Correlation between Angle and Side

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Abstract. Let α be an arbitrary angle in a random spherical triangle Δ and a be the side opposite α. (The sphere has radius 1; vertices of Δ are independent and uniform.) If some other side is constrained to be $\pi/2$, then $\mathrm{E}(\alpha\, a) = 3.05....$ If instead some other angle is fixed at $\pi/2$, then $\mathrm{E}(\alpha\, a) = 2.87....$ In our study of the latter scenario, both Apéry's constant and Catalan's constant emerge. We also review Miles' 1971 proof that $\mathrm{E}(\alpha\, a) = \pi^2/2-2$ when no constraints are in place.

For any planar triangle, long sides are opposite large angles and short sides are opposite small angles. Quantifying this observation for random triangles with either

- independent Gaussian vertices in the plane, or
- independent uniform vertices in a compact convex subset of the plane

seems analytically intractible. We turn attention therefore to random spherical triangles with independent uniform vertices on the unit sphere.

A spherical triangle Δ is a region enclosed by three great circles on the sphere; a great circle is a circle whose center is at the origin. The sides of Δ are arcs of great circles and have length a, b, c. Each of these is $\leq \pi$. The angle α opposite side a is the dihedral angle between the two planes passing through the origin and determined by arcs b, c. The angles β, γ opposite sides b, c are similarly defined. Each of these is $\leq \pi$ too.

Given a random spherical triangle, the univariate density for a is

$$\frac{1}{2}\sin(a), \qquad 0 < a < \pi$$

and

$$\mathrm{E}(a) = \frac{\pi}{2}, \qquad \mathrm{E}(a^2) = \frac{\pi^2}{2} - 2.$$

Further, α is uniformly distributed on $[0, \pi]$ and

$$\mathrm{E}(\alpha) = \frac{\pi}{2}, \qquad \mathrm{E}(\alpha^2) = \frac{\pi^2}{3}.$$

It can be shown that α, b, c are independent random variables; hence $\mathrm{E}(\alpha\, b) = \pi^2/4 = \mathrm{E}(\alpha\, c)$. In contrast, the density for (a, β, γ) is [1]

$$\frac{1}{4\pi}\frac{\sin(\beta)\sin(\gamma)\sin(a)^3}{\left(1-(\cos(\beta)\cos(\gamma)-\sin(\beta)\sin(\gamma)\cos(a))^2\right)^{3/2}}.$$

As special cases, the conditional density for (β, γ) given that $a = \pi/2$ is

$$\frac{1}{2\pi}\frac{\sin(\beta)\sin(\gamma)}{\left(1-\cos(\beta)^2\cos(\gamma)^2\right)^{3/2}};$$

the conditional density for (a, γ) given that $\beta = \pi/2$ is

$$\frac{1}{4}\frac{\sin(\gamma)\sin(a)^3}{\left(1-\sin(\gamma)^2\cos(a)^2\right)^{3/2}};$$

and the unconditional density for (β, γ) is

$$\frac{1}{2\pi}\frac{1}{\sin(\beta)^2\sin(\gamma)^2}\cdot\begin{cases} -\cos(\gamma)\sin(\gamma)+\gamma & \text{if } \beta-\gamma>0 \text{ and } \beta+\gamma<\pi, \\ \pi+\cos(\gamma)\sin(\gamma)-\gamma & \text{if } \beta-\gamma<0 \text{ and } \beta+\gamma>\pi, \\ -\cos(\beta)\sin(\beta)+\beta & \text{if } \beta-\gamma<0 \text{ and } \beta+\gamma<\pi, \\ \pi+\cos(\beta)\sin(\beta)-\beta & \text{if } \beta-\gamma>0 \text{ and } \beta+\gamma>\pi. \end{cases}$$

These facts will be needed later.

1. Univariate Densities

Sides a, b, c are pairwise independent; thus the conditional density for b given $c = \pi/2$ remains unchanged (the sine density on $[0, \pi]$). Angles α, β, γ are uncorrelated but pairwise *dependent*. Therefore the case of two angles, plus two other scenarios involving opposite side and angle, yield interesting new results.

