RESONANCE VARIETIES VIA BLOWUPS OF $\mathbb{P}^2$ AND SCROLLS

HAL SCHENCK

Abstract. Conjectures of Suciu [36] relate the fundamental group of an arrangement complement $M = \mathbb{C}^n \setminus \mathcal{A}$ to the first resonance variety of $H^*(M, \mathbb{Z})$. We describe a connection between the first resonance variety and the Orlik-Terao algebra $C(\mathcal{A})$ of the arrangement. In particular, we show that non-local components of $R^1(\mathcal{A})$ give rise to determinantal syzygies of $C(\mathcal{A})$. As a result, $Proj(C(\mathcal{A}))$ lies on a scroll, placing geometric constraints on $R^1(\mathcal{A})$. The key observation is that $C(\mathcal{A})$ is the homogeneous coordinate ring associated to a nef but not ample divisor on the blowup of $\mathbb{P}^2$ at the singular points of $\mathcal{A}$.

1. Introduction

The fundamental group of the complement M of an arrangement of hyperplanes $\mathcal{A} = \bigcup_{i=1}^d H_i \subseteq \mathbb{C}^n$ is a much studied object. The Lefschetz-type theorem of Hamm-Le [13] implies that taking a generic two dimensional slice of M yields an isomorphism at the level of fundamental groups, so to study $\pi_1(M)$ we may assume $\mathcal{A} \subseteq \mathbb{P}^2$. Even with this simplifying assumption the situation is nontrivial: in [17] Hirzebruch writes "The topology of the complement of an arrangement of lines in $\mathbb{P}^2$ is very interesting, the investigation of the fundamental group very difficult".

Presentations for $\pi_1(M)$ are given by Randell [28], Salvetti [29], Arvola [2], and Cohen-Suciu [3]. Perhaps the most compact of these is the braid monodromy presentation of [3], but even this is quite complicated. Somewhat coarser invariants of $\pi_1(M)$ are the LCS ranks and Chen ranks. For a finitely generated group G, let $G = G_1$ and define a sequence of normal subgroups inductively by $G_k = [G_{k-1}, G]$. This yields an associated Lie algebra

$$gr(G) \otimes \mathbb{Q} := \bigoplus_{k=1}^{\infty} G_k/G_{k+1} \otimes \mathbb{Q},$$

with Lie bracket induced by the commutator. The k-th LCS rank $\phi_k = \phi_k(G)$ is the rank of the k-th quotient. The Chen ranks of a group are the LCS ranks of the maximal metabelian quotient $G/[[G,G],[G,G]]$. Work of Papadima and Suciu [26] shows that the Chen ranks of $\pi_1(M)$ are combinatorially determined; but save for some special classes of arrangements, there are no explicit formulas for either the Chen or LCS ranks. However, there are a beautiful pair of conjectures due to Suciu [36], giving formulas for the LCS and Chen ranks in terms of the first resonance variety $R^1(\mathcal{A})$. The variety $R^1(\mathcal{A})$ is the tangent cone at the origin to the characteristic variety; the study of $R^1(\mathcal{A})$ was pioneered by Falk in [10].

2000 *Mathematics Subject Classification.* 13D02, 52C35, 14J26, 14C20.
Key words and phrases. Rational surface, Line arrangement, syzygy.
Schenck supported by NSF 07–07667, NSA 904-03-1-0006.

In the next section, we review the main subjects of investigation: the Orlik-Solomon algebra $A = H^*(M,\mathbb{Z})$, the Orlik-Terao algebra $C(\mathcal{A})$, the first resonance variety $R^1(\mathcal{A})$, and blowups of $\mathbb{P}^2$ using certain divisors. Our main result is a description of $C(\mathcal{A})$ as the homogeneous coordinate ring of the blowup X of $\mathbb{P}^2$ at the singular points of $\mathcal{A}$, via a specific (nef but not ample) divisor $D_{\mathcal{A}}$. This allows us to give a geometric interpretation of $R^1(\mathcal{A})$ in terms of certain determinantal syzygies; we prove that if $\mathcal{A}$ supports a net, then $Proj(C(\mathcal{A}))$ lies on a scroll.

2. Background

In [23], Orlik and Solomon gave a presentation for the cohomology ring of the complement M of a set of hyperplanes $\mathcal{A} \subseteq \mathbb{C}^n$. A consequence of their work is that the Betti numbers of M are determined by the intersection lattice $L(\mathcal{A})$. This lattice is ranked by codimension: $x \in L_i(\mathcal{A})$ corresponds to a linear space of codimension i which is an intersection of hyperplanes of $\mathcal{A}$. The lattice element $\hat{0}$ corresponds to $\mathbb{C}^n$, and $y \prec x \leftrightarrow x \subsetneq y$. We work with $\mathcal{A}$ central, so $\mathcal{A}$ defines an arrangement in both $\mathbb{C}^n$ and $\mathbb{P}^{n-1}$. We will depict $\mathcal{A}$ projectively, as below:

Example 2.1. The reflecting hyperplanes of the Weyl group of $SL(4)$ are the six hyperplanes in $\mathbb{C}^4$ defined by $V(x_i - x_j)$, $1 \le i < j \le 4$. Projecting along the common subspace (t,t,t,t) yields the *braid arrangement* of six planes containing the origin in $\mathbb{C}^3$, or six lines in $\mathbb{P}^2$:

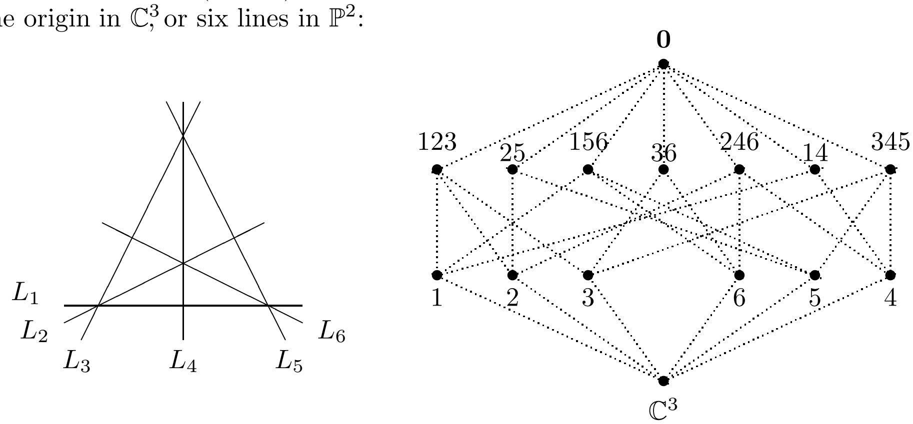

Figure 1. The braid arrangement A_3 and its intersection lattice in $\mathbb{C}^3$

Definition 2.2. *The Möbius function $\mu : L(\mathcal{A}) \longrightarrow \mathbb{Z}$ is defined by*

$$\begin{array}{rcl} \mu(\hat{0}) & = & 1 \\ \mu(t) & = & -\sum\limits_{s \prec t} \mu(s), \text{ if } \hat{0} \prec t \end{array}$$

As noted, the Poincaré polynomial of M is determined by $L(\mathcal{A})$:

$$P(M,t) = \sum_{x \in L(\mathcal{A})} \mu(x) \cdot (-t)^{\operatorname{rank}(x)}.$$

In Example 2.1, $P(M,t) = 1 + 6t + 11t^2 + 6t^3$. For a central arrangement in $\mathbb{C}^n$, $M \simeq \mathbb{C}^* \times (\mathbb{P}^{n-1} \setminus \mathcal{A})$, so by Künneth $P(M,t) = (1+t)P(\mathbb{P}^{n-1} \setminus \mathcal{A}, t)$. For $n = 3$, $b_2(M) = \sum_{p \in L_2(\mathcal{A})} \mu(p)$, where $\mu(p)$ is one less than the number of lines through p.

