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HIGH RANK LINEAR SYZYGIES ON LOW RANK QUADRICS

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Abstract. We study the linear syzygies of a homogeneous ideal $I \subseteq S = Sym_{\Bbbk}(V)$, focussing on the graded betti numbers

$$b_{i,i+1} = \dim_{\Bbbk} Tor_i(S/I, \Bbbk)_{i+1}.$$

For a variety X and divisor D with $V = H^0(D)$, what conditions on D ensure that $b_{i,i+1} \neq 0$? In [4], Eisenbud shows that a decomposition $D \sim A + B$ such that A and B have at least two sections gives rise to determinantal equations (and corresponding syzygies) in I_X; and in [2] conjectures that if I_2 is generated by quadrics of rank ≤ 4, then the last nonvanishing $b_{i,i+1}$ is a consequence of such equations. We describe obstructions to this conjecture and prove a variant. The obstructions arise from toric specializations of the Rees algebra of Koszul cycles, and we give an explicit construction of toric varieties with minimal linear syzygies of arbitrarily high rank. This gives one answer to a question posed by Eisenbud and Koh in [5].

1. Introduction

Let I be a homogeneous ideal in $S = Sym_{\Bbbk}(V)$, with $\Bbbk$ algebraically closed and characteristic zero; we are primarily interested in the case that I is the ideal of an irreducible, nondegenerate variety in $\mathbb{P}(V)$, and in the graded betti numbers

$$b_{ij} = dim_{\Bbbk} Tor_i(S/I, \Bbbk)_j.$$

Definition 1.1. *The length of the 2-linear strand* $2LP(S/I) = \max\{i | b_{i,i+1} \neq 0\}$.

In particular, I has a 2-linear n^{th}–syzygy iff $2LP(S/I) \geq n + 1$.

Example 1.2. The twisted cubic $I_C \subseteq S = \Bbbk[x, y, z, w]$ has resolution

$$0 \longrightarrow S(-3)^2 \xrightarrow{\begin{bmatrix} -z & w \\ y & -z \\ -x & y \end{bmatrix}} S(-2)^3 \xrightarrow{\begin{bmatrix} y^2 - xz & yz - xw & z^2 - yw \end{bmatrix}} S \longrightarrow S/I_C$$

So $2LP(S/I_C) = 2$; in betti diagram notation [4] the b_{ij} for S/I_X are:

total	1	3	2
0	1	–	–
1	–	3	2

The Eagon-Northcott complex gives a free resolution for I_C, which is determined by the maximal minors of the matrix of first syzygies; this simple example provides the key intuition: if $D \sim A + B$, then $h \in H^0(A)$ and $g \in H^0(B)$ yield an element

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$f = g \cdot h \in H^0(D)$. Factoring a divisor of degree three on $\mathbb{P}^1$ into divisors of degree one and two yields a 2×3 matrix for $(s,t) \otimes (s^2, st, t^2)$:

$$\begin{bmatrix} st^2 & t^3 \\ s^2t & st^2 \\ s^3 & s^2t \end{bmatrix}$$

So I_C contains the 2×2 minors of a 2×3 matrix of linear forms. This matrix is of a special type:

Definition 1.3. *A matrix of linear forms is* $1 - generic$ *if it has no zero entry, and cannot be transformed by row and column operations to have a zero entry.*

In [4], Eisenbud shows that for a reduced irreducible nondegenerate linearly normal curve $C \subseteq \mathbb{P}^r$, there is a 1-generic $p \times q$ matrix of linear forms whose 2×2 minors vanish on C iff there exist line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ such that

$$\mathcal{O}_C(1) \simeq \mathcal{L}_1 \otimes \mathcal{L}_2,$$

with $h^0(\mathcal{L}_1) \geq p$ and $h^0(\mathcal{L}_2) \geq q$. Combining this with a result of Eisenbud that the ideal of 2×2 minors of a 1-generic $p \times q$ matrix has $2LP \geq p + q - 3$ (see [12]) leads to:

Conjecture 1.4. [*Eisenbud,* [2]] *Let $\Bbbk$ be algebraically closed of characteristic zero, and $I \subseteq S$ be a prime ideal containing no linear form, such that I_2 is spanned by quadrics of rank at most four. If $2LP(S/I) = n$, then I contains the 2×2 minors of a 1-generic $p \times q$ matrix, with $p + q - 3 = n$.*

With the additional hypotheses that S/I is normal, Gorenstein, of dimension 2 and degree $2r$, Conjecture 1.4 specializes to Green's conjecture [7]. Our first result (Theorem 1.6 below) provides an infinite class of counterexamples to Conjecture 1.4. To state the result we need:

Definition 1.5. *For a homogeneous ideal $I \subseteq S$, let $F_\bullet$ denote the subcomplex of the minimal free resolution of S/I*

$$0 \longleftarrow S/I_2 \longleftarrow S \longleftarrow F_1 \otimes S(-2) \longleftarrow F_2 \otimes S(-3) \longleftarrow F_3 \otimes S(-4) \longleftarrow \cdots$$

Let f be a 2-linear n^{th} syzygy of I (henceforth, "linear n^{th} syzygy"). The rank of f is the dimension of the smallest vector space G such that the diagram below commutes:

$$\begin{array}{ccc} F_n \otimes S(-n-1) & \longleftarrow & F_{n+1} \otimes S(-n-2) \\ \uparrow & & \uparrow \\ G \otimes S(-n-1) & \longleftarrow & f \otimes S(-n-2) \end{array}$$

Theorem 1.6. *For any odd n, there exists an arithmetically Cohen-Macaulay toric ideal generated by n quadrics of rank ≤ 4, with only one linear first syzygy, of rank n. For any even n, there exists an arithmetically Gorenstein toric ideal, generated by n quadrics of rank ≤ 4, and a $\frac{n}{2}$-ic, with only one linear first syzygy, of rank n.*

These ideals are counterexamples to Conjecture 1.4: $2LP(S/I) = 2$ should force $b_{23} \geq 2$. Roughly speaking, the problem is that the condition that I_2 is generated by quadrics of rank at most four does not guarantee that there cannot be 2-linear syzygies (possibly all) of comparatively high rank. Hence, some additional hypothesis, such as requiring a top linear syzygy of low rank, is necessary.

