GLOBAL GENERATION OF THE DIRECT IMAGES OF RELATIVE PLURICANONICAL SYSTEMS

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Abstract

In this article, using the plurisubharmonic variation property of canonical measures (cf. [T7]), we prove that for an algebraic fiber space $f : X \longrightarrow Y$, $f_*\mathcal{O}_X(mK_{X/Y})$ is globally generated on the complement of the discriminant locus of f for every sufficiently large and divisible m. As a byproduct, we prove Iitaka's conjecture on the subadditivity of Kodaira dimensions. MSC: 53C25(32G07 53C55 58E11)

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1 Introduction

Let $f : X \longrightarrow Y$ be a surjective projective morphism between smooth projective varieties with connected fibers. In this paper we shall call such a fiber space an **algebraic fiber space** for simplicity. We set $K_{X/Y} := K_X \otimes f^*K_Y^{-1}$ and call it the relative canonical line bundle of $f : X \to Y$.

Let $f : X \to Y$ be an algebraic fiber space. It is well known that the direct image $f_*\mathcal{O}_X(mK_{X/Y})$ is locally free outside of the discriminant locus (cf. [S2, T5, T9]) and is semipositive for every $m \geqq 1$ in certain algebraic senses (cf. Theorems 1.1 and 1.6 below). But Theorems 1.1 and 1.6 do not imply the existence of nontrivial global sections of $f_*\mathcal{O}_X(mK_{X/Y})$.

The purpose of this article is to prove that $f_*\mathcal{O}_X(mK_{X/Y})$ is globally generated on the complement of the discriminant locus of f for every sufficiently large and divisible m.

The main difficulty to prove the global generation is the fact that the direct image $f_*\mathcal{O}_X(mK_{X/Y})$ is only semipositive and not strictly positive (= ample) in general. The idea of the proof is to distinguish the null direction of the positivity of $f_*\mathcal{O}_X(mK_{X/Y})$ as a Monge-Ampère foliation and to realize the direct image $f_*\mathcal{O}_X(mK_{X/Y})$ (or its certain symmetric power) as the pull back of an ample vector bundle on a certain moduli space via the moduli map.

1.1 Kawamata's semipositivity theorem

In order to clarify what is new in this article, I would like to review briefly the former results and methods on the semipositivity of the direct images of pluricanonical systems in Sections 1.1 and 1.2.

The first result on the semipositivity of the relative pluricanonical system is the following theorem due to Y. Kawamata in 1982.

Theorem 1.1 *([Ka2]) Let $f : X \longrightarrow Y$ be an algebraic fiber space. Suppose that* $\dim Y = 1$. *Then for every positive integer m, $f_*\mathcal{O}_X(mK_{X/Y})$ is a semipositive vector bundle on Y, in the sense that every quotient $\mathcal{Q}$ of $f_*\mathcal{O}_X(mK_{X/Y})$,* $\deg \mathcal{Q} \geqq 0$ *holds.* □

The proof of Theorem 1.1 depends on the variation of Hodge structure due to P.A. Griffiths and W. Schmidt (cf. [G, Sch]). We note that before Theorem 1.1, T. Fujita proved the case of $m = 1$ in [F1] by using the curvature computation of the Hodge metrics of P.A. Griffiths ([G]). In this special case, Fujita gave a singular hermitian metric on the vector bundle $f_*\mathcal{O}_X(K_{X/Y})$ with semipositive curvature in the sense of Griffiths. In contrast to Fujita's result, for $m \geqq 2$, Theorem 1.1 does not give a (singular) hermitian metric on $f_*\mathcal{O}_X(mK_{X/Y})$ with semipositive curvature, because the proof relies on the semipositivity of the curvature of the Finslar metric on $f_*\mathcal{O}_X(mK_{X/Y})$ defined by

$$\text{(1.1.1)} \qquad \parallel \sigma \parallel := \left(\int_{X/Y} |\sigma|^{\frac{2}{m}} \right)^{\frac{m}{2}}$$

which is a singular hermitian metric on the tautological line bundle on $\mathbb{P}((f_*\mathcal{O}_X(mK_{X/Y}))^*)$.

1.2 Viehweg's semipositivity theorem

In 1995 E. Viehweg extended Theorem 1.1 ([V2, Section 6]) in the case of f-semiample relative canonical bundles and constructed quasi-projective moduli spaces of polarized projective manifolds with semiample canonical bundles ([V2]). Since we use Viehweg's idea in this article, we state his result precisely. First we recall several definitions.

Definition 1.2 *Let Y be a quasi-projective scheme, let Y_0 be an open dense suchscheme and let $\mathcal{G}$ be a coherent sheaf on Y, We say that $\mathcal{G}$ is* **globally generated** *over Y_0, if the natural map $H^0(Y, \mathcal{G}) \otimes \mathcal{O}_Y \longrightarrow \mathcal{G}$ is surjective over Y_0.* □

For a coherent sheaf $\mathcal{F}$ and a positive integer a, $S^a(\mathcal{F})$ denotes the a-th symmetric power of $\mathcal{F}$. To measure the positivity of coherent sheaves, we shall introduce the following notion.

Definition 1.3 *Let Y be a quasi-projective reduced scheme, $Y_0 \subseteq Y$ an open dense subscheme and let $\mathcal{G}$ be locally free sheaf on Y, of finite constant rank. Then $\mathcal{G}$ is* **weakly positive** *over Y_0, if for an ample invertible sheaf $\mathcal{H}$ on Y and for a given number $\alpha > 0$ there exists some $\beta > 0$ such that $S^{\alpha\cdot\beta}(\mathcal{G}) \otimes \mathcal{H}^\beta$ is globally generated over Y_0.* □

The notion of weak positivity is a natural generalization of the notion of nefness of line bundles. Roughly speaking, the weak semipositivity of $\mathcal{G}$ over Y_0 means that $\mathcal{G} \otimes \mathcal{H}^\varepsilon$ is $\mathbb{Q}$-globally generated over Y_0 for every $\varepsilon > 0$.

Definition 1.4 *Let $\mathcal{F}$ be a locally free sheaf and let $\mathcal{A}$ be an invertible sheaf, both on a quasi-projective reduced scheme Y. We denote*

$$\text{(1.2.1)} \qquad \mathcal{F} \succeq \frac{b}{a}\, \mathcal{A},$$

if $S^a(\mathcal{F}) \otimes \mathcal{A}^{-b}$ is weakly positive over Y, where a, b are positive integers. □

For a normal variety X, we define the canonical sheaf ω_X of X by

$$\omega_X := i_*\mathcal{O}_{X_{reg}}(K_{X_{reg}}), \tag{1.2.2}$$

where X_{reg} denotes the regular part of X and $i : X_{reg} \longrightarrow X$ denotes the natural injection. The following notion introduced by Viehweg is closely related to the notion of logcanonical thresholds.

Definition 1.5 *Let (X, Γ) be a pair of normal variety X and an effective Cartier divisor Γ. Let $\pi : X' \longrightarrow X$ be a log resolution of (X, Γ) and let $\Gamma' := \pi^*\Gamma$. For a positive integer N we define*

$$\omega_X\left\{\frac{-\Gamma}{N}\right\} = \pi_*\left(\omega_{X'}\left(-\left\lfloor\frac{\Gamma'}{N}\right\rfloor\right)\right) \tag{1.2.3}$$

and

$$\mathcal{C}_X(\Gamma, N) = \mathrm{Coker}\left\{\omega_X\left\{\frac{-\Gamma}{N}\right\} \longrightarrow \omega_X\right\}. \tag{1.2.4}$$

If X has at most rational singularities, one defines :

$$e(\Gamma) = \min\{N > 0 \,|\, \mathcal{C}_X(\Gamma, N) = 0\}. \tag{1.2.5}$$

If $\mathcal{L}$ is an invertible sheaf, X is proper with at most rational singularities and $H^0(X, \mathcal{L}) \neq 0$, then one defines

$$e(\mathcal{L}) = \sup\{e(\Gamma) | \Gamma : \text{ effective Cartier divisor with } \mathcal{O}_X(\Gamma) \simeq \mathcal{L}\}. \tag{1.2.6}$$

□

Now we state the result of E. Viehweg.

Theorem 1.6 *([V2, p.191,Theorem 6.22]) Let $f : X \longrightarrow Y$ be a flat surjective projective Gorenstein morphism of reduced connected quai-projective schemes. Assume that the sheaf $\omega_{X/Y}$ is f-semi-ample and that the fibers $X_y = f^{-1}(y)$ are reduced normal varieties with at most rational singularities. Then one has :*

(1) **Functoriality***: For $m > 0$ the sheaf $f_*\omega^m_{X/Y}$ is locally free of rank $r(m)$ and it commutes with arbitrary base change.*

(2) **Weak semipositivity***: For $m > 0$ the sheaf $f_*\omega^m_{X/Y}$ is weakly positive over Y.*

(3) **Weak semistability***: Let $m > 1, e > 0$ and $\nu > 0$ be chosen so that $f_*\omega^m_{X/Y} \neq 0$ and*

$$e \geqq \sup\left\{\frac{k}{m-1}, e(\omega^k_{X_y}) \;;\; \text{for } y \in Y\right\} \tag{1.2.7}$$

hold. Then

$$f_*\omega^m_{X/Y} \succeq \frac{1}{e \cdot r(k)} \det(f_*\omega^k_{X/Y}) \tag{1.2.8}$$

holds. □

Although Theorem 1.6 assumes the f-semiampleness of $\omega_{X/Y}$, the advantages of this generalization are :

- The base space is of arbitrary dimension.
- The semipositivity is more explicit than the one in Theorem 1.1.
- The comparison of the positivity of $f_*\omega^m_{X/Y}$ and $\det(f_*\omega^m_{X/Y})$ is given.

Later Theorems 1.1 and 1.6 have been extensively used in many other contexts (for example see [Ka3, V2]).

1.3 Analytic Zariski decompositions

To state the main result, we introduce the notion of analytic Zariski decompositions. This notion will be used throughout this article.

