ALGEBRAIC PROPERTIES OF A DISORDERED ASYMMETRIC GLAUBER MODEL

ARVIND AYYER

Abstract. We consider an asymmetric variant of disordered Glauber dynamics of Ising spins on a one-dimensional lattice, where each spin flips according to the relative state of the spin to its left. Moreover, each bond allows for two rates; flips which equalize nearest neighbor spins, and flips which "unequalize" them. In addition, the leftmost spin flips depending on the spin at that site. We explicitly calculate all eigenvalues of the transition matrix for all system sizes and conjecture a formula for the normalization factor of the model. We then analyze two limits of this model, which are analogous to ferromagnetic and antiferromagnetic behavior in the Ising model for which we are able to prove an analogous formula for the normalization factor.

1. Introduction

The Ising model with Glauber dynamics [1] has been an extremely important source for understanding time dependent behavior in the Ising model. Moreover, the simplicity of the model has inspired the creation of a number of related models which have been amenable to rigorous combinatorial and probabilistic techniques. Most of the studies, by far, have been on the probabilistic side, with systems being studied on the continuum, on infinite lattices in arbitrary dimension, or considering asymptotics of large finite lattices. The literature on Glauber dynamics in the Ising model is exceedingly large. When bonds between neighboring sites have random strengths, there seem to be a few studies for large or infinite systems, eg. [2, 3, 4, 5]. We know of very few combinatorial results, eg [6, 7] and they correspond to situations where the bond strengths are not arbitrary. In what follows, we show that a model with Glauber-like dynamics exhibits rich algebraic and combinatorial structure.

We consider a simplified one-dimensional version of the Ising model with spin-flip dynamics, on sites labelled 1 to L, and where there are two kinds of spins, which we label as occupation numbers $\{0, 1\}$. The

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transition rules are given by an asymmetric version of the usual rule. Each site i looks only to the site to its left, $i-1$, and the ith site switches its state with a rate that depends only on whether both sites are in the the same state or not. If the state of both sites is different, the transition occurs with rate α_i (Glauber) and if they are the same, it occurs with rate β_i (anti-Glauber). In other words, α_i's try to homogenize the system and β_i's try to make nearest neighbors opposite. In addition there are boundary interactions at the first site, which flips with rate α_1 if it contains a particle and β_1 if it does not.

This model can be thought of as a voter model [8] within a hierarchical demographic, that is to say individual $i-1$ influences the opinion of individual i, but not vice versa. The rate α_i governs the tendency of $i-1$ being a usual campaigner, whereas β_i governs the tendency of $i-1$ being a double agent.

This model can also be thought of as a variant of the East model studied in [9, 7] on the one dimensional lattice. In the east model, a spin was allowed to flip only if the site to its left was occupied and the rate of the flip depended on whether the site was occupied or not.

Crisanti and Sompolinsky [5] considered a mean-field model of Glauber dynamics on Ising spins with asymmetric bonds with random strengths to understand properties of neural networks. This work is not related to it, but the idea is similar. In their work, asymmetry refers to partial asymmetry, instead of the total asymmetry assumed here.

In an earlier work, the steady state of the border process for this model with $\beta_i = 0$ and $\alpha_i = 1$ for $i > 0$, the asymmetric annihilation process, was obtained [10] by using a transfer matrix ansatz. The spectrum of the transition matrix for α_i unequal was calculated in [11]. This model, even with $\beta_i = 0$ is a clear generalization because one can distinguish here between particles and holes, whereas the former could not. In addition, there is no analog of the rate for β_i in the earlier works because that would have amounted to creation of particles in the bulk.

We will write down the precise statement of the Markov chain in Section 2. The main result of the paper is an explicit formula for the eigenvalues of the Markov chain for any size L, which we prove in Section 3. This leads, in an obvious way, to a formula for the spectral gap of the system.

We will show that the transition matrix of the model exhibits a simple recurrence relation allowing one to express the matrix for a system of size L in terms of that of size $L-1$ in Section 4. We prove a formula for the density in this disordered model in Section 5. We conjecture a formula for the normalization factor for the distribution, informally called the "partition function" in Section 6. Lastly, we consider two special cases, which we naturally label the ferromagnetic and the anti-ferromagnetic limits in Section 7.

2. The Model

We consider a nonequilibrium system on a finite lattice with L sites labelled from 1 to L. Each sites is occupied by one of two spins 1 and 0. The evolution rule in the bulk, for site i depending on the spin at site $i-1$, is given by

$$\begin{aligned} 10 \to 11 \;\text{ and } 01 \to 00 \text{ with rate } \alpha_i\,, \\ 11 \to 10 \;\text{ and } 00 \to 01 \text{ with rate } \beta_i\,. \end{aligned} \tag{1}$$

The evolution of the first site is given by

$$\begin{aligned} 1 \to 0 \text{ with rate } \alpha_1, \\ 0 \to 1 \text{ with rate } \beta_1. \end{aligned} \tag{2}$$

The rule at the left boundary (2) is constructed by supposing that there is a virtual site labelled 0 to the left of the first site which is always empty. The left boundary conditions are deduced from the bulk rules (1) by looking at the second component of the bond.