Theorem 1.7. *Let $\mathbb{k}$ be algebraically closed of characteristic zero, and $I \subseteq S$ be a prime ideal containing no linear form, such that $2LP(S/I) = n+1$.*

(1) *If I has a linear n^{th} syzygy of rank $n+2$, then I contains the 2×2 minors of a 1-generic $2 \times (n+2)$ matrix.*
(2) *If I has a linear n^{th} syzygy of rank $n+3$, then I contains the 4×4 Pfaffians of a skew-symmetric 1-generic $(n+4) \times (n+4)$ matrix.*
(3) *If I has a linear n^{th} syzygy of rank $n+3$* and *is a semigroup ideal, then I contains the 2×2 minors of a 1-generic $p \times (n-p+4)$ matrix.*
(4) *If I has no linear n^{th} syzygies of rank $\leq n+3$, then I does not contain the 2×2 minors of a 1-generic $p \times q$ matrix, with $p+q-3 = n+1$.*

Note that if I has a linear n^{th} syzygy of rank $\leq n+1$, then I cannot be prime. Our main tool in studying Conjecture 1.4 is Koszul homology. Recall that $\mathrm{Tor}_m(S/I,k)_{m+1}$ may be computed as the homology of

$$\bigwedge^{m+1} V \xrightarrow{\partial} \bigwedge^{m} V \otimes V \xrightarrow{\partial} \bigwedge^{m-1} V \otimes (S/I)_2$$

where $V \simeq (S/I)_1 = S_1$. Without the rank conditions, the first two parts of Theorem 1.7 appear in [12].

Perhaps our most interesting result arises from the fact that the ideals which appear in Theorem 1.6 are toric specializations of the Rees algebra of Koszul cycles. Such Rees algebras have been previously studied in [9], [10], [13] and [18]. In [5], Eisenbud and Koh ask "Under what conditions does a module with a linear k^{th} syzygy specialize to one with a linear k^{th} syzygy?" We give one answer to this question: in §4 we prove:

Theorem 1.8. *Let Δ be an oriented n–dimensional pseudomanifold on d–vertices, with top homology class $H_n(\Delta)$. Then there exists a toric variety $X = V(I)$ of dimension d, such that I has a* minimal *linear n^{th}–syzygy of rank d. If $\mathbf{m}$ is the multidegree of this syzygy, then the complex $\Delta_{\mathbf{m}}$ which computes $Tor_n^S(I,\mathbb{k})_{\mathbf{m}}$ is homotopic to Δ.*

It seems reasonable to ask if Conjecture 1.4 holds with additional geometric constraints, and we explore this in §5. For example, taking linear sections of the varieties appearing in Theorem 1.6 yields smooth, projectively normal curves for which Conjecture 1.4 fails, but the associated divisor is special.

2. Linear first syzygies

We start by analyzing the linear first syzygies, computed as the homology of:

$$\bigwedge^{3} V \xrightarrow{\partial_2} \bigwedge^{2} V \otimes V \xrightarrow{\partial_1} V \otimes (S/I)_2$$

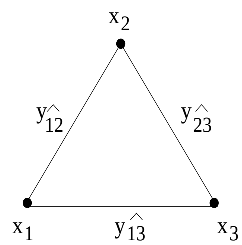
where (unless otherwise noted) $V \simeq S_1$, with a basis element $e_i \in V$ mapping to $x_i \in S$. The differential ∂_1 is defined via

$$\partial_1(e_i \wedge e_j \otimes y_{\widehat{ij}}) = y_{\widehat{ij}} \cdot (x_i e_j - x_j e_i),$$

where $y_{\widehat{ij}}$ is an indeterminate linear form in S (the reason for the $\widehat{\ }$ will appear in §4). A particular class in Koszul homology representing a rank d syzygy will be supported on a subspace W of dimension d:

$$\bigwedge^{2} W \otimes V \subseteq \bigwedge^{2} V \otimes V.$$

Example 2.1. Suppose $\Bbbk^3 \simeq W \subseteq V$, so that an element of $\bigwedge^2 W \otimes V$ may be written as $\omega = e_1 \wedge e_2 y_{\widehat{12}} - e_1 \wedge e_3 y_{\widehat{13}} + e_2 \wedge e_3 y_{\widehat{23}}$.



Thus,

$$\partial(\omega) = y_{\widehat{12}}(x_1e_2 - x_2e_1) - y_{\widehat{13}}(x_1e_3 - x_3e_1) + y_{\widehat{23}}(x_2e_3 - x_3e_2)$$

is a cycle for S/I iff $I' = \langle y_{\widehat{13}}x_3 - y_{\widehat{12}}x_2, y_{\widehat{12}}x_1 - y_{\widehat{23}}x_3, y_{\widehat{23}}x_2 - y_{\widehat{13}}x_1 \rangle \subseteq I$. Now, I' is generated by the 2×2 minors of

$$\begin{bmatrix} x_1 & x_2 & x_3 \\ y_{\widehat{23}} & y_{\widehat{13}} & y_{\widehat{12}} \end{bmatrix}.$$

If $V(I')$ is irreducible and nondegenerate (note that some of the $y_{\widehat{ij}}$ could vanish), then $2LP(S/I') = 2$, so Conjecture 1.4 holds for S/I'. As observed by Schreyer ([16], Lemma 4.3) this process generalizes; let

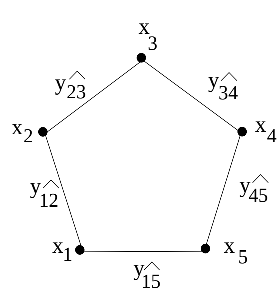
$$\omega = \sum_{1 \le i < j \le n} (-1)^{i+j+1} y_{\widehat{ij}} e_i \wedge e_j. \tag{1}$$

When $d = 4$, this yields Pfaffians, but for $d \ge 5$, interesting behavior occurs. The term $(-1)^{i+j+1}$ simplifies the connection to the Rees algebra in §4.

Example 2.2. If $d = 5$, the quadratic conditions necessary for $\partial(\omega)$ to represent a class in homology have rank eight.

$$\begin{bmatrix} 0 & -y_{\widehat{12}} & y_{\widehat{13}} & -y_{\widehat{14}} & y_{\widehat{15}} \\ y_{\widehat{12}} & 0 & -y_{\widehat{23}} & y_{\widehat{24}} & -y_{\widehat{25}} \\ -y_{\widehat{13}} & y_{\widehat{23}} & 0 & -y_{\widehat{34}} & y_{\widehat{35}} \\ y_{\widehat{14}} & -y_{\widehat{24}} & y_{\widehat{34}} & 0 & -y_{\widehat{45}} \\ -y_{\widehat{15}} & y_{\widehat{25}} & -y_{\widehat{35}} & y_{\widehat{45}} & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = 0$$

Specializing to a simple cycle by setting $y_{\widehat{13}} = y_{\widehat{14}} = y_{\widehat{24}} = y_{\widehat{25}} = y_{\widehat{35}} = 0$



yields an ideal generated by quadrics of rank at most four:

$$\begin{bmatrix} 0 & -y_{\widehat{12}} & 0 & 0 & y_{\widehat{15}} \\ y_{\widehat{12}} & 0 & -y_{\widehat{23}} & 0 & 0 \\ 0 & y_{\widehat{23}} & 0 & -y_{\widehat{34}} & 0 \\ 0 & 0 & y_{\widehat{34}} & 0 & -y_{\widehat{45}} \\ -y_{\widehat{15}} & 0 & 0 & y_{\widehat{45}} & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = 0$$

In contrast to the $d = 4$ case, the resulting ideal is prime, and defines a (singular) 5-fold $X \subseteq \mathbb{P}^9$, with graded betti numbers:

total	1	5	12	10	2
0	1	–	–	–	–
1	–	5	1	–	–
2	–	–	11	10	1
3	–	–	–	–	1

Since $2LP = 2$, Conjecture 1.4 would require that I_X contain the 2×2 minors of a $1-$generic 2×3 matrix. Such a matrix would force existence of a pair of linear first syzygies, so this is a counterexample to Conjecture 1.4. The linear syzygy we obtain is by construction of rank five, so is of rank too high to be a linear first syzygy for an ideal of 2×2 minors.