2.1. **Orlik-Solomon algebra and $R^1(\mathcal{A})$.** The Orlik and Solomon presentation for the cohomology ring of $M = \mathbb{C}^n \setminus \mathcal{A}$ is as follows:

Definition 2.3. *$A = H^*(M, \mathbb{Z})$ is the quotient of the exterior algebra $E = \bigwedge(\mathbb{Z}^d)$ on generators $e_1, \ldots, e_d$ in degree 1 by the ideal generated by all elements of the form $\partial e_{i_1 \ldots i_r} := \sum_q (-1)^{q-1} e_{i_1} \cdots \widehat{e_{i_q}} \cdots e_{i_r}$, for which* $\operatorname{codim} H_{i_1} \cap \cdots \cap H_{i_r} < r$.

Since A is a quotient of an exterior algebra, multiplication by an element $a \in A^1$ gives a degree one differential on A, yielding a cochain complex (A, a):

$$(A,a)\colon \quad 0 \longrightarrow A^0 \xrightarrow{a} A^1 \xrightarrow{a} A^2 \xrightarrow{a} \cdots \xrightarrow{a} A^\ell \longrightarrow 0\ .$$

The complex (A, a) is exact as long as $\sum_{i=1}^n a_i \neq 0$; the *first resonance variety* $R^1(\mathcal{A})$ consists of points $a = \sum_{i=1}^n a_i e_i \leftrightarrow (a_1 : \cdots : a_n)$ in $\mathbb{P}(A^1) \cong \mathbb{P}^{d-1}$ for which $H^1(A, a) \neq 0$. Falk initiated the study of $R^1(A)$ in [10]; among his main innovations was the concept of a *neighborly partition*: a partition Π of $\mathcal{A}$ is neighborly if, for any rank two flat $Y \in L_2(\mathcal{A})$ and any block π of Π,

$$\mu(Y) \leq |Y \cap \pi| \Longrightarrow Y \subseteq \pi,$$

Falk showed that all components of $R^1(A)$ arise from such partitions, and conjectured that $R^1(\mathcal{A})$ was a subspace arrangement. This was proved, essentially simultaneously, by Cohen–Suciu [3] and Libgober–Yuzvinsky [21]; we will return to this in §4.

2.2. **The Orlik-Terao algebra.** In [25], Orlik and Terao introduced a commutative analog of the Orlik-Solomon algebra in order to answer a question of Aomoto.

Definition 2.4. *Let $\mathcal{A} = \cup_{i=1}^d V(\alpha_i) \subseteq \mathbb{P}^n$, and put $R = \mathbb{C}[y_1, \ldots, y_d]$. For each linear dependency $\Lambda = \sum_{j=1}^k c_{i_j} \alpha_{i_j} = 0$, define $f_\Lambda = \sum_{j=1}^k c_{i_j} (y_{i_1} \cdots \hat{y}_{i_j} \cdots y_{i_k})$, and let I be the ideal generated by the f_Λ. The Orlik-Terao algebra $C(\mathcal{A})$ is the quotient of $\mathbb{C}[y_1, \ldots, y_d]$ by I, and the Artinian Orlik-Terao algebra (the main object studied in [25]) is $C(\mathcal{A})/\langle y_1^2, \ldots, y_d^2 \rangle$.*

Example 2.5. Suppose $\mathcal{A} \subseteq \mathbb{P}^2$ is defined by the vanishing of $\alpha_1 = x_1, \alpha_2 = x_2, \alpha_3 = x_3, \alpha_4 = x_1 + x_2 + x_3$. The only relation is $\alpha_1 + \alpha_2 + \alpha_3 - \alpha_4 = 0$, so

$$C(\mathcal{A}) = \mathbb{C}[y_1, \ldots, y_4]/\langle y_2y_3y_4 + y_1y_3y_4 + y_1y_2y_4 - y_1y_2y_3 \rangle.$$

The homogeneous polynomial $y_2y_3y_4 + y_1y_3y_4 + y_1y_2y_4 - y_1y_2y_3$ is irreducible, hence defines a cubic surface in $\mathbb{P}^3$, and a computation shows that the surface has four singular points. A classical result in algebraic geometry is that the linear system of four cubics through six general points in $\mathbb{P}^2$ defines a map from the blowup of $\mathbb{P}^2$ at those points to $\mathbb{P}^3$ whose image is a smooth cubic surface. As the points move into special position the surface acquires singularities, as in this example.

In [34], properties of the Orlik-Terao algebra were studied in relation to 2–formality. An arrangement is 2–formal if any dependency among the linear forms defining the the arrangement can be obtained as a linear combination of dependencies which involve only three of the forms. Among the classes of 2–formal arrangements are $K(\pi, 1)$ arrangements and free arrangements. However, an example of Yuzvinsky [41] shows that 2–formality is not determined by the intersection lattice $L(\mathcal{A})$. The main result of [34] is that 2–formality is determined by the quadratic component of the Orlik-Terao ideal; the key is a computation on the tangent space of $V(I_2) \cap (\mathbb{C}^*)^{d-1}$.

2.3. **Blowups of $\mathbb{P}^2$.** Fix points $p_1, \ldots p_n \in \mathbb{P}^2$, and let

$$X \xrightarrow{\pi} \mathbb{P}^2 \tag{1}$$

be the blow up of $\mathbb{P}^2$ at these points. Then $Pic(X)$ is generated by the exceptional curves E_i over the points p_i, and the proper transform E_0 of a line in $\mathbb{P}^2$. A classical geometric problem asks for a relationship between numerical properties of a divisor $D_m = mE_0 - \sum a_i E_i$ on X, and the geometry of

$$X \xrightarrow{\phi} \mathbb{P}(H^0(D_m)^\vee).$$

First, some basics. Let m and a_i be non-negative, let I_{p_i} denote the ideal of a point p_i, and define

$$J = \bigcap_{i=1}^{n} I_{p_i}^{a_i} \subseteq \mathbb{C}[x, y, z] = S. \tag{2}$$