1.1. Angle β, for Fixed Angle γ. The conditional density for β given that $\gamma = \pi/2$ is

$$\frac{1}{2}\frac{1}{\sin(\beta)^2}\cdot\begin{cases} -\cos(\beta)\sin(\beta)+\beta & \text{if } 0<\beta<\pi/2, \\ \pi+\cos(\beta)\sin(\beta)-\beta & \text{if } \pi/2<\beta<\pi \end{cases}$$

$$= \frac{1}{2}\cdot\begin{cases} -\cot(\beta)+\beta\csc(\beta)^2 & \text{if } 0<\beta<\pi/2, \\ \cot(\beta)+(\pi-\beta)\csc(\beta)^2 & \text{if } \pi/2<\beta<\pi. \end{cases}$$

It follows that

$$\mathrm{E}\left(\beta \,\middle|\, \gamma=\frac{\pi}{2}\right)=\frac{\pi}{2}, \qquad \mathrm{E}\left(\beta^2 \,\middle|\, \gamma=\frac{\pi}{2}\right)=\frac{\pi^2}{2}-\frac{7}{4}\zeta(3)$$

where

$$\zeta(3)=\sum_{k=1}^{\infty}\frac{1}{k^3}$$

is Apéry's constant [4].

1.2. Side c, for Fixed Angle γ. By the Law of Cosines for Sides:

$$\cos(c) = \cos(a)\cos(b) + \sin(a)\sin(b)\cos(\gamma)$$

we obtain

$$\cos(c) = \cos(a)\cos(b)$$

if $\gamma = \pi/2$. Let $u = \cos(a)$, $v = \cos(b)$, $w = u\,v$, $z = \arccos(w)$. Then u, v are independent uniform on $[-1, 1]$, that is, with density

$$f(u,v) = \begin{cases} 1/4 & \text{if } -1 \le u \le 1 \text{ and } -1 \le v \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

By [2, 3], the density of w is

$$g(w) = \int\limits_{-\infty}^{\infty} f\left(t, \frac{w}{t}\right)\frac{1}{|t|}dt = \frac{1}{4}\int\limits_{-1}^{1} \varepsilon(w,t)\frac{1}{|t|}dt$$

where $\varepsilon(w,t) = 1$ if $-1 < w/t < 1$, $\varepsilon(w,t) = 0$ otherwise. We obtain

$$g(w) = -\frac{1}{2}\ln|w|.$$

Since $0 \le z \le \pi$ and

$$\left|\frac{dz}{dw}\right| = \frac{1}{\sqrt{1-w^2}} = \frac{1}{\sin(z)},$$

the density of z is

$$h(z) = \frac{g(\cos(z))}{\frac{1}{\sin(z)}} = -\frac{1}{2}\sin(z)\ln|\cos(z)|.$$

It follows that the conditional density for side c, given $\gamma = \pi/2$, has a singularity at $c = \pi/2$ and

$$\mathrm{E}\left(c \,\middle|\, \gamma = \frac{\pi}{2}\right) = \frac{\pi}{2}, \qquad \mathrm{E}\left(c^2 \,\middle|\, \gamma = \frac{\pi}{2}\right) = -6 + \frac{\pi^2}{2} + 4G$$

where

$$G = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2}$$

is Catalan's constant [5].

1.3. Angle γ, for Fixed Side c. By the Law of Cosines for Angles:

$$-\cos(\gamma) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)\cos(c)$$

we obtain

$$\cos(\gamma) = -\cos(\alpha)\cos(\beta)$$

if $c = \pi/2$. Let $u = \cos(\alpha)$, $v = \cos(\beta)$, $w = -u\,v$, $z = \arccos(w)$. The Jacobian determinant of $(\alpha, \beta) \mapsto (u, v)$ is

$$\begin{vmatrix} -\sin(\alpha) & 0 \\ 0 & -\sin(\beta) \end{vmatrix} = \sin(\alpha)\sin(\beta) = \sqrt{1-u^2}\sqrt{1-v^2}$$

because $0 \leq \alpha \leq \pi$, $0 \leq \beta \leq \pi$. Thus u, v have density

$$\frac{1}{2\pi}\frac{\sqrt{1-u^2}\sqrt{1-v^2}}{\left(1-u^2v^2\right)^{3/2}}\frac{1}{\sqrt{1-u^2}\sqrt{1-v^2}} = \frac{1}{2\pi}\frac{1}{\left(1-u^2v^2\right)^{3/2}}.$$