Remark 1. *There are two symmetries of the model; the first is manifested by interchanging all α_i's and β_i's as well as flipping spins at all odd sites, and the second, by leaving α_1 and β_1 as they are, interchanging all other α_i's and β_i's and flipping spins at all even sites.*

Remark 2. *This model has the property that correlation functions $\langle \eta_{i_1} \dots \eta_{i_n} \rangle$ does not depend on the state of η_{i_n+1} simply because site i_n+1 cannot influence any site less than or equal to i_n Further, as is well-known for Glauber dynamics, correlation functions of n sites like the one above depend only on those of sites less than or equal to n.*

3. Spectrum of the transition matrices

We observe that the characteristic polynomial of the transition matrices factorize into linear factors and has an explicit formula. This is reminiscent of the formula for the asymmetric annihilation process conjectured in [10] and proved in [11]. Let $\mathbb{B}_L$ be the set of binary vectors of size L, that is vectors of length L whose elements are either 0 or 1.

Theorem 1. *The characteristic polynomial of M_L is given by*

$$|M_L - \lambda \mathbb{1}_L| = \prod_{b \in \mathbb{B}_L} \left(\lambda + \sum_{i=1}^{L} b_i(\alpha_i + \beta_i) \right) \tag{3}$$

For example, when $L = 2$, we have

$$|M_2 - \lambda \mathbb{1}_2| = \lambda \, (\lambda + \alpha_1 + \beta_1)\,(\lambda + \alpha_2 + \beta_2)\,(\lambda + \alpha_1 + \alpha_2 + \beta_1 + \beta_2) \tag{4}$$

The proof of Theorem 1 will turn out to be a simpler version of the proof of the eigenvalues in the asymmetric annihilation process [11]. We will first define a slight rearrangement of a Hadamard matrix. Let $\mathbb{B}_L$ be ordered lexicographically. For $b, c \in \mathbb{B}_L$, define the square matrix H_L of size 2^L by

$$H_L = \frac{1}{2^{L/2}} \left((-1)^{\hat{b} \cdot c} \right)_{b,c \in \mathbb{B}_L}, \tag{5}$$

where $\hat{b}$ is the reverse of b. Note that H_L is symmetric because

$$(H_L)_{b,c} = (-1)^{\hat{b} \cdot c} = (-1)^{b \cdot \hat{c}} = (H_L)_{c,b}. \tag{6}$$

This is different from the usual definition in which elements of H_L are written as $(-1)^{b \cdot c}$. For example, the matrix for $L = 3$ is given by

$$H_3 = \frac{1}{2\sqrt{2}} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{pmatrix}. \tag{7}$$

Lemma 2. $H_L^2 = \mathbb{1}_L$.

Proof. We first look at the diagonal terms in H_L^2.

$$(H_L^2)_{b,b} = \sum_{c \in \mathbb{B}_L} (H_L)_{b,c} (H_L)_{c,b} = \frac{1}{2^L} \sum_{c \in \mathbb{B}_L} (-1)^{2\hat{b} \cdot c} = 1. \tag{8}$$

Now suppose $b \neq d$. Then

$$(H_L^2)_{b,d} = \sum_{c \in \mathbb{B}_L} (H_L)_{b,c} (H_L)_{c,d} = \frac{1}{2^L} \sum_{c \in \mathbb{B}_L} (-1)^{(b+d) \cdot \hat{c}}, \tag{9}$$

where the addition of binary vectors is the usual xor-addition: $x+(1-x)=1$ by definition and $x+x=0$ if x is a bit.

Now, for any fixed b,d, we will construct an sign-reversing involution on $\mathbb{B}_L$. Since $b \neq d$, $b+d$ contains at least one entry equal to 1. Consider the leftmost such entry, at position i, say. To any element c, associate another element

$$c \mapsto c' = (c_1,\dots,c_{L-i},1-c_{L-i+1},c_{L-i+2},\dots,c_L).$$

Clearly, this is an involution and moreover,

$$(-1)^{(b+d)\cdot\hat{c}} + (-1)^{(b+d)\cdot\hat{c}'} = 0.$$

We have therefore partitioned $\mathbb{B}_L$, for any fixed $b,d,b\neq d$, such that the number of terms which contribute +1 to the sum is exactly the same as that which contribute -1. Hence $(H_L^2)_{b,d}=0$. □

