respect to the orthogonal decomposition $H = \mathcal{F}_n \oplus \mathcal{F}_n^\perp$:*

$$\Phi(s_i) = \begin{pmatrix} t_i & r_i \\ 0 & v_i \end{pmatrix}.$$

(2) *For Φ as in (1), the quasi-homomorphism $\rho = (\rho^{(0)}, \rho^{(1)})$ in Lemma 8.2 is expressed as*

$$\rho^{(0)}(s_i) = \begin{pmatrix} t_i\hat{\otimes}1 & r_i\hat{\otimes}1 \\ 0 & v_i\hat{\otimes}1 \end{pmatrix}, \quad \rho^{(1)}(s_i) = \begin{pmatrix} t_i\hat{\otimes}1 & r_i\hat{\otimes}z^* \\ 0 & v_i\hat{\otimes}1 \end{pmatrix}.$$

In particular, we have

$$\sum_{i=1}^{n} \rho^{(1)}(s_i)\rho^{(0)}(s_i)^* = (1_H - p_n)\hat{\otimes}1 + p_n\hat{\otimes}z^*.$$

Proof. (1) We first construct $l_n^G : \mathcal{O}_n \to \mathcal{E}_n$ explicitly. Ignoring the G actions, we can find a representation Φ' of $\mathcal{O}_n$ on $\mathcal{F}_n \oplus \mathcal{F}_n$ of the form

$$\Phi'(s_1) = \begin{pmatrix} t_1 & p_n \\ 0 & w_1 \end{pmatrix},$$

$$\Phi'(s_i) = \begin{pmatrix} t_i & 0 \\ 0 & w_i \end{pmatrix}, \quad 2 \leq i \leq n.$$

Using Φ', we define l_n by

$$\begin{pmatrix} l_n(x) & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \Phi'(x) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

and l_n^G by $l_n^G(x) = \int_G \tilde{\alpha}_{g^{-1}} \circ l_n \circ \alpha_g(x) dg$. We have $l_n^G(s_i) = t_i$ for all $1 \leq i \leq n$ by construction.

We show that the Stinespring dilation (Φ, H) of this l_n^G has the desired property. Recall that H is the closure of the algebraic tensor product $\mathcal{O}_n \odot \mathcal{F}_n$ with respect to the inner product

$$\langle x \odot \xi, y \odot \eta \rangle = \langle l_n^G(y^* x)\xi, \eta \rangle,$$

and Φ is given by the left multiplication of $\mathcal{O}_n$. The space $\mathcal{F}_n$ is identified with $1 \odot \mathcal{F}_n$, and the unitary representation π_H is given by $\pi_H(g)(x \odot \xi) = \alpha_g(x) \odot \pi_{\mathcal{F}_n}(g)\xi$. To show that Φ has the desired property, it suffices to show $\|s_i \odot \xi - 1 \odot t_i\xi\| = 0$ for all $\xi \in \mathcal{F}_n$. Indeed,

$$\begin{aligned} &\|s_i \odot \xi - 1 \odot t_i\xi\|^2 \\ &= \langle l_n^G(s_i^* s_i)\xi, \xi \rangle - \langle l_n^G(s_i)\xi, t_i\xi \rangle - \langle l_n^G(s_i^*)t_i\xi, \xi \rangle + \langle t_i\xi, t_i\xi \rangle = 0, \end{aligned}$$

and we get the statement.

(2) The first statement follows from (1) and Lemma 8.2. The Cuntz algebra relation implies

$$p_n r_i = r_i, \quad r_j^* r_i + v_j^* v_i = \delta_{i,j},$$

$$\sum_{i=1}^{n} r_i r_i^* = p_n, \quad \sum_{i=1}^{n} r_i v_i^* = 0, \quad \sum_{i=1}^{n} v_i v_i^* = 1.$$

These relations and the first statement imply the second statement. □

Proof of Theorem 8.1. Thanks to the previous lemma, we may assume that the class $\theta_*(\delta_{q_n}\hat{\otimes}_{\mathbb{C}_1}\boldsymbol{x}) \in KK_G(\mathcal{O}_n, B)$ is given by a quasi-homomorphism $\sigma = (\sigma^{(0)}, \sigma^{(1)})$ from $\mathcal{O}_n$ to B of the form

$$\sigma^{(0)}(s_i) = \begin{pmatrix} t_i\hat{\otimes}1 & r_i\hat{\otimes}1 \\ 0 & v_i\hat{\otimes}1 \end{pmatrix}, \quad \sigma^{(1)}(s_i) = \begin{pmatrix} t_i\hat{\otimes}1 & r_i\hat{\otimes}u_{\psi,\varphi}^* \\ 0 & v_i\hat{\otimes}1 \end{pmatrix},$$

and they satisfy

$$\sum_{i=1}^{n}\sigma^{(1)}(s_i)\sigma^{(0)}(s_i)^* = (1_H - p_n)\hat{\otimes}1 + p_n\hat{\otimes}u^*_{\psi,\varphi}.$$

