SUBORDINATION BY ORTHOGONAL MARTINGALES IN L^p AND ZEROS OF LAGUERRE POLYNOMIALS

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1. Introduction

In this paper we address the question of finding the best L^p-norm constant for martingale transforms with one-sided orthogonality. Let $\mathcal{O} = (\Omega, \mathcal{B}, P)$ be a probability space with filtration $\mathcal{B}$ generated by a two-dimensional Brownian motion B_t. Let $X_t = \int_0^t H_s \cdot dB_s$ and $Y_t = \int_0^t K_s \cdot dB_s$ be two complex-valued martingales on this probability space, such that the quadratic variation of Y runs slower than the quadratic variation of X, i.e. $d\langle Y\rangle_s \le d\langle X\rangle_s$, or equivalently

$$|K_s| = \sqrt{|K_s^1|^2 + |K_s^2|^2} \le \sqrt{|H_s^1|^2 + |H_s^2|^2} = |H_s| \quad \forall \ s.$$

Y is said to be a martingale transform of X that is differentially subordinate to X. If for $1 < p < \infty$, we have $E|X_t|^p < \infty$, then the Burkholder-Davis-Gundy and Doob inequalities (see [RoWi]) imply that $E|Y_t|^p < \infty$ and there exists a universal constant C_p such that $\|Y_t\|_p \le C_p \|X\|_p$. An evident problem then is to find the best constant C_p.

D.L. Burkholder solves this problem entirely in a series of papers in the 1980's, see in particular [Bu1] and [Bu3]. He shows that

$$C_p = p^* - 1, \quad p^* = \max\{p, \frac{p}{p-1}\}. \tag{1.1}$$

His approach (followed in the present paper) is as follows (see [Bu1] section 5 for more general viewpoint). Consider the function $V(x,y) = |y|^p - C_p^p |x|^p$; we wish to find C_p so that for martingales X and Y as above, we always have $EV(X,Y) \le 0$. Now find (if it exists) a majorant-function $U(x,y) \ge V(x,y)$ such that $U(0,0) = 0$ and $U(X,Y)$ is a supermartingale; such a function must exist for the optimal C_p, see Section 2. This then implies

$$EV(X,Y) \le EU(X,Y) \le 0.$$

Burkholder shows that when $C_p = p^* - 1$ such a majorant exists and equals

$$U(x,y) = p\left(1 - \frac{1}{p^*}\right)^{p-1} (|y| - (p^*-1)|x|)(|x| + |y|)^{p-1}, \tag{1.2}$$

and he finds extremals to show that $p^* - 1$ is in fact the best (least) possible constant. To show $U(X,Y)$ is a supermartingale requires that U is a supersolution for a family of PDEs; in this case, it reduces to showing U is a certain biconcave function. Thus Burkholder translates the martingale L^p problem to the calculus-of-variations setting and solves the corresponding obstacle problem. In other work [Bu8], he also shows that this martingale question and its answers are related to the special nature of the range space of the martingales, and obtains specific geometric characterization of all Banach spaces that have finite martingale-transform constant.

2. Burkholder, Bellman and Beurling-Ahlfors

One of the primary applications for Burkholder's theorem has come in Fourier analysis in estimating the L^p norm of the Beurling-Ahlfors transform B, see [BaWa1], [NV1]. This self-adjoint singular-integral operator arises naturally in quasi-conformal mapping theory and PDE, and the knowledge of its L^p-norm, $1 < p < \infty$, would imply important results in these areas. It is a conjecture by Iwaniec [Iw] that the norm constant is $\|B\|_p = p^* - 1$, the same constant as in Burkholder's theorem for martingales. The first major breakthrough in finding the connection between martingale estimates and the Ahlfors–Beurling operator came in [BaWa1] where Bañuelos and Wang show that if a function $f \in L^p(\mathbf{R}^2)$ is extended harmonically as $U_f(x,t)$ to the upper half-space $\mathbf{R}^3_+$, then there exists martingale $X_t = U_f(B_t)$ with martingale transform Y_t satisfying (essentially)

$$X_\tau \approx f(x), \quad E[Y_\tau | B_\tau = x] = Bf(x), \quad \langle Y \rangle \leq 16 \langle X \rangle .$$

Here B_t is 3-dimensional Brownian motion, τ its exit time from $\mathbf{R}^3_+$, and the conditional expectation $E[Y_\tau | B_\tau = x]$ is the average value of Y_τ over paths that exit at x. This then implies (essentially)

$$\|Bf\|_p = \|E[Y_\tau | B_\tau = x]\|_p \leq \|Y_\tau\|_p \leq 4(p^* - 1)\|X_\tau\|_p \leq 4(p^* - 1)\|f\|_p.$$

The first inequality is from Jensen and the second from Burkholder's theorem. Thus we have $\|B\|_p \leq 4(p^* - 1)$.

In a series of papers starting in the late 1990's, Nazarov, Treil, Volberg and others ([NT], [NV1], [NTV8], [DV1], ...) show that the martingale/obstacle problem dealt with by Burkholder fits within a general framework derived from Stochastic control theory, which also works with other questions in harmonic analysis. Here again, a special function $\mathcal{B}$ called the Bellman function is found in relation to the problem, and it usually satisfies certain concavity and boundedness conditions. Burkholder's function is therefore an example of a Bellman function. In fact, the Bellman-function theory establishes that such a function $\mathcal{B}$ necessarily exists for the corresponding optimization problem, and the extremals fully require its concavity and boundedness properties. Using the Bellman-function approach, Nazarov and Volberg [NV1] obtain the better estimate $\|B\|_p \leq 2(p^* - 1)$. We describe how this is done. Given $f \in L^p$ and $g \in L^{p'}$, and denoting their heat-extensions to the upper half-space by f and g again, we can show

$$\left|\int_{\mathbb{C}} Bf\cdot g\right| = \left|2\int_{\mathbb{R}^3_+}(\partial_x+\partial_y)f(\partial_x+\partial_y)g\,dxdydt\right|$$

$$\leq\ 2\int_{\mathbb{R}^3_+}(|\partial_x f||\partial_x g|+|\partial_y f||\partial_y g|+|\partial_x f||\partial_y g|+|\partial_y f||\partial_x g|)dxdydt$$

We wish to bound this integral above by $c(p^*-1)\|f\|_p\|g\|_{p'}$. However we don't understand how to integrate terms like $|\partial_x f||\partial_x g|$, so the idea to find another function above it which can be integrated and whose integral has the required upper bound. Now construct (let $p>2$) the Bellman function $\mathcal{B}$ defined on the domain

$$D_p=\{0<(\xi,\eta,X,Y)\subset\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}\times\mathbb{R}\colon X>\|\xi\|^p, Y>\|\eta\|^q\},$$

that satisfies (essentially)

(1) $0\leq B\leq(p-1)X^{1/p}Y^{1/q}$

(2) $-\left\langle d^2\mathcal{B}\cdot d\xi, d\eta\right\rangle\geq 2|d\xi||d\eta|$

The actual construction (or proof of existence) of $\mathcal{B}$ involves taking supremum of appropriate functions over certain families of martingales, similar to how Burkholder formulates his function in [Bu1]. For more details on Bellman-function construction, refer to [NTV8], [VaVo], [VaVo2].

Define $b:\mathbb{R}^2\times\mathbb{R}_+\to\mathbb{R}_+$ by $b(x,t)=\mathcal{B}(f,g,|f|^p,|g|^p)$ where all input functions are the heat extensions. Let $v=(f,g,|f|^p,|g|^p)$. The boundedness condition on $\mathcal{B}$ implies

$$\begin{aligned}4\pi R^2 b(0,R^2)\quad &\leq\quad (p-1)\left(\int|f|^p e^{\frac{-|x|^2}{4R^2}}\right)^{1/p}\left(\int|g|^p e^{\frac{-|x|^2}{4R^2}}\right)^{1/q}\\ &\to\quad (p-1)\|f\|_p\|g\|_q.\end{aligned}$$

Some clever analysis shows that $4\pi R^2 b(0,R^2)$ is asymptotically (as $R\to\infty$) bounded below by

$$\int\left(\left\langle -d^2\mathcal{B}\partial_x v,\partial_x v\right\rangle+\left\langle -d^2\mathcal{B}\partial_y v,\partial_y v\right\rangle\right).$$

This by the concavity condition on $\mathcal{B}$ is bounded below by

$$\int_{\mathbb{R}^3_+}(|\partial_x f||\partial_x g|+|\partial_y f||\partial_y g|+|\partial_x f||\partial_y g|+|\partial_y f||\partial_x g|)dxdydt.$$

Thus we conclude for $p\geq 2$, $\left|\int Bf\cdot g\right|\leq 2(p-1)\|f\|_p\|g\|_q$. The full result follows for $1<p<2$ by duality.

Following [NV1], Bañuelos and Méndez [BaMH] redo the work done in [BaWa1] but this time with heat extensions and space-time Brownian motion and also obtain $\|B\|_p\leq 2(p^*-1)$.

3. Orthogonal martingales and the Beurling-Ahlfors transform

A complex-valued martingale $Y = Y_1 + iY_2$ is said to be *orthogonal* if the quadratic variations of the coordinate martingales are equal and their mutual covariation is 0:

$$\langle Y_1 \rangle = \langle Y_2 \rangle, \quad \langle Y_1, Y_2 \rangle = 0.$$

In [BaJ1], Bañuelos and Janakiraman make the observation that the martingale associated with the Beurling-Ahlfors transform is in fact an orthogonal martingale. They show that Burkholder's proof in [Bu3] naturally accommodates for this property and leads to an improvement in the estimate of $\|B\|_p$.

