

CANONICAL MODELS FOR BI-ISOMETRIES

H. BERCOVICI, R. G. DOUGLAS, AND C. FOIAS

We dedicate this paper to the memory Israel Gohberg, great mathematician, wonderful human being, friend and teacher to us all.

ABSTRACT. A canonical model, analogous to the one for contraction operators, is introduced for bi-isometries, two commuting isometries on a Hilbert space. This model involves a contractive analytic operator-valued function on the unit disk. Various pureness conditions are considered as well as bi-isometries for which both isometries are shifts. Several families of examples are introduced and classified.

1. INTRODUCTION

It is difficult to overestimate the the importance of the von Neumann-Wold theorem on the structure of isometric operators on Hilbert space. Originally introduced in the study of symmetric operators by von Neumann, it became the foundation for Wold's study of stationary stochastic processes. Later, it was the starting point for the study of contraction operators by Sz.-Nagy and the third author as well as a key ingredient in engineering systems theory. Thus it has had an important role in both pure mathematics and its applications.

For nearly fifty years, researchers have sought a similar structure theory for n -tuples of commuting isometries [4,11,12,15,16,17,19] with varying success. In [2] the authors rediscovered an earlier fundamental result of Berger, Coburn and Lebow [4] on a model for an n -tuple of commuting isometries and carried the analysis beyond what the latter researchers had done. In the course of this study, a very concrete canonical model for bi-isometries emerged; that is for pairs of commuting isometries. This new model is related to the canonical functional model of a contraction, but it displays subtle differences and a new set of challenges. In this paper we take up the systematic presentation and development of this model.

After some preliminaries, we begin in Section 3 by examining the passage from an n -isometry to an $(n + 1)$ -isometry showing that essentially the main ingredient needed is a contraction in the commutant of a completely nonunitary n -isometry. In the case of a bi-isometry, this additional operator can be viewed as a contractive operator-valued analytic function in the unit disk. It is this function that is the heart of our canonical model. We relate the reducing subspaces of an n -isometry to this construction and investigate a variety of notions of "pureness" which generalize the notion of completely nonunitary for contractions and the results of several earlier researchers. (See Section 3 for the details.)

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In Section 4 we specialize to the case $n = 1$, that is to the case of bi-isometries, and study the extension from the first isometry to the pair. The analytic operator function mentioned above then is the characteristic function for the pair. Various relations between the bi-isometry and the characteristic function are investigated. In Section 5, this model is re-examined in the context of a functional model; that is, one in which the abstract Hilbert spaces are realized as Hardy spaces of vector-valued functions on the unit disk. This representation allows one to apply techniques from harmonic analysis in their study. In Section 6, we specialize to bi-shifts or bi-isometries for which both isometries are shift operators. (Note that this use of the term is not the same as that used by earlier authors.)

In Section 7, we return to the functional model for bi-isometries obtaining unitary invariants for them. Finally, in Section 8, several families of bi-isometries are introduced and studied. The results here are not exhaustive but intended to illustrate various aspects of the earlier theory as well as the variety of possibilities presented by bi-isometries.

At the ends of Sections 3 and 4, the connection between intertwining maps and common invariant subspaces for bi-isometries is discussed. This topic has already been considered in [3] and further results will be presented in another paper.

2. PRELIMINARIES ABOUT COMMUTING ISOMETRIES

We will study families $\mathbb{V} = (V_i)_{i \in I}$ of commuting isometric operators on a complex Hilbert space \mathfrak{H} . A (closed) subspace $\mathfrak{M} \subset \mathfrak{H}$ is *invariant* for \mathbb{V} if $V_i \mathfrak{M} \subset \mathfrak{M}$ for $i \in I$; we write $\mathbb{V}|\mathfrak{M} = (V_i|\mathfrak{M})_{i \in I}$ if \mathfrak{M} is invariant. The invariant subspace \mathfrak{M} is *reducing* if \mathfrak{M}^\perp is invariant for \mathbb{V} as well. If \mathfrak{M} is a reducing subspace, we have a decomposition

$$\mathbb{V} = (\mathbb{V}|\mathfrak{M}) \oplus (\mathbb{V}|\mathfrak{M}^\perp),$$

and $\mathbb{V}|\mathfrak{M}$ is called a *direct summand* of \mathbb{V} . The family \mathbb{V} is said to be *unitary* if each $V_i, i \in I$, is a unitary operator. We say that \mathbb{V} is *completely nonunitary* or *cnu* if it has no unitary direct summand acting on a space $\mathfrak{M} \neq \{0\}$. The family \mathbb{V} is *irreducible* if it has no reducing subspaces other than $\{0\}$ and \mathfrak{H} .

The following extension of the von Neumann-Wold decomposition was proved by I. Suciú [20].

Theorem 2.1. *Let \mathbb{V} be a family of commuting isometries on \mathfrak{H} . There exists a unique reducing subspace \mathfrak{M} for \mathbb{V} with the following properties.*

- (1) $\mathbb{V}|\mathfrak{M}$ is unitary.
- (2) $\mathbb{V}|\mathfrak{M}^\perp$ is completely nonunitary.

We recall, for the reader's convenience, the construction of \mathfrak{M} . We simply set

$$\mathfrak{M} = \bigcap_{N=1}^{\infty} \left[\bigcap_{k_1, k_2, \dots, k_N \in I} V_{k_1} V_{k_2} \cdots V_{k_N} \mathfrak{H} \right].$$

Obviously $V_k \mathfrak{M} \supset \mathfrak{M}$ for each k , and the commutativity of \mathbb{V} implies that $V_k \mathfrak{M} \subset \mathfrak{M}$ as well. Thus \mathfrak{M} reduces each V_k to a unitary operator. It is then easily seen that \mathfrak{M} is the largest invariant subspace for \mathbb{V} such that $\mathbb{V}|\mathfrak{M}$ is unitary, and this immediately implies properties (1) and (2), as well as the uniqueness of \mathfrak{M} .

More generally, given a subset $J \subset I$, we will say that \mathbb{V} is *J-unitary* if V_j is a unitary operator for each $j \in J$. The family \mathbb{V} is said to be *J-pure* if it has no

J -unitary direct summand acting on a nonzero space. The preceding result extends as follows.

Theorem 2.2. *Let $\mathbb{V} = (V_i)_{i \in I}$ be a family of commuting isometries on a Hilbert space \mathfrak{H} , and let J be a subset of I . There exists a unique reducing subspace \mathfrak{M}_J for \mathbb{V} with the following properties.*

- (1) $\mathbb{V}|_{\mathfrak{M}_J}$ is J -unitary.
- (2) $\mathbb{V}|_{\mathfrak{M}_J^\perp}$ is J -pure.

Proof. Let us set $\mathbb{V}_J = (V_j)_{j \in J}$ and apply Theorem 2.1 to this family. Thus we can write $\mathfrak{H} = \mathfrak{M} \oplus \mathfrak{N}$, where \mathfrak{M} is reducing for \mathbb{V}_J , $\mathbb{V}_J|_{\mathfrak{M}}$ is unitary, and $\mathbb{V}_J|_{\mathfrak{N}}$ is cnu. Denote by \mathfrak{N}_J the smallest reducing subspace for \mathbb{V} containing \mathfrak{N} , and set $\mathfrak{M}_J = \mathfrak{H} \ominus \mathfrak{N}_J$. Since \mathfrak{M}_J reduces $\mathbb{V}_J|_{\mathfrak{M}}$, it follows immediately that (1) is satisfied. Moreover, if \mathfrak{R} is any reducing subspace for \mathbb{V} such that $\mathbb{V}_J|_{\mathfrak{R}}$ is unitary, then $\mathfrak{R} \subset \mathfrak{M}$ so that $\mathfrak{R} \perp \mathfrak{N}$ and consequently $\mathfrak{R} \perp \mathfrak{N}_J$ as well. We conclude that \mathfrak{M}_J is the largest reducing subspace for \mathbb{V} satisfying condition (1). Property (2), as well as the uniqueness of \mathfrak{M}_J , follow from this observation. \square

Observe that \mathfrak{M}_I is precisely the space \mathfrak{M} in Theorem 2.1, and it is convenient to extend our notation so that $\mathfrak{M}_\emptyset = \mathfrak{H}$. We have then

$$\mathfrak{M}_{J_1 \cup J_2} = \mathfrak{M}_{J_1} \cap \mathfrak{M}_{J_2}, \quad J_1, J_2 \subset I.$$

The spaces \mathfrak{M}_J constructed above are in fact hyperinvariant for \mathbb{V} . Given two families $\mathbb{V}^{(1)} = (V_i^{(1)})_{i \in I}$ and $\mathbb{V}^{(2)} = (V_i^{(2)})_{i \in I}$ of commuting isometries on $\mathfrak{H}^{(1)}$ and $\mathfrak{H}^{(2)}$, respectively, we denote by $\mathcal{I}(\mathbb{V}^{(1)}, \mathbb{V}^{(2)})$ the collection of all bounded linear operators $X : \mathfrak{H}^{(1)} \rightarrow \mathfrak{H}^{(2)}$ satisfying the intertwining relations $XV_i^{(1)} = V_i^{(2)}X$ for every $i \in I$. In the special case $\mathbb{V}^{(1)} = \mathbb{V}^{(2)} = \mathbb{V}$, we use the notation $(\mathbb{V})' = \mathcal{I}(\mathbb{V}, \mathbb{V})$ for the *commutant* of \mathbb{V} .

Proposition 2.3. *Consider two families $\mathbb{V}^{(1)} = (V_i^{(1)})_{i \in I}$ and $\mathbb{V}^{(2)} = (V_i^{(2)})_{i \in I}$ of commuting isometries on $\mathfrak{H}^{(1)}$ and $\mathfrak{H}^{(2)}$, respectively, and a subset $J \subset I$. Denote by $\mathfrak{M}_J^{(p)}$ the reducing subspace for $\mathbb{V}^{(p)}$ provided by the preceding theorem for $p = 1, 2$. Then for every $X \in \mathcal{I}(\mathbb{V}^{(1)}, \mathbb{V}^{(2)})$ we have $X\mathfrak{M}_J^{(1)} \subset \mathfrak{M}_J^{(2)}$.*

Proof. Denote by $\mathfrak{N}_J^{(p)}$ the largest subspace of $\mathfrak{H}^{(p)}$ which reduces $\mathbb{V}^{(p)}$ to a unitary operator. That is,

$$\mathfrak{N}_J^{(p)} = \bigcap_{N=1}^{\infty} \left[\bigcap_{k_1, k_2, \dots, k_N \in J} V_{k_1}^{(p)} V_{k_2}^{(p)} \cdots V_{k_N}^{(p)} \mathfrak{H}^{(p)} \right], \quad p = 1, 2.$$

Given $X \in \mathcal{I}(\mathbb{V}^{(1)}, \mathbb{V}^{(2)})$, it is obvious from this formula that $X\mathfrak{N}_J^{(1)} \subset \mathfrak{N}_J^{(2)}$, and therefore $X^*\mathfrak{N}_J^{(2)\perp} \subset \mathfrak{N}_J^{(1)\perp}$. As noted above, $\mathfrak{M}_J^{(p)\perp}$ is the smallest invariant subspace for $(V_i^{(p)*})_{i \in I}$ containing $\mathfrak{N}_J^{(p)\perp}$; that is,

$$\mathfrak{M}_J^{(p)\perp} = \bigcap_{N=1}^{\infty} \left[\bigcap_{k_1, k_2, \dots, k_N \in I} V_{k_1}^{(p)*} V_{k_2}^{(p)*} \cdots V_{k_N}^{(p)*} \mathfrak{N}_J^{(p)\perp} \right], \quad p = 1, 2.$$

The relations $X^*V_i^{(2)*} = V_i^{(1)*}X^*$ imply now $X^*\mathfrak{M}_J^{(2)\perp} \subset \mathfrak{M}_J^{(1)\perp}$, and this relation is equivalent to the conclusion of the proposition. \square

Another useful result is the existence of a unique minimal unitary extension for every family of commuting isometries [25, Chapter I] (see also [7] for a Banach space version). We review the result briefly.

Theorem 2.4. *Let $\mathbb{V} = (V_i)_{i \in I}$ be a family of commuting isometries on \mathfrak{H} . There exists a family $\mathbb{U} = (U_i)_{i \in I}$ of commuting unitary operators on a Hilbert space $\mathfrak{K} \supset \mathfrak{H}$ with the following properties.*

- (1) \mathfrak{H} is invariant for \mathbb{U} and $\mathbb{U}|_{\mathfrak{H}} = \mathbb{V}$.
- (2) $\mathfrak{K} = \bigvee_{N=0}^{\infty} \left[\bigvee_{k_1, k_2, \dots, k_N \in I} U_{k_1}^* U_{k_2}^* \cdots U_{k_N}^* \mathfrak{H} \right]$.

If \mathbb{U}' is another family of commuting unitary operators on a space $\mathfrak{K}' \supset \mathfrak{H}$ satisfying the analogues of conditions (1) and (2), then there exists a surjective isometry $W : \mathfrak{K} \rightarrow \mathfrak{K}'$ such that $Wh = h$ for $h \in \mathfrak{H}$, and $WU_k = U'_k W$ for $k \in I$.