Then $H^0(D_m)$ is isomorphic to the m^{th} graded piece J_m of J (see [14]). In [5], Davis and Geramita show that if $\gamma(J)$ denotes the smallest degree t such that J_t defines J scheme theoretically, then D_m is very ample if $m > \gamma(J)$, and if $m = \gamma(J)$, then D_m is very ample iff J does not contain m collinear points, counted with multiplicity. Note that $\gamma(J) \leq reg(J)$. Now suppose that $\mathcal{A} = \cup_{i=1}^{d} L_i \subseteq \mathbb{P}^2$, and fix defining linear forms α_i so that $L_i = V(\alpha_i)$. Let X denote the blowup of $\mathbb{P}^2$ at $Sing(\mathcal{A}) = L_2(\mathcal{A})$. The central object of our investigations is the divisor

$$D_{\mathcal{A}} = (d-1)E_0 - \sum_{p_i \in L_2(\mathcal{A})} \mu(p_i) E_i. \tag{3}$$

2.4. **Main results.** For an arrangement $\mathcal{A} \subseteq \mathbb{P}^2$, let

$$X \xrightarrow{\phi_{\mathcal{A}}} \mathbb{P}(H^0(D_{\mathcal{A}})^\vee). \tag{4}$$

We show that $C(\mathcal{A})$ is the homogeneous coordinate ring of $\phi_{\mathcal{A}}(X)$, and that $\phi_{\mathcal{A}}$ is an isomorphism on $\pi^*(\mathbb{P}^2 \setminus \mathcal{A})$, contracts the lines of $\mathcal{A}$ to points, and blows up the singularities of $\mathcal{A}$. Combining results of Proudfoot-Speyer [27] and Terao [38], we bound the Castelnuovo-Mumford regularity of $C(\mathcal{A})$. Finally, we interpret the resonance varieties studied in [3], [10], [12], [21], [32], [42] in terms of linear subsystems of $D_{\mathcal{A}}$, and connect these jump loci to linear syzygies on $C(\mathcal{A})$.

3. Connecting $H^0(D_{\mathcal{A}})$ to the Orlik-Terao algebra

Let $\alpha = \prod_{i=1}^{d} \alpha_i$ and define a map $R = \mathbb{C}[y_1, \ldots, y_d] \longrightarrow \mathbb{C}[1/\alpha_1, \ldots, 1/\alpha_d] = T$. The kernel of this map is the OT ideal (see [34]), so $C(\mathcal{A}) \simeq T$. In [38], Terao proved that the Hilbert series for T is given by

$$HS(T, t) = P\Big(\mathcal{A}, \frac{t}{1-t}\Big). \tag{5}$$

In this section, we show that for $n = 2$, $C(\mathcal{A})$ is the homogeneous coordinate ring of the image of $X \xrightarrow{\phi_{\mathcal{A}}} \mathbb{P}(H^0(D_{\mathcal{A}})^\vee)$, with X as in Equation 1. For brevity, let $l_i = \alpha/\alpha_i$.

Lemma 3.1. *The ideal $L = \langle l_1, \ldots, l_d \rangle$ defines*

$$\bigcap_{p_i \in L_2(\mathcal{A})} I_{p_i}^{\mu(p_i)} \text{ scheme-theoretically.}$$

Proof. Localize at I_p, where $p \in L_2(\mathcal{A})$. Then in S_{I_p}, α_i is a unit if $p \notin V(\alpha_i)$. Without loss of generality, suppose forms $\alpha_1, \dots \alpha_m$ vanish on p, and the remaining forms do not. Thus,

$$L_{I_p} = \langle \alpha_2 \cdots \alpha_m, \alpha_1 \cdot \alpha_3 \cdots \alpha_m, \dots, \alpha_1 \cdots \alpha_{m-1} \rangle.$$

Now note that $I_p^{\mu(p)}$ has $\mu(p)+1$ generators of degree $\mu(p)$. Since $\mu(p) = m-1$ and the forms in L_{I_p} are linearly independent, equality follows. □

Lemma 3.2. *The minimal free resolution of S/L is*

$$0 \longrightarrow S(-d)^{d-1} \xrightarrow{\psi} S(-d+1)^d \xrightarrow{[\, l_1, \quad \cdots \quad , l_d \,]} S \longrightarrow S/L \longrightarrow 0, \text{ where}$$

$$\psi = \begin{bmatrix} \alpha_1 & 0 & \cdots & \cdots & 0 \\ -\alpha_2 & \alpha_2 & 0 & \cdots & \vdots \\ 0 & -\alpha_3 & \ddots & \ddots & \vdots \\ \vdots & 0 & & \ddots & 0 \\ \vdots & \vdots & \ddots & & \alpha_{d-1} \\ 0 & \cdots & \cdots & 0 & -\alpha_d \end{bmatrix}$$

Proof. The columns of ψ are syzygies on L. Since the maximal minors of ψ generate L, the result follows from the Hilbert-Burch theorem and Lemma 3.1. □

Theorem 3.3. $H^0(D_{\mathcal{A}}) \simeq Span_{\mathbb{C}}\{l_1, \dots, l_d\}$ *and* $H^1(D_{\mathcal{A}}) = 0 = H^2(D_{\mathcal{A}})$.

Proof. The remark following Equation 2 shows that $H^0(D_{\mathcal{A}}) \simeq J_{d-1}$. Since $K = -3E_0 + \sum E_i$, by Serre duality

$$H^2(D_{\mathcal{A}}) \simeq H^0((-d-2)E_0 + \sum_{p_i \in L_2(\mathcal{A})} (\mu(p_i)+1)E_i),$$

which is clearly zero. Using that X is rational, it follows from Riemann-Roch that

$$h^0(D_{\mathcal{A}}) - h^1(D_{\mathcal{A}}) = \frac{D_{\mathcal{A}}^2 - D_{\mathcal{A}} \cdot K}{2} + 1.$$

The intersection pairing on X is given by $E_i^2 = 1$ if $i = 0$, and -1 if $i \neq 0$, and

$$E_i \cdot E_j = 0 \text{ if } i \neq j.$$

Thus,

$$\begin{aligned} D_{\mathcal{A}}^2 &= (d-1)^2 - \sum_{p \in L_2(\mathcal{A})} \mu(p)^2 \\ -D_{\mathcal{A}}K &= 3(d-1) - \sum_{p \in L_2(\mathcal{A})} \mu(p), \end{aligned} \tag{6}$$

yielding

$$\begin{aligned} h^0(D_{\mathcal{A}}) - h^1(D_{\mathcal{A}}) &= \frac{\left(d-1\right)^2 - \sum \mu\left(p\right)^2 + 3\left(d-1\right) - \sum \mu\left(p\right)}{2} + 1 \\ &= \binom{d+1}{2} - \sum_{p \in L_2(\mathcal{A})} \binom{\mu(p)+1}{2}. \end{aligned} \tag{7}$$

Double counting the edges between $L_1(\mathcal{A})$ and $L_2(\mathcal{A})$ yields

$$\binom{d}{2} = \sum_{p \in L_2(\mathcal{A})} \binom{\mu(p)+1}{2},$$

hence $h^0(D_{\mathcal{A}}) - h^1(D_{\mathcal{A}}) = d$. From Lemmas 3.1 and 3.2 the Hilbert function satisfies

$$d = HF(\langle l_1, \ldots, l_d \rangle, d-1) = HF(\bigcap_{p_i \in L_2(\mathcal{A})} I_{p_i}^{\mu(p_i)}, d-1).$$