X is arithmetically Cohen-Macaulay with singularities in codimension 3; quotienting by four generic linear forms yields a smooth, projectively normal curve of genus seven in $\mathbb{P}^5$, generated by quadrics of rank ≤ 4. This curve is a projection of a general canonical curve $C \subseteq \mathbb{P}^6$ from a point $p \in C$; $h^1(K_C - p) = 1$ so $K_C - p$ is special. We analyze this in more detail in §5.

Example 2.3. The last explicit example we give is for the $d = 6$ case; specializing to a cycle yields the ideal I_6:

$$\begin{bmatrix} 0 & -y_{\widehat{12}} & 0 & 0 & 0 & -y_{\widehat{16}} \\ y_{\widehat{12}} & 0 & -y_{\widehat{23}} & 0 & 0 & 0 \\ 0 & y_{\widehat{23}} & 0 & -y_{\widehat{34}} & 0 & 0 \\ 0 & 0 & y_{\widehat{34}} & 0 & -y_{\widehat{45}} & 0 \\ 0 & 0 & 0 & y_{\widehat{45}} & 0 & -y_{\widehat{56}} \\ y_{\widehat{16}} & 0 & 0 & 0 & y_{\widehat{56}} & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = 0$$

As with I_4, the ideal I_6 is not prime; $I_6 = \langle x_1, \ldots, x_6 \rangle \cap J_6$, with J_6 prime of codimension 5.

$$J_6 = I_6 + \langle y_{\widehat{12}} y_{\widehat{34}} y_{\widehat{56}} + y_{\widehat{23}} y_{\widehat{45}} y_{\widehat{16}} \rangle,$$

Taking hyperplane sections yields a smooth curve $X \subseteq \mathbb{P}^6$ of degree 21 and genus 21 with betti numbers identical to those of J_6; in particular X is Gorenstein:

total	1	7	22	22	7	1
0	1	–	–	–	–	–
1	–	6	1	–	–	–
2	–	1	21	21	1	–
3	–	–	–	1	6	–
4	–	–	–	–	–	1

To analyze the situation for general d, let Y be a generic $d \times d$ skew symmetric matrix with entries $y_{\widehat{ij}}$ and X a generic column vector with entries x_i. Write I_d for the ideal generated by the entries of $Y \cdot X$, and let $J_d = I_d + \text{Pfaff}(\det(Y))$; when d is odd $J_d = I_d$. When d is even, S/J_d is Gorenstein of deviation two, and was first studied by Huneke and Ulrich in [10]. In [13], Kustin determined the minimal free resolution for both I_d and J_d; in particular both are arithmetically Cohen-Macaulay. The ideals which appear in Theorem 1.6 are obtained from I_d and J_d by specializing all entries of Y above the diagonal to zero, except the supradiagonal entries, and the entry in position $(1, d)$ (the "corner" entry). Write $\mathcal{I}_d$ and $\mathcal{J}_d$ for the specializations, and let y_i denote the entry of the specialization in position $(i, i+1)$ and y_d denote the "corner" entry.

Lemma 2.4. *The specializations described above are of codimension $d - 1$.*

Proof. In lex order on $R = \Bbbk[y_1, \ldots, y_d, x_1, \ldots, x_d]$, the lead monomials of the ideal generated by $(Y \setminus \text{row}_d(Y)) \cdot X$ are relatively prime, so the corresponding elements of I_d generate a complete intersection of codimension $d - 1$. As noted earlier, Kustin's results show that I_d and J_d are Cohen-Macaulay of codimension $d - 1$, so the specialization chosen corresponds to a regular sequence (if it were not, the codimension would be less than $d - 1$) which implies that the codimensions of $\mathcal{I}_d$ and $\mathcal{J}_d$ are also $d - 1$. □

Theorem 1.6 follows from Lemma 2.4. Since the ideals $\mathcal{I}_d$ and $\mathcal{J}_d$ are obtained from I_d and J_d by quotienting with a regular sequence, the graded betti numbers of $\mathcal{I}_d$ and $\mathcal{J}_d$ are identical to those appearing in the Kustin resolutions. In particular, $b_{23}(\mathcal{I}_k) = b_{23}(\mathcal{J}_d) = 1$. After a linear change of coordinates $y_{1d} \mapsto -y_{1d}$ if d is even, $\mathcal{I}_d$ and $\mathcal{J}_d$ are generated by binomials. Letting $\mathbf{e}_i$ denote the i^{th} standard basis vector in $\Bbbk^{d+1}$, the parameterization $x_i = \mathbf{e}_i$, $y_{\widehat{ij}} = \mathbf{e}_0 - \mathbf{e}_i - \mathbf{e}_j$ shows they are toric.

3. Proof of Theorem 1.7

For an n^{th} linear syzygy supported on a d-dimensional subspace $W \subseteq V = S_1$, let B be a basis for W, and for $I \subseteq B$ let

$$x_{\widehat{I}} = \bigwedge_{j \in B \setminus I} x_j.$$

To prove part (1), suppose I has an n^{th} linear syzygy of rank $d = n+2$, supported on W, and let $S = \{x_1, \ldots, x_{n+2}\}$. Then

$$x_{\widehat{i}} = x_1 \wedge \cdots \wedge x_{i-1} \wedge x_{i+1} \wedge \cdots \wedge x_{n+2} \in \bigwedge^{n+1} W.$$

The syzygy is represented by

$$\omega = \sum_{i=1}^{n+2} (-1)^{i+1} x_{\widehat{i}} \otimes y_i \in \bigwedge^{n+1} W \bigotimes V,$$

where y_i is an indeterminate linear form; since $i = [n+2] \setminus \{[n+2] \setminus i\}$, this agrees with the notation of the previous section.

$$\partial((-1)^{i+1} x_{\widehat{i}} \otimes y_i) = (-1)^{i+1} y_i \Big[\sum_{j=1}^{i-1} (-1)^{j-1} x_j x_{\widehat{ij}} + \sum_{j=i+1}^{n+2} (-1)^j x_j x_{\widehat{ij}} \Big],$$

it follows that $\partial(\omega)$ is a class in homology iff the coefficients of all the $x_{\widehat{ij}}$ are in I_2, which occurs exactly when

$$(-1)^{i+1}y_i(-1)^j x_j + (-1)^{j+1}y_j(-1)^{i-1}x_i = 0.$$

Hence, the 2×2 minors of $\phi = \begin{bmatrix} x_1 & \cdots & x_{n+2} \\ y_1 & \cdots & y_{n+2} \end{bmatrix}$ are in I_2, and ω represents a nontrivial class in homology iff some 2×2 minor is nonzero. The assumption that I is prime and nondegenerate implies that ϕ is one–generic.