In equation (2) above, we use the convention that $U_{k_1}^* U_{k_2}^* \cdots U_{k_N}^* \mathfrak{H} = \mathfrak{H}$ when $N = 0$.

The family \mathbb{U} is called the *minimal unitary extension* of \mathbb{V} . In the sequel, we will denote by $\widetilde{\mathbb{V}}$ the minimal unitary extension of \mathbb{V} , and by $\widetilde{\mathfrak{H}}$ the space on which it acts. It is easy to verify the following *commutant extension* result. This can be deduced from the results in [25, Chapter 1], and it is proved in [7] for isometric operators acting on a Banach space.

Theorem 2.5. *Let $\mathbb{V}^{(1)} = (V_i^{(1)})_{i \in I}$ and $\mathbb{V}^{(2)} = (V_i^{(2)})_{i \in I}$ be two families of commuting isometries on $\mathfrak{H}^{(1)}$ and $\mathfrak{H}^{(2)}$, respectively, and denote by $\widetilde{\mathbb{V}^{(1)}}$ and $\widetilde{\mathbb{V}^{(2)}}$ their minimal unitary extensions. The map $Y \mapsto X = Y|_{\mathfrak{H}^{(1)}}$ establishes an isometric bijection between the collection of operators $Y \in \mathcal{I}(\widetilde{\mathbb{V}^{(1)}}, \widetilde{\mathbb{V}^{(2)}})$ such that $Y\mathfrak{H}^{(1)} \subset \mathfrak{H}^{(2)}$ and $\mathcal{I}(\mathbb{V}^{(1)}, \mathbb{V}^{(2)})$.*

Indeed, given $k_1, k_2, \dots, k_N \in I$, a given operator $X \in \mathcal{I}(\mathbb{V}^{(1)}, \mathbb{V}^{(2)})$ easily extends to the space $\widetilde{V_{k_1}^{(1)}}^* \widetilde{V_{k_1}^{(1)}}^* \cdots \widetilde{V_{k_1}^{(1)}}^* \mathfrak{H}^{(1)}$ by setting

$$Y \widetilde{V_{k_1}^{(1)}}^* \widetilde{V_{k_1}^{(1)}}^* \cdots \widetilde{V_{k_1}^{(1)}}^* h = \widetilde{V_{k_1}^{(2)}}^* \widetilde{V_{k_1}^{(2)}}^* \cdots \widetilde{V_{k_1}^{(2)}}^* Xh, \quad h \in \widetilde{V_{k_1}^{(1)}}^* \widetilde{V_{k_1}^{(1)}}^* \cdots \widetilde{V_{k_1}^{(1)}}^* \mathfrak{H}^{(1)},$$

and the corresponding operator $Y \in \mathcal{I}(\widetilde{\mathbb{V}}, \widetilde{\mathbb{V}'})$ is obtained by taking the closure of this extension. This unique extension of X will be denoted \widetilde{X} . If X is isometric or unitary then so is \widetilde{X} . In the particular case $\mathbb{V}^{(1)} = \mathbb{V}^{(2)} = \mathbb{V}$, the operator X belongs to the commutant of \mathbb{V} , and its canonical extension $\widetilde{X} \in (\widetilde{\mathbb{V}})'$ satisfies $\widetilde{X}\mathfrak{H} \subset \mathfrak{H}$.

Irreducible families of commuting isometries have special properties. Theorem 2.1 shows that they are either unitary or cnu. More precisely, we have the following result.

Proposition 2.6. *Let $\mathbb{V} = (V_i)_{i \in I}$ be an irreducible family of commuting isometries on a nonzero Hilbert space \mathfrak{H} . For every $i_0 \in I$, one of the following alternatives occurs.*

- (1) V_{i_0} is a scalar multiple of the identity.
- (2) \mathbb{V} is $\{i_0\}$ -pure.

Proof. Assume that (2) does not occur. Theorem 2.2 implies then that V_{i_0} is unitary. Since the spectral projections of V_{i_0} reduce \mathbb{V} , it follows that the spectrum of V_{i_0} is a singleton, and therefore (1) is true. \square

Proposition 2.7. *Let $\mathbb{V} = (V_i)_{i \in I}$ and $\mathbb{W} = (W_i)_{i \in I}$ be families of commuting isometries on \mathfrak{H} and \mathfrak{K} , respectively. Assume that $\mathfrak{K} \neq \{0\}$ and \mathbb{W} is irreducible. Let $(\mathfrak{M}_\alpha)_{\alpha \in A}$ be a maximal family of pairwise orthogonal, reducing subspaces for \mathbb{V} with the property that $\mathbb{V}|_{\mathfrak{M}_\alpha}$ is unitarily equivalent to \mathbb{W} for every α . If \mathfrak{M} is any reducing subspace for \mathbb{V} with the property that $\mathbb{V}|_{\mathfrak{M}}$ is unitarily equivalent to \mathbb{W} , then $\mathfrak{M} \subset \bigoplus_{\alpha \in A} \mathfrak{M}_\alpha$.*

Proof. This is really a general fact about representations of C^* -algebras. We provide a proof for the sake of completeness. Fix isometric operators $U, U_\alpha : \mathfrak{K} \rightarrow \mathfrak{H}$ such that $U\mathfrak{K} = \mathfrak{M}, U_\alpha\mathfrak{K} = \mathfrak{M}_\alpha, U\mathbb{W} = \mathbb{V}U$ and $U_\alpha\mathbb{W} = \mathbb{V}U_\alpha$. The operator

$$R = \left[I - \sum_{\alpha \in A} P_{\mathfrak{M}_\alpha} \right] U : \mathfrak{K} \rightarrow \mathfrak{H}$$

satisfies the relations $R\mathbb{W} = \mathbb{V}R$ and $R\mathbb{W}^* = \mathbb{V}^*R$, and therefore R^*R commutes with \mathbb{W} and \mathbb{W}^* . We must have then $R^*R = \rho^2 I_{\mathfrak{K}}$ for some $\rho \geq 0$. If $\rho \neq 0$, then the unitary operator $U_0 = R/\rho$ satisfies $U_0\mathbb{W} = \mathbb{V}U_0, U_0\mathbb{W}^* = \mathbb{V}^*U_0, U_0^*U_\alpha = 0$, and therefore $U_0\mathfrak{K}$ is a reducing space orthogonal to each \mathfrak{M}_α , contradicting the maximality of $(\mathfrak{M}_\alpha)_{\alpha \in A}$. Thus $\rho = 0$ and the proposition follows. \square

It follows from this proposition that the reducing subspace

$$\mathfrak{H}_{\mathbb{W}} = \bigoplus_{\alpha \in A} \mathfrak{M}_\alpha$$

does not depend on the particular family $(\mathfrak{M}_\alpha)_{\alpha \in A}$. The restriction $\mathbb{V}|_{\mathfrak{H}_{\mathbb{W}}}$ is an orthogonal sum of copies of \mathbb{W} , while $\mathbb{V}|_{\mathfrak{H}_{\mathbb{W}}^\perp}$ has no restriction to an invariant subspace that is unitarily equivalent to \mathbb{W} .

Proposition 2.8. *Let $\mathbb{V} = (V_i)_{i \in I}, \mathbb{W}_1 = (W_{i1})_{i \in I}$ and $\mathbb{W}_2 = (W_{i2})_{i \in I}$ be families of commuting isometries on $\mathfrak{H}, \mathfrak{K}_1$ and \mathfrak{K}_2 , respectively. Assume that \mathbb{W}_1 and \mathbb{W}_2 are irreducible and not unitarily equivalent. Then the spaces $\mathfrak{H}_{\mathbb{W}_1}$ and $\mathfrak{H}_{\mathbb{W}_2}$ are mutually orthogonal.*

Proof. Let \mathfrak{M}_j be a reducing subspace for \mathbb{V} such that $\mathbb{V}|_{\mathfrak{M}_j}$ is unitarily equivalent to \mathbb{W}_j via a unitary operator $U_j : \mathfrak{K}_j \rightarrow \mathfrak{M}_j$. It will suffice to show that $\mathfrak{M}_1 \perp \mathfrak{M}_2$ or, equivalently, that the operator $R = U_2^*U_1$ is zero. As in the preceding proof, irreducibility shows that $R^*R = \rho^2 I_{\mathfrak{K}_1}$ and $RR^* = \rho^2 I_{\mathfrak{K}_2}$ for some constant $\rho \geq 0$. The assumption $\rho \neq 0$ implies that \mathbb{W}_1 and \mathbb{W}_2 are unitarily equivalent via the unitary operator R/ρ , which is a contradiction. \square

Corollary 2.9. *Let \mathbb{V} be a family of commuting isometries on \mathfrak{H} , and denote by \mathcal{F} a collection of mutually inequivalent irreducible families of commuting isometries such that every irreducible direct summand of \mathbb{V} is equivalent to an element of \mathcal{F} . We have*

$$\mathfrak{H} = \mathfrak{H}_0 \oplus \bigoplus_{\mathbb{W} \in \mathcal{F}} \mathfrak{H}_{\mathbb{W}},$$

where \mathfrak{H}_0 is a reducing subspace for \mathbb{V} such that $\mathbb{V}|_{\mathfrak{H}_0}$ has no irreducible direct summand.

When $\dim \mathfrak{H}_0 > 1$, the family $\mathbb{V}|_{\mathfrak{H}_0}$ is certainly reducible; it just cannot be decomposed into a direct sum of irreducible families. However it can be decomposed into a continuous direct integral of irreducibles if \mathfrak{H} is separable. A concrete example of such a decomposition will be given in Section 8. Direct integrals are also useful

in the proof of the following result, an early variant of which was proved in [20] when I consists of two elements. We refer to [26] for the theory of direct integrals.

Proposition 2.10. *Let $\mathbb{V} = (V_i)_{i \in I}$ be a finite family of commuting isometries on a Hilbert space \mathfrak{H} . We can associate to each subset $J \subset I$ a reducing space \mathfrak{L}_J for \mathbb{V} with the following properties.*

- (1) $\mathfrak{H} = \bigoplus_{J \subset I} \mathfrak{L}_J$.
- (2) $V_j|_{\mathfrak{L}_J}$ is unitary for each $j \in J$.
- (3) $\mathbb{V}|_{\mathfrak{L}_J}$ is $\{j\}$ -pure for each $j \notin J$.

Proof. Since I is finite, \mathfrak{H} can be written as an orthogonal sum of separable reducing subspaces for \mathbb{V} . Thus it is sufficient to consider the case of separable spaces \mathfrak{H} . There exist a probability measure μ on $[0, 1]$, a measurable family $(\mathfrak{H}_t)_{t \in [0, 1]}$ of Hilbert spaces, and a measurable collection $(\mathbb{V}_t)_{t \in [0, 1]} = ((V_{ti})_{i \in I})_{t \in [0, 1]}$ of irreducible families of commuting isometries on \mathfrak{H}_t such that, up to unitary equivalence,

$$\mathfrak{H} = \int_{[0, 1]}^{\oplus} \mathfrak{H}_t d\mu(t), \quad V_i = \int_{[0, 1]}^{\oplus} V_{ti} d\mu(t), \quad i \in I.$$

Proposition 2.6 shows that for each $t \in [0, 1]$ there exists a subset $J(t) \subset I$ such that V_{i_j} is a scalar multiple of the identity if $j \in J(t)$, while \mathbb{V}_t is $\{j\}$ -pure for $j \notin J(t)$. It is easy to verify that the set $\sigma_J = \{t \in [0, 1] : J(t) = J\}$ is measurable for each $J \subset I$. The spaces

$$\mathfrak{L}_J = \int_{\sigma_J}^{\oplus} \mathfrak{H}_t d\mu(t),$$

viewed as subspaces of \mathfrak{H} , satisfy the conclusion of the proposition. \square

3. INDUCTIVE CONSTRUCTION OF COMMUTING ISOMETRIES

In this section it will be convenient to index families of commuting isometries by ordinal numbers. Thus, given an ordinal number n , an n -isometry is simply a family $\mathbb{V} = (V_i)_{0 \leq i < n}$ of commuting isometries on a Hilbert space.