Definition 1.7 *Let M be a compact complex manifold and let L be a holomorphic line bundle on M. A singular hermitian metric h on L is said to be an* **analytic Zariski decomposition**(**AZD** *in short*)*, if the followings hold.*

(1) Θ_h *is a closed positive current.*

(2) *For every $m \geq 0$, the natural inclusion:*

$$H^0(M, \mathcal{O}_M(mL) \otimes \mathcal{I}(h^m)) \to H^0(M, \mathcal{O}_M(mL)) \quad (1.3.1)$$

is an isomorphim. □

Remark 1.8 *If an AZD exists on a line bundle L on a smooth projective variety M, L is pseudoeffective by the condition 1 above.* □

It is known that for every pseudoeffective line bundle on a compact complex manifold, there exists an AZD on F (cf. [T1, T2, D-P-S]). The advantage of the AZD is that we can handle pseudoeffective line bundle L on a compact complex manifold X as a singular hermitian line bundle with semipositive curvature current as long as we consider the ring $R(X, L) := \oplus_{m \geqq 0} H^0(X, \mathcal{O}_X(mL))$.

We also note that there exists a smilar but different notion : *singular hemitian metrics with minimal singularities* introduced by Demailly, Peternell and Schneider (cf. [D-P-S]). A singular hemitian metric with minimal singularities is always an AZD, but in general an AZD need not be a singular hemitian metric with minimal singularities at least in the log canonical case ([T8]) [1]. In this article, we use the notion of AZD's, since the canonical measure (cf. Theorem 2.1) plays the crucial role in this article and the inverse of the canonical measure need not be a singular hermitian metric with minimal singularities.

1.4 Statement of the main results

We note that Theorems 1.1 and 1.6 do not imply the existence of nontrivial global sections of $f_*\mathcal{O}_X(mK_{X/Y})$ for some $m > 0$. In this article we shall prove the global generation of $f_*\mathcal{O}_X(m!K_{X/Y})$ for every sufficiently large m on the complement of the discriminant locus of f. The following is the main result in this article.

[1] Actually this difference is closely related to the abundance conjecture.

Theorem 1.9 *Let $f : X \longrightarrow Y$ be an algebraic fiber space and let Y° be the complement of the discriminant locus of f in Y. Then we have the followings :*

(1) **Global generation**: *There exist positive integers b and m_0 such that for every integer m satisfying $b \,|\, m$ and $m \geqq m_0$, $f_*\mathcal{O}_X(mK_{X/Y})$ is globally generated over Y°.*

(2) **Weak semistability 1**: *Let m be a positive integer such that $f_*\mathcal{O}_X(mK_{X/Y}) \neq 0$. Let r denote* rank $f_*\mathcal{O}_X(mK_{X/Y})$ *and let $X^r := X \times_Y X \times_Y \cdots \times_Y X$ be the r-times fiber product over Y. Let $f^r : X^r \longrightarrow Y$ be the natural morphism.*

Let $\Gamma \in |mK_{X^r/Y} - f^{r*} \det f_*\mathcal{O}_X(mK_{X/Y})|$ be the effective divisor corresponding to the canonical inclusion :
(1.4.1)
$$f^{r*}(\det f_*\mathcal{O}_X(mK_{X/Y})) \hookrightarrow f^{r*}f^r_*\mathcal{O}_{X^r}(mK_{X^r/Y}) \hookrightarrow \mathcal{O}_{X^r}(mK_{X^r/Y}).$$

Then Γ does not contain any fiber $X^r_y (y \in Y^\circ)$ such that if we we define the number δ_0 by

$$\delta_0 := \sup\{\delta \,|\, (X^r_y, \delta \cdot \Gamma_y) \text{ is KLT for all } y \in Y^\circ\}, \tag{1.4.2}$$

then for every $\varepsilon < \delta_0$ and a sufficiently large positive integer d,

$$f_*\mathcal{O}_X(d!K_{X/Y}) \succeq \frac{d!\varepsilon}{(1+m\varepsilon)r} \det f_*\mathcal{O}_X(mK_{X/Y}) \tag{1.4.3}$$

holds over Y°.

(3) **Weak semistability 2**: *There exists a singular hermitian metric $H_{m,\varepsilon}$ on $(1+m\varepsilon)K_{X^r/Y} - \varepsilon \cdot f^{r*} \det f_*\mathcal{O}_X(mK_{X/Y})^{**}$ such that*

(a) *$\sqrt{-1}\,\Theta_{H_{m,\varepsilon}} \geqq 0$ holds on X^r in the sense of current.*

(b) *For every $y \in Y^\circ$, $H_{m,\varepsilon}|X^r_y$ is well defined and is an AZD (cf. Definition 1.3) of*

$$(1+m\varepsilon)K_{X^r/Y} - \varepsilon \cdot f^{r*} \det f_*\mathcal{O}_X(mK_{X/Y})^{**}|X_y. \tag{1.4.4}$$

□

Remark 1.10 *The 3rd assertion implies the 2nd assertion.* □

The major difference between Theorems 1.9 and 1.1 is that in Theorem 1.9 $f_*\mathcal{O}_X(mK_{X/Y})$ is globally generated over the complement of the discriminant locus of f, while Theorem 1.1 implies the semipositivity of $f_*\mathcal{O}_X(mK_{X/Y})$. In this sense Theorem 1.9 is much stronger than Theorem 1.1. The major difference between Theorems 1.9 and 1.6 is (besides the global generation assertion) that in Theorem 1.9, we do not assume the f-semiampleness of $K_{X/Y}$ in Theorem 1.9.

We also have the following log version of Theorem 1.9.

Theorem 1.11 *Let $f : X \to Y$ be an algebraic fiber space and let D be an effective $\mathbb{Q}$ divisor on X such that (X, D) is KLT. Let Y° denote the complement of the discriminant locus of f. We set*

$$Y_0 := \{y \in Y | y \in Y^\circ, (X_y, D_y) \text{ is a KLT pair}\} \tag{1.4.5}$$

(1) **Global generation**: *There exist positive integers b and m_0 such that for every for every integer m satisfying $b \,|\, m$ and $m \geqq m_0$, $m(K_{X/Y}+D)$ is Cartier and $f_*\mathcal{O}_X(m(K_{X/Y}+D))$ is globally generated over Y_0.*

(2) **Weak semistability 1**: *Let m be a positive integer such that $m(K_{X/Y}+D)$ is integral and $f_*\mathcal{O}_X(m(K_{X/Y}+D)) \neq 0$. Let r denote $\mathrm{rank}\, f_*\mathcal{O}_X(\lfloor m(K_{X/Y}+D)\rfloor)$. Let $X^r := X \times_Y X \times_Y \cdots \times_Y X$ be the r-times fiber product over Y and let $f^r : X^r \longrightarrow Y$ be the natural morphism. And let D^r denote the divior on X^r defined by $D^r = \sum_{i=1}^r \pi_i^* D$, where $\pi_i : X^r \longrightarrow X$ denotes the projection: $X^r \ni (x_1, \cdots, x_n) \mapsto x_i \in X$.*

There exists a canonically defined effective divisor Γ (depending on m) on X^r which does not conatin any fiber $X^r_y (y \in Y^\circ)$ such that if we we define the number δ_0 by

$$\delta_0 := \sup\{\delta \,|\, (X^r_y, D^r_y + \delta\Gamma_y) \text{ is KLT for all } y \in Y^\circ\}, \tag{1.4.6}$$

then for every $\varepsilon < \delta_0$ and every sufficiently large positive integer d,

$$f_*\mathcal{O}_X(d!(K_{X/Y}+D)) \succeq \frac{d!\varepsilon}{(1+m\varepsilon)r} \det f_*\mathcal{O}_X(\lfloor m(K_{X/Y}+D)\rfloor) \tag{1.4.7}$$

holds over Y_0.

(3) **Weak semistability 2**: *There exists a singular hermitian metric $H_{m,\varepsilon}$ on*

$$(1+m\varepsilon)(K_{X^r/Y}+D^r) - \varepsilon \cdot f^* \det f_*\mathcal{O}_X(\lfloor m(K_{X/Y}+D)\rfloor)^{**} \tag{1.4.8}$$

such that

(a) $\sqrt{-1}\,\Theta_{H_{m,\varepsilon}} \geqq 0$ holds on X in the sense of current.

(b) For every $y \in Y_0$, $H_{m,\varepsilon}|X^r_y$ is well defined and is an AZD of

$$(1+m\varepsilon)(K_{X^r/Y}+D^r) - \varepsilon \cdot f^{r*} \det f_*\mathcal{O}_X(\lfloor m(K_{X/Y}+D)\rfloor)^{**}|X_y$$

□

The main ingredient of the proof of Theorems 1.9 and 1.11 is the (logarithmic) plurisubharmonic variation property of canonical measures (Theorem 2.5 in [T7]). The new feature of the proof is the use of the Monge-Ampère foliations arising from the canonical measures and the weak semistability of the direct images of relative pluricanonical systems. One may consider these new tools as substitutes of the local Torelli theorem for minimal models with semi-ample canonical divisors in [Ka2].

The scheme of the proof is as follows. For an algebraic fiber space $f : X \longrightarrow Y$ with $\mathrm{Kod}(X/Y) \geqq 0$ (cf. (2.1.3)), we take the relative canonical measure $d\mu_{can,X/Y}$ (see Section 2.5). Then the null distribution of the curvature $\Theta_{d\mu^{-1}_{can,X/Y}}$ of the singular hermitian metric $d\mu^{-1}_{can,X/Y}$ on $K_{X/Y}$ defines a singular Monge-Ampère foliation on X. Here the important fact is that the leaf of the foliation is complex analytic ([B-K]) (although it is not clear that the foliation itself is complex analytic apriori). By using the weak semistability of $f_*\mathcal{O}_X(m!K_{X/Y})$, we prove that this singular foliation actually descends to a singular foliation $\mathcal{G}$ on the base space Y. Let us define the (singular) hermitian metric h_m on $f_*\mathcal{O}_X(m!K_{X/Y})$ defined by

$$h_m(\sigma,\sigma') := \int_{X/Y} \sigma\cdot\overline{\sigma'}\cdot d\mu_{can,X/Y}^{-(m!-1)}. \tag{1.4.9}$$

Then we see that $(f_*\mathcal{O}_X(m!K_{X/Y}), h_m)$ is flat along the leaves of $\mathcal{G}$ on Y. Taking m sufficiently large, we see that the metrized relative canonical model (cf. Definition 3.1 below) of $f : X \longrightarrow Y$ is locally trivial along the leaves. Then we see that the leaves of $\mathcal{G}$ consists of the fiber of the moduli map to the moduli space of relative canonical models marked with the metrized Hodge line bundles. Then the global generation property of $f_*\mathcal{O}_X(mK_{X/Y})$ follows from the Nakai-Moishezon type argument.