The observation after Equation 2 now implies that $h^0(D_{\mathcal{A}}) = d$. □

It follows from Theorem 3.3 that $C(\mathcal{A})$ is the coordinate ring of $\phi_{\mathcal{A}}(X)$. Note also that by Lemmas 3.1 and 3.2, the constant $\gamma(L) = d-1$, so

$$dE_0 - \sum_{p_i \in L_2(\mathcal{A})} \mu(p_i) E_i$$

is very ample, and gives a De Concini-Procesi wonderful model [6]: a compactification $\overline{M}$ of M such that $\overline{M} \setminus M$ is a normal crossing divisor. However, since every line of $\mathcal{A}$ contains exactly $d-1$ points counted with multiplicity, the divisor $D_{\mathcal{A}}$ is not very ample. The description of $C(\mathcal{A})$ makes it obvious that $V(I)$ is an irreducible, nondegenerate rational variety, and by [34] $V(I) \setminus V(y_1 \cdots y_d)$ is smooth. Here is a more explicit description of the map:

Theorem 3.4. *The map* $\phi_{\mathcal{A}}$

(1) *is an isomorphism on* $\pi^*(\mathbb{P}^2 \setminus \mathcal{A})$.
(2) *contracts the lines of* $\mathcal{A}$ *to points on* X.
(3) *takes* E_p *to a rational normal curve of degree* $\mu(p)$.

Proof. For the first part, without loss of generality suppose that $\alpha_1 \cdot \alpha_2 \cdot \alpha_3 = xyz$ and write $L = \prod_{i=4}^{d} \alpha_i$. Then

$$\phi_{\mathcal{A}} = [yzL, xzL, xyL, \ldots].$$

Thus, the first three entries of $\phi_{\mathcal{A}}$ define the Cremona transformation, which is an isomorphism from $\mathbb{P}^2 \setminus V(xyz)$ to itself. Since $\mathbb{P}^2 \setminus \mathcal{A}$ is contained in $\mathbb{P}^2 \setminus V(xyz)$, (1) follows. For (2), suppose p is a point of $V(\alpha_i)$. Since α_i divides l_j for all $j \neq i$, this means $l_j(p) = 0$ if $j \neq i$. Hence $\phi_{\mathcal{A}}(V(\alpha_i))$ is the i^{th} coordinate point of $\mathbb{P}^{d-1}$. The final part follows from the fact that $D_{\mathcal{A}}|_{E_p}$ is a divisor on E_p of degree $D_{\mathcal{A}} \cdot E_p = \mu(p)$, and $E_p \simeq \mathbb{P}^1$. □

3.1. **Castelnuovo–Mumford regularity and graded betti numbers.** The Castelnuovo–Mumford regularity of a coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n$ is usually phrased in terms of vanishing of certain cohomology modules. Letting $N = \oplus_n H^0(\mathcal{F}(n))$, we may [8] rephrase the condition as

Definition 3.5. *For a polynomial ring* R*, a finitely generated, graded* R*–module* N *has Castelnuovo-Mumford regularity* j *if* j *is the smallest number such that* $Tor_i^R(N, \mathbb{C})_{i+j+1} = 0$ *for all* i*. The graded betti numbers of a graded* R*–module* N *are indexed by*

$$b_{ij} = \dim_{\mathbb{C}} \operatorname{Tor}_i^R(N, \mathbb{C})_j.$$

Example 3.6. We revisit Example 2.1. The four triple points yield four quadratic generators for the Orlik-Terao ideal I. These four quadrics generate I (see [34]), and a computation in `Macaulay2` yields the graded betti numbers of $C(\mathcal{A})$:

total	1	4	5	2	
0	1	–	–	–	–
1	–	4	2	–	–
2	–	–	3	2	

This diagram is read as follows: the entry in position (i, j) is simply $b_{i,i+j}$, e.g.

$$\dim_{\mathbb{K}} \mathrm{Tor}_2^R(C(\mathcal{A}), \mathbb{C})_4 = 3.$$

The betti table has a very nice interpretation in terms of Castelnuovo-Mumford regularity: the regularity is the index of the last nonzero row.

Theorem 3.7. *For $\mathcal{A} \subseteq \mathbb{P}^n$, $C(\mathcal{A})$ is n–regular.*

Proof. In [27], Proudfoot and Speyer show that the Orlik-Terao algebra is Cohen-Macaulay (for $n = 2$ this also follows from Theorem 3.3). Thus, there exists a regular sequence on $C(\mathcal{A})$ of $\dim(V(I)) + 1 = n + 1$ linear forms; quotienting by this sequence yields an Artinian ring whose Hilbert series is the numerator of the Hilbert series of $C(\mathcal{A})$. The regularity of an Artinian module is equal to the length of the module, so the result follows from Equation 5. $\square$

It follows easily from Theorem 3.7 and Terao's work in [38] that

Proposition 3.8. *For $\mathcal{A} \subseteq \mathbb{P}^n$ with $|\mathcal{A}| = d$, if $I = I_2$, then*

$$\dim_{\mathbb{C}} Tor_2^R(C(\mathcal{A}), \mathbb{C})_3 = 2\Big(\binom{d}{3} - 1\Big) - \Big(d - 3\Big)\Big(\sum_{p_i \in L_2(\mathcal{A})} \mu(p_i) + 1\Big).$$

Example 3.9. The A_3 arrangement is supersolvable, so by [34] $I = I_2$, and Proposition 3.8 shows there are two linear first syzygies on I. This explains the top row of the betti table in Example 3.6.

4. Nets, syzygies, and scrolls

In [21], Libgober-Yuzvinsky found a surprising connection between nets and the first resonance variety. The approach was further developed by Yuzvinsky in [42], with a beautiful complete picture emerging in Falk and Yuzvinsky's paper on multinets [12]. In this section, we connect nets to the linear syzygies of $C(\mathcal{A})$, and hence to $R^1(\mathcal{A})$. This allows us to give an interpretation of the first resonance variety in terms of the geometry of X_A.

Suppose Z is a subset of the intersection points of $\mathcal{A}$, and let J denote the $|Z| \times d$ incidence matrix of points and lines and E denote a $d \times d$ matrix with every entry one. If $\widehat{Z}$ is the blowup of $\mathbb{P}^2$ at the points of Z, then [21] shows that

$$J^t J - E = Q(\widehat{Z})$$

is the intersection form on $\widehat{Z}$, and is a generalized Cartan matrix. Using the Vinberg classification of such matrices [19], they show that any component of $R^1(A)$ corresponds to a choice of points Z such that $Q(\widehat{Z})$ consists of at least three affine blocks, with no finite or indefinite blocks, and the block sum decomposition of $Q(\widehat{Z})$ yields a neighborly partition. Before going into the details of the connection between multinets, divisors and syzygies, we give a pair of motivating examples.

Example 4.1. The matroid $(9_3)_2$ of Hilbert and Cohn-Vossen is realized below by $\mathcal{A} = V(xyz(x+y)(y+z)(x+3z)(x+2y+z)(x+2y+3z)(2x+3y+3z))$. It has nine triple points and nine double points, thus $P(\mathcal{A}, t) = (1+t)(1+8t+19t^2)$.

