A note on the Petri loci

A. BRUNO – E. SERNESI*

Abstract

Let $\mathcal{M}_g$ be the course moduli space of complex projective nonsingular curves of genus g. We prove that when the Brill-Noether number $\rho(g,r,n)$ is non-negative every component of the Petri locus $P^r_{g,n} \subset \mathcal{M}_g$ whose general member is a curve C such that $W^{r+1}_n(C) = \emptyset$, has codimension one in $\mathcal{M}_g$.

Introduction

Let C be a nonsingular irreducible projective curve of genus $g \geq 2$ defined over $\mathbb{C}$. A pair (L,V) consisting of an invertible sheaf L on C and of an $(r+1)$-dimensional vector subspace $V \subset H^0(L)$, $r \geq 0$, is called *a linear series* of dimension r and degree $n = \deg(L)$, or a g^r_n. If $V = H^0(L)$ then the g^r_n is said to be *complete*.

If (L,V) is a g^r_n then the *Petri map* for (L,V) is the natural multiplication map

$$\mu_0(L,V) : V \otimes H^0(\omega_C L^{-1}) \longrightarrow H^0(\omega_C)$$

The Petri map for L is

$$\mu_0(L) : H^0(L) \otimes H^0(\omega_C L^{-1}) \longrightarrow H^0(\omega_C)$$

Recall that C is called a *Petri curve* if the Petri map $\mu_0(L)$ is injective for every invertible sheaf L on C. By the Gieseker-Petri theorem [5] we know that in $\mathcal{M}_g$, the course moduli space of nonsingular projective curves of genus g, the locus of curves which are not Petri is a proper closed subset P_g, called the *Petri locus*. This locus decomposes into several components, according to the numerical types and to other properties that linear series can have on a curve of genus g. We will say that C is *Petri with respect to g^r_n's* if the Petri map $\mu_0(L,V)$ is injective for every g^r_n (L,V) on C.

We denote by $P^r_{g,n} \subset \mathcal{M}_g$ the locus of curves which are not Petri w.r. to g^r_n's. Then

$$P_g = \bigcup_{r,n} P^r_{g,n}$$

*Both authors are members of GNSAGA-INDAM

The structure of $P^r_{g,n}$ is not known in general: it might a priori have several components and not be equidimensional. If the Brill-Noether number

$$\rho(g,r,n) := g - (r+1)(g-n+r)$$

is nonnegative then it is conjectured that $P^r_{g,n}$ has pure codimension one if it is non-empty. In some special cases this is known to be true (notably in the obvious case $\rho(g,r,n)=0$, and for $r=1$ and $n=g-1$ [10]) .

Denote by $\overline{\mathcal{M}}_g$ the moduli space of stable curves, and let

$$\overline{\mathcal{M}}_g \backslash \mathcal{M}_g = \Delta_0 \cup \cdots \cup \Delta_{[\frac{g}{2}]}$$

be its boundary, in standard notation. In [2] G. Farkas has proved the existence of at least one divisorial component of $P^1_{g,n}$ in case $\rho(g,1,n) \geq 0$ and $n \leq g-1$, using the theory of limit linear series. He found a divisorial component which has a nonempty intersection with Δ_1. Another proof has been given in [1], by degeneration to a stable curve with g elliptic tails. The method of [2] has been extended in [3] to arbitrary r. In this note we take another point of view, which does not rely on degeneration arguments. We prove the following result:

Theorem 0.1 *If $\rho(g,r,n) \geq 0$ then every component of $P^r_{g,n}$ whose general member is a curve C such that $W^{r+1}_n(C) = \emptyset$, has codimension one in $\mathcal{M}_g$.*

Note that a necessary numerical condition for the existence of a curve C as in the statement is that $\rho(g,r+1,n) < 0$. This condition, together with $\rho(g,r,n) \geq 0$ gives:

$$0 \leq \rho(g,r,n) < g - n + 2(r+1)$$

or, equivalently:

$$\frac{r}{r+1} g + r \leq n < \frac{r+1}{r+2} g + r + 1$$

For the proof of the theorem we introduce a modular family $\mathcal{C} \longrightarrow B$ of curves of genus g (see (i) below for the definition) and we use the determinantal description of the relative locus $\mathcal{W}^r_n(\mathcal{C}/B)$ over B and of the naturally defined closed subscheme $\widetilde{P}^r_{g,n} \subset \mathcal{W}^r_n(\mathcal{C}/B)$ whose image in $\mathcal{M}_g$ is $P^r_{g,n}$. Since it is a determinantal locus, every component of $\widetilde{P}^r_{g,n}$ has dimension $\geq 3g-4$. Then a theorem of F. Steffen [9] ensures that every component of $P^r_{g,n}$ has dimension $\geq 3g-4$ as well, thus proving the result.