For part (2), if I has an n^{th} linear syzygy of rank $n+3$, then let W be a vector space of dimension $n+3$ with $B = \{x_1, \ldots, x_{n+3}\}$. The syzygy is represented by

$$\omega = \sum_{1 \le i < j \le n+3} (-1)^{i+j} x_{\widehat{ij}} \otimes y_{ij} \in \bigwedge^{n+1} W \bigotimes V.$$

A computation as above shows that $\partial(\omega)$ is a class in homology when the coefficients of the $x_{\widehat{ijk}}$ are in I_2, which happens if

$$y_{ij}x_k - y_{ik}x_j + y_{jk}x_i \in I_2,$$

so that I_2 contains the 4×4 Pfaffians of

$$\begin{bmatrix} 0 & x_1 & x_2 & \cdots & x_{n+3} \\ -x_1 & 0 & y_{23} & \cdots & y_{2,n+3} \\ -x_2 & -y_{23} & 0 & \cdots & y_{3,n+3} \\ \vdots & 0 & \vdots & 0 & \vdots \\ -x_{n+3} & -y_{2,n+3} & -y_{3,n+3} & \cdots & 0 \end{bmatrix}$$

To see why this last statement holds, consider any 5×5 submatrix of the form

$$\begin{bmatrix} 0 & x_{i_1} & x_{i_2} & x_{i_3} & x_{i_4} \\ -x_{i_1} & 0 & * & * & * \\ -x_{i_2} & * & 0 & * & * \\ -x_{i_3} & * & * & 0 & * \\ -x_{i_4} & * & * & * & 0 \end{bmatrix}$$

A computation shows that if P is the 4×4 Pfaffian of the block $*$, then $x_{i_1} \cdot P \in I$; since I is prime and nondegenerate this implies $P \in I_2$.

For part (3), it follows from (2) that if I has a linear n^{th} syzygy of rank $n+3$, then I_2 contains $\mathrm{Pfaff}_4(N)$ of a skew-symmetric matrix N, so we have polynomials of the form

$$y_{ij}x_k - y_{ik}x_j + y_{jk}x_i \in I_2.$$

Since I is a semigroup ideal, every term in such a polynomial has the same weight with respect to the semigroup. As a semigroup ideal, I is generated by binomials, so one of the y_{ij} must vanish, because otherwise subtracting $y_{ij}x_k - y_{ik}x_j \in I_2$ from $y_{ij}x_k - y_{ik}x_j + y_{jk}x_i$ shows that $y_{jk}x_i \in I_2$, contradicting nondegeneracy of I. Repeating this shows that in the semigroup case the matrix N is of the form

$$\begin{bmatrix} 0 & M \\ -M^t & * \end{bmatrix},$$

where M is a 1–generic $v \times w$ matrix and $v, w \ge 2$. It is easy to show that $I_2(M) \subseteq \mathrm{Pfaff}_4(N)$, and the result follows.

4. Toric specializations of Rees algebras

4.1. **Rees algebras of Koszul cycles.** The skew-symmetric matrices which arose in §2 in conjunction with the linear first syzygies are most naturally studied in the setting of Rees algebras. Let $P = \Bbbk[x_1, \ldots, x_d]$, and let $K_\bullet$ denote the Koszul complex on $\{x_1, \ldots, x_d\}$

$$K_\bullet: \ 0 \to \bigwedge^d(\Bbbk^d) \otimes P(-d) \xrightarrow{\delta_d} \bigwedge^{d-1}(\Bbbk^d) \otimes P(-d+1) \xrightarrow{\delta_{d-1}} \bigwedge^{d-2}(\Bbbk^d) \otimes P(-d+2) \xrightarrow{\delta_{d-2}} \cdots$$

Let $C_i = \mathrm{Im}(\delta_i)$ be the module of i^{th} cycles in $K_\bullet$, and put

$$\mathcal{I}_i = \{I \subseteq \{1, \ldots, d\} \mid |I| = i\}.$$

Then the symmetric algebra on the free module K_i is:

$$S(K_i) = P[y_I \mid I \in \mathcal{I}_i]$$

The presentation

$$K_{i+1} \longrightarrow K_i \longrightarrow C_i \longrightarrow 0$$

gives a presentation for the symmetric algebra of C_i. Let $J_i = \langle z_I \mid I \in \mathcal{I}_{i+1} \rangle$, where if $I = \{a_1 < a_2 < \cdots < a_{i+1}\}$ then

$$z_I = \sum_{j=1}^{i+1} (-1)^{j+1} x_{a_j} y_{I \setminus a_j} \text{ and } S(C_i) \simeq S(K_i)/J_i.$$

For $i = d-2$, any $I \in \mathcal{I}_{d-1}$ is of the form $[d] \setminus k$, so $I \setminus a_j$ has the form

$$I \setminus a_j = [d] \setminus \{k, j\} = \widehat{jk} \text{ for some } j, k.$$

Thus, for $i = d-2$, the z_I are exactly the elements denoted $\partial(\omega)$ in Equation 1, and the reason for the choice of notation in §2. The Rees algebra $R(C_i)$ is $S(C_i)$ modulo the P-torsion. As noted earlier, for $i = d-2$ these algebras were first investigated in [10], and the free resolution of $R(C_{d-2})$ and $S(C_{d-2})$ over $S(K_{d-2})$ was determined by Kustin in [13]. In [9], Herzog, Tang and Zarzuela studied properties of Gröbner and sagbi bases for $R(C_i)$ and $S(C_i)$, concentrating on the cases $i = 2$ and $d-2$; they also conjectured that $R(C_i)$ is Cohen-Macaulay for all i, and that the P-torsion of $S(C_i)$ could be described simply as $0 :_{S(C_i)} x_j$ for any $j \in \{1, \ldots, d\}$.

In [18], Weyman used the geometric method of computing syzygies [19] to prove these conjectures. In fact, Weyman shows that $R(C_i)$ is a normal, Cohen-Macaulay domain, with rational singularities, and gives a representation theoretic description of the free resolution of $R(C_i)$ over $S(K_i)$. In different language, [17] studies $Proj(S(C_i))$, obtaining results on rank two bundles on curves.

As noted in §2, the z_I are quadrics of high rank, so the question is how to specialize the z_I so that they have rank at most four, but where the specialized ideal remains prime.

4.2. **Toric specializations of J_i.** Let Δ be an n–dimensional simplicial complex on d vertices. We associate to Δ an ideal J_Δ which is a specialization of J_{d-n-1}. If Δ is a pseudomanifold, then J_Δ is toric; as a pseudomanifold Δ has a natural top homology class, which corresponds to a minimal syzygy on J_Δ. The specializations in §2 coming from cycles are instances of this construction; motivated by this, we focus on specializations for which the underlying simplicial complex is a pseudomanifold.