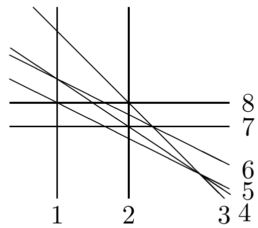

FIGURE 2. An arrangement realizing $(9_3)_2$

The graded betti numbers for $C((9_3)_2)$ are:

total	1	11	75	156	145	66	12
0	1	–	–	–	–	–	–
1	–	9	–	–	–	–	–
2	–	2	75	156	145	66	12

Example 4.2. The $(9_3)_1$ matroid of Hilbert and Cohn-Vossen is realized below by $\mathcal{A} = V(xyz(x-y)(y-z)(x-y-z)(2x+y+z)(2x+y-z)(2x-5y+z))$. It has nine triple points and nine double points, so $P((9_3)_1, t) = P((9_3)_2, t)$.

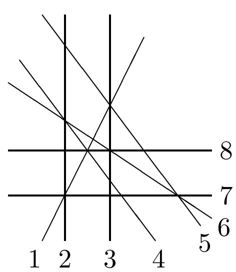

FIGURE 3. An arrangement realizing $(9_3)_1$

However, the graded betti numbers for $C((9_3)_1)$ are:

total	1	13	77	156	145	66	12
0	1	–	–	–	–	–	–
1	–	9	2	–	–	–	–
2	–	4	75	156	145	66	12

The arrangement $(9_3)_1$ possesses a pair of linear first syzygies, while $(9_3)_2$ has no linear first syzygies. An easy check shows that $(9_3)_1$ admits a neighborly partition $|169|258|347|$ and has a corresponding non-local component (see below) in $R^1(A)$, whereas $(9_3)_2$ does not. To better understand the connection between $R^1(A)$ and syzygies, we now review two constructions.

4.1. **Nets and multinets.** It is easy to see that any $p \in L_2(\mathcal{A})$ with $\mu(p) \geq 2$ yields a component $\mathbb{P}^{\mu(p)-1} \subseteq R^1(A)$. Such components are called *local components*. Components which are not of this type are called *essential*. In [42], Yuzvinsky used *nets* to analyze the essential components of $R^1(A)$.

Definition 4.3. *Let $3 \leq k \in \mathbb{Z}$. A k-net in $\mathbb{P}^2$ is a pair $(\mathcal{A}, Z)$ where $\mathcal{A}$ is a finite set of distinct lines partitioned into k subsets $\mathcal{A} = \bigcup_{i=1}^k \mathcal{A}_i$ and Z is a finite set of points, such that:*

(1) *for every $i \neq j$ and every $L \in \mathcal{A}_i,\ L' \in \mathcal{A}_j,\ L \cap L' \in Z$.*
(2) *for every $p \in Z$ and every $i \in \{1, \ldots, k\}$, $\exists$ a unique $L \in A_i$ containing Z.*

Thus, for a k-net, $|A_i| = |L \cap Z|$ for any block A_i and line $L \in \mathcal{A}$; denote this number by m. Following Yuzvinsky, we call m the order of the net, and refer to a k-net of order m as a (k, m)-net; note that $|Z| = m^2$. Yuzvinsky shows in [42] that a net must have $k \in \{3, 4, 5\}$, and improves this in [43] to $k \in \{3, 4\}$.

In [12], Falk and Yuzvinsky extend the notion of a net to a multinet; in a multinet lines may occur with multiplicity. Write $\mathcal{A}_w$ for a multiarrangement, where $w \in \mathbb{N}^d$, and $w(L)$ denotes the multiplicity of a line.

Definition 4.4. *A weak (k, m)-multinet on a multi-arrangement $\mathcal{A}_w$ is a pair (Π, Z) where Π is a partition of $\mathcal{A}_w$ into $k \geq 3$ classes $A_1, \ldots, A_k$, and Z is a set of multiple points, such that*

(1) $\sum_{L \in A_i} w(L) = m$, *independent of i.*
(2) *For every $L \in A_i$ and $L' \in A_j$, with $i \neq j$, $L \cap L' \in Z$.*
(3) *For each $p \in Z$, $\sum_{L \in A_i, p \in L} w(L)$ is a constant n_p, independent of i.*

A multinet is a weak multinet satisfying the additional property

(4) *For $i \in \{1, \ldots, k\}$ and $L, L' \in A_i$, $\exists$ a sequence $L = L_0, L_1, \ldots, L_r = L'$ such that $L_{j-1} \cap L_j \notin Z$ for $1 \leq j \leq r$.*

Example 4.5. The reflection arrangement of type B_3 is depicted below (there is also a line at infinity). Falk and Yuzvinsky show that this arrangement supports a multinet which is not a net: assign weight two to lines $(3, 6, 8)$ and weight one to the remaining lines.

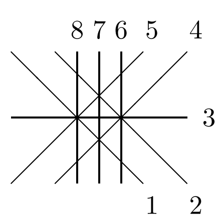

FIGURE 4. The B_3-arrangement

The following lemma of [12] will be useful:

Lemma 4.6. *Suppose $(\mathcal{A}_w, Z)$ is a weak (k, m)-multinet. Then*

(1) $\sum_{L \in \mathcal{A}_w} w(L) = km$.
(2) $\sum_{p \in Z} n_p^2 = m^2$
(3) *For each $L \in \mathcal{A}_w$, $\sum_{p \in Z \cap L} n_p = m$.*

4.2. **Determinantal syzygies and factoring divisors.** One simple way in which linear syzygies can arise comes from a factorization of divisors. First, a definition

Definition 4.7. *A matrix of linear forms is* $1-$ *generic if it has no zero entry, and cannot be transformed by row and column operations to have a zero entry.*

For $Y \subseteq \mathbb{P}^n$ irreducible and linearly normal, if there exist line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ such that $\mathcal{O}_Y(1) = \mathcal{L}_1 \otimes \mathcal{L}_2$ with $h^0(\mathcal{L}_i) = a_i$, then the $a_1 \times a_2$ matrix γ representing the multiplication table

$$H^0(\mathcal{L}_1) \otimes H^0(\mathcal{L}_2) \longrightarrow H^0(\mathcal{O}_Y(1))$$

is 1–generic. More explicitly (see [8]), if

$$H^0(\mathcal{L}_1) = Span_{\mathbb{C}}\{e_1, \dots, e_{a_1}\} \text{ and } H^0(\mathcal{L}_2) = Span_{\mathbb{C}}\{f_1, \dots, f_{a_2}\},$$

then γ has (i,j) entry $e_i \otimes f_j$, corresponding to a linear form on $\mathbb{P}^n$, and elements of the ideal $I_2(\gamma)$ of 2×2 minors of γ vanish on Y. The most familiar example occurs when $a_1 = 2$ and $a_2 = k$. In this case, the minimal free resolution of $I_2(\gamma)$ is an Eagon-Northcott complex. This relates to geometry via scrolls: let Ψ be the locus of points where a 1–generic matrix

$$\gamma = \begin{bmatrix} l_1 & \cdots & l_k \\ m_1 & \cdots & m_k \end{bmatrix}$$

has rank one. If

$$L_{[\lambda:\nu]} = \{p \in \mathbb{P}^n \mid \lambda l_1(p) + \mu m_1(p) = \cdots = \lambda l_k(p) + \mu m_k(p) = 0\},$$

then (see 9.10 of [16])

$$\Psi = \bigcup_{[\lambda:\nu] \in \mathbb{P}^1} L_{[\lambda:\nu]},$$

where $L_{[\lambda:\nu]} \simeq \mathbb{P}^{n-k}$, so Ψ is a union of linear spaces. Geometrically, the zero locus of the 2×2 minors of γ is a scroll which contains $V(I_Y)$.