We consider a special construction which produces an $(n+1)$ -isometry starting from an n -isometry \mathbb{V} on \mathfrak{H} and a contraction $A \in (\mathbb{V})'$; that is, $\|A\| \leq 1$. Observe that the canonical extension $\tilde{A} \in (\tilde{\mathbb{V}})'$ on $\tilde{\mathfrak{H}}$ is then a contraction as well, and therefore we can form the defect operator

$$D_{\tilde{A}} = (I - \tilde{A}^* \tilde{A})^{1/2}$$

and the space $\mathfrak{D} = \overline{D_{\tilde{A}} \tilde{\mathfrak{H}}}$. The space \mathfrak{D} is reducing for $\tilde{\mathbb{V}}$ because $D_{\tilde{A}}$ commutes with $\tilde{\mathbb{V}}$. We form the space

$$\mathfrak{K} = \mathfrak{H} \oplus \mathfrak{D} \oplus \mathfrak{D} \oplus \cdots,$$

and define an $(n+1)$ -isometry $\mathbb{W}_A = (W_k)_{0 \leq k \leq n}$ on \mathfrak{K} as follows. For $0 \leq k < n$ we define

$$W_k = V_k \oplus (\tilde{V}_k|_{\mathfrak{D}}) \oplus (\tilde{V}_k|_{\mathfrak{D}}) \oplus \cdots,$$

while

$$W_n(h \oplus d_0 \oplus d_1 \oplus \cdots) = Ah \oplus D_{\tilde{A}} h \oplus d_0 \oplus d_1 \oplus \cdots$$

if $h \in \mathfrak{H}$ and $d_j \in \mathfrak{D}$ for $j \in \mathbb{N}$. It is easy to verify that \mathbb{W}_A is in fact an $(n+1)$ -isometry. When the operator A is already isometric, we have $\mathfrak{K} = \mathfrak{H}$ and $\mathbb{W}_A = (\mathbb{V}, A)$. In this trivial sense, every $(n+1)$ -isometry is of the form \mathbb{W}_A for some

contraction A commuting with an n -isometry \mathbb{V} . We give now a characterization of $(n+1)$ -isometries which are $\{0 \leq k < n\}$ -pure.

Theorem 3.1. *Let $\mathbb{W} = (W_k)_{0 \leq k \leq n}$ be an $(n+1)$ -isometry on \mathfrak{K} , where $n \geq 1$. The following conditions are equivalent.*

- (1) \mathbb{W} is $\{0 \leq k < n\}$ -pure.
- (2) There exist a cnu n -isometry \mathbb{V} , and a contraction $A \in (\mathbb{V})'$, such that \mathbb{W} is unitarily equivalent to \mathbb{W}_A .

Proof. Assume first that $\mathbb{W} = \mathbb{W}_A$, where A is a contraction in the commutant of the cnu n -isometry \mathbb{V} on \mathfrak{H} . Let \mathfrak{M} be a reducing subspace for \mathbb{W}_A with the property that $W_k|_{\mathfrak{M}}$ is unitary for all $k < n$. Since the cnu direct summand of the n -isometry $(W_k)_{0 \leq k < n}$ is precisely \mathbb{V} viewed as acting on $\mathfrak{H} \oplus \{0\} \oplus \{0\} \oplus \cdots$, we conclude that

$$\mathfrak{M} \subset \{0\} \oplus \mathfrak{D} \oplus \mathfrak{D} \oplus \cdots$$

and therefore $W_n^{*N}h \in \{0\} \oplus \mathfrak{D} \oplus \mathfrak{D} \oplus \cdots$ for every $h \in \mathfrak{M}$ and $N \geq 1$. This is not possible if $h \neq 0$. Indeed, if $h = 0 \oplus 0 \oplus \cdots \oplus 0 \oplus d_N \oplus \cdots$, and the N th component d_N is the first nonzero component of h , then $W_n^{*N}h = D_A d_N \oplus \cdots \notin \mathfrak{M}$ because $D_A d_N \neq 0$.

Conversely, assume that condition (1) is satisfied. Consider the n -isometry $\mathbb{W}' = (W_k)_{0 \leq k < n}$, and the decomposition $\mathfrak{K} = \mathfrak{H} \oplus \mathfrak{H}^\perp$ into reducing subspaces for \mathbb{W}' such that $\mathbb{W}'|_{\mathfrak{H}}$ is cnu and $\mathbb{W}'|_{\mathfrak{H}^\perp}$ is unitary. We denote by $\mathbb{V} = \mathbb{W}'|_{\mathfrak{H}}$ the cnu direct summand of \mathbb{W}' , and define an operator A on \mathfrak{H} by setting $A = P_{\mathfrak{H}} W_n|_{\mathfrak{H}}$. Clearly A is a contraction, and the fact that A commutes with \mathbb{V} follows from the fact that the unitary component \mathfrak{H}^\perp is obviously invariant for W_n , and therefore $A^* = W_n^*|_{\mathfrak{H}}$. Consider next the minimal unitary extension $\widetilde{\mathbb{W}'}$ which can be written as

$$\widetilde{\mathbb{W}'} = \widetilde{\mathbb{V}} \oplus (\mathbb{W}'|_{\mathfrak{H}^\perp})$$

on the space $\widetilde{\mathfrak{H}} \oplus \mathfrak{H}^\perp$, and the unique isometric extension \widetilde{W}_n of W_n in the commutant of $\widetilde{\mathbb{W}'}$. Clearly,

$$\widetilde{W}_n|_{\mathfrak{H}^\perp} = W_n|_{\mathfrak{H}^\perp},$$

and the compression $P_{\widetilde{\mathfrak{H}}} \widetilde{W}_n|_{\widetilde{\mathfrak{H}}}$ is precisely the contractive extension \widetilde{A} of A in the commutant of $\widetilde{\mathbb{V}}$. We show next that \widetilde{W}_n is in fact the minimal isometric dilation of \widetilde{A} . In other words, the smallest invariant subspace \mathfrak{M} for \widetilde{W}_n containing $\widetilde{\mathfrak{H}}$ is $\widetilde{\mathfrak{H}} \oplus \mathfrak{H}^\perp$. To prove this, observe first that, since

$$\mathfrak{M} = \bigvee_{N \geq 0} \widetilde{W}_n^{*N} \widetilde{\mathfrak{H}}$$

and $\widetilde{\mathfrak{H}}$ is invariant for \widetilde{W}_n^* , the space \mathfrak{M} is actually reducing for \widetilde{W}_n . Moreover, \widetilde{W}_i is unitary for $i < n$, and hence the operators \widetilde{W}_i^* and \widetilde{W}_n also commute. Thus \mathfrak{M} is also a reducing space for each \widetilde{W}_i if $i < n$. We conclude that the space $\mathfrak{M}^\perp \subset \mathfrak{H}^\perp$ reduces \mathbb{W} , and $\mathbb{W}'|_{\mathfrak{M}^\perp}$ is unitary. Hypothesis (1) implies that $\mathfrak{M}^\perp = \{0\}$.

With this preparation out of the way, we find ourselves in the familiar territory of minimal isometric dilations [25, Chapter II]. We recall that, up to unitary equivalence, the minimal isometric dilation of the contraction \widetilde{A} is the operator W defined by

$$W(h \oplus d_0 \oplus d_1 \oplus \cdots) = \widetilde{A}h \oplus D_{\widetilde{A}}h \oplus d_0 \oplus d_1 \oplus \cdots$$

on the space $\widetilde{\mathfrak{H}} \oplus \mathfrak{D} \oplus \mathfrak{D} \oplus \cdots$, where $\mathfrak{D} = \overline{D_{\widetilde{A}} \widetilde{\mathfrak{H}}}$. We conclude that there exists a unitary operator $U : \mathfrak{D} \oplus \mathfrak{D} \oplus \cdots \rightarrow \mathfrak{H}^\perp$ such that

$$(I_{\widetilde{\mathfrak{H}}} \oplus U)W = W_n(I_{\widetilde{\mathfrak{H}}} \oplus U).$$

The reader will verify now without difficulty that the operator $I_{\widetilde{\mathfrak{H}}} \oplus U$ provides a unitary equivalence between \mathbb{W}_A and \mathbb{W} . \square

The preceding result shows how any $\{0 \leq k < n\}$ -pure $(n+1)$ -isometry can be constructed from a contraction in the commutant of a cnu n -isometry. General $(n+1)$ -isometries are described using Theorem 2.2 with $J = \{0 \leq k < n\}$. We record the result below, using the lifting concept as in [10, Sec. II.1].

Theorem 3.2. *Let $\mathbb{W} = (W_k)_{0 \leq k < n}$ be an $(n+1)$ -isometry on \mathfrak{K} , where $n \geq 1$. There exist reducing subspaces \mathfrak{K}_0 and \mathfrak{K}_1 for \mathbb{W} with the following properties.*

- (1) $\mathfrak{K}_0 \oplus \mathfrak{K}_1 = \mathfrak{K}$.
- (2) $W_k|_{\mathfrak{K}_1}$ is unitary for every $k < n$.
- (3) $\mathbb{W}|_{\mathfrak{K}_0}$ is unitarily equivalent to \mathbb{W}_A , where A is a contraction in the commutant of a cnu n -isometry \mathbb{V} .

The n -isometry \mathbb{V} on $\mathfrak{H} \subset \mathfrak{K}$ is the cnu part of $\mathbb{W}' = (W_k)_{0 \leq k < n}$, and the operator A is defined by the equivalent relations

$$A = P_{\mathfrak{H}} W_n|_{\mathfrak{H}}, \quad A^* = W_n^*|_{\mathfrak{H}}.$$

In particular, W_n is an isometric lifting of A , and \widetilde{W}_n is an isometric lifting of \widetilde{A} , where the extension \widetilde{W}_n belongs to $(\mathbb{W}')'$ and $\widetilde{A} \in (\widetilde{\mathbb{V}})'$.

Thus, the space \mathfrak{K}_0 is simply the $\{0 \leq k < n\}$ -pure summand of \mathbb{W} .

The operators which intertwine two $(n+1)$ -isometries can also be analyzed in the context of this inductive construction. Indeed, consider $(n+1)$ -isometries $\mathbb{W}^{(p)}$ acting on $\mathfrak{K}^{(p)}$, and the corresponding decompositions

$$\mathfrak{K}^{(p)} = \mathfrak{K}_0^{(p)} \oplus \mathfrak{K}_1^{(p)}, \quad p = 1, 2,$$

provided by Theorem 3.2. In other words, $\mathbb{W}^{(p)}|_{\mathfrak{K}_0^{(p)}}$ is $\{0 \leq k < n\}$ -pure, and $W_k^{(p)}|_{\mathfrak{K}_1^{(p)}}$ is unitary for $0 \leq k < n$. Let us further denote by $\mathfrak{H}^{(p)}$ the cnu part of $\mathfrak{K}^{(p)}$ relative to the n -isometry $\mathbb{W}^{(p)'} = \{W_k^{(p)}\}_{0 \leq k < n}$, and set

$$\mathbb{V}^{(p)} = \mathbb{W}^{(p)'}|_{\mathfrak{H}^{(p)}}, \quad A^{(p)} = P_{\mathfrak{H}^{(p)}} W_n^{(p)}|_{\mathfrak{H}^{(p)}}, \quad p = 1, 2.$$

The minimal unitary extension $\widetilde{\mathbb{W}^{(p)'}}$ of the n -isometry $\mathbb{W}^{(p)'}$ acts on the space

$$\widetilde{\mathfrak{K}^{(p)}} = \widetilde{\mathfrak{K}_0^{(p)}} \oplus \mathfrak{K}_1^{(p)},$$

and we denote by $\widetilde{W_n^{(p)}}$ the canonical extension of $W_n^{(p)}$ to this larger space. We have

$$\widetilde{W_n^{(p)}} = \widetilde{W_n^{(p)}|_{\mathfrak{K}_0^{(p)}}} \oplus (W_n^{(p)}|_{\mathfrak{K}_1^{(p)}})$$

and, as seen above, $\widetilde{W_n^{(p)}|_{\mathfrak{K}_0^{(p)}}}$ is the minimal isometric dilation of the operator $\widetilde{A}^{(p)}$.

Any operator $X \in \mathcal{I}(\mathbb{W}^{(1)}, \mathbb{W}^{(2)})$ can be represented as a matrix

$$X = \begin{bmatrix} X_{00} & X_{01} \\ X_{10} & X_{11} \end{bmatrix},$$

where $X_{ij} \in \mathcal{I}(\mathbb{W}^{(1)}|\mathfrak{K}_j^{(1)}, \mathbb{W}^{(2)}|\mathfrak{K}_i^{(2)})$ for $i, j \in \{0, 1\}$. Theorem 2.4 implies the existence of an extension $\widetilde{X} \in \mathcal{I}(\widetilde{\mathbb{W}}^{(1)}, \widetilde{\mathbb{W}}^{(2)})$. This extension will be represented by a matrix of the form

$$\widetilde{X} = \begin{bmatrix} \widetilde{X}_{00} & X_{01} \\ \widetilde{X}_{10} & X_{11} \end{bmatrix}$$

relative to the decompositions $\widetilde{\mathfrak{K}}^{(p)} = \widetilde{\mathfrak{K}}_0^{(p)} \oplus \widetilde{\mathfrak{K}}_1^{(p)}$.

Proposition 3.3. *With the above notation, the following statements are true.*

- (1) $X_{01} = 0$.
- (2) The operator $Z = P_{\widetilde{\mathfrak{H}}^{(2)}} X_{00} |_{\widetilde{\mathfrak{H}}^{(1)}}$ belongs to $\mathcal{I}(\mathbb{V}^{(1)}, \mathbb{V}^{(2)})$ and $ZA^{(1)} = A^{(2)}Z$.
- (3) The operator $B = P_{\widetilde{\mathfrak{H}}_0^{(2)}} \widetilde{X}$ belongs to $\mathcal{I}(\widetilde{\mathbb{V}}^{(1)}, \widetilde{\mathbb{V}}^{(2)})$, $B\widetilde{W}_n^{(1)} = \widetilde{A}^{(2)}B$, and $B\widetilde{\mathfrak{H}}^{(1)\perp} = \{0\}$.