Definition 4.1. *An oriented pseudomanifold Δ of dimension n on d vertices consists of a set of oriented $(n+1)$-subsets of $\{1,\ldots,d\}$ such that*

(1) *each n-subset of $\{1,\ldots,d\}$ is contained in exactly zero or two elements of Δ; in the latter case the orientations must cancel.*
(2) *Δ is strongly connected: the dual graph $G(\Delta)$ is connected, where $G(\Delta)$ has a vertex for each n–face of Δ, and two vertices are joined by an edge if the corresponding n–faces share an $(n-1)$–face.*

Definition 4.2. *Let Δ be an n–dimensional oriented pseudomanifold on d vertices. $J(\Delta)$ is the specialization of J_{d-n-1} obtained by setting $y_{I\setminus a_j}=0$ if $\widehat{I\setminus a_j}\not\in\Delta$. So $y_{I\setminus a_j}=0$ iff $I\setminus a_j$ is in the Alexander dual $\Delta^*=\{\widehat{\gamma}\mid\gamma\not\in\Delta\}$.*

Example 4.3. If Δ is the boundary of an k–simplex, then Δ has $k+1$ vertices and $k+1$ faces of dimension $k-1$, and J_Δ is the ideal of 2 by 2 minors of a generic 2 by (k+1) matrix. For $k=2$, we have $d=3$, $n=1$. Then J_1 is the ideal

$$\begin{aligned}
z_{12} &= x_1y_2-x_2y_1 &&= x_1y_{\widehat{13}}-x_2y_{\widehat{23}}\\
z_{13} &= x_1y_3-x_3y_1 &&= x_1y_{\widehat{12}}-x_3y_{\widehat{23}}\\
z_{23} &= x_2y_3-x_3y_2 &&= x_2y_{\widehat{12}}-x_3y_{\widehat{13}}
\end{aligned}$$

yielding the 2×2 minors of the 2×3 matrix appearing in Example 2.1. We relabel y_I as $y_{\widehat{[d]\setminus I}}$ to make the connection to the cycle more intuitive.

Lemma 4.4. *With the relabelling introduced above,*

$$J(\Delta)\simeq\langle x_iy_{\widehat{\sigma}}-x_jy_{\widehat{\tau}}\mid\sigma,\tau\in\Delta_n \text{ satisfy } \sigma\setminus\{i\}=\tau\setminus\{j\}\rangle.$$

Proof. The simplicial coboundary map $\partial_n^*:C^{n-1}(\Delta)\to C^n(\Delta)$ has two nonzero entries in each column, $+1$ and -1. Index the source and target of ∂_n^* by complementary faces, so that $C^{n-1}(\Delta)$ is indexed by $\mathcal{I}_{d-n}$ and $C^n(\Delta)$ by $\mathcal{I}_{d-n-1}$. Because Koszul and simplical cohomology agree, choosing an oriented basis for the Koszul cycles to agree with the orientation of Δ and arbitrary orientations for cycles not in Δ yields the result. □

In concrete terms, the ideal J_{d-n-1} consists of column sums of a $\binom{d}{d-n-1}\times\binom{d}{d-n}$ matrix, with entry $(J,I)\in\mathcal{I}_{d-n-1}\times\mathcal{I}_{d-n}$ zero if $J\not\subseteq I$, and entry $(-1)^{j+1}x_{a_j}y_{I\setminus a_j}$ if $I=[a_1,\ldots,a_{d-n}]$ and $J=[a_1,\ldots,\widehat{a_j},\ldots,a_{d-n}]$. Lemma 4.4 replaces $(-1)^{j+1}$ with the sign of $\widehat{I}$ in the boundary of $\widehat{J}$, and y_J with $y_{\widehat{[d]\setminus J}}$.

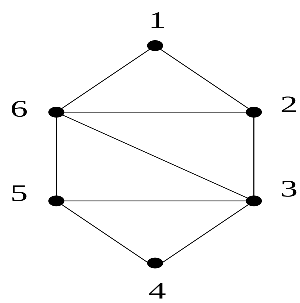
Example 4.5. We revisit Example 2.3, where Δ is the six-cycle with orientation $\{[i,i+1],i\in\{1,\ldots,5\},[6,1]\}$. Since $d=6$ and $n=1$, we consider $J_{d-n-1}=J_4$. Taking lex ordered bases for the Koszul classes yields, for example:

$$\begin{aligned}
z_{23456} &= x_2y_{3456}-x_3y_{2456}+x_4y_{2356}-x_5y_{2346}+x_6y_{2345}\\
&= x_2y_{\widehat{12}}-x_3y_{\widehat{13}}+x_4y_{\widehat{14}}-x_5y_{\widehat{15}}+x_6y_{\widehat{16}}\\
&\rightsquigarrow x_2y_{\widehat{12}}+x_6y_{\widehat{16}}.
\end{aligned}$$

Modifying the basis for Koszul classes as in Lemma 4.4 changes the last expression to $x_2y_{\widehat{12}}-x_6y_{\widehat{16}}$. This explains the linear change of variables $y_{1d}\mapsto -y_{1d}$ in the proof of Theorem 1.6.

$J(\Delta)$ is typically not prime, but if $J_\Delta=J(\Delta):\langle\prod_{i=1}^d x_i\rangle^\infty$, then Lemma 4.10 shows that J_Δ is prime, toric, and is the defining ideal of a Rees algebra.

Example 4.6. Consider the two-dimensional pseudomanifold which is a triangulation of S^2, obtained by coning over the boundary of the complex below with a seventh vertex.



The resulting ideal $J(\Delta)$ is generated by fifteen quadrics; J_Δ defines a toric seven–fold in $\mathbb{P}^{16}$, which is Cohen-Macaulay of degree 73, and has graded betti numbers:

total	1	21	163	447	631	575	377	168	47	8
0	1	–	–	–	–	–	–	–	–	–
1	–	17	19	1	–	–	–	–	–	–
2	–	4	144	444	500	209	8	–	–	–
3	–	–	–	2	131	365	333	93	2	–
4	–	–	–	–	–	1	36	75	45	8

The existence of a linear second syzygy (of rank seven) follows from Theorem 1.8, which we prove next.