4.3. **Connecting nets and determinantal syzygies.** The computation in the proof of Theorem 3.3 and the fact that $h^1(D) \geq 0$ shows that if $D_{\mathcal{A}} = A + B$ with $A = mE_0 - \sum a_i E_i$, then

$$h^0(A) \geq \binom{m+2}{2} - \sum_{p \in L_2(\mathcal{A})} \binom{a_i+1}{2}, \; h^0(B) \geq \binom{d+1-m}{2} - \sum_{p \in L_2(\mathcal{A})} \binom{\mu(p)-a_i+1}{2}.$$

For an arrangement $\mathcal{A}$, if there exists a choice of parameters m and a_i such that $h^0(A) = a \geq 2$ and $h^0(B) = b \geq 3$, then the results of the previous section show that there will exist linear first syzygies on I.

Example 4.8. We revisit Example 4.2. Let $A = 3E_0 - \sum_{\{p|\mu(p)=2\}} E_p$. Clearly $A^2 = AK = 0$, so we can only guarantee that $h^0(A) \geq 1$. In fact, $h^0(A) = 2$, hence $h^1(A) = 1$. To see this, note that a direct computation shows that the space of cubics passing through the nine multiple points of $\mathcal{A}$ is two dimensional. Since

$$Span_{\mathbb{C}}\Big(L_1L_6L_9, L_3L_4L_7, L_2L_5L_8\Big) \subseteq H^0(A),$$

and any two of these are independent, we see that the sections are given by the net. Next, consider the residual divisor $B = D_{\mathcal{A}} - A$. Since $B^2 = 16 - 18 = -2$ and $-BK = 15 - 9 = 6$, we have that $h^0(B) \geq 3$. In fact, equality holds, so I contains the 2×2 minors of a 2×3 matrix of linear forms, explaining the linear syzygies.

Lemma 4.9. *A (k,m) multinet gives a divisor A on X such that $h^0(A)=2$.*

Proof. Let
$$A = mE_0 - \sum_{p\in Z} n_p E_p.$$
Condition (1) of Definition 4.4 implies that for each block A_i of the multinet, $\prod_{L\in A_i} L^{w(L)}$ is homogeneous of degree m, and Condition (3) shows that it vanishes to order exactly n_p on E_p. In particular, this shows that $\prod_{L\in A_i} L^{w(L)} \in H^0(A)$. If all the blocks of the partition were independent, then this would imply that $h^0(A) \geq k$, but it turns out that the sections are all fibers of a pencil of plane curves, which follows from Theorem 3.11 of [12]. □

In [12], Falk and Yuzvinsky show that the following are equivalent:

(1) $R^1(A)$ contains a nonlocal component $\simeq \mathbb{P}^{k-2}$.
(2) $\mathcal{A}$ supports a (k,m) multinet.
(3) $\exists$ a pencil of plane curves with connected fibers, with at least three fibers (loci of) products of linear forms, and $\mathcal{A}$ is the union of all such fibers.

In general, determining the dimension of $h^1(D)$ for $D \in Pic(X)$ is not easy. However, in the special case of a net, there is enough information to give a lower bound for the dimension of the sections of the residual divisor which is often exact.

Theorem 4.10. *If $\mathcal{A}$ is a (k,m) net, then $D_{\mathcal{A}} = A+B$, with*
$$h^0(A) = 2 \text{ and } h^0(B) \geq km - \binom{m+1}{2}.$$

Proof. For a (k,m) net, all lines occur with multiplicity one. Let
$$A = mE_0 - \sum_{p\in Z} E_p.$$
By Lemma 4.9, $h^0(A)=2$. Since $B = D_{\mathcal{A}} - mE_0 + \sum\limits_{p\in Z} E_p$, $\frac{B^2-BK}{2}+1$
$$\begin{array}{rcl} = & \frac{(d-m-1)(d-m-1+3)+2}{2} + (\sum\limits_{p\in L_2(\mathcal{A})} \mu(p)E_p + \sum\limits_{p\in Z} E_p)(\sum\limits_{p\in L_2(\mathcal{A})} (\mu(p)-1)E_p + \sum\limits_{p\in Z} E_p) \\ = & \binom{d+1-m}{2} - \binom{d}{2} + \sum\limits_{p\in Z}\mu(p) \\ = & \binom{m}{2} - d(m-1) + \sum\limits_{p\in Z}\mu(p). \end{array}$$
We now compute that
$$\begin{array}{rcl} \sum\limits_{p\in Z}\mu(p) + |Z| & = & \sum\limits_{p\in Z}(\mu(p)+1) \\ & = & k\sum\limits_{p\in Z} n_p \\ & = & km^2. \end{array}$$
The second line follows since n_p lines from each block A_i pass through p, and there are k blocks. The third line follows from Lemma 4.6 and the fact that $n_p=1$ for a net, hence $n_p^2 = n_p$. Since for a (k,m)–net $|Z| = m^2$,
$$\sum_{p\in Z}\mu(p) = (k-1)m^2 = dm - m^2.$$

Combining this with the previous calculation shows that for a (k,m)–net

$$h^0(B) = h^1(B) + \binom{m}{2} - d(m-1) + dm - m^2 \geq d - \binom{m+1}{2}.$$

Recalling that $km = d$ concludes the proof. □

Corollary 4.11. *If $\mathcal{A}$ is a (k,m) net with $k \geq m$, then I contains the 2×2 minors of a 1-generic $2 \times \left(km - \binom{m+1}{2}\right)$ matrix. Thus the resolution of I contains an Eagon-Northcott complex as a subcomplex.*

Proof. Since $k \in \{3,4\}$, if $(k,m) = (3,2)$ or $(3,3)$ then by Theorem 4.10 $h^0(B) \geq 3$, and if $(k,m) = (4,3)$ or $(4,4)$ then $h^0(B) \geq 6$. Note that the only known example of a 4-net is the $(4,3)$ net corresponding to the Hessian configuration. □

Example 4.12. For the arrangement A_3 appearing in Example 2.1, Z is the collection of multiple points, and

$$A = 2E_0 - \sum_{\{p|\mu(p)=2\}} E_p$$

and

$$B = 3E_0 - \sum_{p \in L_2(\mathcal{A})} E_p.$$

So $d - \binom{m+1}{2} = 6 - 3 = 3$ and I contains the 2×2 minors of a 2×3 matrix.