In a forthcoming paper (in preparation) we will show the existence of a divisorial component of $P^1_{g,n}$ which has a non-empty intersection with Δ_0, when $\rho(g,1,n) \geq 1$.

Proof of Theorem 0.1

In this section we fix g, r, n such that $\rho(g, r, n) \geq 0$ and $\rho(g, r+1, n) < 0$. Consider the following diagram:

$$\begin{array}{ccc} J_n(\mathcal{C}/B) \times_B \mathcal{C} & \longrightarrow & \mathcal{C} \\ \downarrow & & \downarrow f \\ J_n(\mathcal{C}/B) & \xrightarrow{q} & B \end{array} \qquad (1)$$

where:

(i) f is a smooth modular family of curves of genus g parametrized by a non-singular quasi-projective algebraic variety B of dimension $3g-3$. This means that at each closed point $b \in B$ the Kodaira-Spencer map $\kappa_b : T_bB \to H^1(\mathcal{C}(b), T_{\mathcal{C}(b)})$ is an isomorphism. In particular, the functorial morphism

$$\beta : B \longrightarrow \mathcal{M}_g$$

is finite and dominant. The existence of f is a standard fact, see e.g. [6], Theorem 27.2.

(ii) $J_n(\mathcal{C}/B)$ is the relative Picard variety parametrizing invertible sheaves of degree n on the fibres of f.

(iii) For all closed points $b \in B$ the fibre $\mathcal{C}(b)$ satisfies $W_n^{r+1}(\mathcal{C}(b)) = \emptyset$. This condition can be satisfied modulo replacing B by an open neighborhood of $b_0 \in B$ if necessary, because the condition $W_n^{r+1}(\mathcal{C}(b)) = \emptyset$ is open w.r. to $b \in B$.

(iv) We may even assume that any given specific curve C of genus g satisfying $W_n^{r+1}(C) = \emptyset$ appears among the fibres of f. In particular we may assume that the dense subset $\mathrm{Im}(\beta) \subset \mathcal{M}_g$ has a non-empty intersection with all irreducible components of $P^r_{g,n}$ whose general element parametrizes a curve C such that $W_n^{r+1}(C) = \emptyset$.

Let $\mathcal{P}$ be a Poincaré invertible sheaf on $J_n(\mathcal{C}/B) \times_B \mathcal{C}$. Using $\mathcal{P}$ in a well-known way one constructs the relative Brill-Noether scheme

$$\mathcal{W}_n^r(\mathcal{C}/B) \subset J_n(\mathcal{C}/B)$$

Consider the restriction of diagram (1) over $\mathcal{W}_n^r(\mathcal{C}/B)$:

$$\begin{array}{ccc} \mathcal{W}_n^r(\mathcal{C}/B) \times_B \mathcal{C} & \xrightarrow{p_2} & \mathcal{C} \\ p_1 \downarrow & & \downarrow f \\ \mathcal{W}_n^r(\mathcal{C}/B) & \xrightarrow{q} & B \end{array} \qquad (2)$$

Every irreducible component of $\mathcal{W}^r_n(\mathcal{C}/B)$ has dimension $\geq 3g-3+\rho(g,r,n)$ and, since $\rho(g,r,n) \geq 0$, there is a component which dominates B [7, 8]. A closed point $w \in \mathcal{W}^r_n(\mathcal{C}/B)$ represents an invertible sheaf L_w on the curve $\mathcal{C}(q(w))$ such that $h^0(L_w) \geq r+1$. Denoting again by $\mathcal{P}$ the restriction of $\mathcal{P}$ to $\mathcal{W}^r_n(\mathcal{C}/B) \times_B \mathcal{C}$, we have a homomorphism of coherent sheaves on $\mathcal{W}^r_n(\mathcal{C}/B)$, induced by multiplication of sections along the fibres of p_1:

$$\mu_0(\mathcal{P}) : p_{1*}\mathcal{P} \otimes p_{1*}[p_2^*(\omega_{\mathcal{C}/B}) \otimes \mathcal{P}^{-1}] \longrightarrow p_{1*}[p_2^*\omega_{\mathcal{C}/B}]$$

By condition (iii) above, these sheaves are locally free, of ranks $(r+1)(g-n+r)$ and g respectively. Moreover, by definition, at each point $w \in \mathcal{W}^r_n(\mathcal{C}/B)$, the map $\mu_0(\mathcal{P})$ coincides with the Petri map

$$\mu_0(L_w) : H^0(\mathcal{C}(q(w)), L_w) \otimes H^0(\mathcal{C}(q(w)), \omega_{\mathcal{C}(q(w))} L_w^{-1}) \longrightarrow H^0(\mathcal{C}(q(w)), \omega_{\mathcal{C}(q(w))})$$