Theorem 3.1. *(One-sided orthogonality as allowed in Burkholder's proof)*

(1) *(Left-side orthogonality) Suppose $2 \leq p < \infty$. If Y is an orthogonal martingale and X is any martingale such that $\langle Y \rangle \leq \langle X \rangle$, then*

$$\|Y\|_p \leq \sqrt{\frac{p^2 - p}{2}} \|X\|_p. \tag{3.1}$$

(2) *(Right-side orthogonality) Suppose $1 < p < 2$. If X is an orthogonal martingale and Y is any martingale such that $\langle Y \rangle \leq \langle X \rangle$, then*

$$\|Y\|_p \leq \sqrt{\frac{2}{p^2 - p}} \|X\|_p. \tag{3.2}$$

It is not known whether these estimates are the best possible.

The result for right-side orthogonality is stated in [BJV] and not in [BaJ1]. It follows the same lines of proof as for left-side orthogonality. If X and Y are the martingales associated with f and Bf respectively, then Y is orthogonal, $\langle Y \rangle \leq 4 \langle X \rangle$ and hence by (1), we obtain

$$\|Bf\|_p \leq \sqrt{2(p^2 - p)} \|f\|_p \text{ for } p \geq 2. \tag{3.3}$$

By interpolating this estimate $\sqrt{2(p^2 - p)}$ with the known $\|B\|_2 = 1$, Bañuelos and Janakiraman establish the present best estimate in publication:

$$\|B\|_p \leq 1.575(p^* - 1). \tag{3.4}$$

4. New Questions and Main Results

Since B is associated with left-side orthogonality and since we know $\|B\|_p = \|B\|_{p'}$, two important questions are

(1) If $2 \leq p < \infty$, what is the best constant C_p in the left-side orthogonality problem: $\|Y\|_p \leq C_p \|X\|_p$, where Y is orthogonal and $\langle Y \rangle \leq \langle X \rangle$?
(2) Similarly, if $1 < p' < 2$, what is the best constant $C_{p'}$ in the left-side orthogonality problem?

We have separated the two questions since Burkholder's proof (and his function) already gives a good answer when $p \geq 2$. It may be (although we have now some doubts about that) the best possible as well. However no estimate (better than $p-1$) follows from analyzing Burkholder's function when $1 < p' < 2$. Perhaps, we may hope, $C_{p'} < \sqrt{\frac{p^2-p}{2}}$ when $1 < p' = \frac{p}{p-1} < 2$, which would then imply a better estimate for $\|B\|_p$. This paper 'answers' this hope in the negative by finding $C_{p'}$; see Theorem 4.1. We also ask and answer the analogous question of right-side orthogonality when $2 < p < \infty$. In the spirit of Burkholder [Bu8], we believe these questions are of independent interest in martingale theory and may have deeper connections with other areas of mathematics.

Theorem 4.1. *Let $Y = (Y_1, Y_2)$ be an orthogonal martingale and $X = (X_1, X_2)$ be an arbitrary martingale.*

(1) *Let $1 < p' \leq 2$. Suppose $\langle Y \rangle \leq \langle X \rangle$. Then the least constant that always works in the inequality $\|Y\|_{p'} \leq C_{p'}\|X\|_{p'}$ is*

$$C_{p'} = \frac{1}{\sqrt{2}}\frac{z_{p'}}{1-z_{p'}} \tag{4.1}$$

where $z_{p'}$ is the least positive root in $(0,1)$ of the bounded Laguerre function $L_{p'}$.

(2) *Let $2 \leq p < \infty$. Suppose $\langle X \rangle \leq \langle Y \rangle$. Then the least constant that always works in the inequality $\|X\|_p \leq C_p\|Y\|_p$ is*

$$C_p = \sqrt{2}\frac{1-z_p}{z_p} \tag{4.2}$$

where z_p is the least positive root in $(0,1)$ of the bounded Laguerre function L_p.

The Laguerre function L_p solves the ODE

$$sL_p''(s) + (1-s)L_p'(s) + pL_p(s) = 0.$$

These functions are discussed further and their properties deduced in section (6.2); see also [BJV], [C], [CL].

As mentioned earlier, (based however on numerical evidence) we believe in general $\sqrt{\frac{p^2-p}{2}} < C_{p'} < p-1$ and that these theorems cannot imply better estimates for $\|B\|_p$. However based again on numerical evidence, the following conjecture is made.

Conjecture 1. *For $1 < p' = \frac{p}{p-1} < 2$, $C_{p'} = C_p$, or equivalently,*

$$\frac{1}{\sqrt{2}}\frac{z_{p'}}{1-z_{p'}} = \sqrt{2}\frac{1-z_p}{z_p}.$$

It is conjecture relating the roots of the Laguerre functions. Notice that such a statement is not true with the constants from Theorem 3.1, and $\sqrt{\frac{2}{p'^2-p'}} < \sqrt{\frac{p^2-p}{2}}$ for all $p > 2$. So this conjecture (if true) suggests some distinct implications for the two settings. Note on the other hand, that the form of the two sets of constants are very analogous.

Before we embark on the proof of Theorem 4.1, let us mention that there is also the question of both-side orthogonality: what is the best constant when both X and Y are orthogonal martingales? This problem is solved by the authors for $2 < p < \infty$ in [BJV], and the answer is $C_p = \frac{1+s_p}{1-s_p} < \sqrt{\frac{p^2-p}{2}}$ where s_p is the largest root in $[-1, 1]$ of the Legendre function F solving $(1-s^2)F'' - 2sF' + pF = 0$.

5. Proof of Theorem 4.1: Right-side Orthogonality, $2 < p < \infty$

Let us begin with $2 < p < \infty$ (and right-side orthogonality). $X = (X_1, X_2)$ will denote an arbitrary martingale and $Y = (Y_1, Y_2)$ will denote an orthogonal martingale: $\langle Y_1 \rangle = \langle Y_2 \rangle$ and $\langle Y_1, Y_2 \rangle = 0$. In order to make the formulas simpler and the computations easier, we will work initially with the case

$$\langle X \rangle \leq \langle Y_1 \rangle = \frac{1}{2} \langle Y \rangle \tag{5.1}$$

With this condition, the constant corresponding to (4.2) is

$$\frac{1-z_p}{z_p}.$$

Let $\tilde{V}(x, y) = |x|^p - c^p |y|^p$. The objective is to find the best constant c for which there exists a minimal majorant $\tilde{U}(x, y) \geq \tilde{V}(x, y)$, $\tilde{U}(0, 0) \leq 0$, so that for X and Y as above, the process $\tilde{U}(X, Y)$ is a supermartingale. It follows then that $E[\tilde{V}(x, y)] \leq E[\tilde{U}(X, Y)] \leq 0$. This condition is equivalent (by appealing to Itô's formula) to requiring that $\tilde{U}$ has a negative quadratic form, i.e.

$$\sum_{i,j=1}^{2} \tilde{U}_{x_i x_j} d \langle X_i, X_j \rangle + \Delta_y \tilde{U} d \langle Y_1 \rangle + \sum_{i,j=1}^{2} 2\tilde{U}_{x_i y_j} d \langle X_i, Y_j \rangle \leq 0 \tag{5.2}$$

As in the Legendre case [BJV], the function only depends on $|x|$ and $|y|$, hence

$$\tilde{U}(x_1, x_2, y_1, y_2) = U\left(\sqrt{x_1^2 + x_2^2}, \sqrt{y_1^2 + y_2^2}\right) = U(|x|, |y|). \tag{5.3}$$

Let us introduce new vectors:

$$h_1 := \frac{x_1}{|x|} H_1 + \frac{x_2}{|x|} H_2\,,\ h_2 := \frac{x_2}{|x|} H_1 - \frac{x_1}{|x|} H_2 \tag{5.4}$$

$$k := \frac{y_1}{|y|} K_1 + \frac{y_2}{|y|} K_2\,. \tag{5.5}$$

It is an easy but important remark that because of orthogonality of K_1, K_2 and the fact that $\|K_1\| = \|K_2\|$ we have

$$\|k\| = \|K_1\|\,. \tag{5.6}$$

Using direct calculations and the remark (5.6) the condition (5.2) on the quadratic form of U becomes (for $x, y > 0$)

$$U_{xx}\|h_1\|^2 + \frac{U_x}{x}\|h_2\|^2 + 2U_{xy}h_1 \cdot k + (U_{yy} + \frac{U_y}{y})\|k\|^2 \leq 0 \tag{5.7}$$

for all vectors h_1, h_2 and k satisfying

$$\|h_1\|^2 + \|h_2\|^2 \leq \|k\|^2. \tag{5.8}$$

We consider three cases:

Case 1): $U_{xx} - \frac{U_x}{x} < 0$ and $\beta_0 := \frac{|U_{xy}|}{\left|U_{xx} - \frac{U_x}{x}\right|} \leq 1$. Let

$$\beta^2 = \frac{\|h_1\|^2 + \|h_2\|^2}{\|k\|^2}. \tag{5.9}$$

Then we write our expression (5.7) (for $\|k\| > 0$, which is the only interesting case) as

$$\|k\|^2(U_{xx} - \frac{U_x}{x})\left[\left(\frac{\|h_1\|}{\|k\|} - \beta_0\right)^2 - \frac{U_{xy}^2 + (\beta^2\frac{U_x}{x} + (U_{yy} + \frac{U_y}{y}))(\frac{U_x}{x} - U_{xx})}{\left(U_{xx} - \frac{U_x}{x}\right)^2}\right].$$

We want the maximum of this expression, that is the minimum of the expression in brackets. For

$$\beta \in [\beta_0, 1]$$

we can always satisfy $\beta^2 = \frac{\|h_1\|^2+\|h_2\|^2}{\|k\|^2}$, $\frac{\|h_1\|}{\|k\|} = \beta_0$ simultaneously. Putting $\frac{\|h_1\|}{\|k\|} = \beta_0$ we attain the minimum in the brackets. For

$$\beta \in (0, \beta_0)$$

we should make $\frac{\|h_1\|}{\|k\|}$ as close to β_0 as possible under the restriction $\beta^2 = \frac{\|h_1\|^2+\|h_2\|^2}{\|k\|^2}$. The best we can do is to put $h_2 = 0$ to it, or equivalently to (5.7).