Proof. Part (1) follows from Proposition 2.3 applied to the particular case $J = \{0 \leq k < n\}$. The intertwining properties of Z in part (2) follow from the fact that the space $\widetilde{\mathfrak{H}}^{(p)}$ is reducing for $\mathbb{W}^{(p)'}$ and invariant for W_n^* . In other words, we can use the fact that, relative to the decompositions $\mathfrak{K}^{(p)} = \mathfrak{H}^{(p)} \oplus \mathfrak{H}^{(p)\perp}$, the relevant operators have matrices of the form

$$X = \begin{bmatrix} Z & 0 \\ * & * \end{bmatrix}, \quad W_n^{(p)} = \begin{bmatrix} A^{(p)} & 0 \\ * & * \end{bmatrix}, \quad W_k^{(p)} = \begin{bmatrix} V_k^{(p)} & 0 \\ 0 & * \end{bmatrix}, \quad 0 \leq k < n.$$

For part (3) we may assume that $\mathbb{W}^{(p)}$, $p = 1, 2$, are $\{0 \leq k < n\}$ -pure. Hence part (3) follows from similar considerations. \square

In the lifting framework of [10, Sec. II.1], the operator \widetilde{X} is said to be a *lifting* of B , and this lifting is contractive if $\|\widetilde{X}\| \leq 1$. A natural question arises: given a contraction B satisfying the requirements of Proposition 3.3(3), can one construct a contractive lifting $\widetilde{X} \in \mathcal{I}(\widetilde{\mathbb{W}}^{(1)}, \widetilde{\mathbb{W}}^{(2)})$? If one pursues the more modest goal of finding a contractive lifting $\widetilde{X} \in \mathcal{I}(\widetilde{\mathbb{W}}_n^{(1)}, \widetilde{\mathbb{W}}_n^{(2)})$, the answer is in the affirmative, and a parametrization of all such contractive liftings can be extracted from [10, Chapter VI]. We describe the result below, under the additional assumption that $\mathbb{W}^{(2)}$ is $\{0 \leq k < n\}$ -pure. In the notation adopted in this section, this amounts to the requirement that $\mathfrak{K}_1^{(2)} = \{0\}$.

Proposition 3.4. *With the preceding notation, assume that $B \in \mathcal{I}(\widetilde{W}_0^{(1)}, A^{(2)})$ is an operator of norm ≤ 1 . The set of contractive liftings $\widetilde{X} \in \mathcal{I}(\widetilde{W}_1^{(1)}, \widetilde{W}_1^{(2)})$ of B is parametrized by (that is, it is in a canonical bijection with) the set of all contractive analytic functions $R : \mathbb{D} \rightarrow \mathcal{L}(\mathfrak{G}, \mathfrak{G}')$, where the spaces \mathfrak{G} and \mathfrak{G}' are given by the formulas*

$$\begin{aligned} \mathfrak{G} &= \overline{D_B \widetilde{\mathfrak{H}}^{(1)} \ominus D_B \widetilde{W}_1^{(1)} \widetilde{\mathfrak{H}}^{(1)}}, \\ \mathfrak{G}' &= \overline{\left[\overline{(W_1^{(2)} - A^{(2)}) B \widetilde{\mathfrak{H}}^{(1)} \oplus D_B \widetilde{\mathfrak{H}}^{(1)}} \right]} \\ &\quad \ominus \overline{\{(W_1^{(2)} - A^{(2)}) B h^{(1)} \oplus D_B h^{(1)} : h^{(1)} \in \widetilde{\mathfrak{H}}^{(1)}\}}, \end{aligned}$$

and where $D_B = (I - B^*B)^{1/2}$.

One of the liftings considered above will yield an operator $X \in \mathcal{I}(\mathbb{W}^{(1)}, \mathbb{W}^{(2)})$ only when it also satisfies the conditions $\widetilde{X}W_k^{(1)} = W_k^{(2)}\widetilde{X}$ for $0 \leq k < n$, and B itself is subject to the supplementary conditions

$$B\mathfrak{H}^{(1)\perp} = \{0\}, \quad B\mathfrak{H}^{(1)} \subset \mathfrak{H}^{(2)} \quad B \in \mathcal{I}(\widetilde{\mathbb{V}}^{(1)}, \widetilde{\mathbb{V}}^{(2)}).$$

We continue the discussion now under the assumption that the operator B does satisfy these additional conditions. The fact that $B \in \mathcal{I}(\widetilde{\mathbb{V}}^{(1)}, \widetilde{\mathbb{V}}^{(2)})$ is easily seen to imply that $D_B \in \mathcal{I}(\widetilde{\mathbb{V}}^{(1)}, \widetilde{\mathbb{V}}^{(2)})$. Using the notation in Proposition 3.4, these intertwining conditions imply

$$(3.1) \quad \widetilde{V}_k^{(1)}\mathfrak{G} \subset \mathfrak{G} \quad \text{and} \quad (\widetilde{V}_k^{(2)} \oplus \widetilde{V}_k^{(1)})\mathfrak{G}' \subset \mathfrak{G}' \quad \text{for } 0 \leq k < n.$$

Some additional application of techniques from of [10, Chapter VI] yields the following result.

Proposition 3.5. *With the above notation, assume that $\mathbb{W}^{(2)}$ is $\{0 \leq k < n\}$ -pure. The set of contractions in $\mathcal{I}(\mathbb{W}^{(1)}, \mathbb{W}^{(2)})$ can be parametrized by pairs (B, R) , where $B \in \mathcal{I}(\widetilde{W}_n^{(1)}, \widetilde{A}^{(2)})$ is a contraction satisfying the conditions in Proposition 3.3(3) and R is a parameter as in Proposition 3.4 satisfying the additional conditions*

$$(\widetilde{V}_k^{(1)} \oplus \widetilde{V}_k^{(2)})R(z) = R(z)\widetilde{V}_k^{(1)}|_{\mathfrak{G}}, \quad 0 \leq k < n, z \in \mathbb{D}.$$

These results enable one to begin a systematic study of the invariant subspaces of bi-isometries. This study was already started in [3] and it will be continued in a forthcoming paper.

Remark 3.6. We emphasize again that the preceding result does not require that $\mathbb{W}^{(1)}$ is $\{0 \leq k < n\}$ is pure.

4. THE STRUCTURE OF BI-ISOMETRIES

For the remainder of this paper, we focus on bi-isometries $\mathbb{W} = (W_0, W_1)$ on a Hilbert space \mathfrak{K} . In view of Theorem 3.1, Theorem 2.2 takes the following form when $J = \{0\}$.

Proposition 4.1. *Consider a bi-isometry $\mathbb{W} = (W_0, W_1)$ on \mathfrak{K} , let $\mathfrak{K} = \mathfrak{H} \oplus \mathfrak{H}^\perp$ be the von Neumann-Wold decomposition relative to W_0 , so that $V_0 = W_0|_{\mathfrak{H}}$ is a unilateral shift and $W_0|_{\mathfrak{H}^\perp}$ is unitary. Denote by $\widetilde{W}_0 = \widetilde{V}_0 \oplus (W_0|_{\mathfrak{H}^\perp}) \in \mathcal{L}(\widetilde{\mathfrak{K}}) = \mathcal{L}(\widetilde{\mathfrak{H}} \oplus \mathfrak{H}^\perp)$ the minimal unitary extension of W_0 , and denote by $\widetilde{W}_1 \in \mathcal{L}(\widetilde{\mathfrak{K}})$ the unique isometric extension of W_1 which commutes with \widetilde{W}_0 . Define*

$$\mathfrak{M} = \bigvee_{k=0}^{\infty} \widetilde{W}_1^k \widetilde{\mathfrak{H}}, \quad \mathfrak{N} = \widetilde{\mathfrak{K}} \ominus \mathfrak{M}.$$

Then the subspace $\mathfrak{N} \subset \mathfrak{H}^\perp$ is reducing for \mathbb{W} , and $W_0|_{\mathfrak{N}}$ is unitary. Moreover, \mathfrak{N} is the largest reducing subspace for \mathbb{W} with the property that $W_0|_{\mathfrak{N}}$ is unitary.

Corollary 4.2. *With the notation of the preceding result, the following assertions are equivalent.*

- (1) $\mathfrak{N} = \{0\}$.

- (2) \mathbb{W} is $\{0\}$ -pure
- (3) The operator \widetilde{W}_1^* is the minimal coisometric extension of $\widetilde{W}_1^*|\mathfrak{H}$.

The particular case of Proposition 2.10 for bi-isometries can be proved by repeated application of Proposition 4.1. This result was obtained first in the case of doubly commuting isometries in [19]; the general case appears in [11] (see also [17] for another proof).

Corollary 4.3. *Consider a bi-isometry $\mathbb{W} = (W_0, W_1)$ on \mathfrak{K} . There exist unique reducing subspaces $\mathfrak{K}_{00}, \mathfrak{K}_{11}, \mathfrak{K}_{01}, \mathfrak{K}_{10}$ for \mathbb{W} with the following properties.*

- (1) $W_0|\mathfrak{K}_{01}$ is a shift and $W_1|\mathfrak{K}_{01}$ is unitary.
- (2) $W_0|\mathfrak{K}_{10}$ is unitary and $W_1|\mathfrak{K}_{10}$ is a shift.
- (3) $W_0|\mathfrak{K}_{11}$ and $W_1|\mathfrak{K}_{11}$ are unitary.
- (4) There is no nonzero reducing subspace $\mathfrak{N} \subset \mathfrak{K}_{00}$ for \mathbb{W} such that either $W_0|\mathfrak{N}$ or $W_1|\mathfrak{N}$ is unitary.
- (5) $\mathfrak{K} = \mathfrak{K}_{00} \oplus \mathfrak{K}_{01} \oplus \mathfrak{K}_{10} \oplus \mathfrak{K}_{11}$.

Proof. Proposition 4.1 yields a decomposition $\mathfrak{K} = \mathfrak{N}^\perp \oplus \mathfrak{N}$ into reducing subspaces for \mathbb{W} such that $W_0|\mathfrak{N}$ is unitary and there is no reducing subspace $\mathfrak{N}' \subset \mathfrak{N}^\perp$ for \mathbb{W} such that $W_0|\mathfrak{N}'$ is unitary. Apply this result with the pair \mathbb{W} replaced by $(W_1|\mathfrak{N}, W_0|\mathfrak{N})$ and $(W_1|\mathfrak{N}^\perp, W_0|\mathfrak{N}^\perp)$, respectively, to obtain decompositions $\mathfrak{N} = \mathfrak{K}_{10} \oplus \mathfrak{K}_{11}$ and $\mathfrak{N}^\perp = \mathfrak{K}_{00} \oplus \mathfrak{K}_{01}$, respectively, into sums of reducing subspaces such that $W_1|\mathfrak{K}_{11}$ and $W_1|\mathfrak{K}_{01}$ are unitary. Moreover, there is no nontrivial reducing subspace \mathfrak{M} for \mathbb{W} contained in either \mathfrak{K}_{10} or \mathfrak{K}_{00} such that $W_1|\mathfrak{M}$ is unitary. We leave the remaining verifications to the interested reader. \square

Consider a bi-isometry $\mathbb{W} = (W_0, W_1)$ on the Hilbert space \mathfrak{K} . As in Proposition 4.1, we consider the Wold decomposition $\mathfrak{K} = \mathfrak{H} \oplus \mathfrak{H}^\perp$ for W_0 , with

$$\mathfrak{H}_0 = \bigoplus_{k=0}^{\infty} W_0^k \mathfrak{E}, \quad \mathfrak{E} = \ker W_0^* = \mathfrak{K} \ominus W_0 \mathfrak{K},$$

and we set $V_0 = W_0|\mathfrak{H}$ and $A = P_{\mathfrak{H}} W_1|\mathfrak{H}$. Thus, V_0 is a unilateral shift and, as observed earlier, A is a contraction in the commutant of V_0 . We will call (V_0, A) the *characteristic pair* associated to the bi-isometry \mathbb{W} . Thus, the characteristic pair is simply formed by a unilateral shift and a contraction in its commutant. The concept of unitary equivalence for these objects is natural: two such pairs are said to be unitarily equivalent if they are conjugated by a unitary operator (the same for the two operators of the pair).

The pair (W_1, W_0) is also a bi-isometry, and the above procedure associates to it a characteristic pair. The characteristic pairs of (W_0, W_1) and (W_1, W_0) are not unitarily equivalent in general.

For future reference, we restate Theorem 3.1 for the special case $n = 1$, that is, the case of bi-isometries.