We now write down the transition matrix M_L in this language. We use the convention that $(M_L)_{b,c}$ represents the transition rate from $c \to b$, with the end result that $M_L|v_L\rangle = 0$ for the steady state *column* vector $|v_L\rangle$.

$$(M_L)_{b,c} = \begin{cases} \alpha_i, & b+c = 0\dots0\underbrace{1}_{i}0\dots0 \\ & \text{and } c_{i-1}+c_i=1, \\ \beta_i, & b+c = 0\dots0\underbrace{1}_{i}0\dots0 \\ & \text{and } c_{i-1}=c_i, \\ -\displaystyle\sum_{\substack{i=1\\ c_{i-1}+c_i=1}}^{L}\alpha_i - \sum_{\substack{i=1\\ c_{i-1}=c_i}}^{L}\beta_i, & b=c, \\ 0, & \text{otherwise}, \end{cases} \tag{10}$$

Lemma 3. *$H_L M_L H_L$ is lower-triangular.*

Proof. We will show the result by explicitly demonstrating the structure of non-zero terms. By definition,

$$(H_L M_L H_L)_{c,d} = \frac{1}{2^L}\sum_{e,f\in\mathbb{B}_L}(-1)^{c\cdot\hat{e}+f\cdot\hat{d}}(M_L)_{e,f}. \tag{11}$$

We will divide the proof into three parts. We first consider diagonal elements, then certain off-diagonal elements which are non-zero, and finally show that all other off-diagonal elements are zero by introducing another sign-reversing involution just as in Lemma 2.

- $c = d$.

  Then $c \cdot \hat{e} + f \cdot \hat{d} = c \cdot (\hat{e} + \hat{f})$. Suppose e differs from f only at position i, then $(M_L)_{e,f}$ contributes either α_i of β_i and the sign is always the same: $+1$ if $\hat{c}_i = 0$ and -1 otherwise. On the other hand, if $e = f$, this sum is identically 0 and therefore we get a contribution of $-\alpha_i$ if $f_{i-1} + f_i = 1$ and $-\beta_i$ if $f_{i-1} = f_i$ from (10).

  To summarize, we get 2^{L-1} terms contributing $\pm\alpha_i, \pm\beta_i$, all of the same sign, from summands $e \neq f$ and 2^{L-1} terms contributing $-\alpha_i, -\beta_i$ from summands $e = f$. These will clearly cancel if $\hat{c}_i = 0$ and will give $2^{L-1}(-2\alpha_i - 2\beta_i)$ if $\hat{c}_i = 1$. Thus,

$$(H_L M_L H_L)_{c,c} = -\sum_{i=1}^{L} \hat{c}_i(\alpha_i + \beta_i). \tag{12}$$

- $c \neq d$

  For now, we look at conditions on c, d such that $(M_L)_{e,f} = \alpha_i$ and $(-1)^{c\cdot\hat{e}+f\cdot\hat{d}}$ always has the same sign. This happens when e differs from f only at position i and $f_{i-1} + f_i = 1 \Rightarrow e_{i-1} = e_i$. Now,

$$\begin{aligned} c \cdot \hat{e} + f \cdot \hat{d} &= e \cdot \hat{c} + f \cdot \hat{d}, \\ &= e \cdot (\hat{c} + \hat{d}) + (e + f) \cdot \hat{d}, \\ &= f \cdot (\hat{c} + \hat{d}) + (e + f) \cdot \hat{c}. \end{aligned} \tag{13}$$

  Since $e + f$ is fixed, only the former term contributes to the change of sign. As we vary e, this term does not change sign if and only if $(\hat{c} + \hat{d})_{i-1} = (\hat{c} + \hat{d})_i$ and $c_j = d_j$ for all other values of j. Since we are assuming $c \neq d$, the only way this can happen is if $\hat{c} + \hat{d} = 0 \dots 0110 \dots 0$ with 1s in the $i - 1$th and ith position. If this is the case, the sign of all these terms is determined just by $\hat{d}_i$, or equivalently, $\hat{c}_i$. This gives us a total contribution of $2^{L-1}\alpha_i(-1)^{\hat{d}_i}$.

  We also have contributions of α_i from diagonal terms $e = f$ and $e_{i-1} + e_i = 1$ from (10). In that case, $c \cdot \hat{e} + f \cdot \hat{d} = e.(\hat{c} + \hat{d})$. Since we just argued that $(\hat{c} + \hat{d})_{i-1} = (\hat{c} + \hat{d})_i$, diagonal terms always contribute a term of $(-1)^1(-\alpha_i)$, giving us a grand total of $2^{L-1}\alpha_i$.

  Therefore, diagonal contributions will cancel off-diagonal ones if $\hat{d}_i = 1, \hat{c}_i = 0$ and will add to them if $\hat{d}_i = 0, \hat{c}_i = 1$.