Conclusion: in case 1) the negativity of the expression in (5.7) under the cone condition (5.8) is equivalent to

$$U_{xy}^2 + (\beta^2\frac{U_x}{x} + (U_{yy} + \frac{U_y}{y}))(\frac{U_x}{x} - U_{xx}) \leq 0\,,\ \forall \beta \in [\frac{|U_{xy}|}{\left|U_{xx} - \frac{U_x}{x}\right|}, 1]\,. \tag{5.10}$$

and

$$U_{xx}\beta^2 + 2|U_{xy}|\beta + (U_{yy} + \frac{U_y}{y}) \leq 0\,,\ \forall \beta \in (0, \frac{|U_{xy}|}{\left|U_{xx} - \frac{U_x}{x}\right|})\,. \tag{5.11}$$

Case 2): $U_{xx} - \frac{U_x}{x} < 0$ and $\beta_0 := \frac{|U_{xy}|}{\left|U_{xx} - \frac{U_x}{x}\right|} > 1$. We still need the minimum for the expression on brackets above. This means that we should make $\frac{\|h_1\|}{\|k\|}$ as close to β_0 as possible under the restriction $\beta^2 = \frac{\|h_1\|^2 + \|h_2\|^2}{\|k\|^2}$. The best we can do is to put $h_2 = 0$ to it, or equivalently to (5.7).

Conclusion: in case 2) the negativity of the expression in (5.7) under the cone condition (5.8) is equivalent to

$$U_{xx}\beta^2 + 2|U_{xy}|\beta + (U_{yy} + \frac{U_y}{y}) \leq 0\,, \ \forall \beta \in [0,1]\,. \tag{5.12}$$

Case 3): $U_{xx} - \frac{U_x}{x} \geq 0$. Our expression becomes

$$\|k\|^2 (U_{xx} - \frac{U_x}{x}) \left[\left(\frac{\|h_1\|}{\|k\|} + \beta_0 \right)^2 - \frac{U_{xy}^2 + (\beta^2 \frac{U_x}{x} + (U_{yy} + \frac{U_y}{y}))(\frac{U_x}{x} - U_{xx})}{\left(U_{xx} - \frac{U_x}{x} \right)^2} \right].$$

Now we search to attain the maximum of the expression in brackets under the restriction $\beta^2 = \frac{\|h_1\|^2 + \|h_2\|^2}{\|k\|^2}$. The best we can do is to put $h_2 = 0$ to it, or equivalently to (5.7).

Conclusion: in case 3) the negativity of the expression in (5.7) under the cone condition (5.8) is equivalent to

$$U_{xx}\beta^2 + 2|U_{xy}|\beta + (U_{yy} + \frac{U_y}{y}) \leq 0\,, \ \forall \beta \in [0,1]\,. \tag{5.13}$$

Now we see that condition (5.7) can be split into the following two.

Let $0 < \beta < 1$. If $U_{xx} - \frac{U_x}{x} < 0$ and $|U_{xy}| \leq \beta |U_{xx} - \frac{U_x}{x}|$, then

$$U_{xy}^2 + (\beta^2 \frac{U_x}{x} + (U_{yy} + \frac{U_y}{y}))(\frac{U_x}{x} - U_{xx}) \leq 0 \tag{5.14}$$

and in all other cases,

$$U_{xx}\beta^2 + 2|U_{xy}|\beta + (U_{yy} + \frac{U_y}{y}) \leq 0. \tag{5.15}$$

Both these conditions are equivalent when $|U_{xy}| = \beta |U_{xx} - \frac{U_x}{x}|$. Let us look at (5.14) first. If $U_x < 0$, then the maximum value is attained for smallest possible β which is $\frac{|U_{xy}|}{|U_{xx} - \frac{U_x}{x}|}$, hence (5.14) is contained in (5.15) in this case. When $U_x \geq 0$, the maximum value is attained for the largest possible β which is 1. In conclusion, (5.14) can be replaced by

$$\begin{cases} 0 \leq |U_{xy}| \leq \frac{U_x}{x} - U_{xx} \\ \text{and } U_x > 0 \end{cases} \Rightarrow \frac{U_{xy}^2}{\frac{U_x}{x} - U_{xx}} + \frac{U_x}{x} + (U_{yy} + \frac{U_y}{y}) \leq 0 \tag{5.16}$$

The left side of inequality (5.15) is the quadratic function

$$h(\beta) = U_{xx}\beta^2 + 2|U_{xy}|\beta + (U_{yy} + \frac{U_y}{y}).$$

where $|U_{xy}| \geq 0$ and $0 \leq \beta \leq 1$. If $U_{xx} \geq 0$, the maximum occurs at $\beta = 1$. Suppose $U_{xx} < 0$. Then the maximum in $[0, \infty)$ is at $\beta = \frac{-|U_{xy}|}{U_{xx}} \geq 0$. If $\frac{-|U_{xy}|}{U_{xx}} > 1$, then again the maximum of h in $[0, 1]$ is at $\beta = 1$. If $\frac{-|U_{xy}|}{U_{xx}} \leq 1$, then the maximum is at $\beta = \frac{-|U_{xy}|}{U_{xx}}$. Thus inequality (5.15) is equivalent to the following conditions:

$$\begin{cases} U_{xx} \geq 0; \text{ or} \\ -|U_{xy}| \leq U_{xx} \leq 0 \end{cases} \Rightarrow \begin{cases} U_{xx} + 2U_{xy} + (U_{yy} + \frac{U_y}{y}) \leq 0 \\ U_{xx} - 2U_{xy} + (U_{yy} + \frac{U_y}{y}) \leq 0 \end{cases} \tag{5.17}$$

$$U_{xx} \leq -|U_{xy}| \leq 0 \Rightarrow U_{xy}^2 - U_{xx}(U_{yy} + \frac{U_y}{y}) \leq 0 \tag{5.18}$$

6. A simplified setting: $\vec{x} = x_1 \in \mathbb{R}$, $\beta = 1$

In the previous section, we worked with the case when both $\vec{x}, \vec{y} \in \mathbb{R}^2$ and $\beta \in (0, 1]$. If we instead take $\vec{x}$ to be real (hence X a real martingale), then we need only deal with condition (5.15) and hence with (5.17) and (5.18). If we also assume that $\beta = 1$ then $\langle X \rangle = \langle Y_1 \rangle$ by (5.9), (5.6), and $|h_1| = |k|$ as $h_2 = 0$ by (5.4) and by our assumptions that $x_2 = 0, H_2 = 0$. Hence U only needs to satisfy

$$U_{xx} + 2U_{xy} + (U_{yy} + \frac{U_y}{y}) \leq 0 \tag{6.1}$$

$$U_{xx} - 2U_{xy} + (U_{yy} + \frac{U_y}{y}) \leq 0 \tag{6.2}$$

In prior works such as [Bu3], [BaJ1] and [BJV], the best majorant in the simplified setting is also the best in the general case. Hence we may predict the same in our problem and look first for the function U satisfying (6.1) and (6.2). We will proceed as follows.

(1) Use the homogeneity of $U(x, y)$ to reduce the partial differential inequalities to ordinary differential inequalities of a function $g(r)$.
(2) Predict that the optimal U (and g) will solve (with equality) one of the two differential equations, wherever it is above the boundary V. Then solve the easier looking equation, which will be the one with $-U_{xy}$.
(3) We will find the least constant c so that such a majorant based on (6.2) is possible. It will turn out that $U_{xy} \leq 0$ for this solution, hence the $+U_{xy}$ case also is accomplished.

6.1. **Homogeneity and reduction in variables.** The function U satisfies the same homogeneity condition as V: for all $t \in \mathbb{R}$,

$$U(tx, ty) = t^p U(x, y).$$

To see this, suppose U is a suitable majorant of V. Then $U_t(x, y) = \frac{1}{t^p} U(tx, ty) \geq \frac{1}{t^p} V(tx, ty) = V(x, y)$ is also a majorant and as easily checked, satisfies (6.1) and (6.2). Therefore U_t is also a suitable majorant for each $t > 0$. Now take the infimum over all t to get a suitable majorant satisfying the homogeneity condition.

Define

$$g(r) = U(1 - r, r), \qquad 0 < r < 1. \tag{6.3}$$

Then

$$U(x, y) = (x + y)^p U\left(1 - \frac{y}{x + y}, \frac{y}{x + y}\right) = (x + y)^p g\left(\frac{y}{x + y}\right).$$

Substituting this formulation into (6.1) and (6.2) gives the following equivalent conditions on g:

$$\mathcal{D}g(r) = (1-2r)^2 r g''(r) + (4(p-1)(1-2r)r + 1 - r)g'(r) + (4p(p-1)r + p)g(r) \leq 0 \tag{6.4}$$

$$\mathcal{L}_p g(r) = r g''(r) + (1 - r)g'(r) + p g(r) \leq 0 \tag{6.5}$$

The operator $\mathcal{L}_p$ is the Laguerre operator, the equation $\mathcal{L}_p f = 0$ is the Laguerre equation and its solutions are the Laguerre functions. Function g also should be above its obstacle v.

$$g(r) \geq v(r) = (1 - r)^p - c^p r^p \tag{6.6}$$

Finally note that

$$\frac{U_{xy}}{(x + y)^{p-2}} = -r(1 - r)g''(r) + (p - 1)(1 - 2r)g'(r) + p(p - 1)g(r) \tag{6.7}$$

Since $v(0) = 1$ for all c, g must have $g(0) \geq 1$. As g is the minimal function possible, it is likely that it solves either $\mathcal{D}g = 0$ or $\mathcal{L}_p g = 0$ wherever $g > v$. We consider first the simpler operator $\mathcal{L}_p$ and attempt to construct g from its solutions.