Definition 4.7. *Let*

$$\mathcal{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_q\} \subseteq \mathbb{Z}^p$$

and let A be the matrix with i^{th} column $\mathbf{a}_i$. The toric ideal

$$I_{\mathcal{A}} = \langle x^\alpha - x^\beta \mid \alpha, \beta \in \mathbb{N}^q \textit{ and } \alpha - \beta \in ker(A)\rangle.$$

To any $\mathbf{m} \in \mathbb{Z}^p$ we associate a simplicial complex

$$\Delta_{\mathbf{m}}(\mathcal{A}) = \{I \subseteq \{1, \ldots, q\} \mid \mathbf{m} - \sum_{i \in I} \mathbf{a}_i \textit{ lies in } \mathcal{A}\}.$$

A result of Hochster shows that

$$\widetilde{H}_j(\Delta_{\mathbf{m}}(\mathcal{A}), \Bbbk) = Tor_j(I_{\mathcal{A}}, \Bbbk)_{\mathbf{m}}.$$

For a proof and more details on $I_{\mathcal{A}}$ see Chapter 9 of [14]. Let Δ be an oriented n–dimensional pseudomanifold on d–vertices, and for each $i \in \{1, \ldots, d\}$, assign the vertex v_i the weight $\mathbf{e}_i \in \mathbb{Z}^{d+1}$, and for a facet $\sigma = \{v_{i_1}, \ldots, v_{i_{n+1}}\} \in \Delta_n$,

$$wt(\sigma) = \mathbf{e}_0 - \sum_{j=1}^{n+1} \mathbf{e}_{i_j}.$$

For the remainder of this section,

$$\mathcal{A} = \{\mathbf{e}_1, \ldots, \mathbf{e}_d, wt(\sigma) \mid \sigma \in \Delta_n\} \subseteq \mathbb{Z}^{d+1}.$$

Lemma 4.8. *Let Δ be an oriented n–dimensional pseudomanifold on d–vertices, with top homology class $H_n(\Delta)$. Then Δ is homotopic to $\Delta_{\mathbf{e}_0}(\mathcal{A})$*

Proof. Consider the complex $\Delta_{\mathbf{e}_0}(\mathcal{A})$. Let $\tau = \{i_1, \ldots, i_k\}$ define a $(k-1)$–face of Δ. Since Δ is a pseudomanifold, there is an n–simplex $\{i_1, \ldots, i_k, i_{k+1}, \ldots, i_{n+1}\}$ containing τ, so

$$\mathbf{e}_0 - \mathbf{e}_{i_1} - \cdots - \mathbf{e}_{i_k} = \mathbf{e}_0 - \sum_{j=1}^{n+1} \mathbf{e}_{i_j} + \sum_{j=k+1}^{n+1} \mathbf{e}_{i_j}, \tag{2}$$

hence $\tau \in \Delta_{\mathbf{e}_0}(\mathcal{A})$, which implies that $\Delta \subseteq \Delta_{\mathbf{e}_0}(\mathcal{A})$. On the other hand,

$$\mathbf{e}_0 - (\mathbf{e}_0 - \mathbf{e}_{i_1} - \cdots - \mathbf{e}_{i_k}) = \sum_{j=1}^{k} \mathbf{e}_{i_j} \in \mathcal{A}, \tag{3}$$

so the cone over every $(k-1)$–face of Δ is also in $\Delta_{\mathbf{e}_0}(\mathcal{A})$. This is most easily visualized as adding a cone vertex (corresponding to the variable y_σ) over every n–face σ of Δ. Any $\sum_{i \in I} \mathbf{a}_i$ such that $I \in \Delta_{\mathbf{e}_0}(\mathcal{A})$ can have at most one $\mathbf{a}_i$ with $\mathbf{e}_0$ coefficient one. If there are no such $\mathbf{a}_i$, then I must correspond to an element of Δ_n. Otherwise, let $k \in I$ with $\mathbf{a}_k = \mathbf{e}_0 - \sum_{j \in \sigma} \mathbf{e}_j$. Then

$$\mathbf{e}_0 - \sum_{i \in I} \mathbf{a}_i = \mathbf{e}_0 - (\mathbf{e}_0 - \sum_{j \in \sigma} \mathbf{e}_j) - (\sum_{i \in I' = I \setminus k} \mathbf{a}_i) \in \mathcal{A},$$

which can only occur if $I' \subseteq \sigma$. Thus, all faces of $\Delta_{\mathbf{e}_0}(\mathcal{A})$ are described by Equations 2 and 3. Since each cone over a face σ (the faces appearing in Equation 3) is homotopic to the face σ, the result follows. □

Definition 4.9. *Let $L_\Delta \subseteq S = \Bbbk[x_1, \ldots, x_d, \{y_\sigma \mid \sigma \in \Delta_n\}]$ be the defining ideal of the Rees algebra $P[I_{\Delta^*} t]$, where I_{Δ^*} is the Stanley-Reisner ideal of the Alexander dual of Δ, and the map $S \to P[I_{\Delta^*} t]$ is defined via $y_\sigma \mapsto t \prod_{i \notin \sigma} x_i$.*

Lemma 4.10. *Let Δ be an oriented n–dimensional pseudomanifold on d–vertices. Then $J_\Delta = L_\Delta = I_\mathcal{A}$ is a prime toric ideal.*

Proof. Since the Stanley-Reisner ideal I_{Δ^*} of the Alexander dual of Δ is

$$I_{\Delta^*} = \langle \prod_{i \notin \sigma} x_i \mid \sigma \in max(\Delta) \rangle,$$

the following inclusions are immediate:

$$J(\Delta) \subseteq L_\Delta \subseteq I_\mathcal{A}.$$

Let $f = \prod_{i=1}^{d} x_i$. Since Δ is strongly connected, we have

$$J(\Delta) S_f = I_\mathcal{A} S_f,$$

Therefore

$$J(\Delta) S_f = L_\Delta S_f = I_\mathcal{A} S_f.$$

By contracting back to S, noting that L_Δ and $I_\mathcal{A}$ are prime, and $J(\Delta) S_f \cap S = J_\Delta$, we obtain the desired equalities. □

Combining Lemma 4.10 with Lemma 4.8 yields Theorem 1.8. In Example 4.6, the linear syzygy constructed from Theorem 1.8 generates $Tor_3(S/J_\Delta, \Bbbk)_4$, but in general its dimension can be larger.

Proposition 4.11. *Let Δ be an oriented n–dimensional pseudomanifold on d–vertices, with top homology class $H_n(\Delta)$. Suppose $\Delta' \subseteq \Delta$ is a bipyramid on vertices x_1, x_2 over a $n-1$–dimensional complex Δ''. Then if $|\Delta''_{n-1}| = k$, J_Δ contains the two by two minors of a matrix of the form:*

$$\begin{bmatrix} x_1 & y_1 & y_2 & \cdots & y_k \\ x_2 & y'_1 & y'_2 & \cdots & y'_k \end{bmatrix}$$

Proof. Let $\Delta''_{n-1} = \{\sigma_1, \ldots, \sigma_k\}$. Each σ_i yields a pair of n–faces of Δ: the cone of σ with x_1 and x_2. Associate to the cone of σ_i with x_2 the variable y_i, and to the cone of σ_i with x_1 the variable y'_i. Then

$$\deg(x_1 y'_i) = \mathbf{e}_1 + (\mathbf{e}_0 - \sum_{i \in \sigma} \mathbf{e}_i - \mathbf{e}_1) = \mathbf{e}_2 + (\mathbf{e}_0 - \sum_{i \in \sigma} \mathbf{e}_i - \mathbf{e}_2) = \deg(x_2 y_i).$$

A similar argument shows that $\deg(y_i y'_j) = \deg(y_j y'_i)$. □

The point is that if $\dim_{\Bbbk} Tor_n(S/J_\Delta, \Bbbk)_{n+1} = 1$, then Δ cannot contain a bipyramid over an $n-2$–dimensional Δ'' with $|\Delta''_{n-2}| \geq n$. By Example 4.3, this also implies that Δ is not the boundary of an n–simplex.