  By a very similar reasoning, we will get a contribution of $-\beta_i$ under exactly the same circumstances. The relation $(\hat{c}+\hat{d})_{i-1} =$

$(\hat{c}+\hat{d})_i$ remains the same, but the roles are of c and d are reversed, giving us an overall negative sign.

Assuming all other off-diagonal terms are zero,

$$(H_L M_L H_L)_{c,d} = \alpha_i - \beta_i$$

if and only if c and d differ only in the $i-1$th and ith position and moreover $d_i = 0$ and $c_i = 1$. Therefore $d < c$ in the lexicographic order and the matrix is lower triangular.

- Involution.

  To complete the proof, we have to show that when c and d do not satisfy $\hat{c}+\hat{d} = 0\dots0110\dots0$ with 1s in the $i-1$th and ith position, $(H_L M_L H_L)_{c,d} = 0$. To do this, we introduce another sign-reversing involution just as in Lemma 2. For a fixed c, d and any pair (e, f) which gives a contribution of α to $(H_L M_L H_L)_{c,d}$, say, we find another pair (e', f') which also give the same contribution but such that $c\cdot\hat{e}+f\cdot\hat{d} = c\cdot\hat{e}'+f'\cdot\hat{d}+1$. Notice that (e, f) satisfies the condition that $\hat{e}$ and $\hat{f}$ differ only in the ith position and that $f_{i-1}+f_i = 1$.

  This is easily done. Since $c \neq d$, find the smallest value of j, $j \neq i, i-1$ such that $\hat{c}_j+\hat{d}_j = 1$. Then choose (e', f') as

$$\begin{aligned} e \mapsto & e' = (e_1, \dots, e_{L-j}, 1-e_{L-j+1}, e_{L-j+2}, \dots, e_L), \\ f \mapsto & f' = (f_1, \dots, f_{L-j}, 1-f_{L-j+1}, f_{L-j+2}, \dots, f_L). \end{aligned} \tag{14}$$

  Then $e+f = e'+f'$ and $e.(\hat{c}+\hat{d}) = e'.(\hat{c}+\hat{d}) \pm 1$ and we are done. Notice that we need $j \neq i-1, i$ to ensure that $e_{i-1} = e_i$ remains intact. If $\hat{c}$ differs from $\hat{d}$ only at these two positions, then we already proved that there can be no such involution.

  □

We have now done all the work necessary to write down a proof of the main result.

Proof of Theorem 1: Since H_L is symmetric and $H_L^2 = 1$ by Lemma 2, $H_L M_L H_L$ has the same characteristic polynomial as M_L. But the former is lower triangular. The characteristic polynomial is therefore

$$|M_L - \lambda \mathbb{1}_L| = \prod_{i=1}^{2^L} (-\lambda - (H_L M_L H_L)_{i,i}). \tag{15}$$

Using the representation of the rows and columns using boolean vectors of size L and using (12), we have the required result. ■

Corollary 4. *The spectral gap for the system of size L is given by*

$$\min_{i=1}^{L}(\alpha_i + \beta_i). \tag{16}$$

Proof. The proof directly follows from the explicit construction of eigenvalues in Theorem 1. □

4. Recurrence Relation for the Transition Matrices

We will prove that there is a recursion of order one among the transition matrices. Let $M_L(\alpha_1, \dots, \alpha_L; \beta_1, \dots, \beta_L)$ represent the transition matrix for the system of size L with parameters α_i and β_i.

Theorem 5. *Let $\mathbb{1}$ denote the identity matrix of size 2^{L-1}. Then M_L can be expressed in 2×2 block-diagonal form as*

$$M_L(\alpha_1, \dots, \alpha_L; \beta_1, \dots, \beta_L) = \left(\begin{array}{c|c} M_{1,1} & \alpha_1 \mathbb{1} \\ \hline \beta_1 \mathbb{1} & M_{2,2} \end{array}\right), \tag{17}$$

where

$$\begin{aligned} M_{1,1} &= M_{L-1}(\alpha_2, \alpha_3, \dots, \alpha_L; \beta_2, \beta_3, \dots, \beta_L) - \beta_1 \mathbb{1}, \\ M_{2,2} &= M_{L-1}(\beta_2, \alpha_3, \dots, \alpha_L; \alpha_2, \beta_3, \dots, \beta_L) - \alpha_1 \mathbb{1}, \end{aligned} \tag{18}$$

with the initial matrix for $L = 1$ given by

$$M_1(\alpha_1; \beta_1) = \begin{pmatrix} -\beta_1 & \alpha_1 \\ \beta_1 & -\alpha_1 \end{pmatrix}. \tag{19}$$

Proof. The matrix for size 1 given in (19) is easy to check. The matrix recursion (17) is proved by looking at transitions that binary vectors beginning with different two-bit vectors undergo. Let v, v' denote binary vectors of length L and w, w' denote vectors of length $L-1$. There are then four possibilities of transitions depending on the first bits of v and v', namely $v = 0w$ or $v = 1w$ and similarly for v'.