Proposition 4.4. *Let $V_0 \in \mathcal{L}(\mathfrak{H})$ be a unilateral shift, and $A \in \{V_0\}'$ a contraction. Denote by $\widetilde{V}_0 \in \mathcal{L}(\widetilde{\mathfrak{H}}_0)$ the minimal unitary extension of V_0 , let $\widetilde{A} \in \{\widetilde{V}_0\}'$ be the extension of A , and set $D = (I - \widetilde{A}^* \widetilde{A})^{1/2}$, $\mathfrak{D} = (D\widetilde{\mathfrak{H}}_0)^-$.*

- (1) The space \mathfrak{D} is reducing for \widetilde{V}_0 .
- (2) Define the Hilbert space

$$\mathfrak{K} = \mathfrak{H} \oplus \mathfrak{D} \oplus \mathfrak{D} \oplus \cdots,$$

and the operators $W_0, W_1 \in \mathcal{L}(\mathfrak{K})$ by

$$W_0(h \oplus d_0 \oplus d_1 \oplus \cdots) = V_0 h \oplus \widetilde{V}_0 d_0 \oplus \widetilde{V}_0 d_1 \oplus \cdots,$$

$$W_1(h \oplus d_0 \oplus d_1 \oplus \cdots) = Ah \oplus Dh \oplus d_0 \oplus d_1 \oplus \cdots.$$

Then (W_0, W_1) is a $\{0\}$ -pure bi-isometry whose characteristic pair is unitarily equivalent to (V_0, A) .

We collect in the following statement some basic properties of the characteristic pair. These follow immediately from the results in Section 3.

Proposition 4.5. *Let $\mathbb{W} = (W_0, W_1)$ and $\mathbb{W}' = (W'_0, W'_1)$ be two bi-isometries with characteristic pairs (V_0, A) and (V'_0, A') , respectively.*

- (1) *The characteristic pair of $\mathbb{W} \oplus \mathbb{W}'$ is $(V_0 \oplus V'_0, A \oplus A')$.*
- (2) *If \mathbb{W} is unitarily equivalent to \mathbb{W}' , then (V_0, A) is unitarily equivalent to (V'_0, A') .*
- (3) *Assume in addition that \mathbb{W} and \mathbb{W}' are $\{0\}$ -pure. If (V_0, A) is unitarily equivalent to (V'_0, A') , then \mathbb{W} is unitarily equivalent to \mathbb{W}' .*
- (4) *For every pair (V_0, A) , where V_0 is a unilateral shift and $A \in \{V_0\}'$ is a contraction, there exists a bi-isometry \mathbb{W} such that (V_0, A) is the characteristic pair associated to \mathbb{W} . This bi-isometry can be chosen to be $\{0\}$ -pure.*

Part (2) of this proposition characterizes the reducing subspaces of a $\{0\}$ -pure bi-isometry in terms of its characteristic pair. General invariant subspaces of a bi-isometry are not characterized as easily. One difficulty is the fact that the restriction of a $\{0\}$ -pure bi-isometry to an invariant subspace is not always $\{0\}$ -pure. Assume then that we start with a $\{0\}$ -pure bi-isometry \mathbb{W} on \mathfrak{K} , $\mathfrak{K}' \subset \mathfrak{K}$ is an invariant subspace for \mathbb{W} , and $\mathbb{W}' = \mathbb{W}|_{\mathfrak{K}'}$. The inclusion operator $X \in \mathcal{L}(\mathfrak{K}', \mathfrak{K})$ is obviously an isometry in $\mathcal{I}(\mathbb{W}', \mathbb{W})$. Conversely, given an isometric intertwining between bi-isometries $X \in \mathcal{I}(\mathbb{W}'^{(1)}, \mathbb{W})$, the range of X is an invariant subspace for \mathbb{W} . Thus the description of invariant subspaces for bi-isometries can be achieved by understanding the structure of isometric operators intertwining two bi-isometries. In the terminology of Proposition 3.5, one needs to find the *parameters* R which give rise to isometric liftings \widetilde{X} of a given contraction B . We presented in [3] some general results concerning this problem, and further results will appear in a forthcoming paper.

5. FUNCTIONAL REPRESENTATION

The data in a characteristic pair (V_0, A) on \mathfrak{H} can alternately be encoded in a contractive analytic operator-valued function on the unit disk \mathbb{D} . Set $\mathfrak{E} = \mathfrak{H} \ominus V_0 \mathfrak{H}$, and define operators $\Theta_k \in \mathcal{L}(\mathfrak{E})$ as follows:

$$\Theta_k = P_{\mathfrak{E}} V_0^{*k} A|_{\mathfrak{E}}, \quad k \geq 0.$$

We can then associate to the pair (V_0, A) the operator-valued analytic function

$$\Theta(z) = \sum_{k=0}^{\infty} z^k \Theta_k = P_{\mathfrak{E}} (I - zV_0^*)^{-1} A|_{\mathfrak{E}}, \quad |z| < 1.$$

When (V_0, A) is the characteristic pair of a bi-isometry \mathbb{W} , Θ will be called the *characteristic function* of \mathbb{W} ; we will use the notation $\Theta = \Theta_{\mathbb{W}}$ when it is necessary. If Θ is the characteristic function of $\mathbb{W} = (W_0, W_1)$, then its coefficients satisfy

$$\Theta_k = P_{\mathfrak{E}} V_0^{*k} A|_{\mathfrak{E}} = P_{\mathfrak{E}} W_0^{*k} P_{\mathfrak{H}_0} W_1|_{\mathfrak{E}} = P_{\mathfrak{E}} W_0^{*k} W_1|_{\mathfrak{E}}, \quad k \geq 0,$$

since $P_{\mathfrak{E}}W_0^{*k}P_{\mathfrak{H}_0} = P_{\mathfrak{E}}W_0^{*k}$. In particular, the constant coefficient

$$\Theta_0 = \Theta(0) = P_{\mathfrak{E}}W_1|_{\mathfrak{E}} = (W_1^*|_{\mathfrak{E}})^*$$

is precisely the pivotal operator associated with the pair (W_1, W_0) , as defined in [2]. For the convenience of the reader, we recall that the adjoint of the *pivotal operator* associated with a bi-isometry (W_0, W_1) is defined as $W_0^*|_{\ker(W_1^*)}$.

The operator $\Theta(z)$ is a contraction for $|z| < 1$; in fact

$$\sup_{|z| < 1} \|\Theta(z)\| = \|A\|,$$

where $A^* = W_1^*|\mathfrak{H}$. Unitary equivalence of characteristic functions is defined in the natural way: Θ is unitarily equivalent to Θ' if $U\Theta(z) = \Theta'(z)U$, $z \in \mathbb{D}$, for some unitary operator U . It may be useful to contrast this notion of unitary equivalence with the weaker notion of *coincidence*, which is the appropriate concept in the study of functional models for contractions [25]. Two operator-valued analytic functions Θ and Θ' are said to coincide if there exist unitary operators U and V such that $U\Theta(z) = \Theta'(z)V$ for all $z \in \mathbb{D}$.

Proposition 4.5 can now be reformulated as follows.

Corollary 5.1. *Let \mathbb{W} and \mathbb{W}' be two bi-isometries with characteristic functions Θ and Θ' , respectively.*

- (1) *The characteristic function of $\mathbb{W} \oplus \mathbb{W}'$ is given by $\Theta(z) \oplus \Theta'(z)$ for $z \in \mathbb{D}$.*
- (2) *If \mathbb{W} is unitarily equivalent to \mathbb{W}' , then Θ is unitarily equivalent to Θ' .*
- (3) *Assume in addition that \mathbb{W} and \mathbb{W}' are $\{0\}$ -pure. If Θ is unitarily equivalent to Θ' then \mathbb{W} is unitarily equivalent to \mathbb{W}' .*
- (4) *For every contractive analytic function $\Theta : \mathbb{D} \rightarrow \mathcal{L}(\mathfrak{E})$, there exists a $\{0\}$ -pure bi-isometry \mathbb{W} such that $\Theta_{\mathbb{W}}$ is unitarily equivalent to Θ .*

In order to translate the result of Proposition 4.4 into function theoretical terms we need some notation. First, given a separable, complex Hilbert space \mathfrak{E} , we denote as usual by $H^2(\mathfrak{E})$ the Hilbert space of all square summable power series with coefficients in \mathfrak{E} . Given a contractive analytic function $\Theta : \mathbb{D} \rightarrow \mathcal{L}(\mathfrak{E})$, the analytic Toeplitz operator $T_{\Theta} \in \mathcal{L}(H^2(\mathfrak{E}))$ is defined simply as pointwise multiplication by Θ . The particular case $\Theta(z) = zI_{\mathfrak{E}}$ yields the unilateral shift $S_{\mathfrak{E}}$. The minimal unitary extension of $S_{\mathfrak{E}}$ is the bilateral shift $U_{\mathfrak{E}}$ on the Hilbert space $L^2(\mathfrak{E})$ of all square summable Laurent series with coefficients in \mathfrak{E} . The extension of T_{Θ} which commutes with $U_{\mathfrak{E}}$ is the Laurent operator L_{Θ} with symbol Θ .

Now, the space $L^2(\mathfrak{E})$ can also be viewed as the space of square integrable \mathfrak{E} -valued functions $f : \mathbb{T} = \partial\mathbb{D} \rightarrow \mathfrak{E}$. When viewed in this manner, the operator L_{Θ} is given by

$$(L_{\Theta}f)(\zeta) = \Theta(\zeta)f(\zeta)$$

for almost every $\zeta \in \mathbb{T}$, where the strong operator limit

$$\Theta(\zeta) = \lim_{r \uparrow 1} \Theta(r\zeta)$$

exists almost everywhere. Similarly, the operator $D = (I - L_{\Theta}^*L_{\Theta})^{1/2}$ is given as a multiplication operator by the strongly measurable operator-valued function

$$\Delta(\zeta) = (I - \Theta(\zeta)^*\Theta(\zeta))^{1/2}, \quad \zeta \in \mathbb{T}.$$

The infinite sum

$$(D\tilde{\mathfrak{H}})^- \oplus (D\tilde{\mathfrak{H}})^- \oplus (D\tilde{\mathfrak{H}})^- \oplus \dots$$

appearing in Proposition 4.4 can then be identified with $H^2((L_\Delta L^2(\mathfrak{E}))^-)$. The elements in this space can be viewed as functions of two variables $(w, \zeta) \in \mathbb{D} \times \mathbb{T}$, analytic in w and measurable in ζ .

We are now ready to reformulate Proposition 4.4.

Proposition 5.2. *Let $\Theta : \mathbb{D} \rightarrow \mathcal{L}(\mathfrak{E})$ be a contractive analytic function, and set $\Delta(\zeta) = (I - \Theta(\zeta)^* \Theta(\zeta))^{1/2}$, $\zeta \in \mathbb{T}$.*

- (1) *The space $(L_\Delta L^2(\mathfrak{E}))^-$ is reducing for $U_{\mathfrak{E}}$.*
- (2) *Define the Hilbert space*

$$\mathfrak{K} = H^2(\mathfrak{E}) \oplus H^2((L_\Delta L^2(\mathfrak{E}))^-),$$

and the operators $W_0, W_1 \in \mathcal{L}(\mathfrak{K})$ by

$$W_0(f \oplus g) = a \oplus b, \quad W_1(f \oplus g) = c \oplus d,$$

where

$$a(z) = zf(z), \quad b(w, \zeta) = \zeta g(w, \zeta),$$

$$c(z) = \Theta(z)f(z), \quad d(w, \zeta) = \Delta(\zeta)f(\zeta) + wg(w, \zeta)$$

for $z, w \in \mathbb{D}$ and $\zeta \in \mathbb{T}$. Then (W_0, W_1) is a $\{0\}$ -pure bi-isometry whose characteristic function is unitarily equivalent to Θ .

We will use the notation $\mathbb{W}(\Theta) = (W_0, W_1)$ for the bi-isometry described in the preceding statement. The mapping $\Theta \mapsto \mathbb{W}(\Theta)$ establishes a bijection between unitary equivalence classes of contractive analytic functions $\Theta : \mathbb{D} \rightarrow \mathcal{L}(\mathfrak{E})$ and unitary equivalence classes of $\{0\}$ -pure bi-isometries \mathbb{W} . The formulas given for $\mathbb{W}(\Theta)$ allow, in principle, explicit calculations. A first instance is the following result.

Proposition 5.3. *Let $\Theta : \mathbb{D} \rightarrow \mathcal{L}(\mathfrak{E})$ be a contractive analytic function, and denote $(W_0, W_1) = \mathbb{W}(\Theta)$.*

- (1) *The operator W_1 is unitary if and only if Θ is a constant unitary operator, that is, $\Theta(z) \equiv \Theta(0)$, and $\Theta(0)$ is a unitary operator in $\mathcal{L}(\mathfrak{E})$.*
- (2) *The following conditions are equivalent:*
 - (a) *$\mathbb{W}(\Theta)$ is $\{1\}$ -pure.*
 - (b) *the contraction $\Theta(0)$ is completely nonunitary.*

Proof. If W_1 is unitary, W_0 must be a unilateral shift, and therefore $V = S_{\mathfrak{E}}$, and $W = T_{\Theta}$. It is well-known that T_{Θ} is unitary if and only if Θ is a constant unitary operator.