Question 4.12. *What are necessary and sufficient conditions on a pseudomanifold Δ so that $\dim_{\Bbbk} Tor_n(S/J_\Delta, \Bbbk)_{n+1} = 1$? What are necessary and sufficient conditions on a pseudomanifold Δ so that Conjecture 1.4 holds for S/J_Δ?*

Any one–dimensional Δ which is a cycle on five or more vertices gives J_Δ with $\dim_{\Bbbk} Tor_2(S/J_\Delta, \Bbbk)_3 = 1$. For the toric varieties produced from two–dimensional pseudomanifolds on at most six vertices, there are no counterexamples to Conjecture 1.4. However, Example 4.6 may be specialized to yield a toric counterexample:

Example 4.13. Identifying the vertices labeled 1 and 4 in Example 4.6 and saturating yields an ideal which defines a toric six–fold in $\mathbb{P}^{15}$, which is Cohen-Macaulay of degree 56, and has graded betti numbers:

total	1	25	177	549	816	676	449	255	67	5
0	1	–	–	–	–	–	–	–	–	–
1	–	19	30	1	–	–	–	–	–	–
2	–	6	147	546	788	484	45	–	–	–
3	–	–	–	2	28	192	404	255	64	3
4	–	–	–	–	–	–	–	–	3	2

This specialization does not correspond to a pseudomanifold; there is an edge of Δ (connecting vertex $1 = 4$ and vertex 7), which lies on four two-faces.

Question 4.14. *If Δ is a pseudomanifold, which specializations of J_Δ are toric? If $\dim_{\Bbbk} Tor_n(S/J_\Delta, \Bbbk)_{n+1} = 1$, which specializations of J_Δ are toric,* and *preserve uniqueness of the top linear syzygy?*

5. Examples with $2LP = 2$

In this final section, we consider geometric reasons for the failure of Conjecture 1.4. We restrict our attention to the case where $2LP = 2$, and focus on two concrete classes of examples: curves and toric surfaces. In Example 2.2, the divisor which was used to embed the curve was special. The next result shows that this is not an isolated phenomenon. Recall that normal generation means S/I_C is projectively normal, and normal presentation means I_C is generated by quadrics.

Proposition 5.1. *If D is a very ample divisor on C such that $C \subseteq \mathbb{P}(H^0(D)^*)$ is normally generated and normally presented, and $b_{23}(S/I_C) = 1$, then D is special.*

Proof. Let $r+1 = h^0(\mathcal{O}_C(D))$ and $d = deg(D)$. The assumption on normal generation means that $h^1(\mathcal{I}_C(t)) = 0$ for all $t \geq 0$, yielding the exact sequence

$$0 \longrightarrow H^0(\mathcal{I}_C(t)) \longrightarrow H^0(\mathcal{O}_{\mathbb{P}^r}(t)) \longrightarrow H^0(\mathcal{O}_C(t)) \longrightarrow 0.$$

If D is nonspecial, then $h^1((\mathcal{O}_C(tD)) = 0$ for all $t \geq 1$, which for $t = 1$ implies:

$$g = d - r.$$

For $t = 2$, Riemann-Roch and the short exact sequence above shows that

$$h^0(\mathcal{I}_C(2)) = \binom{r+2}{2} - 2d - 1 + g.$$

Since I_C is generated by quadrics, having a single linear syzygy implies that

$$h^0(\mathcal{I}_C(3)) = (r+1) \cdot h^0(\mathcal{I}_C(2)) - 1 = \binom{r+3}{3} - 3d - 1 + g.$$

Eliminating g from these equations shows that

$$d = \frac{r^3 - r - 3}{3(r-1)},$$

which has no integral solutions. □

Taking this as our cue, we study nonspecial D, such that $C \subseteq \mathbb{P}(H^0(D)^*)$ is normally generated and presented, with $2LP(S/I_C) = 2$. When $\deg(D) \geq 2g+1+a$, a result of Schreyer ([3], Theorem 8.17) shows that $2LP(S/I_C) \geq a + \lfloor \frac{g}{2} \rfloor$, which is forced due to a factorization of D. This implies Conjecture 1.4 holds if $2LP(S/I_C) = 2$ and $\deg(D) \geq 2g+1+a$. Slightly modifying Schreyer's proof yields:

Lemma 5.2. *If $\mathcal{L}$ is a very ample line bundle on C such that*

$$\deg \mathcal{L} = g + \left\lceil \frac{g}{2} \right\rceil + a, \;\; a \geq 1,$$

then $2LP(S/I_C) \geq a - 1$.

Proof. Brill–Noether implies that if $m = 1 + \lceil \frac{g}{2} \rceil$, then C carries a g^1_m. Let D be a divisor in this system. Then

$$h^0(\mathcal{L}(-D)) = g + \left\lceil \frac{g}{2} \right\rceil + a - (1 + \left\lceil \frac{g}{2} \right\rceil) + 1 - g + h^1(\mathcal{L}(-D)).$$

If X is a curve embedded by a complete linear series $|\,\mathcal{A}\,|$ and B is a divisor on X such that $h^0(B) \geq s+1$, then if $h^0(\mathcal{A}(-B)) \geq t+1$, I_X contains the 2×2 minors of a $(s+1) \times (t+1)$ matrix. Theorem 8.12 of [3] implies that

$$2LP(S/I_X) \geq s + t - 1.$$

Applying this to $\mathcal{L}$ and D yields the lemma. If $\mathcal{L}(-D)$ is special, then in fact $2LP(S/I_C) \geq a$. □

Lemma 5.3. *If D is a very ample, nonspecial divisor on C of degree d, such that $C \subseteq \mathbb{P}(H^0(D)^*)$ is normally generated and normally presented and $2LP(S/I_C) = 2$, then the graded betti numbers of S/I_C are given by:*

total	*1*	b_2	$\cdots$	$\cdots$	$\cdots$	b_{r+1}
0	*1*	–	–	–	–	–
1	–	b_2	b_3	–	–	–
2	–	–	b_4	b_5	$\cdots$	b_{r+1}

where

$$\begin{aligned} b_2 &= \tbinom{d-g}{2} - g \\ b_3 &= (d-g-1)(\tbinom{d-g-1}{2} - g) - \tbinom{d-g-1}{3} \end{aligned}$$

and when $i \geq 4$,

$$b_i = \binom{d-g-1}{i} - (d-g-1)\binom{d-g-1}{i-1} + g\binom{d-g-1}{i-2}$$

Proof. The hypothesis of projective normality means that all values of the Hilbert function of S/I_C can be computed from $H^0(D)$. As in the proof of Proposition 5.1, Riemann-Roch and the hypothesis that D is nonspecial yield these values. The assumption that I_C is generated by quadrics and that $2LP(S/I_C) = 2$ means that there are no overlaps in the resolution, hence the Hilbert function in fact determines the resolution. □

Since the b_i are positive, the hypotheses of Lemma 5.3 impose very strong constraints on D: for curves of genus at most six, the only possible values for genus and degree are:

genus	0	1	2	3	4	5	6
degree	3	5	6	8	9	11	12

Lemma 5.2 implies that Conjecture 1.4 holds for all such pairs. For a curve of genus seven, a divisor satisfying the hypotheses of Lemma 5.3 must have degree 13 or 14. In the latter case, Conjecture 1.4 again holds by Lemma 5.2.