1.5 Iitaka's conjecture

In this subsection, we apply Theorem 1.9 to Iitaka's conjecture. The following conjecture by S. Iitaka ([I]) is well known.

Conjecture 1.12 *(Iitaka's conjecture) Let $f : X \longrightarrow Y$ be an algebraic fiber space. Then*

$$\mathrm{Kod}(X) \geqq \mathrm{Kod}(Y) + \mathrm{Kod}(X/Y) \tag{1.5.1}$$

holds, where $\mathrm{Kod}(X), \mathrm{Kod}(Y)$ *denote the Kodaira dimension (cf. (2.1.1)) of* X, Y *repsectively and* $\mathrm{Kod}(X/Y)$ *denotes the relative Kodaira dimension as (2.1.3).* □

The typical examples of algebraic fiber spaces are Iitaka fibrations, Albanese maps, the universal families over fine moduli spaces. Especially the Iitaka fibration $f : X \longrightarrow Y$ has the property that $\mathrm{Kod}(X/Y) = 0$. Hence Conjecture 1.12 reduces the birational classification of X to the study of families of varieties with Kodaira dimension 0 and the study of the base sace Y with $\mathrm{Kod}(Y) \leqq \mathrm{Kod}(X)$. Conjecture 1.12 is considered to be one of the key for the birational classification of projective varieties . For detailed explanation and references, see the survey article [M] for example.

In [Ka2] Kawamata solved Conjecture 1.12 in the case of $\dim Y = 1$ by using Theorem 1.1. And if $\mathrm{Kod}(Y) = \dim Y$,i.e., Y is of general type, then Conjecture 1.12 can be easily deduced from Theorem 1.1. And in the case that a general fiber of $f : X \longrightarrow Y$ is of general type, Conjecture 1.12 has been solved (cf. [V1, Ko]). And in [Ka2], Kawamata reduced Conjecture 1.12 to the completion of the minimal model program (MMP). Hence by the completion of MMP in dimension 3 (see [K-M] for example), Conjecture 1.12 has been solved in the case of $\dim X = 3$.

As an immediate consequence of Theorem 1.9, we give an affirmative answer to Iitaka's conjecture.

Theorem 1.13 *Conjecture 1.12 holds.* □

Proof of Theorem 1.13. Let $f : X \longrightarrow Y$ be an algebraic fiber space. If $\mathrm{Kod}(Y)$ or $\mathrm{Kod}(X/Y)$ is $-\infty$, Conjecture 1.12 certainly holds. Hence we assume that $\mathrm{Kod}(Y)$ and $\mathrm{Kod}(X/Y)$ are nonnnegative. We note that there exists a natural morphism :

$$H^0(Y, f_*\mathcal{O}_X(mK_{X/Y})) \otimes H^0(Y, \mathcal{O}_Y(mK_Y)) \rightarrow H^0(X, \mathcal{O}_X(mK_X)). \tag{1.5.2}$$

Then by Theorem 1.9, we have that

$$\limsup_{m \longrightarrow \infty} \frac{\log \dim H^0(Y, f_*\mathcal{O}_X(mK_{X/Y}))}{\log m} \geqq \mathrm{Kod}(X/Y) \tag{1.5.3}$$

holds. Hence we see that

$$\mathrm{Kod}(X) \geqq \mathrm{Kod}(Y) + \mathrm{Kod}(X/Y) \tag{1.5.4}$$

holds. □

Remark 1.14 *The optimal form of Iitaka's conjecture is:*

$$\mathrm{Kod}(X) \geqq \mathrm{Kod}(Y) + \max\{\mathrm{Kod}(X/Y), \mathrm{Var}(f)\}. \tag{1.5.5}$$

At this moment, I do not know the proof. □

The organization of this article is as follows. In Section 2, we review the canonical measures intorduced in [S-T, T7]. Especially the logarithmic subhrmonicity of the canonical meaures (cf. [T7, T8]) is explained. Using the logarithmic subharmonicity and Viehweg's idea, we prove the weak semistability of the direct images of relative pluri log canonical systems for a family of KLT pairs. In Section 3, we construct the moduli space of the metrized canonical models of KLT pairs. The construction is rather standard, but technical. In Section 4, we analyse the Monge-Ampère foliation assuming the regularity results of canonical measures which is proven Section 6 below. In Section 5, we complete the proof of the main results assuming the regularity rusults in Section 6. In Section 6, we prove the regularity of canonical measures by the dynamical construction of canonical measures and Hörmander's L^2-estimate of $\bar{\partial}$-operators. In Section 7, we provide several technical results which are used in Section 4.

The order of contents may be a little bit irregular. But I hope that to put off the technical stuffs later makes the scheme of the proof clear.

Notations

- For a real number a, $\lceil a \rceil$ denotes the minimal integer greater than or equal to a and $\lfloor a \rfloor$ denotes the maximal integer smaller than or equal to a.
- Let X be a projective variety and let D be a Weil divsor on X. Let $D = \sum d_i D_i$ be the irreducible decomposition. We set

$$\lceil D \rceil := \sum \lceil d_i \rceil D_i, \lfloor D \rfloor := \sum \lfloor d_i \rfloor D_i. \tag{1.5.6}$$

- Let $f : X \longrightarrow Y$ be an algebraic fiber space and let D be a $\mathbb{Q}$-divisor on X. Let

$$D = D^h + D^v \tag{1.5.7}$$

be the decomposition such that an irreducible component of $\operatorname{Supp} D$ is contained in $\operatorname{Supp} D^h$ if and only if it is mapped onto Y. D^h is the horizontal part of D and D^v is the vertical part of D.

- Let (X, D) be a pair of a normal variety and a $\mathbb{Q}$-divisor on X. Suppose that $K_X + D$ is $\mathbb{Q}$-Cartier. Let $f : Y \longrightarrow X$ be a log resolution. Then we have the formula :

$$K_Y = f^*(K_X + D) + \sum a_i E_i,$$

where E_i is a prime divisor and $a_i \in \mathbb{Q}$. The pair (X, D) is said to be **subKLT**(resp. **subLC**), if $a_i > -1$ (resp. $a_i \geqq -1$) holds for every i. (X, D) is said to be **KLT** (resp. **LC**), if (X, D) is subKLT(resp. subLC) and D is effective.

- Let X be a projective variety and let $\mathcal{L}$ be an invertible sheaf on X. $\mathcal{L}$ is said to be semiample, if there exists a positive integer m such that $|\mathcal{L}^{\otimes m}|$ is base point free.

- $f : X \longrightarrow Y$ be a morphism between projective varieties. Let $\mathcal{L}$ be an invertible sheaf on X. $\mathcal{L}$ is said to be f-semiample, if for every $y \in Y$, $\mathcal{L}|f^{-1}(y)$ is semiample.

- Let L be a $\mathbb{Q}$-line bundle on a compact complex manifold X, i.e., L is a formal fractional power of a genuine line bundle on X. A singular hermitian metric h on L is given by

$$h = e^{-\varphi} \cdot h_0,$$

where h_0 is a C^∞ hermitian metric on L and $\varphi \in L^1_{loc}(X)$ is an arbitrary function on X. We call φ the weight function of h with respect to h_0. We note that h makes sense, since a hermitian metric is a real object.

The curvature current Θ_h of the singular hermitian $\mathbb{Q}$-line bundle (L, h) is defined by

$$\Theta_h := \Theta_{h_0} + \partial\bar{\partial}\varphi,$$

where $\partial\bar{\partial}\varphi$ is taken in the sense of current. We define the multiplier ideal sheaf $\mathcal{I}(h)$ of (L, h) by

$$\mathcal{I}(h)(U) := \{f \in \mathcal{O}_X(U);\ |f|^2 e^{-\varphi} \in L^1_{loc}(U)\},$$

where U runs open subsets of X.

- A singular hermitian line bundle (L, h) is said to be **pseudoeffective**, if $\sqrt{-1}\Theta_h$ is a closed semipositive current.

- For a closed positive $(1,1)$ current T, T_{abc} denotes the abosolutely continuous part of T.

- For a Cartier divisor D, we denote the corresponding line bundle by the same notation. Let D be an effective $\mathbb{Q}$-divisor on a smooth projective variety X. Let a be a positive integer such that aD is Cartier. We identify D with a formal a-th root of the line bundle aD. We say that σ is a multivalued global holomorphic section of D with divisor D, if σ is a formal a-th root of a global holomorphic section of aD with divisor aD. And $1/|\sigma|^2$ denotes the singular hermitian metric on D defined by

$$\frac{1}{|\sigma|^2} := \frac{h_D}{h_D(\sigma, \sigma)},$$

where h_D is an arbitrary C^∞ hermitian metric on D.

- For a singular hermitian line bundle (F, h_F) on a compact complex manifold X of dimension n. $K(X, K_X + F, h_F)$ denotes (the diagonal part of) the Bergman kernel of $H^0(X, \mathcal{O}_X(K_X + F) \otimes \mathcal{I}(h_F))$ with respect to the L^2-inner product:

(1.5.8) $$(\sigma, \sigma') := (\sqrt{-1})^{n^2} \int_X h_F \cdot \sigma \wedge \bar{\sigma}',$$

i.e.,

(1.5.9) $$K(X, K_X + F, h_F) = \sum_{i=0}^{N} |\sigma_i|^2,$$

where $\{\sigma_0, \cdots, \sigma_N\}$ is a complete orthonormal basis of $H^0(X, \mathcal{O}_X(K_X + F) \otimes \mathcal{I}(h_F))$. It is clear that $K(X, K_X + F, h_F)$ is independent of the choice of the complete orthonormal basis.