(1) $0w \to 1w'$ (resp. $1w \to 0w'$):
These transitions are only possible if $w = w'$ and then occurs with rate β_1 (resp. α_1). This is why the $(2,1)-$th (resp. $(1,2)-$th) block of M_L is $\beta_1 \mathbb{1}$ (resp. $\alpha_1 \mathbb{1}$).

(2) $0w \to 0w'$ (resp. $1w \to 1w'$):
In this case, all the transitions for the system of size $L - 1$ go through but with rates whose indices are one more than that of the smaller system simply because one extra site has been added to the left. For example, what occurred with rate α_2 in the system of size $L-1$ occurs with rate α_3 now. Because of the extra transitions in the off-diagonal blocks (described above), we must add negative terms to the diagonal so that column

sums are zero. Therefore we subtract $\beta_1 \mathbb{1}$ (resp. $\alpha_1 \mathbb{1}$) from $M_{1,1}$ (resp. $M_{2,2}$).

□

For later consideration, we also express the recursion relation using a second order recurrence relation. Expressing the transition matrix for the system of size $L-1$ as

$$M_L(\alpha_1, \dots, \alpha_L; \beta_1, \dots, \beta_L) =$$

$$\left(\begin{array}{c|c|c|c} M_{1,1}- & \alpha_2 \mathbb{1} & \alpha_1 \mathbb{1} & 0 \\ (\beta_1+\beta_2)\mathbb{1} & & & \\ \hline \beta_2 \mathbb{1} & M_{2,2}- & 0 & \alpha_1 \mathbb{1} \\ & (\beta_1+\alpha_2)\mathbb{1} & & \\ \hline \beta_1 \mathbb{1} & 0 & M_{1,1}- & \beta_2 \mathbb{1} \\ & & (\alpha_1+\alpha_2)\mathbb{1} & \\ \hline 0 & \beta_1 \mathbb{1} & \alpha_2 \mathbb{1} & M_{2,2}- \\ & & & (\alpha_1+\beta_2)\mathbb{1} \end{array}\right), \tag{20}$$

where

$$\begin{aligned} M_{1,1} &= M_{L-2}(\alpha_3, \alpha_4, \dots, \alpha_L; \beta_3, \beta_4, \dots, \beta_L), \\ M_{2,2} &= M_{L-2}(\beta_3, \alpha_4, \dots, \alpha_L; \alpha_3, \beta_4, \dots, \beta_L). \end{aligned} \tag{21}$$

Remark 3. *Notice that the transition matrix M_L is invariant under the transformation $M_L \leftrightarrow (M_L)^T$ and $\alpha \leftrightarrow \beta$.*

5. Density in the Steady State

We compute the density of 1's in the steady state at site k in a system of size larger than k. Using Remark 2, this density is completely independent of the size of the system.

Let S_k be the set of subsets of $[k] = \{1, \dots, k\}$ with an odd number of elements, and for $s \in S_k$, let $\bar{s} = S_k \setminus s$.

Lemma 6. *The density of 1's at site k is given by*

$$\langle \eta_k \rangle = \frac{\displaystyle\sum_{s \in S_k} \prod_{i \in s} \beta_i \prod_{j \in \bar{s}} \alpha_j}{\displaystyle\prod_{j=1}^{k} (\alpha_j + \beta_j)}. \tag{22}$$

Proof. The master equation for the density of 1's at site 1 is

$$\frac{d}{dt} \langle \eta_1 \rangle = \beta_1 \langle 1 - \eta_1 \rangle - \alpha_1 \langle \eta_1 \rangle = 0, \tag{23}$$

which implies $\eta_1 = \frac{\beta_1}{(\alpha_1+\beta_1)}$, which is consistent with the (22) with $k=1$.

For $k > 1$, the master equation is

$$\frac{d}{dt}\langle \eta_k \rangle = \beta_k \langle (1-\eta_{k-1})(1-\eta_k) \rangle + \alpha_k \langle \eta_{k-1}(1-\eta_k) \rangle \\ - \alpha_k \langle (1-\eta_{k-1})\eta_k \rangle - \beta_k \langle \eta_{k-1}\eta_k \rangle = 0, \tag{24}$$

which on simplifying gives

$$\langle \eta_k \rangle = \frac{\alpha_k \langle \eta_{k-1} \rangle + \beta_k \langle 1-\eta_{k-1} \rangle}{\alpha_k + \beta_k}. \tag{25}$$

One can check that (22) satisfies this recurrence. □

6. The Normalization Factor

Consider the system of size L. For any configuration η, we can express its steady state probability as a numerator divided by a denominator. We define the normalization factor $Z_L(\vec{\alpha}, \vec{\beta})$ as the least common multiple of the denominators of the steady state probabilities for all configurations η such that the greatest common divisor of the corresponding numerators is 1.