By [2, 3], the density of w is

$$g(w) = \int\limits_{-\infty}^{\infty} f\left(t, \frac{w}{t}\right)\frac{1}{|t|}dt = \frac{1}{2\pi}\frac{1}{\left(1-w^2\right)^{3/2}}\int\limits_{-1}^{1}\varepsilon(w,t)\frac{1}{|t|}dt$$

where $\varepsilon(w, t) = 1$ if $-1 < w/t < 1$, $\varepsilon(w, t) = 0$ otherwise. We obtain

$$g(w) = -\frac{1}{\pi}\frac{\ln|w|}{\left(1-w^2\right)^{3/2}}$$

and hence the density of z is

$$h(z) = \frac{g(\cos(z))}{\frac{1}{\sin(z)}} = -\frac{1}{\pi}\frac{\ln|\cos(z)|}{\sin(z)^2}.$$

It follows that the conditional density for angle γ, given $c = \pi/2$, has a singularity at $\gamma = \pi/2$ and

$$\mathrm{E}\left(\gamma \,\middle|\, c = \frac{\pi}{2}\right) = \frac{\pi}{2}, \qquad \mathrm{E}\left(\gamma^2 \,\middle|\, c = \frac{\pi}{2}\right) = \frac{\pi^2}{4} + \ln(2)^2.$$

This completes our quick survey of univariate densities, for a fixed side or angle.

2. Bivariate Moments

We evaluate $\mathrm{E}(\alpha\, a\,|\, b = \pi/2)$ and $\mathrm{E}(\alpha\, a\,|\,\beta = \pi/2)$ here, giving precise numerics for the former and exact symbolics for the latter.

2.1. (Angle α, Side a), for Fixed Side b. The Law of Cosines for Sides:

$$\cos(a) = \cos(b)\cos(c) + \sin(b)\sin(c)\cos(\alpha)$$

can be expressed as

$$w = u\,v + \sqrt{1-u^2}\sqrt{1-v^2}\cos(\theta)$$

where $u = \cos(b)$, $v = \cos(c)$, $w = \cos(a)$, $\theta = \alpha$. Then u, v, θ are independent; u, v, w are uniform on $[-1,1]$ in the unconditional case and θ is uniform on $[0,\pi]$. Fix $0 \le b \le \pi/2$ for simplicity, then $0 \le u \le 1$. Solving for v in terms of u, w, θ we obtain two solutions

$$\varphi(u,w,\theta) = \frac{u\,w + |\cos(\theta)|\sqrt{(1-u^2)\,(u^2-w^2+(1-u^2)\cos(\theta)^2)}}{u^2+(1-u^2)\cos(\theta)^2},$$

$$\psi(u,w,\theta) = \frac{u\,w - |\cos(\theta)|\sqrt{(1-u^2)\,(u^2-w^2+(1-u^2)\cos(\theta)^2)}}{u^2+(1-u^2)\cos(\theta)^2}$$

assuming

$$u^2 - w^2 + (1-u^2)\cos(\theta)^2 > 0$$

and, further,

$$(w > -u \text{ and } \theta < \pi/2) \text{ or } (w < -u \text{ and } \theta > \pi/2)$$

for φ and

$$(w > u \text{ and } \theta < \pi/2) \text{ or } (w < u \text{ and } \theta > \pi/2)$$

for ψ. Observe that the domains for φ, ψ overlap when

$$(w > u \text{ and } \theta < \pi/2) \text{ or } (w < -u \text{ and } \theta > \pi/2),$$

that is, the transformation is one-to-one for $(w,\theta) \in [-u,u] \times [0,\pi]$ and two-to-one otherwise. Also, the Jacobian determinant of $(v,\theta) \mapsto (w,\theta)$ is

$$\delta(u,v,\theta) = u - \frac{\sqrt{1-u^2}v\cos(\theta)}{\sqrt{1-v^2}}.$$

Let

$$\xi(u,\theta) = \sqrt{u^2+(1-u^2)\cos(\theta)^2}$$

for convenience, then $\mathrm{E}(\alpha\, a\,|\,b)$ is equal to [2]

$$\frac{1}{2\pi}\int_0^{\pi/2}\int_{-u}^{u}\frac{\theta\arccos(w)}{|\delta(u,\varphi(u,w,\theta),\theta)|}dw\,d\theta + \frac{1}{2\pi}\int_{\pi/2}^{\pi}\int_{-u}^{u}\frac{\theta\arccos(w)}{|\delta(u,\psi(u,w,\theta),\theta)|}dw\,d\theta$$

$$+\frac{1}{2\pi}\int_{0}^{\pi/2}\int_{u}^{\xi(u,\theta)}\left(\frac{1}{|\delta(u,\varphi(u,w,\theta),\theta)|}+\frac{1}{|\delta(u,\psi(u,w,\theta),\theta)|}\right)\theta\arccos(w)dw\,d\theta$$

$$+\frac{1}{2\pi}\int_{\pi/2}^{\pi}\int_{-\xi(u,\theta)}^{-u}\left(\frac{1}{|\delta(u,\varphi(u,w,\theta),\theta)|}+\frac{1}{|\delta(u,\psi(u,w,\theta),\theta)|}\right)\theta\arccos(w)dw\,d\theta.$$