6.2. **The Laguerre functions.** Just as for the Legendre case in [BJV], the Laguerre equation

$$xy'' + (1 - x)y' + py = 0 \tag{6.8}$$

has solutions that are linear combinations of two independent solutions L_p and $\tilde{L}_p$.

$$L_p(x) = \quad 1 - px + \tfrac{p(p-1)}{4}x^2 + \cdots + (-1)^n \tfrac{p(p-1)\cdots(p-n+1)}{n!^2} x^n + \ldots \tag{6.9}$$

$$\tilde{L}_p(x) = \quad L_p(x) \log \tfrac{1}{|x|} + H(x) \tag{6.10}$$

H is analytic in a neighborhood of 0. Evidently, $L_p(x)$ is a bounded analytic function in $[0, 1]$ and $\tilde{L}_p$ is unbounded near 0. Let z_p denote the smallest zero of L_p in the interval $[0, 1]$.

Lemma 6.1. *Among all the solutions of Laguerre equation* (6.8)*, the maximum of their smallest zero in* $[0,1]$ *is* z_p*.*

Proof. Notice that $\tilde{L}_p(0) = +\infty$. Consider Wronskian $W(x) = \tilde{L}_p'(x)L_p(x) - L_p'(x)\tilde{L}_p(x)$. Substituting (6.10) gives

$$W(x) = \frac{-L_p^2}{x} + H'L_p - L_p'H,$$

which is < 0 for x close to 0. Since $W'(x) = -\frac{1-x}{x}W(x)$, W preserves sign in $[0,1]$ and is negative. At z_p, since L_p changes sign from positive to negative, we have $L_p'(z_p) < 0$ and

$$W(z_p) = -L_p'(z_p)\tilde{L}_p(z_p) = |L_p'(z_p)|\tilde{L}_p(z_p)$$

Since $W < 0$, it follows that $\tilde{L}_p(z_p) < 0$. Now consider $f_3 = c_1L_p + c_2\tilde{L}_p$ for $c_2 > 0$. Then $f_3(z_p) < 0$ and $f_3(x) > 0$ for x close to 0. Therefore f_3 has a zero in $(0, z_p)$. The same arguments work for $c_2 < 0$. $\square$

Lemma 6.2. *Function* L_p *is strictly convex in* $(0, z_p]$ *for* $1 < p < \infty$*.* L_p *is strictly concave in* $(0, z_p]$ *for* $0 < p < 1$*.*

Proof. From Laguerre's equation and from differentiating it, we get

$$xL_p'' + (1-x)L_p' + pL_p = 0, \tag{6.11}$$

$$xL_p''' + (2-x)L_p'' + (p-1)L_p' = 0. \tag{6.12}$$

Let $x_1 > 0$ be the first point > 0 where $L_p''(x_1) = 0$. Suppose $x_1 < z_p$. Then $L_p(x_1) > 0$ and from (6.11), it follows that $L_p'(x_1) < 0$. Then (6.12) implies $L_p'''(x_1) > 0$ and so L_p'' is strictly increasing (from $-$ to $+$) at x_1. But this is not possible, since $L_p''(0) = \frac{p(p-1)}{2} > 0$. Therefore $x_1 > z_p$ and L_p is strictly convex in $(0, z_p]$. The same proof shows that L_p is strictly concave in $(0, z_p]$ for $0 < p < 1$. $\square$

Let us make the following important conclusion from Lemma (6.1).

Lemma 6.3. *Let* $c_2 \neq 0$ *and* $c = \frac{1-z_p}{z_p}$*. The obstacle function* $v_c(s)$ *cannot be below any* $f = c_1L_p + c_2\tilde{L}_p$ *in the interval* $(0, z_p]$*.*

Proof. This follows from the following facts: $v_c(z_p) = 0$, $v_c > 0$ in $(0, z_p)$, $\lim_{s\to 0+}|f(s)| = \infty$, and by Lemma (6.1), $f(x) = 0$ for some $x \in (0, z_p]$. $\square$

Remark 6.1. *It will be shown that* $c = \frac{1-z_p}{z_p}$ *is a valid choice with a Laguerre majorant in* $[0, z_p]$ *of the form* aL_p*. It will also be clear due to Lemma* (6.3) *that no other Laguerre functions can be used in our method to obtain better constant. Henceforth we will no longer mention* $\tilde{L}_p$*.*

Recall $L_p(0) = 1 = v_c(0)$. For c very large and $a > 1$, we will have

$$aL_p(r) > v_c(r)\ \forall r \in [0,1]$$

Next lower the value of c and thereby raise the function v_c. When the boundary $v_c(r)$ first meets the Laguerre function $aL_p(r)$, they will have the same tangent lines. Suppose they first meet at point $r = s$. We would like to have $g = v$ on one side of s (where $\mathcal{L}_p v_c \le 0$) and $g = L_p$ on the other side. This patched together function would then become the candidate majorant; finally adjust constants a and c so that it yields the best possible constant.

Computing with v_c shows that $\mathcal{L}_p v_c \le 0$ when s is greater than

$$m_p = \frac{(p-1)^{\frac{1}{p-2}}}{(p-1)^{\frac{1}{p-2}} + (pc^p)^{\frac{1}{p-2}}} \tag{6.13}$$

Thus the meeting point must be greater than this value and also satisfy:

$$\begin{cases} (1-s)^p - c^p s^p = aL_p(s) \\ -p(1-s)^{p-1} - pc^p s^{p-1} = aL_p'(s) \end{cases} \tag{6.14}$$

Alternatively, this can be transformed as:

$$\begin{cases} -pc^p s^{p-1} = apL_p + a(1-s)L_p' \\ -p(1-s)^{p-1} = -apL_p + asL_p' \end{cases}$$

Dividing the two equations gives

$$\frac{c^p s^{p-1}}{(1-s)^{p-1}} = \frac{pL_p + (1-s)L_p'}{-pL_p + sL_p'}$$

which implies

$$\begin{aligned} c^p &= \frac{(1-s)L_p' + pL_p}{sL_p' - pL_p}\frac{(1-s)^{p-1}}{s^{p-1}} \\ &= \frac{(1-s)^p L_p' + p(1-s)^{p-1}L_p}{s^p L_p' - ps^{p-1}L_p} := F(s). \end{aligned} \tag{6.15}$$

As a remark (when p is an integer), note that the denominator term $sL_p' - pL_p$ is up to a constant equal to L_{p-1}; similarly the numerator term can be corresponded with the associated Laguerre function $L_{p-1}^{(1)}$.

So for each possible s, there is a corresponding c^p value such that the two functions will meet at that s. We wish to find the s value that will minimize this c^p. To do this, we will differentiate the function and find its critical points. However, let us observe first that the function $\frac{(1-s)L_p'+pL_p}{sL_p'-pL_p}$ will have a singularity in $(0,1)$ if $sL_p'(s) - pL_p(s) = 0$ for some s. Since this term is $\tilde{c}L_{p-1}$, we know that this does happen for integral values of p and expect in general case as well.

Differentiating:

$$0 = \frac{d}{ds}\left[\frac{(1-s)^p L_p' + p(1-s)^{p-1}L_p}{s^p L_p' - ps^{p-1}L_p}\right]$$

We must therefore set

$$(s^pL_p'-ps^{p-1}L_p)((1-s)^pL_p'+p(1-s)^{p-1}L_p)'-((1-s)^pL_p'+p(1-s)^{p-1}L_p)(s^pL_p'-ps^{p-1}L_p)'=0$$

Some calculations:

$$\begin{aligned}
&((1-s)^pL_p'+p(1-s)^{p-1}L_p)'\\
=\ & -p(1-s)^pL_p'+(1-s)^pL_p''-p(p-1)(1-s)^{p-2}L_p+p(1-s)^{p-1}L_p'\\
=\ & (1-s)^pL_p''-p(p-1)(1-s)^{p-2}L_p.
\end{aligned}$$

$$\begin{aligned}
A &= (s^pL_p'-ps^{p-1}L_p)((1-s)^pL_p'+p(1-s)^{p-1}L_p)'\\
&= s^p(1-s)^pL_p''L_p'-ps^{p-1}(1-s)^pL_p''L_p\\
&\quad -p(p-1)s^p(1-s)^{p-2}L_p'L_p+p^2(p-1)s^{p-1}(1-s)^{p-2}L_p^2
\end{aligned}$$

$$\begin{aligned}
&(s^pL_p'-ps^{p-1}L_p)'=ps^{p-1}L_p'+s^pL_p''-p(p-1)s^{p-2}L_p-ps^{p-1}L_p'\\
=\ & s^pL_p''-p(p-1)s^{p-1}L_p
\end{aligned}$$

$$\begin{aligned}
B &= ((1-s)^pL_p'+p(1-s)^{p-1}L_p)(s^pL_p'-ps^{p-1}L_p)'\\
&= s^p(1-s)^pL_p''L_p'+ps^p(1-s)^{p-1}L_p''L_p\\
&\quad -p(p-1)s^{p-2}(1-s)^pL_p'L_p-p^2(p-1)s^{p-2}(1-s)^{p-1}L_p^2
\end{aligned}$$

$$\begin{aligned}
A-B &= ps^{p-2}(1-s)^{p-2}L_p[s(s-1)L_p''+(p-1)(1-2s)L_p'+p(p-1)L_p]\\
&= ps^{p-2}(1-s)^{p-2}L_p(s)\mathcal{H}L_p(s)
\end{aligned}$$

Therefore

$$\begin{aligned}
F'(s) &= \frac{ps^{p-2}(1-s)^{p-2}L_p(s)\mathcal{H}L_p(s)}{(s^pL_p'-ps^{p-1}L_p)^2}\\
&= \frac{p(1-s)^{p-2}}{s^p}\frac{L_p(s)\mathcal{H}L_p(s)}{(sL_p'-pL_p)^2} \qquad (6.16)
\end{aligned}$$

Applying Laguerre's equation will show below that $\mathcal{H}L_p=-s[(p-s)L_p''+(p-1)L_p']$ and so

$$F'(s)=-\frac{p(1-s)^{p-2}}{s^{p-1}}\frac{L_p(s)[(p-s)L_p''+(p-1)L_p']}{(sL_p'-pL_p)^2} \qquad (6.17)$$

So the critical points of the function $F(s)$ in (6.15) are included among $s=1$, the zero-points of L_p and the positive zero-points of $\mathcal{H}L_p$. Moreover F has singularities at $s=0$ and at the zero-points of the function $s^pL_p'-ps^{p-1}L_p$.