We have been able to find a conjecture for Z_L but have not been able to prove the formula in general. In certain special cases discussed in Section 7, the following conjecture does reduce correctly.

Conjecture 1. *The normalization factor for a system of size L with rates α_i and β_i is given by*

$$Z_L(\vec{\alpha}, \vec{\beta}) = \prod_{i=1}^{L} (\alpha_i + \beta_i) \prod_{1 \le i < j \le L} (\alpha_i + \beta_i + \alpha_j + \beta_j). \tag{26}$$

7. Special Cases: Ferromagnetic and Antiferromagnetic models

We now analyze two limits of this model which depend on two parameters, $\alpha_1 = \alpha$ and $\beta_1 = \beta$. Both these limits can be interpreted as nonequilibrium analogs of the ferromagnetic and antiferromagnetic Ising models respectively, hence the terminology.

The strategy for the analysis of both cases is similar. We will first use Theorem 5 to write a first order recurrence for the transition matrices. We will then show that the transfer matrix ansatz [10] holds, using which we will express the steady state probabilities of the system of size L in terms of the system of size $L-1$.

We first recall the definition of the transfer matrix ansatz. A family of Markov processes satisfies the transfer matrix ansatz if there exist matrices T_L for all sizes L such that

$$M_{L+1}T_L = T_L M_L\,. \tag{27}$$

We also impose that this equality is nontrivial in the sense that

$$M_{L+1}T_L \neq 0. \tag{28}$$

The rectangular transfer matrices or conjugation matrices T_L can be interpreted as a consequence of the integrability of the model.

For the system we consider here, the transition matrices are of size 2^L. It is most convenient for us to take the naturally ordered basis of binary sequences of size L. For example, when $L = 2$, the ordered list is $(00, 01, 10, 11)$.

Using the transfer matrices, we will calculate the normalization factor Z_L for both these models. Letting $|v_1\rangle$ be the Perron-Frobenius eigenvector of the system of size one, we can define

$$Z_L = \langle 1_L | T_{L-1} T_{L-2} \cdots T_1 | v_1 \rangle, \tag{29}$$

where $\langle 1_L|$ is the row vector of size 2^L composed of 1's.

For these special cases, we will explicitly calculate the density, the recurrences for the transition matrix and the transfer matrix, as well as the formula for the normalization factor. We will omit all the proofs since they are quite straightforward, if somewhat tedious. The recurrences for the transition matrix follow from the recurrence (20) and the others can be provided without too much difficulty using induction. The technique of proof is the same as was used in [10].

7.1. **Ferromagnetic limit.** We consider the case $\alpha_i = 1$ and $\beta_i = 0$ for $i > 1$. The densities are easily calculated. The only term that contributes to the sum in (22) is the subset $\{1\} \in S_k$. Therefore,

$$\langle \eta_k \rangle = \frac{\beta}{\alpha + \beta}, \tag{30}$$

for all k. Therefore all densities are identical, which is reminiscent of the ferromagnetic limit where all spins prefer to be aligned in the same direction.

Corollary 7. *Let $\mathbb{1}$ denote the identity matrix of size 2^{L-2}. Suppose the transition matrix for size $L-1$ is written in block-diagonal form as*

$$M^{(F)}_{L-1} = \left(\begin{array}{c|c} M^{(F)}_{11} & \alpha \mathbb{1} \\ \hline \beta \mathbb{1} & M^{(F)}_{22} \end{array}\right). \tag{31}$$

Then

$$M_L^{(F)} = \left(\begin{array}{c|c|c|c} M_{11}^{(F)} & \mathbb{1} & \alpha\mathbb{1} & 0 \\ \hline 0 & \begin{array}{c} M_{22}^{(F)} - \\ (1-\alpha+\beta)\mathbb{1} \end{array} & 0 & \alpha\mathbb{1} \\ \hline \beta\mathbb{1} & 0 & \begin{array}{c} M_{11}^{(F)} - \\ (1-\beta+\alpha)\mathbb{1} \end{array} & 0 \\ \hline 0 & \beta\mathbb{1} & \mathbb{1} & M_{22}^{(F)} \end{array}\right), \tag{32}$$

where $M_L^{(F)}$ *is written as a* 2×2 *block matrix with each block made up of matrices of size* 2^{L-1}. *The initial matrix for* $L = 1$ *is given by*

$$M_1^{(F)} = \begin{pmatrix} -\beta & \alpha \\ \beta & -\alpha \end{pmatrix}. \tag{33}$$