In the event $b=\pi/2$, we have $u=0$,

$$\varphi(0,w,\theta)=\frac{\sqrt{-w^2+\cos(\theta)^2}}{|\cos(\theta)|}=-\psi(0,w,\theta),$$

$$\delta(0,v,\theta)=-\frac{v\cos(\theta)}{\sqrt{1-v^2}},\qquad \xi(0,\theta)=|\cos(\theta)|,$$

$$\begin{aligned}\frac{1}{|\delta(0,\varphi(0,w,\theta),\theta)|}+\frac{1}{|\delta(0,\psi(0,w,\theta),\theta)|} &= \frac{\sqrt{1-\varphi^2}}{\varphi\,|\cos(\theta)|}+\frac{\sqrt{1-\psi^2}}{(-\psi)|\cos(\theta)|}\\ &= \frac{2\sqrt{1-\varphi^2}}{\varphi\,|\cos(\theta)|}\end{aligned}$$

which becomes

$$\frac{2\sqrt{1-\frac{-w^2+\cos(\theta)^2}{\cos(\theta)^2}}}{\sqrt{-w^2+\cos(\theta)^2}}=\frac{2|w|}{|\cos(\theta)|\sqrt{-w^2+\cos(\theta)^2}}$$

and therefore $\mathrm{E}(\alpha\,a\,|\,b=\pi/2)$ is equal to

$$\frac{1}{\pi}\int_{0}^{\pi/2}\int_{0}^{\cos(\theta)}\frac{\theta\,w\arccos(w)}{\cos(\theta)\sqrt{-w^2+\cos(\theta)^2}}dw\,d\theta+\frac{1}{\pi}\int_{\pi/2}^{\pi}\int_{\cos(\theta)}^{0}\frac{\theta\,w\arccos(w)}{\cos(\theta)\sqrt{-w^2+\cos(\theta)^2}}dw\,d\theta.$$

This can be reduced to a single integral:

$$\frac{1}{4}\int_{0}^{\pi}\left[2-{}_2F_1\left(\tfrac{1}{2},\tfrac{1}{2},2,\cos(\theta)^2\right)\cos(\theta)\right]\theta\,d\theta=3.0538319164380270202505577...$$

involving the following Gauss hypergeometric function:

$$\begin{aligned}{}_2F_1\left(\tfrac{1}{2},\tfrac{1}{2},2,x\right) &= \frac{1}{\pi}\sum_{n=0}^{\infty}\frac{\Gamma(n+1/2)^2}{\Gamma(n+2)}\frac{x^n}{n!}\\ &= \frac{4}{\pi}\left[\frac{1}{x}\int_{0}^{\pi/2}\sqrt{1-x\sin(t)^2}\,dt+\left(1-\frac{1}{x}\right)\int_{0}^{\pi/2}\frac{1}{\sqrt{1-x\sin(t)^2}}\,dt\right].\end{aligned}$$

Despite a connection to complete elliptic integrals [6], this unfortunately seems to be as far as we can go.

2.2. (Angle α, Side a), for Fixed Angle β. The Law of Cosines for Angles:

$$-\cos(\alpha)=\cos(\beta)\cos(\gamma)-\sin(\beta)\sin(\gamma)\cos(a)$$

can be expressed as

$$w=-u\,v+\sqrt{1-u^2}\sqrt{1-v^2}\cos(\theta)$$

where $u=\cos(\beta)$, $v=\cos(\gamma)$, $w=\cos(\alpha)$, $\theta=a$. Fix $0\leq\beta\leq\pi/2$ for simplicity, then $0\leq u\leq 1$. Solving for v in terms of u, w, θ we obtain two solutions $\varphi(-u,w,\theta)$, $\psi(-u,w,\theta)$ as before. Also, the Jacobian determinant of $(v,\theta)\mapsto(w,\theta)$ is