2 Canonical measures

In this section we review the definition and the basic properties of canonical measures which plays the key role[2] in the proof of Theorems 1.9 and 1.11.

The canonical measure is a natural generalization of Kähler-Einstein volume form to the case of projective varieties with nonnegative Kodaira dimension (cf. [S-T, T7]). The basic properties of the canonical measure are :

(1) It is completely determined by the complex structure of the variety and is birationally invariant.

(2) It is C^∞ on a on a nonempty Zariski open subset of the variety and satisfies a Monge-Ampère equation on a Zariski open subset on the base space of the Iitaka fibration (cf. Section 2.1).

[2]Probably we may use the Narashimhan-Simha volume form ([N-S]) instead of canonical measures to prove Theorem 1.9 and 1.11. For a smooth projective vairiety X with nonnegative Kodaira dimension and a positive integer m, the m-th Narashimhan-Simha volume form K_m^{NS} is defined by

(2.0.10) $$K_m^{NS}(x) := \{|\sigma|^{\frac{2}{m}}(x) | \sigma \in H^0(X, \mathcal{O}_X(mK_X)), \int_X |\sigma|^{\frac{2}{m}} = 1\}.$$

The advantage of the Narashimhan-Simha volume form is that its construction is much simpler than the one of the canonical measure. But on the other hand, it seems to be hard to prove the regularity of the Narashimhan-Simha measure.

(3) The logarithm of the measure is plurisubharmonic under projective deformations.

For the detailed account, see [S-T, T7, T8]. The canonical measure is defined on an arbitrary KLT pair with nonnegative logarithmic Kodaira dimension ([T8]).

2.1 Iitaka fibration

To construct the canonical measure, we need to consider the Iitaka fibration. The Iitaka fibration is the most naive way to extract the positivity of the canonical bundle on a smooth projective variety with nonnegative Kodaira dimension.

Let X be a smooth projective variety. We define the Kodaira dimension of X by

$$\text{Kod}(X) := \limsup_{m\to\infty} \frac{\log \dim H^0(X, \mathcal{O}_X(mK_X))}{\log m}. \tag{2.1.1}$$

More generally for a KLT pair (X, D), we define *the Kodaira dimension* $\text{Kod}(X, D)$ of (X, D) by

$$\text{Kod}(X, D) := \limsup_{m\to\infty} \frac{\log \dim H^0\left(X, \mathcal{O}_X(\lfloor m(K_X + D)\rfloor)\right)}{\log m}. \tag{2.1.2}$$

Similarly for an algebraic fiber space $f : X \to Y$, we define *the relative Kodaira dimension* $\text{Kod}(X/Y)$ by

$$\text{Kod}(X/Y) := \text{Kod}(F), \tag{2.1.3}$$

where F is a general fiber of f.

Let X be a smooth projective variety with $\text{Kod}(X) \geqq 0$. Then for a sufficiently large $m > 0$, the complete linear system $|m!K_X|$ gives a rational fibration (with connected fibers) :

$$f : X - \cdots \to Y. \tag{2.1.4}$$

We call $f : X - \cdots \to Y$ the **Iitaka fibration** of X.

The Iitaka fibration is independent of the choice of the sufficiently large m up to birational equivalence. See [I] for detail. In this sense the Iitaka fibration is unique. By taking a suitable modification, we may assume that f is a morphism and Y is smooth.

The Iitaka fibration $f : X \to Y$ satisfies the following properties:

(1) For a general fiber F, $\text{Kod}(F) = 0$ holds,

(2) $\dim Y = \text{Kod}(X)$.

2.2 Relative Iitaka fibrations

The Iitaka fibration can be easily generalized to the relative setting. This generalization will be used to analyze the variation of canonical measures on a projective faimily.

Let $f : X \longrightarrow Y$ be an algebraic fiber space, i.e., X, Y are smooth projective varieties and f is a proper surjective morphism with connected fibers.

Let m be a sufficiently large positive integer and we set $F_m := f_*\mathcal{O}_X(m!K_{X/Y})^{**}$. For $x \in X$, we set

$$ev_x : F_{m,f(x)} \longrightarrow mK_{X/Y} \tag{2.2.1}$$

be the evaluation map. We define the relative canonical map :

$$g : X - \cdots \to \mathbb{P}(F_m^*) \tag{2.2.2}$$

by

$$g(x) := \{[v^*] |\, v^* \in F^*_{m,f(x)},\, v^*|\mathrm{Ker}\ ev_x = 0\}. \tag{2.2.3}$$

Let Z be the image of g. Then we have the commutative diagram :

$$\begin{array}{ccc} X & \xrightarrow{\quad g \quad} & Z \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle h} & \\ & Y & \end{array} \tag{2.2.4}$$

For a sufficiently large m, we see that a general fiber F of $g : X - \cdots \to Z$ is connected and $\mathrm{Kod}(F) = 0$. We call $g : X - \cdots \to Z$@**a@relative Iitaka fibration**. By taking a suitable modification of X, we may assume that g is a morphism.

Let $f : X \longrightarrow Y$ be an algebraic fiber space and let $g : X \longrightarrow Z$ be a relative Iitaka fibration associated with $f_*\mathcal{O}_X(m!K_{X/Y})$. Taking a suitable modification we may and do assume the followings :

(1) g is a morphism.

(2) Z is smooth.

(3) $g_*\mathcal{O}_X(m!K_{X/Z})^{**}$ is a line bundle on Z for every sufficiently large m.

Let $h : Z \longrightarrow Y$ be the natural morphism.

This construction can be easily generalized to the case of a KLT pair (X, D) with algebraic fiber space structure $f : X \to Y$.

Also by the finite generation of the log canonical ring for a KLT pair ([B-C-H-M]), we may take Z be be a family of logcanonical models at least on a nonempty Zariski open subset of Y. In this case Z has singularities. We call such a triangle (2.2.4) or the family $h : Z \to Y$, **the relative canonical model**.

2.3 Hodge line bundles associated with Iitaka fibrations

Let $f : X \to Y$ be an Iitaka fibration such that X, Y are smooth and f is a morphism. Then by [F-M, p.169,Proposition 2.2], $f_*\mathcal{O}_X(m!K_{X/Y})^{**}$ is locally free on Y for every sufficiently large m, where $**$ denotes the double dual. Since $f : X \to Y$ is an Iitaka fibration, a general fiber is of Kodaira dimension 0 and the direct image $f_*\mathcal{O}_X(m!K_{X/Y})$ is of rank 1 for every sufficiently large m. We define the $\mathbb{Q}$-line bundle $L_{X/Y}$ on Y by

$$L_{X/Y} := \frac{1}{m!} f_*\mathcal{O}_X(m!K_{X/Y})^{**}. \tag{2.3.1}$$

We note that $L_{X/Y}$ is independent of a sufficiently large m (cf. [F-M, Section 2]). Let us fix such a m. Let Y° denote the complement of the discriminant locus of $f : X \longrightarrow Y$. Then $L_{X/Y}$ carries the natural singular hermitian metric $h_{L_{X/Y}}$ defined by

$$h^{m!}_{L_{X/Y}}(\sigma,\sigma)_y := \left(\int_{X_y} |\sigma|^{\frac{2}{m!}}\right)^{m!}, \tag{2.3.2}$$

where $y \in Y^\circ, X_y := f^{-1}(y)$ and $\sigma \in m!L_{X/Y,y}$. $h_{L_{X/Y}}$ is defined on $L_{X/Y}|Y^\circ$ apriori. But by the theory of variation of Hodge structures ([Sch]), $h_{L_{X/Y}}$ extends to a singular hermitian metric on $L_{X/Y}$. It is known that $h_{L_{X/Y}}$ has semipositive curvature in the sense of current ([Ka2]).

2.4 Definition of canonical measures and the existence

Now we define the canonical semipositive current on a smooth projective variety of nonnegative Kodaira dimension. Let $f : X \longrightarrow Y$ be the Iitaka fibration such that some positive multiple of the Hodge $\mathbb{Q}$-line bundle $L_{X/Y}$ defined as in the last subsection is locally free.

Theorem 2.1 *(cf. [T7, Theorem 1.5] and [S-T, Theorem B.2]) In the above notations, there exists a unique singular hermitian metric on h_K on $K_Y + L_{X/Y}$ such that*

(1) *h_K is an AZD of $K_Y + L_{X/Y}$,*

(2) *f^*h_K is an AZD of K_X,*

(3) *h_K is C^∞ on a nonempty Zariski open subset U,*

(4) *$\omega_Y = \sqrt{-1}\,\Theta_{h_K}$ is a Kähler form on U,*

(5) *$-\mathrm{Ric}_{\omega_Y} + \sqrt{-1}\,\Theta_{h_{L_{X/Y}}} = \omega_Y$ holds on U, where $h_{L_{X/Y}}$ denotes the Hodge metric defined as (2.3.2).* □

The above equation:

$$-\mathrm{Ric}_{\omega_Y} + \sqrt{-1}\,\Theta_{h_{L_{X/Y}}} = \omega_Y \tag{2.4.1}$$

is similar to the Kähler-Einstein equation :

$$-\mathrm{Ric}_{\omega_Y} = \omega_Y. \tag{2.4.2}$$

The correction term $\sqrt{-1}\,\Theta_{h_{L_{X/Y}}}$ reflects the isomorphism :

$$R(X, K_X)^{(a)} = R(Y, K_Y + L_{X/Y})^{(a)} \tag{2.4.3}$$

for some positive integer a, where for a graded ring $R := \oplus_{i=0}^{\infty} R_i$ and a positive integer b, we set

$$R^{(b)} := \oplus_{i=0}^{\infty} R_{bi}. \tag{2.4.4}$$

Now we shall define the canonical measure.