6.3. **The case** $s=1$**.** Since $F(1)=0$ and we know that $c>0$, it is clear that the critical point 1 cannot be the first touching point. In fact, since

$L_p(1) \neq 0$ and $v_0(1) = (1-1)^p = 0$, the only way that such a meeting could happen is if $a = 0$. So this case is eliminated.

So we look to the other cases for the correct critical point.

6.4. **The function $\mathcal{H}L_p(s)$.** The function $\mathcal{H}L_p(s)$ is known to us. See (6.7). If we set

$$\begin{aligned}\mathcal{U}(x,y) &= (x+y)^p \mathcal{U}(\frac{x}{x+y}, \frac{y}{x+y}) \\ &= (x+y)^p L_p(\frac{y}{x+y}),\end{aligned}$$

then

$$\mathcal{H}L_p(s) = \mathcal{U}_{xy}(1-s, s). \tag{6.18}$$

(We could also have defined in terms of x as opposed to y, and the conclusion would be essentially same.)

Lemma 6.4. $\mathcal{H}L_p(s) = s(sL'_p)'''$

Proof.

$$\begin{aligned}(sL'_p)' &= sL''_p + L'_p = sL'_p - pL_p \\ (sL'_p)'' &= (sL'_p - pL_p)' = sL''_p - (p-1)L'_p = -[(p-s)L'_p + pL_p] \\ (sL'_p)''' &= -[(p-s)L''_p + (p-1)L'_p] \end{aligned} \tag{6.19}$$

Now consider $\mathcal{H}L_p(s) = s(s-1)L''_p + (p-1)(1-2s)L'_p + p(p-1)L_p$. Subtracting $0 = (p-1)[sL''_p + (1-s)L'_p + pL_p]$ from this gives

$$\mathcal{H}L_p(s) = -s[(p-s)L''_p + (p-1)L'_p] \tag{6.20}$$

Comparing (6.20) and (6.19) proves the lemma. □

Lemma 6.5. $(sL'_p)' = -pL_{p-1}$

Proof. Differentiating the Laguerre equation

$$(sL'_p)' - sL'_p + pL_p = 0 \tag{6.21}$$

gives

$$(sL'_p)'' - (sL'_p)' + pL'_p = 0. \tag{6.22}$$

Multiply this by s and differentiate again to get

$$s(sL'_p)''' + (1-s)(sL'_p)'' + (p-1)(sL'_p)' = 0.$$

This shows that $(sL'_p)'$ solves the Laguerre equation with constant $p-1$ and hence is equal to γL_{p-1}. Since $(sL'_p)'(0) = -p$, $\gamma = -p$. □

Theorem 6.1. *For $2 < p < \infty$, $\mathcal{H}L_p < 0$ in $(0, z_{p-1}]$. For $1 < p < 2$, $\mathcal{H}L_p > 0$ in $(0, z_{p-1}]$.*

Proof. By Lemma (6.2), L_{p-1} is strictly convex in $[0, z_{p-1}]$. Therefore by Lemma (6.5), $(sL'_p)'$ is strictly concave in $[0, z_{p-1}]$, and by Lemma (6.4), $\mathcal{H}L_p < 0$ in $(0, z_{p-1}]$. Similarly, the $1 < p < 2$ case follows from the fact that L_{p-1} is strictly concave in $[0, z_{p-1}]$. □

Lemma 6.6. *(Ordering of the roots)* $z_p < z_{p-1}$

Proof. Since z_{p-1} is the root of L_{p-1}, by Lemma (6.5), $(sL'_p)'(z_{p-1}) = 0$. Then by (6.21), $-z_{p-1}L'_p(z_{p-1}) + pL_p(z_{p-1}) = 0$. Since $L'_p < 0$ and $L_p > 0$ in $(0, z_p]$, it follows that $z_{p-1} > z_p$. □

Note that since $\mathcal{H}L_p$ corresponds with U_{xy}, this implies that the Laguerre function satisfies both (6.4) and (6.5) in $[0, z_p]$, and the majorant U derived from L_p in the corresponding regions will satisfy (6.1) and (6.2). Some more facts on Laguerre functions are recorded next.

Lemma 6.7. $L_p < L_{p-1}$ *on* $(0, z_{p-1}]$*. Also* L_p *has exactly one root (at* z_p*) in* $[0, z_{p-1}]$*.*

Proof. Suppose $L_p(x) = L_{p-1}(x)$ for some $x \in (0, z_{p-1})$. Since $L_{p-1}(s) = -\frac{1}{p}(sL'_p)'$ by Lemma (6.5), it follows from the Laguerre equation (6.21) that

$$-pL_{p-1}(x) - xL'_p(x) + pL_p(x) = 0$$

and hence

$$xL'_p(x) = -pL_{p-1}(x) + pL_p(x) = 0.$$

In fact, $L_p(x) = L_{p-1}(x)$ if and only if $L'_p(x) = 0$. But $L'_p < 0$ in $(0, z_p]$. Therefore $x > z_p$. Now $L_{p-1} > 0$ in (z_p, z_{p-1}) and $L_p < 0$ in some interval $(z_p, z_p + \epsilon)$; so their meeting point must be where $L'_p = 0$ and $L_{p-1} = L_p < 0$, in particular $after$ z_{p-1}. This implies that L_p remains negative till after z_{p-1} and hence its only root in $(0, z_{p-1}]$ is at z_p □

Corollary 6.1. L_p *is convex in* $(0, z_{p-1})$*.*

Proof. Lemma (6.2) shows this in $(0, z_p)$. If $L''_p(x) = 0$ for some $x < z_{p-1}$, then since $L_p(x) < 0$, the Laguerre equation implies $L'_p(x) > 0$ which means (as $L'_p(0) < 0$) there must be a point $y \in (z_p, x)$ such that $L'_p(y) = 0$. This cannot be as shown in previous lemma. □

Lemma 6.8. *(Location of root) For* $1 < p < \infty$*,* $0 < z_p < 1$ *and* $L_p(1) < 0$*.*

Proof. By Lemma 6.6, it suffices to show for $1 < p \le 2$. For $p = 2$ $z_2 = 2 - \sqrt{2} < 1$, so assume $1 < p < 2$. Assume also without the loss of generality that $L_p(1) \neq 0$, for otherwise $0 = \mathcal{L}_p L_p(1) = L''_p(1) > 0$, a contradiction. Since $L_1(s) = 1 - s$, $\mathcal{L}_p L_1 = sL''_1 + (1-s)L'_1 + pL_1 = (p-1)L_1 \ge 0$. Consider $f = L_1 - L_p$ which is also a subsolution for $\mathcal{L}_p$. Then $f(0+) > 0$. If $z_p > 1$, then $f(1) < 0$, and hence there must be some point $x \in (0,1)$ where $f(x) = 0$ and $f'(x) \le 0$. But then $\mathcal{L}_p f(x) = -xL''_p(x) + (1-x)f'(x) < 0$ since $L''_p > 0$ in $(0, z_p)$, a contradiction. Therefore $0 < z_p < 1$.

Now consider the situation that $0 < z_p < 1$ but $L_p(1) > 0$. Then L_p must have two roots in $(0,1)$ and there is a point $x \in (0,1)$ such that $L_p(x) < 0$ and $L'_p(x) = 0$. But this means (see Lemma 6.7) $L_{p-1}(x) = L_p(x) < 0$. Since $0 < \gamma = p - 1 < 1$ and $L_\gamma(0+) > L_1(0+)$, there is a point $y \in (0,1)$ such that $g(y) = L_1(y) - L_\gamma(y) = 0$ and $g'(y) \ge 0$ (in fact, $g(0+) < 0$, $g(x) > 0$ as we just saw). We know $L''_\gamma(y) < 0$ from Lemma

6.2, therefore $\mathcal{L}_\gamma g(y) = -yL''_\gamma(y) + (1-y)g'(y) > 0$. However, a direct application shows $\mathcal{L}_\gamma g = \mathcal{L}_\gamma L_1 = (\gamma - 1)L_1 < 0$. Contradiction; therefore $L_p(1) < 0$. $\square$

Remark 6.2. *A stronger result than Lemma 6.7 should be true: we expect $f = \frac{d}{dp}L_p < 0$ in $(0, z_{p-1})$. In fact, this function satisfies $\mathcal{L}_p f = -L_p$ and hence is a supersolution of the Laguerre equation, with $f(0) = 0$, $f'(0) = -1$ and $f''(0) = \frac{2p-1}{2}$. Moreover $-\mathcal{L}_p[\partial_p^k f] = \partial_p^{k-1} f$ and hence $-\mathcal{L}_p$ plays approximately the role of $\int dp$ on these functions.*

6.5. **Consequences of the lemmas/theorem.** From the results in the previous subsection and considering (6.16), we can draw the following conclusions on $F(s)$ from (6.15).