$$\delta(u,v,\theta)=-u-\frac{\sqrt{1-u^2}v\cos(\theta)}{\sqrt{1-v^2}}.$$

In the event $\beta=\pi/2$, we have $u=0$ and an identical formula for $|\delta(0,\varphi,\theta)|^{-1}+|\delta(0,\psi,\theta)|^{-1}$ follows. The distinction with earlier calculations arises from the density

$$\frac{1}{4}\frac{\sqrt{1-v^2}\sin(\theta)^3}{\left(1-(1-v^2)\cos(\theta)^2\right)^{3/2}}\frac{1}{\sqrt{1-v^2}}=\frac{1}{4}\frac{\sin(\theta)^3}{\left(1-(1-v^2)\cos(\theta)^2\right)^{3/2}}$$

for (v,θ). Substituting φ in place of v, we obtain

$$\frac{1}{4}\frac{\sin(\theta)^3}{\left(1-\left[1-\frac{-w^2+\cos(\theta)^2}{\cos(\theta)^2}\right]\cos(\theta)^2\right)^{3/2}}=\frac{1}{4}\frac{\sin(\theta)^3}{\left(1-w^2\right)^{3/2}}$$

and therefore $\mathrm{E}(\alpha\,a\,|\,\beta=\pi/2)$ is equal to

$$\frac{1}{2}\int\limits_0^{\pi/2}\int\limits_0^{\cos(\theta)}\frac{\theta\,w\arccos(w)}{\cos(\theta)\sqrt{-w^2+\cos(\theta)^2}}\frac{\sin(\theta)^3}{\left(1-w^2\right)^{3/2}}dw\,d\theta$$
$$+\frac{1}{2}\int\limits_{\pi/2}^{\pi}\int\limits_{\cos(\theta)}^{0}\frac{\theta\,w\arccos(w)}{\cos(\theta)\sqrt{-w^2+\cos(\theta)^2}}\frac{\sin(\theta)^3}{\left(1-w^2\right)^{3/2}}dw\,d\theta.$$

This can be reduced to a single integral:

$$\begin{aligned}\frac{\pi}{4}\int\limits_0^{\pi}\theta\,\tan(\theta)\left[\cos(\theta)+\sin(\theta)-1\right]d\theta &= \frac{\pi}{4}\left[2+(1+\ln(2))\,\pi-4G\right]\\ &= 2.870878761423354258374234 0...\end{aligned}$$

using the fact that

$$\begin{aligned}&\int\arccos(w)\frac{w}{\left(1-w^2\right)^{3/2}\sqrt{-w^2+\cos(\theta)^2}}dw\\ =\;&-\frac{1}{\sin(\theta)^2}\left(\arccos(w)\sqrt{\frac{-w^2+\cos(\theta)^2}{1-w^2}}+\int\frac{\sqrt{-w^2+\cos(\theta)^2}}{1-w^2}dw\right)\end{aligned}$$

and

$$\left.\begin{array}{ll} \left.\arccos(w)\sqrt{\dfrac{-w^2+\cos(\theta)^2}{1-w^2}}\right|_{w=0}^{\cos(\theta)} & \text{if } 0\leq\theta\leq\pi/2, \\ \left.\arccos(w)\sqrt{\dfrac{-w^2+\cos(\theta)^2}{1-w^2}}\right|_{\cos(\theta)}^{w=0} & \text{if } \pi/2\leq\theta\leq\pi \end{array}\right\} = -\frac{\pi}{2}\cos(\theta),$$

$$\left.\begin{array}{ll} \displaystyle\int_0^{\cos(\theta)} \frac{\sqrt{-w^2+\cos(\theta)^2}}{1-w^2}dw & \text{if } 0\leq\theta\leq\pi/2, \\ \displaystyle\int_{\cos(\theta)}^{0} \frac{\sqrt{-w^2+\cos(\theta)^2}}{1-w^2}dw & \text{if } \pi/2\leq\theta\leq\pi \end{array}\right\} = \frac{\pi}{2}\left(1-\sin(\theta)\right).$$

We have not attempted to extend these formulas for $\beta \neq \pi/2$. It is intriguing that quadrantal triangles ($b = \pi/2$) should present an unevaluated integral 3.05... while right-angled triangles ($\beta = \pi/2$) give an integral 2.87... expressible in closed-form.