Definition 2.2 *([S-T, T7, T8]) The current ω_Y on Y constructed in Theorem 2.1 is said to be* **the canonical Kähler current** *of the Iitaka fibration $f : X \longrightarrow Y$. Also $\omega_X := f^*\omega_Y$ is said to be* **the canonical semipositive current** *on X. We define the measure $d\mu_{can}$ on X by*

$$d\mu_{can} := \frac{1}{n!} f^* \left(\omega_Y^n \cdot h_{L_{X/Y}}^{-1} \right) \tag{2.4.5}$$

and is said to be **the canonical measure***, where n denotes* $\dim Y$*. Here we note that ω_Y^n is a degenerate volume form on Y and $f^* h_{X/Y}^{-1}$ is considered to be a relative (degenerate) volume form on $f : X \to Y$ (cf. (2.3.2)), hence $f^* \left(\omega_Y^n \cdot h_{L_{X/Y}}^{-1} \right)$ is considered to be a degenerate volume form on X.* □

We also have the log version of Theorem 2.1 which plays a crucial role not only in the proof of Theorem 1.11, but also in the one of Theorem 1.9.

Let (X, D) be a KLT pair such that X is smooth projective. We assume that $\mathrm{Kod}(X, D) \geqq 0$, i.e., for every $m >> 1$, $|m!(K_X + D)| \neq \emptyset$. Let

$$f : X \longrightarrow Y \tag{2.4.6}$$

be a log Iitaka fibration of (X, D). After modifications, we may assume the followings:

(1) X,Y are smooth and f is a morphism with connected fibers.

(2) $\mathrm{Supp}\, D$ is a divisor with normal crossings.

(3) There exists an effective divisor Σ on Y such that f is smooth over $Y - \Sigma$, $\mathrm{Supp}\, D^h$ is relatively normal crossings over $Y - \Sigma$ and $f(D^v) \subset \Sigma$, where D^h, D^v denote the horizontal and the vertical component of D respectively.

(4) There exists a positive integer m_0 such that $f_*\mathcal{O}_X \left(m!(K_{X/Y} + D) \right)^{**}$ is a line bundle on Y for every $m \geqq m_0$ ([F-M, p.175,Proposition 4.2]).

We note that adding effective exceptional $\mathbb{Q}$-divisors does not change the log canonical ring. Similarly as (2.3.1) we define the $\mathbb{Q}$-line bundle $L_{X/Y,D}$ on Y by

$$L_{X/Y,D} = \frac{1}{m!} f_*\mathcal{O}_X(m!(K_{X/Y} + D))^{**}. \tag{2.4.7}$$

$L_{X/Y,D}$ is independent of the choice of a sufficiently large m ([F-M, p.169,Proposition 2.2]). Let us fix such a m. Similarly as before we shall define the singular hermitian metric on $L_{X/Y,D}$ by

$$h_{L_{X/Y,D}}^{m!}(\sigma, \sigma)(y) := \left(\int_{X_y} |\sigma|^{\frac{2}{m!}} \right)^{m!}, \tag{2.4.8}$$

where $y \in Y - \Sigma$, $X_y := f^{-1}(y)$ and $\sigma \in m! L_{X/Y,D,y}$. We note that since (X, D) is KLT, $h_{L_{X/Y,D}}$ is well defined. As before $h_{L_{X/Y,D}}$ has semipositive curvature in the sense of current (cf. [Ka3, B-P]). By the same strategy as in the proof of Theorem 1.9, we have the following KLT version of Theorem 2.1.

Theorem 2.3 *([T7, T8]) In the above notations, there exists a unique singular hermitian metric on h_K on $K_Y + L_{X/Y,D}$ and a nonempty Zariski open subset U of Y such that*

(1) *h_K is an AZD of $K_Y + L_{X/Y,D}$,*

(2) *f^*h_K is an AZD of $K_X + D$,*

(3) *h_K is C^∞ on U,*

(4) *$\omega_Y = \sqrt{-1}\,\Theta_{h_K}$ is a Kähler form on U,*

(5) *$-\mathrm{Ric}_{\omega_Y} + \sqrt{-1}\,\Theta_{h_{L_{X/Y,D}}} = \omega_Y$ holds on U.* □

Remark 2.4 *In Theorem 2.3, the metric h_K depends only on the logcanonical ring of (X, D). Hence adding effective exceptional $\mathbb{Q}$-divisors does not affect h_K and ω_Y essentially.* □

We define **the canonical measure** $d\mu_{can}$ **of the KLT pair** (X, D) by

$$d\mu_{can} := \frac{1}{n!} f^* \left(\omega_Y^n \cdot h_{L_{X/Y,D}}^{-1} \right), \tag{2.4.9}$$

where $n = \dim Y$.

2.5 Relative canonical measures

In the previous subsection, we have introduced the (log) canonical measure on a KLT pair (X, D) with nonnegative Kodaira dimension. In this subsection, we consider the variation of canonical measures on an algebraic fiber space. Let $f : X \longrightarrow Y$ be an algebraic fiber space and let D be an effective $\mathbb{Q}$-divisor such that (X, D) is KLT. Let Y° denote the complement of the discriminant locus of $f : X \longrightarrow Y$. For a general $y \in Y^\circ$, (X_y, D_y) is a KLT pair. We denote the set : $\{y \in Y^\circ | (X_y, D_y) \text{ is KLT}\}$ by Y_0. We assume that $\mathrm{Kod}(X_y, D_y) \geqq 0$ holds for $y \in Y_0$. By [T9], we see that $h^0(X_y, \mathcal{O}_{X_y}(m(K_{X_y} + D_y)))$ is constant over Y_0 for every $m > 0$ such that mD is Cartier and $f_*\mathcal{O}_Y(m(K_{X/Y} + D))$ is locally free over Y_0 for such a m. Then by Theorem 2.3, we may define the canonical measure $d\mu_{can,y}$ of (X_y, D_y). The family $\{d\mu_{can,y}^{-1}\}_{y \in Y_0}$ defines a singular hermitian metric h_K on $K_{X/Y} + D$. The following theorem asserts that h_K has semipositive curvature[3].

Theorem 2.5 *([T7, Theorem 4.1]) Let $f : X \longrightarrow Y$ be an algebraic fiber space. And let D be an effective divisor on X such that (X, D) is KLT. Suppose that $f_*\mathcal{O}_Y\left(\lfloor m(K_{X/Y} + D)\rfloor\right) \neq 0$ for some $m > 0$. Then there exists a singular hermitian metric h_K on $K_{X/Y} + D$ such that*

(1) $\omega_{X/Y} := \sqrt{-1}\,\Theta_{h_K}$ is semipositive on X,

(2) For a general smooth fiber $X_y := f^{-1}(y)$ such that (X_y, D_y) is KLT, $h_K|X_y$ is $d\mu_{can,(X_y,D_y)}^{-1}$, where $d\mu_{can,(X_y,D_y)}$ denotes the canonical measure on (X_y, D_y). In particular $\omega_{X/Y}|X_y$ is the canonical semipositive current on (X_y, D_y) constructed as in Theorem 1.9. □

[3]Of course the main assertion is the semipositivity of the curvature in horizontal direction with respect to $f : X \longrightarrow Y$

We call

$$d\mu_{can,(X,D)/Y} := h_K^{-1} \tag{2.5.1}$$

the relative log canonical measure for the family of KLT pairs $f : (X, D) \to Y$. Theorem 2.5 is the direct consequence of the dynamical construction of the canonical measures (cf. [T7, Theorem 1.7]) and the plurisubharmonic variation property of Bergman kernels ([B2, T4, B-P]).

2.6 Weak semistability

In this subsection we prove the 2nd and the 3rd assertions in Theorems 1.9 and 1.11. The proof follows closely the one of Theorem 1.6 in [V2]. But we replace the use of branched coverings in [V2] by the use of Theorem 2.3. This enables us to get rid of the assumption that $K_{X/Y}$ is f-semiample.

Let us start the proof. Let $f : X \longrightarrow Y$ be an algebraic fiber space. And let Y° be the complement of the discriminant locus of f. And let $X^\circ := f^{-1}(Y^\circ)$. We set $r = \operatorname{rank} f_*\mathcal{O}_X(mK_{X/Y})$ and let $X^r := X \times_Y X \times_Y \cdots \times_Y X$ denote the r-times fiber product over Y and let $f^r : X^r \to Y$ be the natural morphism. Then we have the natural morpshim:

$$\det f_*\mathcal{O}_X(mK_{X/Y}) \to \otimes^r f_*\mathcal{O}_X(mK_{X/Y}) = f_*^r\mathcal{O}_{X^r}(mK_{X^r/Y}). \tag{2.6.1}$$

Hence we have the canonical global section

$$\gamma \in \Gamma\left(X, f^{r*}(\det f_*\mathcal{O}_X(mK_{X/Y}))^{-1} \otimes \mathcal{O}_{X^r}(mK_{X/Y})\right). \tag{2.6.2}$$

Let Γ denote the zero divisor of γ. It is clear the Γ does not contain any fiber over Y°. Now we set

$$\delta_0 := \sup\{\delta > 0 | (X_y^r, \delta \cdot \Gamma_y) \text{ is KLT for every } y \in Y^\circ\}. \tag{2.6.3}$$