To prove (2), assume first that $W_1|_{\mathfrak{N}}$ is unitary for some nonzero reducing subspace \mathfrak{N} of $\mathbb{W}(\Theta)$. Applying part (1) of Proposition 5.1 and part (1) of this proposition, which has already been proved, shows that we can write $\Theta = \Theta' \oplus \Theta''$, with Θ'' a constant unitary operator acting on a nonzero space. In particular $\Theta(0)$ has a nontrivial unitary direct summand. Conversely, assume that $\Theta(0)$ is not completely nonunitary, so that its restriction to some nonzero invariant subspace \mathfrak{E}_0 is a unitary operator. The contractive analytic function $\Theta_0 : \mathbb{D} \rightarrow \mathcal{L}(\mathfrak{E}_0)$ defined by $\Theta_0(z) = P_{\mathfrak{E}_0} \Theta(z)|_{\mathfrak{E}_0}$ is such that $\Theta_0(0)$ is unitary. The maximum principle implies that Θ_0 is constant, and \mathfrak{E}_0 reduces each $\Theta(z)$ to Θ_0 . A second application of part (1) of Proposition 5.1, as well as the already proved part (1) of this proposition, shows that $W_1|_{\mathfrak{N}}$ is unitary for some nonzero reducing subspace \mathfrak{N} of $\mathbb{W}(\Theta)$. \square

If $\mathbb{W} = \mathbb{W}(\Theta)$, one can also calculate the characteristic function of $\check{\mathbb{W}} = (W_1, W_0)$, whose coefficients are

$$(I - W_1 W_1^*) W_1^{*k} W_0 | \text{ran}(I - W_1 W_1^*), \quad k \geq 0.$$

Thus this function is given by

$$(I - W_1 W_1^*) (I - z W_1^*)^{-1} W_0 | \text{ran}(I - W_1 W_1^*), \quad z \in \mathbb{D}.$$

In these formulas we use the abbreviation ‘ran’ for the range of an operator.

6. THE STRUCTURE OF BI-SHIFTS

Consider a bi-isometry $\mathbb{W} = (W_0, W_1)$. As seen earlier, the operators W_0 and W_1 do not need to be cnu, even if \mathbb{W} is $\{0\}$ -pure and $\{1\}$ -pure. In this section we study bi-isometries for which both W_0 and W_1 are cnu, and such bi-isometries will be called *bi-shifts*. Clearly bi-shifts are both $\{0\}$ -pure and $\{1\}$ -pure. Note that the bi-shifts described in [11] are, in our terminology, doubly commuting bi-shifts; see Proposition 6.4 below.

Proposition 6.1. *Assume that the bi-isometry \mathbb{W} is both $\{0\}$ -pure and $\{1\}$ -pure. The following conditions are equivalent.*

- (1) \mathbb{W} is a bi-shift.
- (2) $W_0^{*n} \rightarrow 0$ and $W_1^{*n} \rightarrow 0$ as $n \rightarrow \infty$ in the strong operator topology.
- (3) The characteristic function $\Theta_{\mathbb{W}}$ is inner (that is, $\Theta_{\mathbb{W}}(\zeta) \in \mathcal{L}(\mathfrak{E})$ is an isometry for almost every $\zeta \in \mathbb{T}$) and it enjoys the following property:
 - (*) There exists no inner function $\Omega : \mathbb{D} \rightarrow \mathcal{L}(\mathfrak{F}, \mathfrak{E})$ such that $\mathfrak{F} \neq \{0\}$ and

$$\Theta_{\mathbb{W}}(z)\Omega(z) = \Omega(z)U, \quad z \in \mathbb{D},$$

with a unitary operator $U \in \mathcal{L}(\mathfrak{F})$.

Proof. The proposition is almost immediate, but we provide the brief argument below in order to illustrate the use of the results in the preceding section. The equivalence between (1) and (2) follows from the fact that an isometry is cnu if and only if it is a unilateral shift. Assume next that (2) holds so that, in particular, W_0 has no unitary part. With the notation of the preceding sections, $V_0 = W_0$, and $A = W_1$, so that \mathbb{W} serves as its own characteristic pair. Passing to the functional model, we identify W_0 with the unilateral shift $S_{\mathfrak{E}}$, in which case $W_1 = T_{\Theta}$ for some operator valued function Θ . The function Θ must then be inner because T_{Θ} is an isometry. Assume now that a function Ω exists with the properties in (*). Then it follows that $T_{\Theta} | \Omega H^2(\mathfrak{F})$ is a unitary operator, unitarily equivalent to $T_U \in \mathcal{L}(H^2(\mathfrak{F}))$. This contradicts the assumption that (2) holds, and we conclude that (3) is true. Finally, assume that (3) holds, but (2) does not. Since $S_{\mathfrak{E}}$ is completely nonunitary, the operator T_{Θ} must have a unitary part. The nonzero space

$$\mathfrak{M} = \bigcap_{n=0}^{\infty} W_1^n \mathfrak{F} = \bigcap_{n=0}^{\infty} T_{\Theta}^n H^2(\mathfrak{E})$$

on which this unitary part acts is obviously invariant for $S_{\mathfrak{E}}$, and the Beurling-Lax-Halmos theorem implies that $\mathfrak{M} = \Omega H^2(\mathfrak{F})$ for some inner function $\Omega : \mathbb{D} \rightarrow \mathcal{L}(\mathfrak{F}, \mathfrak{E})$ with $\mathfrak{F} \neq \{0\}$. The operator $T_{\Omega}^{-1} T_{\Theta} T_{\Omega}$ is then a unitary operator in the commutant of $S_{\mathfrak{F}}$, and such operators are of the form T_U for some unitary operator $U \in \mathcal{L}(\mathfrak{F})$. We conclude that $T_{\Theta} T_{\Omega} = T_{\Omega} T_U$, contrary to (3). \square

Remark 6.2. The example $\Theta(z) \equiv I_{\mathfrak{E}}$, $z \in \mathbb{D}$, shows that condition (*) is needed in the third statement of the preceding proposition.

The preceding result shows that the construction of bi-shifts requires the construction of appropriate inner functions $\Theta : \mathbb{D} \rightarrow \mathcal{L}(\mathfrak{E})$. We start with some simple examples. Fix a nonzero Hilbert space \mathfrak{E} and an inner function $\vartheta \in H^\infty$. We can then form the bi-isometry $\mathbb{W}(\vartheta \otimes I_{\mathfrak{E}}) = (S_{\mathfrak{E}}, T_{\vartheta \otimes I_{\mathfrak{E}}})$. This is easily seen to be a bi-shift provided that ϑ is not constant.

Proposition 6.3. *Given two nonconstant inner functions $\vartheta_1, \vartheta_2 \in H^\infty$, the bi-shifts $\mathbb{W}(\vartheta_1 \otimes I_{\mathfrak{E}})$ and $\mathbb{W}(\vartheta_2 \otimes I_{\mathfrak{E}})$ are quasi-similar if and only if $\vartheta_1 = \vartheta_2$.*

Proof. Let $X \in \mathcal{I}(\mathbb{W}(\vartheta_1 \otimes I_{\mathfrak{E}}), \mathbb{W}(\vartheta_2 \otimes I_{\mathfrak{E}}))$ be a quasi-affinity. We have $X \in (S_{\mathfrak{E}})'$, and therefore we have $X = T_{\Xi}$ for some outer function $\Xi : \mathbb{D} \rightarrow \mathcal{L}(\mathfrak{E})$. The relation $XT_{\vartheta_1 \otimes I_{\mathfrak{E}}} = T_{\vartheta_2 \otimes I_{\mathfrak{E}}}X$ implies then

$$(\vartheta_1(z) - \vartheta_2(z))\Xi(z) = \Xi(z)(\vartheta_1(z) \otimes I_{\mathfrak{E}}) - (\vartheta_2(z) \otimes I_{\mathfrak{E}})\Xi(z), \quad z \in \mathbb{D}.$$

The operator $\Xi(z)$ has dense range for every $z \in \mathbb{D}$, and we conclude that $\vartheta_1 = \vartheta_2$. The converse is immediate. \square

As pointed out earlier, the bi-isometries $\mathbb{W}(\Theta_1)$ and $\mathbb{W}(\Theta_2)$, $\Theta_1, \Theta_2 : \mathbb{D} \rightarrow \mathfrak{E}$, are unitarily equivalent if and only if the functions Θ_1 and Θ_2 are unitarily equivalent, that is, $U\Theta_1(z) = \Theta_2(z)U$ for a unitary operator U independent of $z \in \mathbb{D}$. Similarity of the two bi-isometries requires the existence of an invertible outer function $\Psi : \mathbb{D} \rightarrow \mathcal{L}(\mathfrak{E})$ such that $\Psi(z)\Theta_1(z) = \Theta_2(z)\Psi(z)$ for all $z \in \mathbb{D}$.

Another important family of bi-shifts is defined on the Hardy space $H^2(\mathbb{D}^2) \otimes \mathfrak{F}$ by the formula $\mathbb{W}_{\mathfrak{F}} = (W_0, W_1)$, where

$$(W_j f)(z_0, z_1) = z_j f(z_0, z_1), \quad f \in H^2(\mathbb{D}^2) \otimes \mathfrak{F}, (z_0, z_1) \in \mathbb{D}^2.$$

This class of bi-isometries has a simple characterization. Parts of the following proposition are known. We include a brief argument for the reader's convenience.

Proposition 6.4. *Assume that the bi-isometry \mathbb{W} is both $\{0\}$ -pure and $\{1\}$ -pure. The following conditions are equivalent.*

- (1) \mathbb{W} is unitarily equivalent to $\mathbb{W}_{\mathfrak{F}}$ for some Hilbert space \mathfrak{F} .
- (2) \mathbb{W} is doubly commuting, that is, $W_0 W_1^* = W_1^* W_0$.
- (3) The characteristic function $\Theta_{\mathbb{W}}$ is a constant isometry.
- (4) The pivotal operator of (W_1, W_0) is an isometry.
- (5) The pivotal operator of \mathbb{W} is an isometry.

Proof. It is immediate that (1) implies (2). For the remainder of the argument we identify \mathbb{W} with $\mathbb{W}(\Theta)$, where $\Theta : \mathbb{D} \rightarrow \mathcal{L}(\mathfrak{E})$ is a contractive analytic function. Thus \mathbb{W} acts on the space \mathfrak{K} described in Proposition 5.3. Assume now that (2) holds. In this case the kernel of W_0^* must be a reducing subspace for W_1 . This kernel consists of functions in \mathfrak{K} of the form $e \oplus 0 \oplus 0 \oplus \dots$, with $e \in \mathfrak{E}$ a constant. Since

$$W_1(e \oplus 0 \oplus 0 \oplus \dots) = \Theta e \oplus \Delta e \oplus 0 \oplus \dots,$$

we deduce immediately that Θ is constant and $\Delta = 0$, so that (3) is true. Assume now that (3) holds, so that Θ is a constant isometry. It follows that $\Theta(0)$ is in particular an isometry. Condition (4) follows because $\Theta(0)$ is the pivotal operator of the pair (W_1, W_0) . Assume that (4) holds, so that $\Theta(0)$ is an isometry. Then it follows from the maximum principle that $\Theta(z) = \Theta(0)$ for all z . In particular, the

function Θ is inner, and hence $W_0 = S_{\mathfrak{E}}$ and $W_1 = T_{\Theta}$. Note that any orthogonal decomposition $\Theta(0) = \Theta_1 \oplus \Theta_2$ yields a decomposition $T_{\Theta} = T_{\Theta_1} \oplus T_{\Theta_2}$. If Θ_1 is unitary, the operator T_{Θ_1} is unitary as well, and therefore Θ_1 must act on the space $\{0\}$ because \mathbb{W} was assumed to be $\{1\}$ -pure. We deduce that $\Theta(0)$ is cnu, and thus it is unitarily equivalent to $S_{\mathfrak{F}}$ for some Hilbert space \mathfrak{F} , and in this case $\mathbb{W}(\Theta)$ is unitarily equivalent to $\mathbb{W}_{\mathfrak{F}}$.

So far we have proved that conditions (1–4) are equivalent. The equivalence of (5) with these conditions follows from the symmetry of (2). \square

The example of the constant function $\Theta(z) \equiv I$, $z \in \mathbb{D}$, shows why the assumption that \mathbb{W} is both $\{0\}$ -pure and $\{1\}$ -pure is needed in the preceding proposition.

If two isometries are quasi-similar and one of them is a shift, then the other one is a shift as well. It follows that a bi-isometry quasi-similar to a bi-shift must also be a bi-shift. We conclude this section with some simple properties of those bi-shifts which are similar to $\mathbb{W}_{\mathfrak{F}}$ for some \mathfrak{F} .