We set $\tilde{\sigma}^{(0)} = \sigma^{(0)}\oplus\varphi$, $\tilde{\sigma}^{(1)} = \sigma^{(1)}\oplus\psi$, which are unital homomorphisms from $\mathcal{O}_n$ to $\mathbb{B}((H\oplus\mathbb{C})\hat{\otimes}B)$. Then $\tilde{\sigma} = (\tilde{\sigma}^{(0)}, \tilde{\sigma}^{(1)})$ is a quasi-homomorphism with

$$\sum_{i=1}^{n}\tilde{\sigma}^{(1)}(s_i)\tilde{\sigma}^{(0)}(s_i)^* = \big((1_H - p_n)\hat{\otimes}1 + p_n\hat{\otimes}u^*_{\psi,\varphi}\big)\oplus(1_{\mathbb{C}}\hat{\otimes}u_{\psi,\varphi}),$$

which is denoted by u. Then we can construct a norm continuous path $\{u_t\}_{t\in[0,1]}$ of unitaries in $\mathbb{C}1 + \mathbb{K}(H\oplus\mathbb{C})^G\otimes B^G$ satisfying $u(0) = u$ and $u(1) = 1$. Let $\tilde{\sigma}^{(0)}_t = \tilde{\sigma}^{(0)}$, and let $\tilde{\sigma}^{(1)}_t$ be the homomorphism from $\mathcal{O}_n$ to $\mathbb{B}((H\oplus\mathbb{C})\hat{\otimes}B)$ determined by $\tilde{\sigma}^{(1)}_t(s_i) = u(t)\tilde{\sigma}^{(0)}_t(s_i)$. Then $\tilde{\sigma}_t = (\tilde{\sigma}^{(0)}_t, \tilde{\sigma}^{(1)}_t)$ gives a homotopy of quasi-homomorphisms connecting $\tilde{\sigma}$ and $\tilde{\sigma}_1 = (\tilde{\sigma}^{(0)}, \tilde{\sigma}^{(0)})$. This shows $[\tilde{\sigma}] = 0$ in $KK_G(\mathcal{O}_n, B)$, and so $\theta_*(\delta_{q_n}\hat{\otimes}_{\mathbb{C}_1}\boldsymbol{x}) = KK_G(\psi) - KK_G(\varphi)$. □

Remark 8.4. The above argument shows that there exists a short exact sequence

$$0\to \mathrm{Coker}(1 - K_{1-*}(\hat{\beta}_{\pi_\alpha}))\to KK^*_G(\mathcal{O}_n, B)\to \mathrm{Ker}(1 - K_*(\hat{\beta}_{\pi_\alpha}))\to 0.$$

Remark 8.5. From (8.1), we obtain the 6-term exact sequence (see [16, Theorem 4.9]),

$$\begin{array}{ccccc} KK^0_G(B,\mathbb{C}) & \xrightarrow{1-\hat{\otimes}[H_n]} & KK^0_G(B,\mathbb{C}) & \longrightarrow & KK^0_G(B,\mathcal{O}_n) \\ \uparrow & & & & \downarrow \\ KK^1_G(B,\mathcal{O}_n) & \longleftarrow & KK^1_G(B,\mathbb{C}) & \xleftarrow{1-\hat{\otimes}[H_n]} & KK^1_G(B,\mathbb{C}) \end{array}.$$

In particular, we have the following exact sequence by setting $B = \mathbb{C}$:

$$\begin{array}{ccccccc} 0 & \longrightarrow & K_1(\mathcal{O}_n\rtimes_\alpha G) & \longrightarrow & & & \\ K^G_0(\mathbb{C}) & \xrightarrow{1-\hat{\otimes}[H_n]} & K^G_0(\mathbb{C}) & \longrightarrow & K_0(\mathcal{O}_n\rtimes_\alpha G) & \longrightarrow & 0 \end{array}.$$

Let $\iota_\alpha : C^*(G)\to\mathcal{O}_n\rtimes_\alpha G$ be the embedding map, let (π, H_π) be an irreducible representation of G, and let

$$e(\pi)_{ij} = \dim\pi\int_G \overline{\pi(g)_{ij}}\lambda_g dg\in C^*(G).$$

Then the canonical isomorphism from $K^G_0(\mathbb{C})$ onto $K_0(C^*(G))$ sends the class of (π, H_π) in $K^G_0(\mathbb{C})$ to $[e(\overline{\pi})_{11}]\in K_0(C^*(G))$. Thus we have the exact sequence

$$0\longrightarrow K_1(\mathcal{O}_n\rtimes_\alpha G)\longrightarrow \mathbb{Z}\hat{G}\xrightarrow{1-[\overline{\pi_\alpha}]}\mathbb{Z}\hat{G}\longrightarrow K_0(\mathcal{O}_n\rtimes_\alpha G)\longrightarrow 0,$$

where $[\pi]\in\mathbb{Z}\hat{G}$ is sent to $K_0(\iota_\alpha)([e(\pi)_{11}])\in K_0(\mathcal{O}_n\rtimes G)$. With the identification of $K_*(\mathcal{O}_n\rtimes_\alpha G)$ and $K_*(\mathcal{O}_n^G)$, this recovers the formula of $K_*(\mathcal{O}_n^G)$ obtained in [11], [14].

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