Let us take a positive rational number $\varepsilon < \delta_0$. Then we have that there exists the relative canonical measure $d\mu_{can,(X^r,\Delta)}$ on $f : (X^r, \Delta) \longrightarrow Y$ as in Theorem 2.3. By the logarithmic plurisubharmonicity of the canonical measure (Theorem 2.5), we see that

$$\sqrt{-1}\partial\bar{\partial}\log d\mu_{can,(X^r,\Delta)/Y} \geqq 0 \tag{2.6.4}$$

holds on X in the sense of current. We set

$$H_{m,\varepsilon} := d\mu_{can,(X^r,\Delta)/Y}^{-1}. \tag{2.6.5}$$

Then $H_{m,\varepsilon}$ is a singular hermitian metric on

$$(1 + m\varepsilon)K_{X^r/Y} - \varepsilon \cdot f^{r*}\det f_*\mathcal{O}_X(mK_{X/Y}) \tag{2.6.6}$$

with semipositive curvature current by Theorem 2.5 and $H_{m,\varepsilon}|X_y^r$ is an AZD of

$$(1 + m\varepsilon)K_{X_y^r} - \varepsilon \cdot f^{r*}\det f_*\mathcal{O}_X(mK_{X/Y})|X_y \tag{2.6.7}$$

for every $y \in Y^\circ$. Hence by [B-P], we have that

$$f_*^r\mathcal{O}_{X^r}(K_{X^r/Y} + \ell(1 + m\varepsilon)K_{X^r/Y}) \succeq \ell\varepsilon\det f_*\mathcal{O}_X(mK_{X/Y}) \tag{2.6.8}$$

holds for every positive integer ℓ such that $\ell\varepsilon$ is an integer. Since

$$f^r_*\mathcal{O}_{X^r}(K_{X^r/Y} + \ell(1+m\varepsilon)K_{X^r/Y}) = f_*\mathcal{O}_X(r(1+\ell(1+m\varepsilon))K_{X/Y}) \tag{2.6.9}$$

holds, we have that

$$f_*\mathcal{O}_X(r(1+\ell(1+m\varepsilon))K_{X/Y}) \succeq \ell\varepsilon \det f_*\mathcal{O}_X(mK_{X/Y}) \tag{2.6.10}$$

holds. By [B-C-H-M], we have that for every sufficiently large integer a, the natural morphism:

$$\otimes^k f_*\mathcal{O}_X(a!K_{X/Y}) \to f_*\mathcal{O}_X(ka!K_{X/Y}) \tag{2.6.11}$$

is surjective for every $k \geqq 0$. Hence dividing the both sides of (2.6.10) by $\ell(1+m\varepsilon)$ and letting ℓ tend to infinity, by the surjection (2.6.11) we have that for every sufficiently large positive integer d,

$$f_*\mathcal{O}_X(d!K_{X/Y}) \succeq \frac{d!\,\varepsilon}{(1+m\varepsilon)r} \det f_*\mathcal{O}_X(mK_{X/Y})^{**} \tag{2.6.12}$$

holds. □

3 Moduli spaces of metrized canonical models

So far we have completed the proof of the 2nd and the 3rd assertions in Theorems 1.9 and 1.11 (cf. Section 2.6). To prove the 1st assertion of Theorems 1.9 or 1.11, we need to use the moduli space of metrized canonical models.

In this section, we shall construct the moduli space of metrized canonical models (cf. Definition 3.1) and prove that it is an algebraic space in the sense of [Ar]. Here we shall explain only the abosolute case, i.e., we do not explain the case of KLT pairs for simplicity. The general case follows from the similar argument. Hence we omit it.

3.1 Metrized canonical models

Let X be a smooth projective variety with $\mathrm{Kod}(X) \geqq 0$. By [B-C-H-M], we see that the canonical ring: $R(X, K_X) := \oplus_{m=0}^{\infty}\Gamma(X, \mathcal{O}_X(mK_X))$ is finitely generated. Then

$$Y := \mathrm{Proj}\, R(X, K_X) \tag{3.1.1}$$

is called the canonical model of X. Then Y has only canonical singularities and the Hodge $\mathbb{Q}$-line bundle $L_{X/Y}$ is defined on Y (cf. Section 2.3). Unless X is of general type, the canonical model Y does not reflect the full informaion of the canonical ring $R(X, K_X)$. The full information of the canonical ring is recovered from K_Y and $L_{X/Y}$ by the isomorphism:

$$R(X, K_X) \simeq R(Y, K_Y + L_{X/Y})^{(a)}, \tag{3.1.2}$$

where a is the minimal positive integer such that $f_*\mathcal{O}_X(aK_{X/Y}) \neq 0$. by using the Hodge $\mathbb{Q}$-line bundle $L_{X/Y}$. Hence it is natural to consider the pair $(Y, L_{X/Y})$ instead of Y. But to describe the semipositivity of $f_*\mathcal{O}_X(mK_{X/Y})$, even the pair $(Y, L_{X/Y})$ is not enough. Hence we consider the triple $(Y, (L_{X/Y}, h_{L_{X/Y}}))$ instead of Y, where $(L_{X/Y}, h_{L_{X/Y}})$ is the Hodge $\mathbb{Q}$-line bundle on Y with the Hodge metric $h_{L_{X/Y}}$ (cf. Section 2.3). We call the pair $(L_{X/Y}, h_{L_{X/Y}})$ the metrized Hodge $\mathbb{Q}$-line bundle. Then we may recover the canonical ring $R(X, K_X)$ from $(Y, (L_{X/Y}, h_{L_{X/Y}}))$ by the isomorphism (3.1.2). Moreover we may recover the canonical Kähler current ω_Y by solving the equation:

$$-\operatorname{Ric}_{\omega_Y} + \sqrt{-1}\,\Theta_{h_{L_{X/Y}}} = \omega_Y \tag{3.1.3}$$

in terms of the dynamical construction as in [T7] (cf. Theorem 6.3 below).

Definition 3.1 *The pair $(Y, (L_{X/Y}, h_{L_{X/Y}}))$ above is said to be the* **metrized canonical model** *of X.* □

Hereafter we shall construct the moduli space of the metrized canonical models.

3.2 Construction of the moduli space and the statement of the result

Let $f : X \longrightarrow Y$ be an algebraic fiber space and let Y° denote the complement of the discriminant locus of f.

$$\begin{array}{ccc} X & \xrightarrow{\;g\;} & Z \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle h} & \\ & Y & \end{array}$$

be the relative Iitaka fibration such that Z is the relative canonical model on Y° and $g : X \longrightarrow Z$ is a morphism. Let $(L_{X/Z}, h_{L_{X/Z}})$ be the Hodge $\mathbb{Q}$-line bundle on Z. We consider the set

$$\mathcal{U} := \{(Z_y, (L_{X/Z}, h_{L_{X/Z}})|Z_y) | y \in Y^\circ\}. \tag{3.2.1}$$

Let a be the minimal positive integer such that $aL_{X/Z}$ is Cartier. We define the equivalence relation $\sim$ on $\mathcal{U}$ by

$$(Z_y, (L_{X/Z}, h_{L_{X/Z}})|Z_y) \sim (Z_{y'}, (L_{X/Z}, h_{L_{X/Z}})|Z_{y'}), \tag{3.2.2}$$

if and only if there exists a biholomorphism: $\varphi : Z_y \longrightarrow Z_{y'}$ and a bundle isomorphism: $\tilde{\varphi} : aL_{X/Z}|Z_y \longrightarrow aL_{X/Z}|Z_{y'}$ such that the following commutative diagram :

$$\begin{array}{ccc} aL_{X/Z}|Z_y & \xrightarrow{\;\tilde{\varphi}\;} & aL_{X/Z}|Z_{y'} \\ \downarrow & & \downarrow \\ Z_y & \xrightarrow[\;\varphi\;]{} & Z_{y'} \end{array}$$

and

$$\tilde{\varphi}^*(h_{L_{X/Z}}|Z_{y'}) = h_{L_{X/Z}}|Z_y \tag{3.2.3}$$

hold. Then we define the set $\mathcal{M}$ by

$$\mathcal{M} := \mathcal{U}/\sim \tag{3.2.4}$$

and call it the moduli space of metrized canonical models associated with $f : X \longrightarrow Y$.

At this moment it is not clear the $\mathcal{M}$ has a complex structure. In this section we start to prove the following theorem.

Theorem 3.2 *The moduli space of metrized canonical models $\mathcal{M}$ (associated with $f : X \longrightarrow Y$) has a structure of quasiprojective variety.* $\square$

3.3 Topological structure on $\mathcal{M}$

To endow the topology and the complex structure on $\mathcal{M}$, first we identify $\mathcal{M}$ with a quotient of certain subset of a Hilbert scheme.

Lemma 3.3 *There exists a positive integer m_0 such that for every $m \geqq m_0$ and $(Z_y, (L_{X/Z}, h_{L_{X/Z}})|Z_y) \in \mathcal{U}$, the complete linear system $|am(K_{Z_y} + L_{X/Z})|$ embeds Z_y into a projective space $\mathbb{P}^{N(m)}$, where $N(m)$ is a positive integer independent of $y \in Y^\circ$.* $\square$

Let m_0 be a positive integer as in Lemma 3.3 and let m be a positive integer greater than or equal to m_0. Let $(Z_y, (L_{X/Z}, h_{L_{X/Z}})|Z_y) \in \mathcal{U}$ be an arbitrary point and let ω_{Z_y} denote the canonical Kähler current on Z_y (cf. Definition 2.2). Let $\{\sigma_0, \cdots, \sigma_{N(m)}\}$ be an orthonormal basis of $H^0(Z_y, \mathcal{O}_{Z_y}(am(K_Y + L_{X/Z}|_{Z_y})))$ with respect to the inner product:

$$(\sigma, \sigma') := \int_{Z_y} h^{am}_{L_{X/Z}} (\omega^n_{Z_y})^{-(am-1)} \cdot \sigma \cdot \overline{\sigma'}. \tag{3.3.1}$$

Let $[\Phi_m(Z_y)]$ denote the Hilbert point corresponding to the embedding:

$$\Phi_m(z) := [\sigma_0(z) : \cdots : \sigma_{N(m)}](z \in Z_y). \tag{3.3.2}$$

We consider the set

$$\mathcal{U}_m := \{[\Phi_m(Z_y)] | (Z_y, (L_{X/Z}, h_{L_{X/Z}})|Z_y) \in \mathcal{U}\}, \tag{3.3.3}$$

where $\Phi_m(Z_y)$ runs all the choice of orthonormal basis $\{\sigma_0, \cdots, \sigma_{N(m)}\}$. We set

$$\mathcal{U}_\infty := \prod_{m=m_0}^{\infty} \mathcal{U}_m \tag{3.3.4}$$

and

$$G_\infty := \prod_{m=m_0}^{\infty} PU(N(m) + 1), \tag{3.3.5}$$

where for a positive integer k, $PU(k + 1)$ denotes the projective unitary group acting on $\mathbb{P}^k$.