Let σ_L denote the matrix of size 2^L with 1's on the antidiagonal and zeros everywhere else. For example,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{34}$$

Lemma 8. *Define rectangular matrices* $T_L^{(F)}$ *of size* $2^{L+1} \times 2^L$ *using the following recurrence relation.*

$$T_1^{(F)} = \begin{pmatrix} 1-\beta+\alpha & \alpha \\ 0 & \alpha \\ \beta & 0 \\ \beta & 1-\alpha+\beta \end{pmatrix}, \tag{35}$$

and define $T_L^{(F)}$ *by the following recursion. If*

$$T_{L-1}^{(F)} = \left(\begin{array}{c|c} T_{11}^{(F)} & T_{12}^{(F)} \\ \hline 0 & T_{22}^{(F)} \\ \hline T_{31}^{(F)} & 0 \\ \hline T_{41}^{(F)} & T_{42}^{(F)} \end{array}\right). \tag{36}$$

then

$$T_L^{(F)} = \left(\begin{array}{c|c|c|c} 2T_{11}^{(F)} & T_{11}^{(F)} & 2T_{12}^{(F)} & T_{12}^{(F)} \\ \hline 0 & \sigma_{L-2}T_{11}^{(F)}\sigma_{L-2} & 0 & T_{22}^{(F)} \\ \hline 0 & 0 & T_{12}^{(F)} & 0 \\ \hline 0 & 0 & T_{22}^{(F)} & 2T_{22}^{(F)} \\ \hline 2T_{31}^{(F)} & T_{31}^{(F)} & 0 & 0 \\ \hline 0 & T_{41}^{(F)} & 0 & 0 \\ \hline T_{31}^{(F)} & 0 & \sigma_{L-2}T_{42}^{(F)}\sigma_{L-2} & 0 \\ \hline T_{41}^{(F)} & 2T_{41}^{(F)} & T_{42}^{(F)} & 2T_{42}^{(F)} \end{array}\right). \tag{37}$$

Then the family of Markov processes given by the transition matrices $\{M_L^{(F)}\}$ *satisfy the transfer matrix ansatz with* $T_L^{(F)}$ *defined above, ie* $M_L^{(F)}T_{L-1}^{(F)} = T_{L-1}^{(F)}M_{L-1}^{(F)}$.

Using Lemma 8, we can compute the normalization factor $Z_L^{(F)}(\alpha,\beta)$ using (29) by setting

$$|v_1\rangle = \begin{pmatrix} \beta \\ \alpha \end{pmatrix}. \tag{38}$$

For example, $Z_1^{(F)} = \alpha+\beta$. We find a remarkable property of the partition function of the system, namely its super-extensive growth with the size of the system.

Corollary 9. *The partition function of the system of size* L *is given by*

$$Z_L^{(F)} = 2^{\binom{L-1}{2}}(\alpha+\beta)(1+\alpha+\beta)^{L-1}. \tag{39}$$

7.2. **Antiferromagnetic limit.** We consider the case $\alpha_i = 0$ and $\beta_i = 1$ for $i > 1$. Now the only terms that can contribute to the sum in (22) are the subset $\bar{s} = \phi, \{1\} \in S_k$. Therefore, the density depends on the parity of the site. If k is odd (resp. even), $s = [k]$ (resp. $s = \{2,\dots,k\}$) is the only contribution,

$$\langle \eta_k \rangle = \begin{cases} \frac{\beta}{\alpha+\beta}, & k \text{ odd}, \\ \frac{\alpha}{\alpha+\beta}, & k \text{ even}. \end{cases} \tag{40}$$

This is reminiscent of the antiferromagnetic Ising model in which neighboring sites prefer to be aligned opposite to one another.

Corollary 10. *Let $\mathbb{1}$ denote the identity matrix of size 2^{L-2}. Suppose the transition matrix for size $L-1$ is written in block-diagonal form as*

$$M_{L-1}^{(A)} = \left(\begin{array}{c|c} M_{11}^{(A)} & \alpha\mathbb{1} \\ \hline \beta\mathbb{1} & M_{22}^{(A)} \end{array}\right). \tag{41}$$