(1) $F(s)$ has exactly one critical point in $[0, z_{p-1}]$ and it is at z_p. The minimum value of F in $[0, z_{p-1}]$ is $F(z_p) = \frac{(1-z_p)^p}{z_p^p}$.
(2) $\lim_{s\to 0+} F(s) = \lim_{s\to z_{p-1}-} F(s) = +\infty$.
(3) Although not proved here, it is expected that $F < 0$ in $(z_{p-1}, 1)$ and z_p is the unique root of L_p in $(0, 1)$.

Since $F(s) = c^p$, for each value of $c > c_{\min}$, there are exactly two points x_1 and x_2 in $[0, z_{p-1}]$ ($x_1 < z_p < x_2$) and value(s) $a_1 = a(x_1)$ and $a_2 = a(x_2)$ such that $a_i L_p$ and v_c touch at $s = x_i$. Let $a_0 = a(z_p)$. Then $a_1 < a_0 < a_2$. This follows from the following lemma.

Lemma 6.9. *The function $a(s)$ is increasing in $(0, z_{p-1})$.*

Proof. Since $a(s)L_p(s) = v_{c(s)}(s)$,

$$a'(s)L_p(s) + a(s)L'_p(s) = \frac{d}{ds}(v_{c(s)})(s) = v'_c(s) - s^p \frac{d}{ds}(c(s)^p).$$

Next since $v'_c(s) = a(s)L'_p(s)$, it follows that

$$a'(s)L_p(s) = -s^p \frac{d}{ds}(c(s)^p).$$

By (6.16), Lemma (6.7) and Theorem 6.1,

$$\frac{d}{ds}(c(s)^p) = F'(s) = \frac{p(1-s)^{p-2}}{s^p} \frac{L_p(s)\mathcal{H}L_p(s)}{(sL'_p - pL_p)^2} \tag{6.23}$$

is negative in $(0, z_p)$ and positive in (z_p, z_{p-1}), and in particular has the opposite sign of L_p. Therefore it follows that

$$a'(s) = \frac{-p(1-s)^{p-2}\mathcal{H}L_p(s)}{(sL'_p - pL_p)^2} > 0 \tag{6.24}$$

in $(0, z_{p-1})$. $\square$

(1) So for each $s \in (0, z_p)$, there exists unique values of c and a such that touching happens. Call (a, c, s) a touching-triple.

(2) For each $c > c(z_p)$, there exists exactly two points in $(0, z_{p-1})$: $s_1 < z_p < s_2$, and two values $a_1 < a(z_p) < a_2$, such that (a_1, c, s_1) and (a_2, c, s_2) are touching triples.
(3) For each $a \in (1, a(z_p)]$, there exists a unique $c \geq c(z_p)$ and unique $s \in (0, z_p]$ such that (a, c, s) is a touching triple. For each $a \in (a(z_p), \infty]$, there exists a unique $c \geq c(z_p)$ and unique $s \in (z_p, z_{p-1})$ such that (a, c, s) is a touching triple.

Recall $a_0 = a(z_p)$. If we start with large c so that $v_c < a_0 L_p$ and slowly decrease c, the function v_c can meet $a_0 L_p$ at z_p only when $c = \frac{1-z_p}{z_p}$, in which case it would be a touching point. If for a larger c', in the course of changing c from very large values to $c = \frac{1-z_p}{z_p}$, $v_{c'}$ happened to meet $a_0 L_p$ at some $x \in (0, z_p)$, then that x also must be a touching point. But we have shown that $a(x) < a(z_p) = a_0$. So this cannot happen and we have

Theorem 6.2. *For $c = \frac{1-z_p}{z_p}$, the function v_c touches $a_0 L_p$ at z_p and $v_c < a_0 L_p$ in $(0, z_p)$.* $\square$

Define

$$g(s) = \begin{cases} a_0 L_p(s), & 0 < s \leq z_p \\ v_c(s), & z_p < s \leq 1. \end{cases} \tag{6.25}$$

It can be easily checked that g is a supersolution of the Laguerre equation $\mathcal{L}_p g \leq 0$, hence satisfies (6.5). Function g also satisfies (6.4) since by

$$\mathcal{D}_p g = \begin{cases} \mathcal{L}_p L_p + 4s\mathcal{H} L_p = -4ps^2 L''_{p-1} < 0, 0 < s < z_p \\ \mathcal{L}_p v_c < 0, z_p \leq s < 1. \end{cases} \tag{6.26}$$

The second line follows immediately from (6.18). Thus we have a constant

$$c = \frac{1 - z_p}{z_p}$$

for which there exists a valid majorant g of the obstacle $v_c(s) = (1-s)^p - c^p s^p$ satisfying (6.4) and (6.5). The majorant $U(x, y) = (x+y)^p g(\frac{y}{x+y})$ will then satisfy (6.1) and (6.2).

6.6. **Sharpness of the constant.** It remains to prove that for any $c < \frac{1-z_p}{z_p}$, there is no majorant of v_c that will satisfy (6.4) and (6.5). Since the Bellman function (which has the best constant) must satisfy the corresponding quadratic form inequalities, it will follow that our constant is sharp. Note that for $c < \frac{1-z_p}{z_p}$, $v_c(z_p) > 0$. So any possible supersolution f of the Laguerre equation, satisfying $f \geq v_c$ must have $f(z_p) > 0$.

Suppose there is a supersolution $f \geq g$ with $\mathcal{L}_p f \leq 0$ and $f(z_p) > 0$. Then the root of f occurs at some $s_p > z_p$. Let $a < 1$ be the first value for which af and g meet at some point x in $[0, z_p)$. (If there was a meeting point to begin with, just start the argument from here on.) If $x \in (0, z_p)$ then the meeting point is a touching point; note importantly that $x < z_p$

since $af(z_p) > 0$ for all $a > 0$. Since $af > g$ away from x, it follows that $af''(x) \geq g''(x)$ and hence

$$xaf''(x) + (1-x)af'(x) + paf(x) \geq xg''(x) + (1-x)g'(x) + pg(x) = 0.$$

Since f is a Laguerre supersolution, the only possibility is if $\mathcal{L}_p f(x) = 0$ and $af''(x) = g''(x)$. Now consider $h = af - g$ in a small interval $(x - \epsilon, x + \epsilon)$. Since af is supersolution and g a solution of the Laguerre equation, it follows that h is also a supersolution. Assume without the loss of generality that $\mathcal{L}_p h(x+) < 0$, for otherwise h is locally a solution with its value and of its first two derivatives zero at a point. This implies (by differentiating the Laguerre equation repeatedly) that all its derivatives are zero at that point, hence the function is identically 0. Therefore to avoid this, we must have $\mathcal{L}_p h(x+) < 0$. But then, observe $h(x) = h'(x) = h''(x) = 0$ and $h(x+) > 0$. Since $h(x+) > 0$, we must have $h'(x+) > 0$, and since $h'(x+) > 0$, we have $h''(x+) > 0$. In particular, $\mathcal{L}_p h(x+) > 0$; contradiction. Therefore, there is no such meeting point $x \in (0, s_p)$.

As for the case when the first meeting point is $x = 0$, if it is a touching point, then the above proof works to show this cannot happen. If it is not a touching point, then $af'(0) > g'(0)$. But then,

$$0 \cdot af''(0) + (1-0)af'(0) + paf(0) = af'(0) + paf(0) =$$

$$af'(0) + pg(0) > g'(0) + pg(0) = 0,$$

which means $\mathcal{L}_p f(0) > 0$. This contradicts the assumption that f is a supersolution. So we have proved there is no supersolution $f \geq g$ with $f(z_p) > g(z_p)$. Therefore all supersolutions $\geq g$ must be 0 at z_p; it follows that g is the best majorant.

7. The general case, $2 \leq p < \infty$

Let $c_0 = \frac{1-z_p}{z_p}$. In section (6), it is shown that if $Y = (Y_1, Y_2)$ is an orthogonal martingale and X any real martingale satisfying $\langle X \rangle = \langle Y_i \rangle$, then

$$\|X\|_p \leq \left(\frac{1 - z_p}{z_p} \right) \|Y\|_p, \tag{7.1}$$

where z_p is the closest-to-zero root of the bounded-in-$(0,1)$ Laguerre function L_p of index p. To show this, we started with $V(|x|, |y|) = |x|^p - c^p |y|^p$ and found the best-constant majorant $U(|x|, |y|)$ satisfying the required quadratic-form inequalities (6.1) and (6.2). Now we turn to the general cases where X is a complex valued martingale and $\langle X \rangle \leq \langle Y_i \rangle$. U will have to satisfy (5.16) and (5.18) respectively, in addition to (6.1) and (6.2) (or more precisely, in addition to (5.17)). We will show that the function U obtained from the simple setting also works in the general case. Henceforth, U will denote this function, and g will be its corresponding one-dimensional version in (6.25).

Recall the conditions (5.16) and (5.18):

$$\begin{cases} 0 \le |U_{xy}| \le \frac{U_x}{x} - U_{xx} \\ \text{and } U_x > 0 \end{cases} \Rightarrow U_{xy}^2 + (\frac{U_x}{x} + (U_{yy} + \frac{U_y}{y}))\left(\frac{U_x}{x} - U_{xx}\right) \le 0 \tag{7.2}$$

$$U_{xx} \le -|U_{xy}| \le 0 \Rightarrow U_{xy}^2 - U_{xx}(U_{yy} + \frac{U_y}{y}) \le 0. \tag{7.3}$$

We will show in the following lemma that both these requirements are satisfied trivially.