Lemma 3.4 *([Ti, Ze]) Let $(Z_y, (L_{X/Z}, h_{L_{X/Z}})|_{Z_y}) \in \mathcal{U}$ be an arbitrary point and let $\{\sigma_0, \cdots, \sigma_{N(m)}\}$ be the orthonormal basis of $H^0(Z_y, \mathcal{O}_{Z_y}(am(K_Y + L_{X/Z})))$ as above. Then the Bergman kernel:*

$$K_{am} := \sum_{i=0}^{N(m)} |\sigma_i|^2 \tag{3.3.6}$$

satisfies the identity:

$$(\omega^n_{Z_y})^{-1} \cdot h_{L_{X/Z}}|_{Z_y} := \lim_{m \longrightarrow \infty} K_{am}^{-\frac{1}{am}} \tag{3.3.7}$$

compact uniformly with respect to the C^∞-topology on the complement of the discriminant locus of $g|_{X_y} : X_y \longrightarrow Z_y$. $\square$

By Lemma 3.4, we have the natural identification:

$$\mathcal{M} := \mathcal{U}_\infty / G_\infty, \tag{3.3.8}$$

where G_∞ acts on $\mathcal{U}_\infty$ in the natural manner. Hence $\mathcal{M}$ has a natural topological space structure with respect to the quotient topology.

3.4 Complex structure on $\mathcal{M}$

Although $\mathcal{U}_m$ does not have a natural complex structure apriori, we may endow a natural complex structure on $\mathcal{M}$ using the variation of Hodge structure and the logarithmic deformation.

The reason is that the Hodge $\mathbb{Q}$-line bundle $(L_{X/Z}, h_{L_{X/Z}})$ is nothing but the pull back of the universal line bundle on the period domain by the (reduced) period map. But since the Hodge line bundle $L_{X/Z}$ is not a genuine line bundle, we need to take a cyclic covering to define the period map. This makes the proof a little bit more complicated.

First we shall define the period map on a family of a metrized canonical model. Let $f : X \longrightarrow Y$ be an algebraic fiber space with $\mathrm{Kod}(X/Y) \geqq 0$. Let $g : X \longrightarrow Z$ be the relative Iitaka fibration with respect to $f : X \longrightarrow Y$ as above and we set

$$k := \dim X/Z = \dim X - \dim Z. \tag{3.4.1}$$

Let F_z denote the fiber of $g : X \longrightarrow Z$ over $z \in Z$. Let Z° denote the complement of the discriminant locus of $g : X \longrightarrow Z$. Then $\mathrm{Kod}(F_z) = 0$ holds for every $z \in Z^\circ$. Let a be a minimal positive integer such that $|aK_{F_z}| \neq \emptyset$ for every $z \in Z^\circ$. Then for every $z \in Z^\circ$ there exists a nonzero element $\eta_z \in \Gamma(F_z, \mathcal{O}_{F_z}(aK_{F_z}))$ and let

$$\mu_z : \tilde{F}_z \longrightarrow F_z \tag{3.4.2}$$

be the normalization of the cyclic cover which uniformize $\sqrt[a]{\eta_z}$. Let us consider the family $\{\tilde{F}_z\}_{z \in Z^\circ}$. This family is not well defined over Z°, but it defines a family

$$\tilde{f} : \tilde{X}^\circ \longrightarrow \tilde{Z}^\circ \tag{3.4.3}$$

over the finite unramified covering

$$\varpi : \tilde{Z}^\circ \longrightarrow Z^\circ \tag{3.4.4}$$

corresponding to the monodromy representation of the fundamental group

$$\pi_1(Z^\circ) \longrightarrow \mathbb{Z}/a\mathbb{Z}. \tag{3.4.5}$$

We take a $\mathbb{Z}/a\mathbb{Z}$ equivariant resolution $Z^{(a)} \longrightarrow \tilde{Z}^\circ$ and let

$$g^{(a)} : X^{(a)} \longrightarrow Z^{(a)} \tag{3.4.6}$$

be the resulting family of the cyclic a-coverings. We set

(3.4.7) $$U := \text{the complement of the discriminant locus of } g^{(a)}$$

and let

(3.4.8) $$g_U : (g^{(a)})^{-1}(U) \longrightarrow U$$

be the restriction of $g^{(a)}$. Let $\mathbb{E} \to U$ be the local system $R^k g_{U*}\mathbb{C}$ and let $\{\mathbb{F}^p\}_{p=0}^k$ be the Hodge filtration of $\mathbb{E}$. Then we have the period map

(3.4.9) $$\Phi : U \longrightarrow \Gamma \backslash D$$

associated with the variation of Hodge structures, where D is the period domain and Γ denotes the image of the monodromy representation of $\pi_1(U)$ to $\mathrm{Aut}(D)$. In this geometric variation of Hodge structures, it is known that Γ acts on D properly discontinuously ([G]). Hence $\Gamma \backslash D$ is a complex space. Let $\overline{U}$ be the completion of U such that the boundary $B := \overline{U} - U$ is a divisor with normal crossings. Then by [Del], the quasi canonical extension $\overline{\mathbb{E}}$ of $\mathbb{E} \otimes \mathcal{O}_U$ exists, i.e., $\overline{\mathbb{E}}$ is a locally free sheaf with the Gauss-Manin connection:

(3.4.10) $$\nabla : \overline{\mathbb{E}} \longrightarrow \Omega^1_{\overline{U}}(\log B) \otimes \overline{\mathbb{E}}$$

such that the real part of the eigenvalues of the residues around components of B lie in $[0, 1)$. Since we have assumed that B is a divisor with normal crossings, the Hodge filtration $\{\mathbb{F}^p\}$ extends as a filtration $\{\mathcal{F}^p\}$ of $\overline{\mathbb{E}}$ by subbundles. Then the metrized Hodge $\mathbb{Q}$-line bundle $(L_{X/Z}, h_{L_{X/Z}})$ corresponds to the Hodge bundle $\mathbb{F}^k$ induced by the period map Φ. Moreover the metric $h_{L_{X/Z}}$ is induced by the Hodge metric on the universal Hodge bundle on the period domain D. Here the Hodge metric is induced from the Hodge bilinear form.

Let Z^0 be the maximal Zariski open subset of Z such that $h_{L_{X/Z}}|_Z$ is locally bounded. We note that Z^0 may be much larger than the complement of the discriminant locus of $g : X \longrightarrow Z$. We note that $h_{L_{X/Z}}|U$ extends smoothly across the component B_i such that the Picard-Lefschetz transformations on $\mathbb{F}^k$ are of finite order around B_i ([G, Sch]). Let B^0 be the union of the irreducible components of B such that the Picard-Lefschetz transforms on $\mathbb{F}^k$ around the components are of infinite order. Let $\varpi : \overline{U} \longrightarrow Z$ be the natural morphism. Then $Z^0 = \varpi(\overline{U} - B^0)$ holds (cf.[Sch]). We set

(3.4.11) $$S = \varpi(B^0).$$

We consider the pair of the pairs:

(3.4.12) $$\left((Z, S), \varpi_*(\overline{\mathbb{E}}, \mathcal{F}^k)\right).$$

Then by the above construction we have the following lemma.

Lemma 3.5 $\mathcal{U} := \{(Z_y, (L_{X/Z}, h_{L_{X/Z}})|Z_y) | y \in Y^\circ\}$ *is bijective to the set of quadruples:*

(3.4.13) $$\mathcal{U}^* := \{\left((Z_y, S_y), \varpi_{y*}(\overline{\mathbb{E}}_y, \mathcal{F}^k_y)\right) | y \in Y^\circ\}.$$

□

Proof. Since S_y is the polar locus of $h_{L_{X/Z}}|_{Z_y}$, $(Z_y, (L_{X/Z}, h_{L_{X/Z}})|Z_y)$ determines the pair (Z_y, S_y). Since $(L_{X/Z}, h_{L_{X/Z}})$ is determined by the period map:

$$\Phi : U \longrightarrow \Gamma \backslash D,$$

$(Z_y, (L_{X/Z}, h_{L_{X/Z}})|Z_y)$ determines the quadruple $\big((Z_y, S_y), \varpi_{y*}(\overline{\mathbb{E}}_y, \mathcal{F}_y^k)\big)$. Conversely, since $\mathbb{E}_y$ is a flat vector bundle with the natural bilinear form, the quadruple $\big((Z_y, S_y), \varpi_{y*}(\overline{\mathbb{E}}_y, \mathcal{F}_y^k)\big)$ determines the pair $(Z_y, (L_{X/Z}, h_{L_{X/Z}})|Z_y)$. This completes the proof. □

We define the equivalence relation $\sim$ on $\mathcal{U}^*$ by

$$\big((Z_y, S_y), \varpi_{y*}(\overline{\mathbb{E}}_y, \mathcal{F}_y^k)\big) \sim \big((Z_{y'}, S_{y'}), \varpi_{y'*}(\overline{\mathbb{E}}_{y'}, \mathcal{F}_{y'}^k)\big)$$

if and only if there exist a biholomorphism

$$\varphi : (Z_y, S_y) \longrightarrow (Z_{y'}, S_{y'}) \tag{3.4.14}$$

and a sheaf isomorphism

$$\tilde{\varphi} : (\varpi_y)_*(\overline{\mathbb{E}}_y, \mathcal{F}_y^k) \longrightarrow (\varpi_{y'})_*(\overline{\mathbb{E}}_{y'}, \mathcal{F}_{y'}^k) \tag{3.4.15}$$

which covers φ which induced by an isomorphism of the flat vector bundles

$$\mathbb{E}|_{W_y} \longrightarrow \mathbb{E}|_{W_{y'}}, \tag{3.4.16}$$

where $W_y, W_{y'}$ are some nonempty Zariski open subsets of U_y and $U_{y'}$ (cf. (3.4.7)) respectively.