Consider the degeneracy scheme:

$$\widetilde{P}^r_{g,n} := D_{(r+1)(g-n+r)-1}(\mu_0(\mathcal{P})) \subset \mathcal{W}^r_n(\mathcal{C}/B)$$

which is supported on the locus of $w \in \mathcal{W}^r_n(\mathcal{C}/B)$ such that $\mu_0(L_w)$ is not injective. Since $\widetilde{P}^r_{g,n}$ is defined by a determinantal condition, all its components have dimension at least

$$\dim[\mathcal{W}^r_n(\mathcal{C}/B)] - [g - (r+1)(g-n+r) + 1] = 3g - 4$$

If we restrict diagram (2) over any $b \in B$ and we let $C = \mathcal{C}(b)$, we obtain:

$$\begin{array}{ccc} W^r_n(C) \times C & \xrightarrow{\pi_2} & C \\ \downarrow{\scriptstyle \pi_1} & & \\ W^r_n(C) & & \end{array}$$

and the map $\mu_0(\mathcal{P})$ restricts over $W^r_n(C)$ to

$$m_P : \pi_{1*}P \otimes \pi_{1*}[\pi_2^*\omega_C \otimes P^{-1}] \longrightarrow H^0(C, \omega_C) \otimes \mathcal{O}_{W^r_n}$$

where $P = \mathcal{P}_{|W^r_n(C)\times C}$ is a Poincaré sheaf on $W^r_n(C) \times C$. Observe that the dual of the source of m_P is an ample vector bundle (compare [4], §2), while the target is a trivial vector bundle. This implies that the vector bundle

$$\left[p_{1*}\mathcal{P} \otimes p_{1*}[p_2^*(\omega_{\mathcal{C}/B}) \otimes \mathcal{P}^{-1}]\right]^\vee \otimes p_{1*}[p_2^*\omega_{\mathcal{C}/B}]$$

is q-relatively ample. Therefore we can apply Theorem 0.3 of [9] to deduce that every irreducible component of $q(\widetilde{P}^r_{g,n}) \subset B$ has dimension $\geq 3g-4$. Since f is a modular family, it follows that every irreducible component of $\overline{\beta(q(\widetilde{P}^r_{g,n}))} \subset \mathcal{M}_g$ has dimension $\geq 3g-4$ as well. But $\overline{\beta(q(\widetilde{P}^r_{g,n}))} \subset P^r_{g,n} \neq \mathcal{M}_g$ and therefore all the components of $\overline{\beta(q(\widetilde{P}^r_{g,n}))}$ are divisorial. Since, by (iv), $\overline{\beta(q(\widetilde{P}^r_{g,n}))}$ is the union of all the components of $P^r_{g,n}$ whose general element parametrizes a curve C such that $W^{r+1}_n(C) = \emptyset$, the theorem is proved. □

References

[1] A. Castorena, M. Teixidor i Bigas: Divisorial components of the Petri locus for pencils, *J. Pure Appl. Algebra* 212 (2008), 1500–1508.

[2] G. Farkas: Gaussian maps, Gieseker-Petri loci and large theta-characteristics, *J. reine angew. Mathematik* 581 (2005), 151-173.

[3] G. Farkas: Rational maps beween moduli spaces of curves and Gieseker-Petri divisors, *Journal of Algebraic Geometry* 19 (2010), 243-284.

[4] W. Fulton - R. Lazarsfeld: On the connectedness of degeneracy loci and special divisors, *Acta Math.* 146 (1981), 271-283.

[5] D. Gieseker: Stable curves and special divisors, *Inventiones Math.* 66 (1982), 251-275.

[6] R. Hartshorne: *Deformation Theory,* Springer GTM vol.257 (2010).

[7] Kempf G.: Schubert methods with an application to algebraic curves, *Publication of Mathematisch Centrum*, Amsterdam 1972.

[8] Kleiman S., Laksov D.: On the existence of special divisors, *Amer. Math. J.* 94 (1972), 431-436.

[9] F. Steffen: A generalized principal ideal theorem with an application to Brill-Noether theory. *Inventiones Math.* 132 (1998), 73-89.

[10] M. Teixidor: The divisor of curves with a vanishing theta null, *Compositio Math.* 66 (1988), 15-22.

ADDRESS OF THE AUTHORS:
Dipartimento di Matematica, Università Roma Tre
Largo S. L. Murialdo 1, 00146 Roma, Italy.

bruno@mat.uniroma3.it
sernesi@mat.uniroma3.it