Example 5.4. On a curve of genus seven, a divisor D of degree 13 satisfying the hypotheses of Lemma 5.3 has graded betti numbers

total	1	8	30	46	30	7
0	1	–	–	–	–	–
1	–	8	5	–	–	–
2	–	–	25	46	30	7

For a general curve, $D = K + p_1 + p_2 + p_3 + p_4 - q_1 - q_2 - q_3$ has such betti numbers. A computation shows that all linear syzygies have rank ≥ 5, and that I_C can be generated by quadrics of rank ≤ 4. So Conjecture 1.4 can fail even if D satisfies the hypotheses of Lemma 5.3.

Nevertheless, there are classes of objects where the constraints of Lemma 5.3 are strong enough to prove Conjecture 1.4. Let X be a complete toric surface and D a very ample divisor on X. Then $H^0(D)$ corresponds to the set of integral points of a lattice polygon P, and S/I_X is Cohen-Macaulay and three-regular, so we may repeat the analysis above. Let $v(P)$ denote the volume of P, $\partial(P)$ the number of lattice points on the boundary of P, and $i(P)$ the number of interior lattice points of P; write X_P for the projective embedding of X determined by P.

Proposition 5.5. *If X_P is a toric surface generated by quadrics and $2LP(S/I_{X_P}) = 2$, then Conjecture 1.4 holds.*

Proof. Pick's theorem shows that for a divisor D on a toric surface,

$$D^2 = 2v(P) = 2i(P) + \partial(P) - 2.$$

The surface X_P is projectively normal [1], so if $C = X_P \cap H$ for a general hyperplane section, the graded betti numbers of C and X are identical. Riemann-Roch and adjunction show that the genus of C is equal to $i(P)$, and so C is embedded by a divisor of degree $2g+1+\partial(P)-3$. Slicing with a hyperplane as in [15] and applying Green's theorem [8] shows that

$$Tor_i(S/I_C, \Bbbk)_{i+2} = 0$$

for all $i \leq \partial(P) - 3$; in [11], Koelman shows that if $Tor_1(S/I_C, \Bbbk)_3 = 0$ and $Tor_2(S/I_C, \Bbbk)_4 \neq 0$, then $\partial(P) = 4$. By Lemma 5.2 if $\mathcal{L}$ is a line bundle on C of degree greater than $g + \lceil \frac{g}{2} \rceil + 4$, then $2LP(S/I_C) \geq 3$, so the assumption that $2LP(S/I_C) = 2$ implies that

$$2g + 1 + \partial(P) - 3 = \deg(D|_D) \leq g + \left\lceil \frac{g}{2} \right\rceil + 3,$$

hence $\lfloor \frac{i(P)}{2} \rfloor + \partial(P) \leq 5$. This means either $\partial(P) = 4$ and $i(P) \in \{0, \ldots, 3\}$ or $\partial(P) = 5$ and $i(P)$ is 0 or 1. There are only finitely many such polygons, and a check verifies the conjecture. □

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References

[1] W. Bruns, J. Gubeladze, N.V. Trung, *Normal polytopes, triangulations, and Koszul algebras*, J. Reine Angew. Math. **485** (1997), 123–160.

[2] D. Eisenbud, *Green's conjecture: an orientation for algebraists*, in "Free resolutions in commutative algebra and algebraic geometry" (Sundance, UT, 1990), 51–78, Res. Notes Math., 2, Jones and Bartlett, Boston, MA, 1992

[3] D. Eisenbud, *The geometry of syzygies*, Graduate Texts in Mathematics, vol. 229, Springer-Verlag, Berlin-Heidelberg-New York, 2005.

[4] D. Eisenbud, *Linear sections of determinantal varieties*, Amer. J. Math. **110** (1988), no. 3, 541–575.

[5] D. Eisenbud, J. Koh, *Some linear syzygy conjectures*, Adv. Math. **90** (1991), 47–76.

[6] D. Eisenbud, J. Koh, M. Stillman, *Determinantal equations for curves of high degree*, Amer. J. Math. **110** (1988), 513–539.

[7] M. Green, *Quadrics of rank four in the ideal of a canonical curve*, Invent. Math. **75** (1984), 85–104.

[8] M. Green, *Koszul cohomology and the geometry of projective varieties*, J. Differential Geom. **19** (1984), 125–171.

[9] J. Herzog, Z. Tang, S. Zarzuela, *Symmetric and Rees algebras of Koszul cycles and their Gröbner bases*, Manuscripta Math. **112** (2003), 489–509.

[10] C. Huneke, B. Ulrich, *Divisor class groups and deformations*, Amer. J. Math. **107** (1986), 1265–1303.

[11] R. Koelman, *A criterion for the ideal of a projectively embedded toric surface to be generated by quadrics*, Beiträge Algebra Geom. **34** (1993), 57–62.

[12] J. Koh, M. Stillman, *Linear syzygies and line bundles on an algebraic curve*, J. Algebra **125** (1989), 120–132.

[13] A. Kustin, *The minimal free resolutions of the Huneke-Ulrich deviation two Gorenstein ideals*, J. Algebra **100** (1986), 265–304.

[14] E. Miller, B. Sturmfels, *Combinatorial commutative algebra*, Graduate Texts in Mathematics, vol. 227, Springer-Verlag, Berlin-Heidelberg-New York, 2005.
[15] H. Schenck, *Lattice polygons and Green's theorem*, Proc. Amer. Math. Soc. **132** (2004) 3509–3512.
[16] F-O. Schreyer, *A standard basis approach to syzygies of canonical curves*, J. Reine Angew. Math. **421** (1991), 83–123.
[17] H. von Bothmer, *Generic syzygy schemes*, J. Pure App. Alg., **208** (2007), 867–876.
[18] J. Weyman, *Application of the geometric technique of calculating syzygies to Rees algebras*, J. Algebra **276** (2004), 776–793.
[19] J. Weyman, *Cohomology of vector bundles and syzygies*, London Mathematical Society Lecture Note Series, Cambridge, 2003.

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