5. Connection to Derivations

In this section, we show that the generators of the Jacobian ideal of $\mathcal{A} \subseteq \mathbb{P}^2$ are contained in $H^0(D_{\mathcal{A}})$, and that the associated projection map $X \to \mathbb{P}^2$ has degree $\sum_{p\in L_2(\mathcal{A})} \mu(p) - |\mathcal{A}| + 1$. This relates X to one of the fundamental objects in arrangement theory: the module $D(\mathcal{A})$ of derivations tangent to $\mathcal{A}$.

Definition 5.1. $D(\mathcal{A}) = \{\theta \mid \theta(\alpha_i) \in \langle \alpha_i \rangle$ *for all* α_i *such that* $V(\alpha_i) \in \mathcal{A}\}$.

The module $D(\mathcal{A})$ is a graded $S = \mathbb{C}[x_0,\ldots,x_n]$ module, and over a field of characteristic zero, $D(\mathcal{A}) \simeq E \oplus D_0(\mathcal{A})$, where E is the Euler derivation and $D_0(\mathcal{A})$ corresponds to the module of syzygies on the Jacobian ideal J_α of the defining polynomial $\alpha = \prod_{i=1}^d \alpha_i$ of $\mathcal{A}$. An arrangement $\mathcal{A}$ is *free* if $D(\mathcal{A}) \simeq \oplus S(-a_i)$; the a_i are called the *exponents* of $\mathcal{A}$. Terao's theorem [37] is that if $D(\mathcal{A}) \simeq \oplus S(-a_i)$, then $P(M,t) = \prod(1+a_i t)$. Supersolvable arrangements are free, so Example 2.1 is of this type, and

$$P(A_3,t) = (1+t)(1+2t)(1+3t)$$

On the other hand, the arrangement $\mathcal{A}$ of Example 2.5 is not free, and $P(\mathcal{A},t)$ does not factor. However, it is shown in [30] that for $\mathcal{A} \subseteq \mathbb{P}^2$ the Poincare polynomial is $(1+t)\cdot c_t(D_0^\vee)$, where c_t is the Chern polynomial and $D_0^\vee$ is the dual of the rank two vector bundle associated to D_0. An easy localization argument [31] shows that in this case, the Jacobian ideal is a local complete intersection with

$$deg(J_\alpha) = \sum_{p\in L_2(\mathcal{A})} \mu(p)^2.$$

For $\mathcal{A} \subseteq \mathbb{P}^n$ with $n \geq 3$, some generalizations are possible, see [22].

Proposition 5.2. *For an arrangement $\mathcal{A} \subseteq \mathbb{P}^2$,*

$$J_\alpha \subseteq L = \langle l_1, \ldots, l_d \rangle = \langle \frac{\alpha}{\alpha_1}, \ldots, \frac{\alpha}{\alpha_d} \rangle.$$

Proof. By Lemma 3.2,

$$L = \bigcap_{p \in L_2(\mathcal{A})} I_p^{\mu(p)} \subseteq \mathbb{C}[x, y, z].$$

The ideal J_α is generated in degree $d-1$. The result of [31] mentioned above implies that at any point $p \in L_2(\mathcal{A})$, the localization $(J_\alpha)_p$ is a local complete intersection: changing coordinates so that $p = (0:0:1)$, and writing $\mathcal{A} = V(L_0 L_1)$ with L_0 the product of the defining linear forms which vanish at p and L_1 the product of the remaining forms, we have

$$(J_\alpha)_p = \langle \partial(L_0)/\partial_x, \partial(L_0)/\partial_y \rangle.$$

In particular, both generators are of degree $\mu(p)$, so in the primary decomposition of J_α, the primary component associated to $I(p)$ is contained in $I_p^{\mu(p)}$. Now,

$$Sing(\mathcal{A}) = \bigcup_{I(p) \in Ass(\sqrt{J_\alpha})} V(I(p)).$$

If Q_p is the $I(p)$-primary component of J_α, then

$$\bigcap_{I(p) \in S} Q_p \subseteq L.$$

Since the saturation of J_α with respect to $\langle x, y, z \rangle$ is the left hand side, and J_α is generated in degree $d-1$, the result follows. □

The inclusion $W = J_\alpha \subseteq H^0(D_\mathcal{A})$ corresponds to an induced map

$$\begin{tikzcd} \mathbb{P}^2 \setminus V(\alpha) \arrow[r, "\phi_\mathcal{A}"] \arrow[dr, "\psi"'] & \mathbb{P}(H^0(D_\mathcal{A})) \arrow[d, "\pi"] \\ & \mathbb{P}(W) \end{tikzcd}$$

Proposition 5.3. *The degree of π is $\sum_{p \in L_2(\mathcal{A})} \mu(p) - |\mathcal{A}| + 1$.*

Proof. In [7], Dimca and Papadima show that on a projective hyperplane complement $\mathbb{P}^n \setminus V(\alpha)$, the degree of the gradient map ψ

$$\mathbb{P}^n \setminus V(\alpha) \xrightarrow{\left[\ \partial(\alpha)/\partial x_0 : \quad \cdots \quad : \partial(\alpha)/\partial x_n\ \right]} \mathbb{P}^n$$

is equal to $b_n(\mathbb{P}^n \setminus V(\alpha))$; for a configuration of $\mathcal{A} \subseteq \mathbb{P}^2$ this means the degree of the gradient map is

$$\sum_{p \in L_2(\mathcal{A})} \mu(p) - |\mathcal{A}| + 1.$$

By Theorem 3.4, ϕ_A is an isomorphism on $\mathbb{P}^2 \setminus V(\alpha)$, and the result follows. □

Concluding Remarks and Questions

(1) To study $Tor_i^R(C(A), \mathbb{C})_{i+1}$, it suffices to restrict to the case of line arrangements. This follows since the quadratic generators of $C(A)$ depend only on $L_2(\mathcal{A})$, hence taking the intersection of $\mathcal{A}$ with a generic $\mathbb{P}^2$ leaves these generators (and relations among them) unchanged. The graded betti numbers for I_2 are not combinatorial invariants; it would be interesting to understand how the geometry of $\mathcal{A}$ governs b_{ij}, even for $j = i + 1$.
(2) Can freeness of $D(\mathcal{A})$ be related to the surface X and divisor $D_{\mathcal{A}}$? It seems possible that there is a connection between $D_{\mathcal{A}}$ and multiarrangements, studied recently in [1], [39], [40].
(3) For $n \geq 3$, is $C(\mathcal{A})$ the homogeneous coordinate ring of a blowup of $\mathbb{P}^n$ along a locus related to the arrangement $\mathcal{A}$? Since $C(\mathcal{A})$ is Cohen-Macaulay, if this is true, then combining Riemann-Roch with Terao's result would yield a formula for the global sections of $D_{\mathcal{A}}$.

Acknowledgements Computations were performed using Macaulay2, by Grayson and Stillman, and available at: `http://www.math.uiuc.edu/Macaulay2/`. Many of the examples in this paper appear in Suciu's survey [36]; that paper is also an excellent reference for the many questions on resonance varieties not treated here. This work began during a visit to the Fields institute for a conference in honour of Peter Orlik.