Lemma 7.1. *For $x > 0$, $U_x > 0$ and $-U_{xx} < \frac{U_x}{x} - U_{xx} < |U_{xy}|$.*

Observe that this lemma implies that the 'if' parts of (7.2) and (7.3) do not happen for $x > 0$. The special case when $x = 0$ is also simple. Since x corresponds with $1-r$, $x = 0$ corresponds with $r = 1$, where $g = v_c$ and hence $U = V$. Both V_{xx} and $\frac{V_x}{x}$ are 0 when $x = 0$, from which (7.2) and (7.3) follow. Now for the proof the lemma:

Proof. When $U = V$, we have $V_x = px^{p-1} > 0$ and

$$-V_{xx} = -p(p-1)x^{p-2} < -p(p-2)x^{p-2} = \frac{V_x}{x} - V_{xx} < 0 = |V_{xy}|$$

as stated. So assume $U > V$ where U corresponds with the Laguerre function $a_0 L_p$ on $[0, z_p)$. A simple computation shows that $U_x(1-s, s) = [pg(s) - sg'(s)] > 0$ since g and $-g'$ are positive in $(0, z_p)$. Therefore the first inequality $-U_{xx} < \frac{U_x}{x} - U_{xx}$ is true. It remains to show

$$\frac{U_x}{x} - U_{xx} < |U_{xy}|. \tag{7.4}$$

By (6.18), $U_{xy} \sim \mathcal{H}L_p < 0$ in $(0, z_p)$, so $|U_{xy}| = -U_{xy}$ where $U > V$. Therefore (7.4) is equivalent to

$$\frac{U_x}{x} - U_{xx} + U_{xy} < 0 \text{ where } U > V. \tag{7.5}$$

Evaluating at $(1-s, s)$, we have

$$\begin{aligned} \frac{U_x}{x} &= \left[\frac{pg(s)}{1-s} - \frac{s}{1-s}g'(s)\right]; && (7.6) \\ U_{xx} &= s^2 g'' - 2(p-1)sg' + p(p-1)g \\ &= s^2 g'' - (p-1)sg' - (p-1)sg'' - (p-1)g' \\ &= -s(p-1-s)g'' - (p-1)(1+s)g' && (7.7) \\ U_{xy} &= \mathcal{H}g = -s(p-s)g'' - s(p-1)g' && (7.8) \end{aligned}$$

Using (7.6), (7.7) and (7.8), condition (7.5) can be restated as

$$\frac{pg - sg'}{1-s} - sg'' + (p-1)g' < 0 \tag{7.9}$$

Substituting $-sg'' = pg + (1-s)g'$ and multiplying through by $1-s$ gets

$$\begin{aligned}
& pg - sg' + p(1-s)g + (1-s)^2 g' + (p-1)(1-s)g' \\
= \;& p(2-s)g + (s^2 - (p+2)s + p)g' \\
= \;& -(2-s)sg'' - (2-s)(1-s)g' + (s^2 - (p+2)s + p)g' \\
= \;& -s(2-s)g'' + [(p-1)s + (p-2)]g'.
\end{aligned}$$

Since g'' and g are > 0 in $(0, z_p)$, we have $(1-s)g' < -sg''$. This means

$$\begin{aligned}
-s(2-s)g'' + [(p-1)s + (p-2)]g' &= \frac{-s(2-s)(1-s)g'' + [(p-1)s + (p-2)](1-s)g'}{1-s} \\
&\le (-sg'')\frac{s^2 + (p-4)s + p}{1-s}. \qquad (7.10)
\end{aligned}$$

Since $s^2 + (p-4)s + p \ge s^2 - 2s + p \ge s^2 - 2s + 2 > 0$, it follows that (7.10) is < 0 and hence (7.9) holds. $\square$

8. Left-side Orthogonality, $1 < p < 2$

In this section, we show that the same methods extend to the case of left-side orthogonality when $1 < p < 2$. Again for the sake of simplicity, we work with the case

$$\langle Y_i \rangle = \frac{1}{2} \langle Y \rangle \le \langle X \rangle \qquad (8.1)$$

With this condition, the constant corresponding to (4.1) is

$$\frac{z_p}{1 - z_p}.$$

Let us begin with the special case when $\langle X \rangle = \langle Y_i \rangle$. The obstacle functions are

$$V(x,y) = y^p - c^p x^p, \qquad v_c(s) = s^p - c^p(1-s)^p \qquad (8.2)$$

and the majorants

$$U(x,y) = (x+y)^p g\left(\frac{y}{x+y}\right), \quad g(s) = U(1-s, s)$$

will have to satisfy the quadratic form inequalities (6.1) and (6.2), and (6.4) and (6.5). The function v_c starts with value $-c^p$ at $s = 0$ and increases to 1 at $s = 1$. It is convex in the beginning and concave in the end. Moreover $\mathcal{D}_p v_c = \mathcal{L}_p v_c(s)$ is ≤ 0 for $s \ge \frac{1}{1+C}$, for some positive C. So we may expect that the majorant g equals some βL_p in $[0, \alpha]$ and equals v_c in $(\alpha, 1]$, where α is a touching point. Since $L_p(0) = 1$ and $L_p(1) < 0$, we must have $\beta = -a$ for some $a > 0$. We come to a corresponding version of (6.14):

$$\begin{cases} s^p - c^p(1-s)^p = -aL_p(s) \\ ps^{p-1} + pc^p(1-s)^{p-1} = -aL_p'(s) \end{cases} \qquad (8.3)$$

Letting $\tilde{c} = \frac{1}{c}$ and $\tilde{a} = \frac{a}{c^p}$, this is transformed to

$$\begin{cases} (1-s)^p - \tilde{c}^p s^p = \tilde{a} L_p(s) \\ -p(1-s)^{p-1} - p\tilde{c}^p s^{p-1} = \tilde{a} L_p'(s) \end{cases} \tag{8.4}$$

Note that (8.4) has the same form as (6.14), hence we have

$$\tilde{c}^p = F(s) = \frac{(1-s)^p L_p' + p(1-s)^{p-1} L_p}{s^p L_p' - p s^{p-1} L_p}, \qquad c^p = G(s) = \frac{1}{F(s)}. \tag{8.5}$$

Since $G'(s) = \frac{-F'(s)}{F(s)^2}$, by (6.16), we have

$$G'(s) = \frac{p(1-s)^{p-2}}{s^p} \frac{L_p(s)(-\mathcal{H} L_p(s))}{F(s)^2 (s L_p' - p L_p)^2} \tag{8.6}$$

By Theorem 6.1, $\mathcal{H} L_p > 0$ in $[0, z_{p-1}]$, hence the function $G(s)$ for $1 < p < 2$ exhibits the same behavior in the interval $(0, z_{p-1})$ as F did for $p > 2$. It decreases in $(0, z_p)$, reaches its minimum at z_p and increases back to ∞ at z_{p-1}. And this implies that as in Lemma (6.9), we have

$$-a'(s) > 0 \text{ in } (0, z_{p-1}). \tag{8.7}$$

Thus the analysis for $1 < p < 2$ continues exactly the same as before; the obstacle v_c and the Laguerre function $-a_0 L_p$ $(a_0 = a(z_p))$ touch at their common zero, when the best constant

$$c = \frac{z_p}{1 - z_p}. \tag{8.8}$$

The best majorant satisfying the required quadratic form inequalities is

$$g(s) = \begin{cases} -a_0 L_p(s), & 0 < s \le z_p \\ v_c(s), & z_p < s \le 1. \end{cases} \tag{8.9}$$

8.1. **Sharpness of the constant.** The method used to prove sharpness for $2 < p < \infty$ does not apply directly for $1 < p < 2$. We will instead follow the Wronskian-technique from [BJV]. It is proved in greater generality in that paper, but here we will focus on the Laguerre case alone. If for $c < \frac{z_p}{1-z_p}$ there exists a bounded supersolution $h \ge v_c$ of $\mathcal{L}_p$, then $h(z_p) > 0$. The following lemma proves that such a supersolution cannot exist. As stated in Section 2, the Bellman function theory requires the existence of the Bellman function, so its non-existence implies that the hypothesis $EV_c(X, Y) \le 0$ for all appropriate X, Y, is not true for this constant c. Thus it follows that $\frac{z_p}{1-z_p}$ is the best constant.

Lemma 8.1. *Let h be a supersolution on $[0, z_p]$, that is $sh''(s) + (1-s)h'(s) + ph(s) \le 0$, such that $h(z_p) > 0$. Then $h(0) = -\infty$. In particular, h is not a majorant of v_c for any c.*

Proof. Recall that the Laguerre equation $sf'' + (1-s)f' + pf = 0$ has two independent solutions: a function f_1 that is bounded in $[0,1]$ (and has $f_1(z_p) = 0$) and an unbounded $f_2(x) = -f_1(x)\log|x| + H(x)$ where H is analytic. Assume $f_1(0) = -1$, then $f_2(0) = -\infty$. Consider the Wronskians:

$$W(x) = f_2'(x)f_1(x) - f_1'(x)f_2(x)$$

$$\tilde{W}(x) = h'(x)f_1(x) - f_1'(x)h(x).$$

Since h is a supersolution and f_1, f_2 are solutions, we can write in $(0,1)$

$$\tilde{W}' \le \frac{s-1}{s}\tilde{W},$$

$$W' = \frac{s-1}{s}W.$$

Combining we get

$$\tilde{W}' \le \frac{W'}{W}\tilde{W}. \tag{8.10}$$

Now observe that $W \approx -\infty$ when $x \approx 0$ since $f_1(0) = -1$, $f_2' \approx \frac{1}{x} - f_1'(0)\log x \sim \frac{1}{x}$ and $f_2 \approx \log x$. Since Wronskian preserves sign, $W(x) < 0$ in $(0, z_p]$. Therefore (8.10) can be rewritten (after division by $W < 0$) as

$$(\tilde{W}/W)' \ge 0, \text{ on } (0, z_p]. \tag{8.11}$$

Notice $\tilde{W}(z_p) = -h(z_p)f_1'(z_p) < 0$ and $W(z_p) = -f_2(z_p)f_1'(z_p) < 0$; these follow from $h(z_p) > 0$, $f_1'(z_p) > 0$ and $f_2(z_p) > 0$. (To see why $f_2(z_p) > 0$, observe that by Lemma (6.1) the first root a of f_2 is $< z_p$, so if $f_2(z_p) < 0$ then it has a second root b in (a, z_p) where $f_1(b) < 0$ and $f_2'(b) < 0$. This means $W(b) > 0$, a contradiction.) Denote $\kappa = \frac{h(z_p)}{f_2(z_p)} > 0$. Inequality (8.11) shows that

$$\frac{\tilde{W}(x)}{W(x)} \le \kappa$$

on $[0, z_p]$. Using the negativity of W we get

$$\tilde{W} \ge \kappa W, \text{ on } (0, z_p]. \tag{8.12}$$

Notice that $\left(\frac{h}{f_1}\right)' = \frac{\tilde{W}}{f_1^2}$, $\left(\frac{f_2}{f_1}\right)' = \frac{W}{f_1^2}$. Then (8.12) means

$$\left(\frac{h}{f_1}\right)' \ge \kappa\left(\frac{f_2}{f_1}\right)', \text{ on } (0, z_p].$$