3. Unconstrained Scenario

Miles [1] proved that

$$\mathrm{E}((\alpha+\beta+\gamma-\pi)(a+b+c)) = \frac{3}{2}\pi^2 - 6$$

where $\alpha+\beta+\gamma-\pi$ is the area V of the spherical triangle and $a+b+c$ is perimeter S. (The notation V, S appears to be traditional.) By preceding correlation results,

$$3\,\mathrm{E}(\alpha\,a) + 6\left(\frac{\pi^2}{4}\right) - 3\pi\left(\frac{\pi}{2}\right) = 3\,\mathrm{E}(\alpha\,a) + 6\,\mathrm{E}(\alpha\,b) - 3\pi\,\mathrm{E}(a) = \frac{3}{2}\pi^2 - 6$$

hence $\mathrm{E}(\alpha\,a) = \pi^2/2 - 2$. It remains to verify Miles' argument.

Up to now, our random spherical triangles have been built using independent uniform vertices. From now on, they will be built using independent uniform great circles. By duality, $\mathrm{E}(V\,S) = 3\pi^2/2 - 6$ under either convention.

Let k independent uniform great circles be placed on the unit sphere. The number of polygonal cells determined is $k^2 - k + 2$ almost always. Randomly select one of the cells (endowed with equal weighting) and denote the density for (V, S) by $f_k(v, s)$. For example, if $k = 2$, then [7]

$$f_2(v,s) = \frac{1}{4}\sin\left(\frac{v}{2}\right)\delta(s-2\pi) \qquad \text{if } 0 \leq v \leq 2\pi$$

and δ is the Dirac delta function. No formulas for $f_k(v,s)$ are known for $k \geq 3$, although when $k = 3$ marginal densities for V and for S are well-understood [6].

Let the cells be labeled randomly by the integers 1, 2, 3, …, $k^2 - k + 2$. It is not allowed, for example, to specify that cell 1 cover the north pole and that cells 2, 3 be adjacent to it. The labeling must be independent of all features of the tessellation. Hence, for the preceding experiment, a cell was selected merely by generating a uniform integer $j \in [1,\ k^2 - k + 2]$. This is the most basic sampling technique.

We wish to examine alternative methods for selecting a cell. Suppose that the weighting is proportional to cellular area. Let C_j denote the event that a uniform point falls in cell j, where $1 \leq j \leq k^2 - k + 2$. If the volume V_j of the cell is v, then the probability of C_j is $v/(4\pi)$; unconditionally it is $\mathrm{E}_k(V)/(4\pi)$. The density for (V,S) here is

$$\begin{aligned} g_k(v,s) &= \mathrm{P}_k\left\{V_j \in [v, v+dv] \text{ and } S_j \in [s, s+ds] \mid C_j\right\} \\ &= \frac{\mathrm{P}_k\left\{V_j \in [v, v+dv] \text{ and } S_j \in [s, s+ds] \text{ and } C_j\right\}}{\mathrm{P}_k\left\{C_j\right\}} \\ &= \frac{\mathrm{P}_k\left\{C_j \,|V_j \in [v, v+dv] \text{ and } S_j \in [s, s+ds]\right\}\ f_k(v,s)}{\mathrm{P}_k\left\{C_j\right\}} \\ &= \frac{(v/(4\pi))\, f_k(v,s)}{\mathrm{E}_k(V)/(4\pi)} = \frac{v\, f_k(v,s)}{\mathrm{E}_k(V)}; \end{aligned}$$

thus

$$v\, g_k(v,s) = \frac{v^2\, f_k(v,s)}{\mathrm{E}_k(V)}. \tag{1}$$

Suppose instead that the weighting is proportional to cellular perimeter. A uniform great circle hits $2k$ cells almost always; we then choose one of these cells at random. Let C'_j denote the event that a uniform great circle hits cell j and cell j is subsequently chosen. If the perimeter S_j of the cell is s, then the probability of C'_j is $(s/(2\pi))(1/(2k))$; unconditionally it is $\mathrm{E}_k(S)/(4\pi k)$. The density for (V,S) here is