Proposition 6.5. *Let $\mathbb{W} = \mathbb{W}(\Theta)$ be a bi-shift, where $\Theta : \mathbb{D} \rightarrow \mathcal{L}(\mathfrak{E})$ is an inner analytic function. Assume further that \mathbb{W} is similar to $\mathbb{W}_{\mathfrak{F}}$ for some Hilbert space \mathfrak{F} . Then the following assertions are true.*

- (1) *The pivotal operator is similar to a unilateral shift.*
- (2) *There exists a bounded analytic function $\Omega : \mathbb{D} \rightarrow \mathcal{L}(\mathfrak{E})$ such that*

$$\Omega(z)\Theta(z) = I, \quad z \in \mathbb{D}.$$

- (3) *The operator $\Theta(z)$ is similar to a unilateral shift for every $z \in \mathbb{D}$.*

Proof. We argue first that two similar bi-isometries have similar pivotal operators. Indeed, assume that $X \in \mathcal{I}(\mathbb{W}^{(1)}, \mathbb{W}^{(2)})$ is an invertible operator. We have then $X \ker W_0^{(1)*} = \ker W_0^{(2)*}$, and this implies that $X|_{\ker W_0^{(1)*}}$ is an invertible operator intertwining the two pivotal operators. Now, the pivotal operator of $\mathbb{W}_{\mathfrak{F}}$ is a shift, and the preceding observation implies (1). By symmetry, we also deduce that $\Theta(0)$ is similar to a shift, and then (2) follows from the main result of [24]. To verify (3), we observe that the bi-shift $\mathbb{W}_{\mathfrak{F}}$ is unitarily equivalent to $\mathbb{W}(\Theta_1)$, where $\Theta_1(z) \equiv S$ for $z \in \mathbb{D}$, with $S \in \mathcal{L}(\mathfrak{E})$ a unilateral shift. Let $X \in \mathcal{I}(\mathbb{W}(\Theta), \mathbb{W}(\Theta_1))$ be an invertible operator. We have $X \in (W_0)'$, and therefore the operator X is of the form $X = T_{\Xi}$ for some bounded analytic function $\Xi \in \mathcal{L}(\mathfrak{E})$. The fact that X is invertible implies that $X(z)$ is invertible for every $z \in \mathbb{D}$, and the relation $XT_{\Theta} = T_{\Theta_1}X$ shows that $\Theta(z)$ is similar to $S = \Theta_1(z)$. The proposition is proved. \square

A different approach to the similarity between a contraction and an isometry is described in [14]. This approach may also be useful in the study of similarities between bi-shifts.

Conditions (1) and (2) in the above proposition are not sufficient to imply the similarity of $\mathbb{W}(\Theta)$ to a bi-shift of the form $\mathbb{W}_{\mathfrak{F}}$, as shown by the following example.

Example 6.6. Define $\Theta(z) \in \mathcal{L}(\ell^2)$ using the infinite matrix

$$\Theta(z) = \begin{bmatrix} \frac{3}{5}\varphi(z) & 0 & 0 & 0 & \cdots \\ \frac{4}{5}z & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \cdots \end{bmatrix}, \quad z \in \mathbb{D},$$

where $\varphi \in H^\infty$ is an inner function such that $\varphi(0) \neq 0$. The operator $\Theta(0)$ has the eigenvalue $3\varphi(0)/5$ and therefore it is not similar to a shift. However Θ satisfies condition (2) in the preceding proposition. One left inverse is given by

$$\Omega(z) = \begin{bmatrix} \frac{5}{3\varphi(0)} & \eta(z) & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ 0 & 0 & 0 & 0 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}, \quad z \in \mathbb{D},$$

with

$$\eta(z) = \frac{4}{5z} \left[1 - \frac{\varphi(z)}{\varphi(0)} \right], \quad z \in \mathbb{D}.$$

The reader will verify without difficulty that $\mathbb{W}(\Theta)$ is indeed a bi-shift.

7. THE UNITARY INVARIANTS OF A FUNCTIONAL MODEL

Bi-isometries $\mathbb{W} = (W_0, W_1)$ with the property that the product W_0W_1 is a shift were classified, up to unitary equivalence, in [4]; see also [2]. The parameters in that classification are pairs (U, P) , where U is a unitary operator on a Hilbert space \mathfrak{D} , and P is an orthogonal projection on \mathfrak{D} . In this section we consider the characteristic functions of such bi-isometries. The bi-isometry $\mathbb{W} = (W_0, W_1)$ associated to the pair (U, P) acts on $H^2(\mathfrak{D})$ and is defined by

$$(W_0f)(z) = U(zP + P^\perp)f(z), \quad (W_1f)(z) = (P + zP^\perp)U^*f(z), \quad f \in H^2(\mathfrak{D}), z \in \mathbb{D}.$$

The space \mathfrak{D} is identified with the space $\ker(W_0W_1)^*$ of constant functions in $H^2(\mathfrak{D})$, while the range of P^\perp is identified with $\ker W_1^*$. For a constant function $f_0 \in \mathfrak{D}$ we have

$$(7.1) \quad Uf_0 = W_0f_0, \quad f_0 \in P^\perp\mathfrak{D},$$

while for $f_0 \in P\mathfrak{D}$ we have

$$W_0f_0 = zUf_0 = W_0W_1Uf_0.$$

Therefore the vector $f_0 = W_1Uf_0$ is in the range of W_1 , and we find that

$$(7.2) \quad Uf_0 = W_1^*f_0, \quad f_0 \in P\mathfrak{D}.$$

From this we easily conclude that

$$\ker W_0^* = UP\mathfrak{D} = W_1^*P\mathfrak{D}.$$

By reversing the order of these observations we easily deduce the following result.

Proposition 7.1. *Let $\mathbb{W} = (W_0, W_1)$ be a bi-isometry on \mathfrak{H} . Define spaces*

$$\mathfrak{D} = \ker(W_0W_1)^*, \quad \mathfrak{E} = \ker W_0^*, \quad \mathfrak{F} = \ker W_1^*.$$

- (1) *We have $\mathfrak{D} = \mathfrak{E} \oplus W_0\mathfrak{F} = W_1\mathfrak{E} \oplus \mathfrak{F}$.*
- (2) *The operator $U : \mathfrak{D} \rightarrow \mathfrak{D}$ defined by*

$$U(W_1e + f) = e + W_0f \quad e \in \mathfrak{E}, f \in \mathfrak{F},$$

is unitary.

- (3) *The bi-isometry associated with the pair $(U, P_{W_1\mathfrak{E}})$ on \mathfrak{D} is unitarily equivalent to the cnu part of \mathbb{W} .*

For further calculation, it is convenient to replace the space \mathfrak{D} by the external direct sum $\mathfrak{E} \oplus \mathfrak{F}$ via the identification $\Phi : e \oplus f \mapsto W_1 e + f$. With this identification we obviously have

$$\Phi^* P \Phi = \begin{bmatrix} I_{\mathfrak{E}} & 0 \\ 0 & 0 \end{bmatrix}.$$

Corollary 7.2. *With the notation of Proposition 7.1, we have*

$$\Phi^* U \Phi = \begin{bmatrix} W_1^* |_{\mathfrak{E}} & W_1^* W_0 |_{\mathfrak{F}} \\ (I - W_1 W_1^*) |_{\mathfrak{E}} & (I - W_1 W_1^*) W_0 |_{\mathfrak{F}} \end{bmatrix}.$$

Proof. For a vector $e \in \mathfrak{E}$ we have

$$U \Phi(e \oplus 0) = U W_1 e = e = W_1 W_1^* e + (I - W_1 W_1^*) e,$$

and this is precisely the decomposition of this vector as an element of the space $W_1 \mathfrak{E} \oplus \mathfrak{F}$. Therefore

$$\Phi^* U \Phi(e \oplus 0) = W_1^* e \oplus (I - W_1 W_1^*) e.$$

To verify the identity involving the second column, we use a similar calculation:

$$U \Phi(0 \oplus f) = U f = W_0 f = W_1 W_1^* W_0 f + (I - W_1 W_1^*) W_0 f, \quad f \in \mathfrak{F}.$$

In these calculations we made use of (7.1) and (7.2). \square

Let us consider now a contractive analytic function $\Theta : \mathbb{D} \rightarrow \mathcal{L}(\mathfrak{E})$ and the functional model $\mathbb{W}(\Theta) = (W_0, W_1)$. In order to identify the space \mathfrak{F} , it will be useful to recall a few facts from the theory of functional models of contraction operators. Let us introduce the auxiliary space

$$\mathfrak{K} = H^2(\mathfrak{E}) \oplus (L_{\Delta} L^2(\mathfrak{E}))^{-},$$

which can be viewed as a subspace of $\mathfrak{H} = H^2(\mathfrak{E}) \oplus H^2((L_{\Delta} L^2(\mathfrak{E}))^{-})$. Obviously, the space \mathfrak{K} is reducing for W_0 . The space

$$\mathfrak{G} = \{\Theta u \oplus \Delta u : u \in H^2(\mathfrak{E})\}$$

is invariant for W_0 , and therefore

$$\mathfrak{H}(\Theta) = \mathfrak{K} \ominus \mathfrak{G}$$

is invariant for W_0^* . The compression of W_0 to this space is denoted $S(\Theta)$, and it is called the functional model associated with Θ . It is known that $S(\Theta)$ is a completely nonunitary contraction, and the characteristic function of $S(\Theta)$ coincides (in the sense defined in [25]) with the purely contractive part of the function Θ .

A vector $u \oplus v \in \mathfrak{K}$ belongs to $\mathfrak{H}(\Theta)$ if and only if the measurable function $\Theta^* u + \Delta v$ is orthogonal to $H^2(\mathfrak{E})$. In other words, we have a Fourier expansion

$$\Theta^* u + \Delta v = \sum_{n=-1}^{\infty} \zeta^n e_n,$$

with $e_n \in \mathfrak{E}$. We will use the notation $(\Theta^* u + \Delta v)_{-1}$ for e_{-1} .

Lemma 7.3. *Viewed as a subspace of \mathfrak{H} , we have $\mathfrak{H}(\Theta) = \mathfrak{F}$. Moreover, $S(\Theta)$ is precisely the pivotal operator associated with the bi-isometry $\mathbb{W}(\Theta)$:*

$$S(\Theta)^* = W_0^* |_{\mathfrak{F}}.$$

Proof. In order to identify \mathfrak{F} , we consider its orthogonal complement which is easily calculated as

$$\mathfrak{F}^\perp = W_1 \mathfrak{H} = \mathfrak{G} \oplus W_1 H^2((L_\Delta L^2(\mathfrak{E}))^-).$$

The conclusion $\mathfrak{H}(\Theta) = \mathfrak{F}$ then follows because

$$\mathfrak{H} = \mathfrak{K} \oplus W_1 H^2((L_\Delta L^2(\mathfrak{E}))^-).$$

The identification of the pivotal operator follows now from the fact that $\mathfrak{H}(\Theta) = \mathfrak{F}$ is invariant for W_0^* . \square

Proposition 7.4. *Let $\Theta : \mathbb{D} \rightarrow \mathcal{L}(\mathfrak{E})$ be a contractive analytic function, and $\mathbb{W}(\Theta) = (W_0, W_1)$ the corresponding model bi-isometry. Then $\mathbb{W}(\Theta)$ is unitarily equivalent to the bi-isometry associated with the pair (U, P) of operators on $\mathfrak{E} \oplus \mathfrak{H}(\Theta)$ defined as follows:*

$$(7.3) \quad \begin{aligned} U(e \oplus 0) &= \Theta(0)^* e \oplus [(e - \Theta(0)^* e) \oplus (-\Delta \Theta(0)^* e)], \quad e \in \mathfrak{E}, \\ U(0 \oplus (u \oplus v)) &= (\Theta^* u + \Delta v)_{-1} \oplus S(\Theta)(u \oplus v), \quad u \oplus v \in \mathfrak{H}(\Theta), \end{aligned}$$

and

$$P = \begin{bmatrix} I_{\mathfrak{E}} & 0 \\ 0 & 0 \end{bmatrix}.$$

Proof. This proof amounts to an identification of the matrix entries in Corollary 7.2. It is convenient to regard \mathfrak{H} as an infinite orthogonal sum

$$\mathfrak{H} = H^2(\mathfrak{E}) \oplus (L_\Delta L^2(\mathfrak{E}))^- \oplus (L_\Delta L^2(\mathfrak{E}))^- \oplus \cdots,$$

relative to which the operator W_1 has the matrix

$$W_1 = \begin{bmatrix} T_\Theta & 0 & 0 & \cdots \\ L_\Delta |_{H^2(\mathfrak{E})} & 0 & 0 & \cdots \\ 0 & I_{(L_\Delta L^2(\mathfrak{E}))^-} & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

We now apply the formulas in Corollary 7.2 to calculate the entries of the matrix U explicitly. Thus, for $e \in \mathfrak{E}$, which is viewed now as a subspace of \mathfrak{H} , we obtain by applying the matrix above

$$W_1^* e = T_\Theta^* e = P_{H^2(\mathfrak{E})} \Theta^* e = \Theta(0)^* e,$$

and

$$(I - W_1 W_1^*) e = e - W_1 \Theta(0)^* e.$$

If $u \oplus v \in \mathfrak{H}(\Theta)$ then clearly

$$(I - W_1 W_1^*) W_0(u \oplus v) = P_{\mathfrak{H}(\Theta)} W_0(u \oplus v) = S(\Theta)(u \oplus v).$$