Then

$$M_L^{(A)} = \left(\begin{array}{c|c|c|c} M_{11}^{(A)} - \mathbb{1} & 0 & \alpha\mathbb{1} & 0 \\ \hline \mathbb{1} & \begin{array}{c} M_{22}^{(A)} - \\ (\alpha-\beta)\mathbb{1} \end{array} & 0 & \alpha\mathbb{1} \\ \hline \beta\mathbb{1} & 0 & \begin{array}{c} M_{11}^{(A)} - \\ (\beta-\alpha)\mathbb{1} \end{array} & \mathbb{1} \\ \hline 0 & \beta\mathbb{1} & 0 & M_{22}^{(A)} - \mathbb{1} \end{array}\right), \tag{42}$$

where $M_L^{(A)}$ is written as a 2×2 block matrix with each block made up of matrices of size 2^{L-1}. The initial matrix for $L=1$ is given by

$$M_1^{(A)} = \begin{pmatrix} -\beta & \alpha \\ \beta & -\alpha \end{pmatrix}. \tag{43}$$

Lemma 11. *Define rectangular matrices $T_L^{(A)}$ of size $2^{L+1}\times 2^L$ using the following recurrence relation.*

$$T_1^{(A)} = \begin{pmatrix} 0 & \alpha \\ 1-\beta+\alpha & \alpha \\ \beta & 1-\alpha+\beta \\ \beta & 0 \end{pmatrix}, \tag{44}$$

and if

$$T_{L-1}^{(A)} = \left(\begin{array}{c|c} 0 & T_{12}^{(A)} \\ \hline T_{21}^{(A)} & T_{22}^{(A)} \\ \hline T_{31}^{(A)} & T_{32}^{(A)} \\ \hline T_{41}^{(A)} & 0 \end{array}\right). \tag{45}$$

then

$$T_L^{(A)} = \left(\begin{array}{c|c|c|c} 0 & 0 & 0 & T_{12}^{(A)} \\ \hline 0 & 0 & 2T_{22}^{(A)} & T_{22}^{(A)} \\ \hline \sigma T_{21}^{(A)}\sigma & 2\sigma T_{21}^{(A)}\sigma & T_{12}^{(A)} & 2T_{12}^{(A)} \\ \hline T_{21}^{(A)} & 0 & T_{22}^{(A)} & 0 \\ \hline 0 & T_{31}^{(A)} & 0 & T_{32}^{(A)} \\ \hline 2T_{41}^{(A)} & T_{41}^{(A)} & 2\sigma T_{32}^{(A)}\sigma & \sigma T_{32}^{(A)}\sigma \\ \hline T_{31}^{(A)} & 2T_{31}^{(A)} & 0 & 0 \\ \hline T_{41}^{(A)} & 0 & 0 & 0 \end{array}\right). \tag{46}$$

Then the family of Markov processes given by the transition matrices $\{M_L^{(A)}\}$ *satisfy the transfer matrix ansatz with* $T_L^{(A)}$ *defined above, ie* $M_L^{(A)} T_{L-1}^{(A)} = T_{L-1}^{(A)} M_{L-1}^{(A)}$.

Using Lemma 8, we can compute the normalization factor $Z_L^{(A)}(\alpha, \beta)$ using (29) by using the same formula for $|v_1\rangle$ as in the ferromagnetic case, (38). We find that the normalization factor is exactly the same as for the ferromagnetic model.

Corollary 12. *The partition function of the system of size L is given by*

$$Z_L^{(A)} = 2^{\binom{L-1}{2}} (\alpha + \beta)(1 + \alpha + \beta)^{L-1}. \tag{47}$$

7.3. **Relation between these special cases.** The transition matrix $M_L^{(X)}$ for $X = F, A$ is invariant, in addition to the transposition symmetry, under the transformation

$$(M_L^{(X)})_{i,j} \leftrightarrow (M_L^{(X)})_{2^L+1-i, 2^L+1-j} \text{ and } \alpha \leftrightarrow \beta \tag{48}$$

in the usual matrix coordinate notation where i, j vary from 1 to 2^L. Similarly, the transfer matrix $T_L^{(X)}$ is invariant under the transformation

$$(T_L^{(X)})_{i,j} \leftrightarrow (T_L^{(X)})_{2^{L+1}+1-i, 2^L+1-j} \text{ and } \alpha \leftrightarrow \beta \tag{49}$$

where this time i varies from 1 to 2^{L+1} and j varies from 1 to 2^L. The normalization factors for the two systems are exactly the same, $Z_L^{(F)} = Z_L^{(A)}$. One also sees that the recurrence relations for the transition matrices and the transfer matrices are similar.

The reason that there are so many similarities between the two models is that one can establish an exact correspondence between the ferromagnetic and antiferromagnetic limits defined previously using Remark 1. In this case, we use the second part of the remark. As a consequence of this coupling, time-dependent correlation functions in the ferromagnetic model are related to those in the antiferromagnetic model.

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E-mail address: ayyer@math.ucdavis.edu

Institut de Physique Théorique, C. E. A. Saclay, 91191 Gif-sur-Yvette Cedex, France

Department of Mathematics, Mathematical Sciences Building, One Shields Ave, University of California, Davis CA 95616