References

[1] T. Abe, H. Terao, M. Wakefield, *The e-multiplicity and addition-deletion theorems for multiarrangements*, J. Lond. Math. Soc. **77** (2008), 335–348.
[2] W. Arvola, *The fundamental group of the complement of an arrangement of complex hyperplanes*, Topology **31** (1992), 757–765.
[3] D. Cohen, A. Suciu, *The braid monodromy of plane algebraic curves and hyperplane arrangements*, Comment. Math. Helv. **72** (1997), 285–315.
[4] D. Cohen, A. Suciu, *Characteristic varieties of arrangements*, Math. Proc. Cambridge Phil. Soc. **127** (1999), 33–53.
[5] E. Davis, A. Geramita, *Birational morphisms to $\mathbb{P}^2$: an ideal-theoretic perspective*, Math. Ann. **279** (1988), 435–448.
[6] C. De Concini, C. Procesi, *Wonderful models of subspace arrangements.* Selecta Math. **1** (1995), no. 3, 459–494.
[7] A. Dimca, S. Papadima, *Hypersurface complements, Milnor fibers and higher homotopy groups of arrangments*, Ann. Math. **158** (2003), 473–507.
[8] D. Eisenbud, *The geometry of syzygies*, Graduate Texts in Mathematics, vol. 229, Springer-Verlag, Berlin-Heidelberg-New York, 2005.
[9] D. Eisenbud, *Linear sections of determinantal varieties*, Amer. J. Math. **110** (1988), no. 3, 541–575.
[10] M. Falk, *Arrangements and cohomology*, Ann. Combin. **1** (1997), 135–157.
[11] M. Falk, *Resonance varieties over fields of positive characteristic*, IMRN **3** (2007)
[12] M. Falk, S. Yuzvinsky, *Multinets, resonance varieties, and pencils of plane curves*, Compos. Math. **143** (2007), 1069–1088 MR 2339840
[13] H. Hamm, D. T. Lê, *Un théorème de Zariski du type de Lefschetz*, Ann. Sci. École Norm. Sup. **6** (1973), 317–366.
[14] B. Harbourne, *Problems and progress: survey on fat points in $\mathbb{P}^2$*, Queens Papers in Pure and Applied Mathematics **123** (2002), 85-132.
[15] B. Harbourne, *Free resolutions of fat point ideals on $\mathbb{P}^2$*. J. Pure Appl. Algebra **125** (1998), 213–234.
[16] J. Harris, *Algebraic Geometry*, Graduate Texts in Mathematics, vol. 133, Springer-Verlag, Berlin-Heidelberg-New York, 1992.
[17] F. Hirzebruch, *Arrangements of lines and algebraic surfaces*, In: Arithmetic and Geometry, vol. II, Progress in Math., vol. 36, Birkhäuser, Boston, 1983, pp. 113–140.

[18] P. Lima-Filho, H. Schenck, *The holonomy Lie algebra of subarrangements of A_n*, IMRN, (2009), 1421-1432.

[19] V. Kac, *Infinite dimensional Lie algebras*, Cambridge University Press, 1990.

[20] J. Koh, M. Stillman, *Linear syzygies and line bundles on an algebraic curve*, J. Algebra **125** (1989), 120–132.

[21] A. Libgober, S. Yuzvinsky, *Cohomology of the Orlik-Solomon algebras and local systems*, Compositio Math. **121** (2000), 337–361.

[22] M. Mustaţă, H. Schenck, *The module of logarithmic p-forms of a locally free arrangement*, J. Algebra, **241** (2001), 699-719.

[23] P. Orlik, L. Solomon, *Combinatorics and topology of complements of hyperplanes*, Invent. Math. **56** (1980), 167–189.

[24] P. Orlik, H. Terao, *Arrangements of Hyperplanes*, Grundlehren Math. Wiss., Bd. 300, Springer-Verlag, Berlin-Heidelberg-New York, 1992.

[25] P. Orlik, H. Terao, *Commutative algebras for arrangements*, Nagoya Math Journal, **134** (1994), 65-73.

[26] S. Papadima, A. Suciu, *Chen Lie algebras*, IMRN **2004:21** (2004), 1057–1086.

[27] N. Proudfoot, D. Speyer, *A broken circuit ring*, Beiträge zur Algebra und Geometrie, **47** (2006), 161-166.

[28] R. Randell, *The fundamental group of the complement of a union of complex hyperplanes*, Invent. Math. **69** (1982), 103–108.

[29] M. Salvetti, *Topology of the complement of real hyperplanes in $\mathbb{C}^N$*, Invent. Math. **88** (1987), 603–618.

[30] H. Schenck, *A rank two vector bundle associated to a three arrangement, and its Chern polynomial*, Advances in Mathematics, **149** (2000), 214-229.

[31] H. Schenck, *Elementary modifications and line configurations in $\mathbb{P}^2$*, Commentarii Mathematici Helvetici, **78** (2003), 447-462.

[32] H. Schenck, A. Suciu, *Lower central series and free resolutions of hyperplane arrangements*, Trans. Amer. Math. Soc. **354** (2002), 3409–3433.

[33] H. Schenck, A. Suciu, *Resonance, linear syzygies, Chen groups, and the Bernstein-Gelfand-Gelfand correspondence*, Trans. Amer. Math. Soc. **358** (2006), 2269-2289.

[34] H. Schenck, S. Tohaneanu, *The Orlik-Terao algebra and 2-formality*, Math. Res. Lett.**16** (2009), 171–182.

[35] R. Stanley, *Supersolvable lattices*, Algebra Universalis **2** (1972), 197–217.

[36] A. Suciu, *Fundamental groups of line arrangements: Enumerative aspects*, Contemporary Math. **276** (2001), AMS, Providence, RI, 43–79.

[37] H. Terao, *Generalized exponents of a free arrangement of hyperplanes and Shepard-Todd-Brieskorn formula*, Inventiones Mathematicae **63** (1981), 159–179.

[38] H. Terao, *Algebras generated by reciprocals of linear forms*, J. Algebra, **250** (2002), 549–558.

[39] M. Yoshinaga, *Characterization of a free arrangement and conjecture of Edelman and Reiner*, Invent. Math. **157** (2004), no. 2, 449–454.

[40] M. Yoshinaga, *On the freeness of* 3*-arrangements*, Bull. London Math. Soc. **37** (2005), no. 1, 126–134.

[41] S. Yuzvinsky, *First two obstructions to the freeness of arrangements*, Transactions of the A.M.S., **335** (1993), 231-244.

[42] S. Yuzvinsky, *Realization of finite Abelian groups by nets in $\mathbb{P}^2$*, Compos. Math. **140** (2004), 1614–1624.

[43] S. Yuzvinsky, *A new bound on the number of special fibers in a pencil of plane curves*, Proc. Amer. Math. Soc. **137** (2009), 1641–1648.

Schenck: Mathematics Department, University of Illinois, Urbana, IL 61801, USA
E-mail address: schenck@math.uiuc.edu