For the first direct summand in the right-hand side of (7.3), let us write $W_0(u \oplus v) = u' \oplus v'$ and note that

$$W_1^* W_0(u \oplus v) = W_1^*(u' \oplus v') = P_{H^2(\mathfrak{E})}(\Theta^* u' + \Delta v').$$

If we write the Fourier expansion

$$\Theta^* u + \Delta v = \sum_{n=-1}^{\infty} \zeta^n e_n,$$

then

$$\Theta^*u' + \Delta v' = \sum_{n=-1}^{\infty} \zeta^{n+1} e_n,$$

and the projection of this function onto $H^2(\mathfrak{E})$ is precisely $e_{-1} = (\Theta^*u + \Delta v)_{-1}$, as stated. \square

8. EXAMPLES OF IRREDUCIBLE TWO-ISOMETRIES AND DIRECT INTEGRAL DECOMPOSITIONS

For a single isometry, that is, when I has only one element, it follows from the von Neumann–Wold theorem that there is, up to unitary equivalence, only one nonunitary irreducible isometry. However, when I has two or more elements there are many irreducible families of commuting isometries which do not consist of unitary operators. We will illustrate this in the case of bi-isometries $\mathbb{V} = (V_0, V_1)$. We recall that a complete unitary invariant of a completely nonunitary bi-isometry $\mathbb{V} = (V_0, V_1)$ is given by a pair (U, P) , where U is a unitary operator on some Hilbert space \mathfrak{D} , and P is an orthogonal projection on \mathfrak{D} . The bi-isometry determined by (U, P) acts on $H^2(\mathfrak{D})$ as follows:

$$(V_0 f)(z) = U(zP + P^\perp)f(z), \quad (V_1 f)(z) = (P + zP^\perp)U^*f(z), \quad f \in H^2(\mathfrak{D}), z \in \mathbb{D},$$

where $P^\perp = I_{\mathfrak{E}} - P$. The bi-isometry \mathbb{V} is irreducible if and only if the pair (U, P) is irreducible. Note for further use that the product $V_0 V_1$ is precisely multiplication by the variable z . (These unitary invariants classify more general bi-isometries than the completely nonunitary ones; see [4, 2].)

For our illustration we will let U be the bilateral shift on the space L^2 of all square integrable functions on the unit circle \mathbb{T} ; thus

$$(Uf)(\zeta) = \zeta f(\zeta), \quad f \in L^2, \zeta \in \mathbb{T}.$$

We will denote by $e_j(\zeta) = \zeta^j$ the standard orthonormal basis in L^2 , and for every set $A \subset \mathbb{Z}$ of integers we denote by Q_A the orthogonal projection onto the space generated by $\{e_j : j \in A\}$. In this case V_0 and V_1 are uniquely determined by the relations $V_1 e_{n+1} = e_n$ if $n \in A$ and $V_0 e_n = e_{n+1}$ if $n \notin A$.

Proposition 8.1. *Two pairs (U, Q_A) , (U, Q_B) are unitarily equivalent if and only if there exists $n \in \mathbb{Z}$ such that*

$$B = \{i + n : i \in A\}.$$

Proof. Sufficiency is obvious: if $B = A + n$ then the operator U^n implements the unitary equivalence of the two pairs. Conversely, assume that there is a unitary operator Φ on L^2 such that $\Phi U = U \Phi$ and $U Q_A = Q_B U$. There exists then a function $\varphi \in L^\infty$ such that $|\varphi| = 1$ almost everywhere and $\Phi f = \varphi f$ for every $f \in L^2$. The fact that φe_i is in the range of Q_B for $i \in A$ means that

$$(\varphi, e_{j-i}) = (\varphi e_i, e_j) = 0, \quad i \in A, j \notin B.$$

Similarly, φe_i is in the range of Q_B^\perp if $i \notin A$, so that

$$(\varphi, e_{j-i}) = 0, \quad i \notin A, j \in B.$$

We deduce that there exists at least one integer n not in the set $\{j - i : (i, j) \in (A \times (\mathbb{Z} \setminus B)) \cup ((\mathbb{Z} \setminus A) \times B)\}$. The function e_n will then have the property that $e_{n+i} = e_n e_i$ is in the range of Q_B if $i \in A$, and it is in the range of Q_B^\perp if $i \notin A$. Therefore $B = A + n$. \square

Corollary 8.2. *The pair (U, Q_A) is reducible if and only if A is a periodic set, that is, $A = A + n$ for some nonzero integer n .*

Proof. The pair (U, Q_A) is reducible if and only if it commutes with a unitary which is not a scalar multiple of the identity. The argument in the proof of the preceding proposition shows that such a unitary can be chosen to be multiplication by e_n for some $n \in \mathbb{Z} \setminus \{0\}$. \square

We see therefore that there is a continuum of mutually inequivalent irreducible bi-isometries. Indeed, there is a continuum of subsets of \mathbb{Z} , and only countably many of them are periodic.

Quite interestingly, the bi-isometry associated with (U, Q_A) can be described very explicitly. Consider the space $L^2(\mathbb{T}^2) = L^2 \otimes L^2$ and its standard orthonormal basis

$$e_{ij}(\zeta_0, \zeta_1) = \zeta_0^i \zeta_1^j, \quad i, j \in \mathbb{Z}, \zeta_0, \zeta_1 \in \mathbb{T}.$$

Multiplication by the two variables defines a bi-isometry $\mathbb{V} = (V_0, V_1)$ on $L^2(\mathbb{T}^2)$; actually V_0 and V_1 are unitary. We will look at proper nonempty subsets $\Gamma \subset \mathbb{Z}^2$ with the property that the space \mathfrak{H}_Γ generated by $\{e_{ij} : (i, j) \in \Gamma\}$ is invariant for \mathbb{V} . In other words, $(i + n, j + m) \in \Gamma$ if $(i, j) \in \Gamma$ and $n, m \geq 0$ or, equivalently, $\Gamma + \mathbb{N}^2 \subset \Gamma$. We define the boundary $\partial\Gamma$ of Γ to consist of those pairs $(i, j) \in \Gamma$ such that $(i - 1, j - 1)$ does not belong to Γ . For each integer n , there exists a unique point $\gamma_n = (i_n, j_n) \in \partial\Gamma$ such that $i_n - j_n = n$. Uniqueness is obvious by the definition of $\partial\Gamma$; existence follows from the fact that $\emptyset \neq \Gamma \neq \mathbb{Z}^2$. The difference $\gamma_{n+1} - \gamma_n = (i_{n+1} - i_n, j_{n+1} - j_n)$ is either $(1, 0)$ or $(0, -1)$. We can then define the set $A_\Gamma \subset \mathbb{Z}$ by

$$A_\Gamma = \{n \in \mathbb{Z} : \gamma_{n+1} - \gamma_n = (0, -1)\}.$$

Geometrically, A_Γ is the union of the vertical segments in $\partial\Gamma$, omitting the lower endpoint of each one. The following result is an easy exercise.

Proposition 8.3. *For every subset $A \subset \mathbb{Z}$ there exists a nonempty subset $\Gamma \subset \mathbb{Z}^2$ such that $\Gamma + \mathbb{N}^2 \subset \Gamma$ and $A_\Gamma = A$. We have $A_{\Gamma+(p,q)} = A_\Gamma + p - q$ for all $(p, q) \in \mathbb{Z}^2$.*

Proposition 8.4. *Let Γ be a nonempty proper subset of \mathbb{Z}^2 such that \mathfrak{H}_Γ is invariant for \mathbb{V} . The bi-isometry associated with the invariants (U, Q_{A_Γ}) is unitarily equivalent to $\mathbb{V}|_{\mathfrak{H}_\Gamma}$.*

Proof. The space $\mathfrak{H}_{\partial\Gamma} = \mathfrak{H}_\Gamma \ominus VW\mathfrak{H}_\Gamma$ can be identified with L^2 by mapping e_{γ_n} to e_n . Denote by U_0 the unitary operator on $\mathfrak{H}_{\partial\Gamma}$ which corresponds to the shift on L^2 ; in other words, $U_0 e_{\gamma_n} = e_{\gamma_{n+1}}$. Since $V_0 V_1$ corresponds with multiplication by z , it is clear that $\mathfrak{H}_{\partial\Gamma}$ can be identified with $H^2(\mathfrak{H}_{\partial\Gamma})$. Therefore, we only need to show that $V_0 e_{\gamma_n} = V_0 V_1 e_{\gamma_{n+1}}$ if $n \in A_\Gamma$ and $V_0 e_{\gamma_n} = e_{\gamma_{n+1}}$ if $n \notin A_\Gamma$. This however is immediate from the definition of A_Γ and the remark preceding Proposition 8.1. \square

A direct consequence of this proposition is the following:

Corollary 8.5. *Let Γ and Γ' be two nonempty proper subsets of \mathbb{Z}^2 such that \mathfrak{H}_Γ and $\mathfrak{H}_{\Gamma'}$ are invariant for \mathbb{V} .*

- (1) *The bi-isometries $\mathbb{V}|_{\mathfrak{H}_\Gamma}$ and $\mathbb{V}|_{\mathfrak{H}_{\Gamma'}}$ are unitarily equivalent if and only if $\Gamma' = \Gamma + \gamma$ for some $\gamma \in \mathbb{Z}^2$.*
- (2) *The bi-isometry $\mathbb{V}|_{\mathfrak{H}_\Gamma}$ is reducible if and only if $\Gamma = \Gamma + \gamma$ for some $\gamma \in \mathbb{Z}^2 \setminus \{(0, 0)\}$.*

Two particular sets Γ yielding irreducible bi-isometries were considered in [11, 17]. The first is $\Gamma = \mathbb{N}^2$, for which $A_\Gamma = \{n : n < 0\}$. The restriction $\mathbb{V}|_{\mathfrak{H}_\Gamma}$ is a doubly commuting bi-shift. The second is $\Gamma = (\mathbb{Z} \times \mathbb{N}) \cup (\mathbb{N} \times \mathbb{Z})$, for which $A_\Gamma = \mathbb{N}$. The corresponding restriction of \mathbb{V} was called a modified bi-shift in these works. The modified bi-shift can be seen to be the dual of the doubly commuting bi-shift in the sense of [6].

The bi-isometries of the form $\mathbb{V}|_{\mathfrak{H}_\Gamma}$ were considered earlier in [18]. They have the special property that the range projections of the isometries in the multiplicative semigroup they generate commute with each other. The case $\Gamma \subset \mathbb{N}^2$ was also considered in [8] from the point of view of Hilbert modules over the bidisk algebra.

We now illustrate the decomposition of a bi-isometry into a direct integral of irreducibles with the particular case provided by the set $A = 2\mathbb{Z}$. In this case, the commutant of the pair (U, Q_A) is the algebra generated by U^2 , and this operator is a unitary operator with uniform multiplicity 2 relative to the usual arclength measure on \mathbb{T} . This is realized upon using the identification

$$\Phi : L^2 \oplus L^2 \rightarrow L^2$$

defined by

$$(\Phi(f \oplus g))(\zeta) = f(\zeta^2) + \zeta g(\zeta^2), \quad \zeta \in \mathbb{T}.$$

The operator $\Phi^*U\Phi$ is simply multiplication by the matrix-valued function

$$U_0(\zeta) = \begin{bmatrix} 0 & \zeta \\ 1 & 0 \end{bmatrix}, \quad \zeta \in \mathbb{T},$$

while $\Phi^*Q_A\Phi$ is multiplication by the constant matrix

$$P_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

In other words, we have the decomposition

$$(U, P) = \int_{\mathbb{T}}^{\oplus} (U_0(\zeta), P_0) |d\zeta|,$$

and it is clear that the pairs $(U_0(\zeta), P_0)$ are irreducible and mutually inequivalent. This corresponds with a direct integral decomposition of the corresponding bi-isometry. The reader will have no difficulty verifying that the bi-isometry associated with $(U_0(\zeta), P_0)$ is of the form $(\zeta S, S)$, where S is a unilateral shift of multiplicity one.

The general case of a set A such that $A = A + n$, with $n > 2$, lends itself to a similar analysis, with

$$U_0(\zeta) = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & \zeta^{n-1} \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \zeta & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & \zeta^{n-2} & 0 \end{bmatrix}, \quad \zeta \in \mathbb{T},$$

and P_0 a diagonal projection. The diagonal elements $(\alpha_1, \alpha_2, \dots, \alpha_n)$ of this projection are defined by setting $\alpha_i = 1$ if $i \in A$ and $\alpha_i = 0$ otherwise. The pair $(U_0(\zeta), P_0)$ is irreducible provided that n is the smallest positive period of A .

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RGD AND CF: DEPARTMENT OF MATHEMATICS, TEXAS A&M UNIVERSITY, COLLEGE STATION,
TX 77843

E-mail address: `rdouglas@math.tamu.edu`, `foias@math.tamu.edu`