Hence,

$$\frac{h}{f_1} \ge \kappa\frac{f_2}{f_1} + \text{const}, on (0, z_p].$$

Since $f_2 \to -\infty$ as $x \to 0$ and $f_1(0) = -1$, it follows that $\frac{h}{f_1} \to +\infty$. Again, since $f_1(0) = -1$, this means $h(x) \to -\infty$ when $x \to 0$. Lemma is proved. $\square$

8.2. **The general case.** For $1 < p < 2$, the general quadratic form requirement is

$$U_{xx}\|h_1\|^2 + \frac{U_x}{x}\|h_2\|^2 + 2U_{xy}h_1 \cdot k + (U_{yy} + \frac{U_y}{y})\|k\|^2 \leq 0 \tag{8.13}$$

where

$$\|k\|^2 \leq \|h_1\|^2 + \|h_2\|^2. \tag{8.14}$$

Setting $\|k\| = 1$ $a = \|h_1\|$ and $\beta = (\|h_1\|^2 + \|h_2\|^2)^{1/2}$, we require for all

$$\beta \geq 1 \text{ and } a \leq \beta$$

$$-\left(\frac{U_x}{x} - U_{xx}\right)a^2 + 2|U_{xy}|a + \frac{U_x}{x}\beta^2 + \left(U_{yy} + \frac{U_y}{y}\right) \leq 0. \tag{8.15}$$

8.2.1. *The case X is real-valued and $\langle Y_i \rangle \leq \langle X \rangle$.* In this case, $h_2 = 0$, hence (5.16) becomes for $\beta \geq 1$,

$$U_{xx}\beta^2 + 2|U_{xy}|\beta + (U_{yy} + \frac{U_y}{y}) \leq 0. \tag{8.16}$$

When $U = V$, $U_{xx} = -p(p-1)x^{p-2} < 0$, therefore the maximum value occurs for minimal $\beta = 1$. It is shown in the previous section that U satisfies this special case.

Where $U > V$, we have for $U(x,y) = (x+y)^p g(\frac{y}{x+y}) = (x+y)^p g(r)$,

$$\begin{aligned}
U_x &= (x+y)^{p-1}[pg - rg'], && (8.17)\\
U_{xx} &= (x+y)^{p-2}[r^2g'' - 2(p-1)rg' + p(p-1)g], && (8.18)\\
U_{xy} &= (x+y)^{p-2}[-r(1-r)g'' + (p-1)(1-2r)g' + p(p-1)g], && (8.19)\\
U_y &= (x+y)^{p-1}[pg + (1-r)g'], && (8.20)\\
U_{yy} &= (x+y)^{p-2}[(1-r)^2g'' + 2(p-1)(1-r)g' + p(p-1)g]. && (8.21)
\end{aligned}$$

Observe that since $g < 0$, $g' > 0$ and $g'' < 0$ in $(0, z_p)$, it follows that $U_{xx} < 0$ where $U > V$. Therefore the maximum in (8.16) occurs when

$$\beta = \frac{-|U_{xy}|}{U_{xx}},$$

provided this value exceeds 1; otherwise we take $\beta = 1$. The $\beta = 1$ case is the special case of the previous section, so (8.16) is satisfied in this case. What if $\frac{-|U_{xy}|}{U_{xx}} > 1$? Then the maximum value becomes

$$\begin{aligned}
&U_{xx}\left(\frac{-U_{xy}}{U_{xx}}\right)^2 + 2|U_{xy}|\frac{-|U_{xy}|}{U_{xx}} + (U_{yy} + \frac{U_y}{y})\\
= \;&\frac{|U_{xy}|^2}{-U_{xx}} + (U_{yy} + \frac{U_y}{y}) \leq 0.
\end{aligned}$$

U solves $U_{xx} - 2U_{xy} + U_{yy} + \frac{U_y}{y} = 0$, so the above inequality becomes

$$\begin{aligned}
U_{xy}^2 + (2U_{xy} - U_{xx})(-U_{xx}) &= U_{xy}^2 - 2U_{xy}U_{xx} + U_{xx}^2\\
&= (U_{xy} - U_{xx})^2 \leq 0.
\end{aligned}$$

Clearly, this is not true in general. Therefore we must ensure that

$$\frac{-|U_{xy}|}{U_{xx}} \leq 1? \tag{8.22}$$

Since $U_{xy} \sim -\mathcal{H}L_p \leq 0$, this is equivalent to showing $\frac{U_{xy}}{U_{xx}} \leq 1$ or

$$U_{xx} - U_{xy} \leq 0. \tag{8.23}$$

This is equivalent to showing

$$rg'' - (p-1)g' \leq 0.$$

This is true because $g'' < 0$ and $g' > 0$ in $(0, z_p)$. In conclusion, we see that U *always* satisfies (8.16). This completes the case when X is real-valued and $\langle Y_i \rangle \leq \langle X \rangle$.

8.2.2. *The case when X is complex-valued and $\langle Y_i \rangle \leq \langle X \rangle$.* Now we wish to deal with (8.15) in full generality. If $\frac{U_x}{x} - U_{xx} \leq 0$, then the maximum value occurs when $a = \beta$, and we return to the case of (8.16), which is true. So now suppose $\frac{U_x}{x} - U_{xx} > 0$. This can happen only if $U > V$ since $\frac{V_x}{x} - V_{xx} = -p(2-p)x^{p-2} < 0$. So suppose $U > V$. The maximum in $[0, \infty)$ occurs for

$$a_* = \frac{U_{xy}}{\frac{U_x}{x} - U_{xx}}.$$

If $\beta \leq a_*$, then the maximum in $[0, \beta]$ occurs at β and we return to (8.16). If $a_* < \beta$, then the maximum is at $a = a_*$, and we have

$$|U_{xy}| < \beta\left(\frac{U_x}{x} - U_{xx}\right) \Rightarrow U_{xy}^2 + \left(\frac{U_x}{x}\beta^2 + U_{yy} + \frac{U_y}{y}\right)\left(\frac{U_x}{x} - U_{xx}\right) \leq 0. \tag{8.24}$$

We know that $U_x \sim pg - rg' < 0$, so the maximum occurs for the minimal possible β. If $1 \leq a_*$, then $\beta = a_* = a$ and we return to (8.16). If $a_* < 1$, then $\beta = 1$, and (8.24) is equivalent to

$$|U_{xy}| < \left(\frac{U_x}{x} - U_{xx}\right) \Rightarrow U_{xy}^2 + \left(\frac{U_x}{x} + U_{yy} + \frac{U_y}{y}\right)\left(\frac{U_x}{x} - U_{xx}\right) \leq 0. \tag{8.25}$$

However since $U_{yy} + \frac{U_y}{y} = -U_{xx} + 2U_{xy}$, the right-side value is equal to $\left(\frac{U_x}{x} - U_{xx} + U_{xy}\right)^2$ which is ≥ 0. So the only way that (8.25) can be true is if

$$|U_{xy}| \geq \frac{U_x}{x} - U_{xx}, \text{ or } \frac{U_x}{x} - U_{xx} + U_{xy} \leq 0?$$

Lemma 8.2. *Where $U > V$, $\frac{U_x}{x} - U_{xx} + U_{xy} \leq 0$.*

Proof. Using (8.17), (8.18) and (8.19), we can write this condition in terms of g as

$$\begin{aligned} &\frac{pg - rg'}{1-r} + \big(-p(p-1)g + 2(p-1)rg' - r^2 g''\big) \\ &\qquad\qquad + \big(-r(1-r)g'' + (p-1)(1-2r)g' + p(p-1)g\big) \leq 0 \\ \Rightarrow\quad & pg - rg' - r(1-r)g'' + (p-1)(1-r)g' \leq 0. \end{aligned} \tag{8.26}$$

Since

$$U_{xy} \sim -r(1-r)g'' + (p-1)(1-2r)g' + p(p-1)g \leq 0,$$

we have $-r(1-r)g'' \leq -(p-1)(1-2r)g' - p(p-1)g$ and so (8.26) is

$$\begin{aligned} \leq\quad & pg - rg' + \big(-(p-1)(1-2r)g' - p(p-1)g\big) + (p-1)(1-r)g' \\ =\quad & pg - rg' + (p-1)rg' - p(p-1)g = (2-p)[pg - rg']. \end{aligned}$$

Since $g \leq 0$ and $g' > 0$ in $[0, z_p]$, the lemma is proved (and the generalization to the case X complex is complete). $\square$

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