$$\begin{aligned} h_k(v,s) &= \mathrm{P}_k\left\{V_j \in [v, v+dv] \text{ and } S_j \in [s, s+ds] \mid C'_j\right\} \\ &= \frac{\mathrm{P}_k\left\{V_j \in [v, v+dv] \text{ and } S_j \in [s, s+ds] \text{ and } C'_j\right\}}{\mathrm{P}_k\left\{C'_j\right\}} \\ &= \frac{\mathrm{P}_k\left\{C'_j \,|V_j \in [v, v+dv] \text{ and } S_j \in [s, s+ds]\right\}\ f_k(v,s)}{\mathrm{P}_k\left\{C'_j\right\}} \\ &= \frac{(s/(4\pi k))\, f_k(v,s)}{\mathrm{E}_k(S)/(4\pi k)} = \frac{s\, f_k(v,s)}{\mathrm{E}_k(S)}; \end{aligned}$$

thus

$$v\,h_k(v,s) = \frac{v\,s\,f_k(v,s)}{\mathrm{E}_k(S)}. \tag{2}$$

Here is an equivalent definition of C_j which is more compatible with that of C'_j. The intersection of two independent uniform great circles (two diametrically-opposed points z and $-z$) hits two cells almost always; we then choose one of these cells at random. The new vertex $\pm z$ has four new adjacent cells; upon integrating both sides of (1), it becomes clear that

$$4\,\mathrm{E}_{k+2}(V) = \frac{\mathrm{E}_k(V^2)}{\mathrm{E}_k(V)}.$$

In the same way, with regard to C'_j, the new arc forms the boundary between two new adjacent cells; upon integrating both sides of (2), it becomes clear that

$$2\,\mathrm{E}_{k+1}(V) = \frac{\mathrm{E}_k(V\,S)}{\mathrm{E}_k(S)}.$$

Therefore

$$\frac{\mathrm{E}_{k-1}(V^2)}{\mathrm{E}_{k-1}(V)} = 2\frac{\mathrm{E}_k(V\,S)}{\mathrm{E}_k(S)}$$

and, setting $k = 3$,

$$\mathrm{E}_3(V\,S) = \frac{1}{2}\frac{\mathrm{E}_2(V^2)}{\mathrm{E}_2(V)}\,\mathrm{E}_3(S) = \frac{1}{2}\frac{2\,(\pi^2-4)}{\pi}\frac{3\pi}{2} = \frac{3}{2}\pi^2 - 6$$

as was to be shown.

For $k = 3$, the number N of cellular vertices is 3 almost always. For $k = 4$, the number N is 3 with probability 4/7 and 4 with probability 3/7. Recursive equations in k for second order moments of V, S, N appear in [1, 8] which vastly generalize our discussion here.

4. Acknowledgement

I am grateful to Richard Cowan for providing the clearer version of Miles' proof that appears here. Much more relevant material can be found at [9, 10], including experimental computer runs that aided theoretical discussion here.

5. Addendum

M. Larry Glasser reduced the integral 3.05... to an expression

$$\frac{\pi^2}{2} - \frac{4G}{\pi} - \frac{2}{\pi}\,{}_4F_3\left(\frac{1}{2},\frac{1}{2},1,1;\frac{3}{2},\frac{3}{2},\frac{3}{2};1\right)$$

where

$$ {}_4F_3\left(\frac{1}{2},\frac{1}{2},1,1;\frac{3}{2},\frac{3}{2},\frac{3}{2};x\right)=\frac{\sqrt{\pi}}{8}\sum_{n=0}^{\infty}\frac{\Gamma(n+1/2)^2\Gamma(n+1)^2}{\Gamma(n+3/2)^3}\frac{x^n}{n!}. $$

He and Jonathan Borwein independently found that

$$ {}_4F_3\left(\frac{1}{2},\frac{1}{2},1,1;\frac{3}{2},\frac{3}{2},\frac{3}{2};1\right)=\int_0^{\pi/2}\frac{\mathrm{Li}_2(\sin(\theta))-\mathrm{Li}_2(-\sin(\theta))}{2}d\theta $$

where Li_2 is the dilogarithm function. Let $\mathrm{agm}(x,y)$ denote the common limit of sequences $\{a_n\}$ and $\{b_n\}$ defined via [11]

$$ a_0=x, \qquad b_0=y, \qquad a_n=\frac{a_{n-1}+b_{n-1}}{2}, \qquad b_n=\sqrt{a_{n-1}b_{n-1}} \quad \text{for } n\geq 1. $$

David Broadhurst's preferred integral for 3.05... is

$$ -\int_{\pi/2}^{\pi}\frac{\sin(\theta)+\theta\cos(\theta)}{\mathrm{agm}(1,\sin(\theta))}d\theta $$

because it permits quick high-precision numerical computation.

References

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