Lemma 3.6 *$\mathcal{M}^* := \mathcal{U}^*/\sim$ has a structure of an algebraic space in the sense of [Ar].* □

Proof of Lemma 3.6. Let m_0 be a sufficiently large positive integer such that $m_0!(K_{Z_y} + L_{X/Z}|_{Z_y})$ is Cartier and $|m_0!(K_{Z_y} + L_{X/Z}|_{Z_y})|$ is very ample for every $y \in Y^\circ$. We set $N := \dim|m_0!(K_{Z_y} + L_{X/Z}|_{Z_y})|$. If we fix a basis of $H^0(Z_y, \mathcal{O}_{Z_y}(m_0!(K_Z + L_{X/Z}|_{Z_y})))$, then the basis gives an embedding:

$$\phi : Z_y \longrightarrow \mathbb{P}^N.$$

and the images $\phi(Z_y), \phi(S_y)$ define points in the Hilbert scheme $\mathrm{Hilb}_{\mathbb{P}^N}$ of $\mathbb{P}^N$. Hence the linear system $|m_0!(K_{Z_y} + L_{X/Z}|_{Z_y})|$ gives an $PGL(N+1, \mathbb{C})$ orbit in $\mathrm{Hilb}_{\mathbb{P}^N} \times \mathrm{Hilb}_{\mathbb{P}^N}$. We denote the union of the orbits in $\mathrm{Hilb}_{\mathbb{P}^N} \times \mathrm{Hilb}_{\mathbb{P}^N}$ by $\mathcal{V}$ and let

$$\pi : (\mathcal{Z}, \mathcal{S}) \longrightarrow \mathcal{V} \tag{3.4.17}$$

be the universal family.

Next we consider the pair $(\overline{\mathbb{E}}_y, \mathcal{F}_y^k)$. Let $\mathcal{O}_{Z_y}(1)$ denote $\mathcal{O}_{Z_y}(m_0!(K_{Z_y} + L_{X/Z}|_{Z_y}))$. For a positive integer ℓ, we set

$$(\varpi_y)_*\overline{\mathbb{E}}_y(\ell) := (\varpi_y)_*\overline{\mathbb{E}}_y \otimes \mathcal{O}_{Z_y}(\ell) \text{ and } (\varpi_y)_*\mathcal{F}_y^k(\ell) := (\varpi_y)_*\mathcal{F}_y^k \otimes \mathcal{O}_{Z_y}(\ell).$$

Then for every $\ell \geqq 0$, we have the canonical inclusion:

$$H^0(Z_y, (\varpi_y)_*\mathcal{F}_y^k(\ell)) \hookrightarrow H^0(Z_y, (\varpi_y)_*\overline{\mathbb{E}}_y(\ell)). \tag{3.4.18}$$

We denote $PGL(N+1,\mathbb{C})$ by G. If $(Z_v, S_v), (Z_{v'}, S_{v'})(v, v' \in \mathcal{V})$ are in the same orbit of G, then an element $g \in G$ induces a biholomorphism between (Z_v, S_v) and $(Z_{v'}, S_{v'})$ and an isomorphism of the flat vector bundles $\mathbb{E}_v$ and $\mathbb{E}_{v'}$ on the cyclic covers and induces the isomorphism between $(\varpi_v)_*\overline{\mathbb{E}}_v$ and $(\varpi_{v'})_*\overline{\mathbb{E}}_{v'}$. The latter isomorphisms are unique up to the action of $\mathbb{Z}/a\mathbb{Z}$ and the $\mathbb{C}^*$-action. But since $\mathcal{F}^k_v(v \in \mathcal{V})$ is $\mathbb{Z}/a\mathbb{Z}$-equivariant subsheaf of $\mathbb{E}_v$, in spite of the umbiguity of the isomorphism, any such isomorphism maps the subspace $H^0(Z_v, (\varpi_v)_*\mathcal{F}^k_v(\ell)) \subset H^0(Z_v, (\varpi_v)_*\overline{\mathbb{E}}_v(\ell))$ to the same subspace of $H^0(Z_{v'}, (\varpi_{v'})_*\mathcal{F}^k_{v'}(\ell))$. We set

$$\mathcal{M}' := \mathcal{V}/G. \tag{3.4.19}$$

Then by the construction, $\mathcal{M}'$ is an algebraic space. And $\{(\varpi_v)_*\overline{\mathbb{E}}_v(\ell) | v \in \mathcal{V}\}$ decends to a coherent sheaf $\mathcal{E}$ on $\mathcal{M}'$ and the image of the inclusion (3.4.18) determines the subsheaf $(\varpi_v)_*\mathcal{F}^k$ for every $v \in \mathcal{V}$, if we take ℓ sufficiently large. Let us fix such ℓ. Hence by the properness of the period map ([G]), $\mathcal{M}^*$ is a locally closed subset (in Zariski topology) of the Grassmann bundle

$$\mathcal{G} \to \mathcal{M}' \tag{3.4.20}$$

associated with $\mathcal{E}$ with fiber $Gr(e,a)$, where $e = \operatorname{rank}\mathcal{E}$. Hence $\mathcal{M}^*$ is an algebraic space. □

By Lemma 3.5, we see that there exists a homeomorphism between $\mathcal{M}$ and $\mathcal{M}^*$. Hence we have the complex structure on $\mathcal{M}$ by Lemma 3.6.

3.5 Separatedness

To ensure the existence of $\mathcal{M}$ as a Hausdorff complex space, the following lemma is essential.

Lemma 3.7 *Let $f : (X, D) \longrightarrow \Delta$ and $f : (X', D') \longrightarrow \Delta$ be flat projective families of KLT pairs with nonnegative Kodaira dimension over the unit open disk Δ in $\mathbb{C}$. Let $\Delta^* := \Delta - \{0\}$ denote the punctured disk. And let $h : (Y, (L_{X/Y}, h_{L_{X/Y}})) \to \Delta$, $h' : (Y', (L'_{X/Y}, h_{L'_{X/Y}})) \to \Delta$ be the corresponding family of metrized pairs. Suppose that there exists an equivalence*

$$\varphi : (Y, (L_{X/Y}, h_{L_{X/Y}}))|\Delta^* \to (Y', (L'_{X/Y}, h_{L'_{X/Y}}))|\Delta^* \tag{3.5.1}$$

of the families over Δ^* in the sense of (3.2.2). Then φ extends uniquely to an equivalence between $(Y, (L_{X/Y}, h_{L_{X/Y}}))$ and $(Y', (L'_{X/Y}, h_{L'_{X/Y}}))$. □

Proof of Lemma 3.7. Let φ_s denote the restriction of φ to Y_s $(s \in \Delta^*)$. Let ω_s denotes the canonical Kähler current on Y_s constructed as in Theorem 2.1. Then by the equation (2.4.1), we see that $\varphi_s : Y_s \longrightarrow Y'_s(s \in \Delta^*)$ is an isometry between the Kähler spaces (Y_s, ω_{Y_s}) and (Y'_s, ω'_s). Then by Ascoli-Arzela's theorem and Montel's theorem, we can easily see that φ_s converges to an isometry

$$\varphi_0 : (Y_{0,reg}, \omega_{Y_0}) \to (Y'_{0,reg}, \omega_{Y'_0}) \tag{3.5.2}$$

and is holomorphic. This means that φ extends uniquely to a biholomorphism between Y and Y'.

The correspondence of the Hodge line bundles is obtained as follows. By (3.5.2), we have the equality:

$$\varphi_0^* \, \omega_{Y_0'}^n = \omega_{Y_0}^n.$$

Then by the equation (2.4.1), we obtain that

$$\varphi_0^* \, \Theta_{h_{L'_{X/Y}}} = \Theta_{h_{L_{X/Y}}}$$

holds on Y_0. Hence we see that φ extends uniquely to an equivalence between $(Y, (L_{X/Y}, h_{L_{X/Y}}))$ and $(Y', (L'_{X/Y}, h_{L'_{X/Y}}))$. $\square$

By Lemma 3.7, we see that $\mathcal{M}$ is separable. Then by the construction, we see that $\mathcal{M}$ is an separable algebraic space in the sense of Artin (cf. [Ar]). So far we have proven the following:

Proposition 3.8 *$\mathcal{M}$ is a separable algebraic space.* $\square$

4 Descent of the Monge-Ampère foliation

Let $f : X \to Y$ be an algebraic fiber space such that $\mathrm{Kod}(X/Y) \geqq 0$. Then we have the relative canonical measure $d\mu_{can,X/Y}$ as in Theorem 2.5. Then by Theorem 2.5, $\omega_{X/Y} := \sqrt{-1}\,\Theta_{d\mu_{can,X/Y}^{-1}}$ is a closed semipositive current on X which is generically C^∞ by Theorem 6.1 below. Then $\omega_{X/Y}$ defines a (possibly) singular foliation on X whose leaves are complex analytic. In this section, we analyse this foliation.

4.1 Weak semistability and Monge Ampère foliations

Let $f : X \longrightarrow Y$ be a surjective projective morphism of smooth projective varieties with connected fibers such that $\mathrm{Kod}(X/Y) \geqq 0$. Let m be a positive integer and let

$$E_m := f_*\mathcal{O}_X(mK_{X/Y}). \tag{4.1.1}$$

We assume that $E_m \neq 0$. Let

$$r := \mathrm{rank}\, E_m. \tag{4.1.2}$$

Let h_m be the (singular) hermitian metric on E_m defined by

$$h_m(\sigma, \tau) := \int_{X/Y} \sigma \cdot \bar{\tau} \cdot h_{K,X/Y}^{m-1}. \tag{4.1.3}$$

Then since $h_{K,X/Y}|X_y$ is an AZD of K_{X_y} for every $y \in Y^\circ$. We see that h_m is a locally bounded hermitian metric on $E_m|Y^\circ$. h_m defines an hermitian metric $\det h_m$ on $\det E_m$ and is locally bounded on Y°. By [T4] or [B-P], we see that $\sqrt{-1}\,\Theta_{\det h_m}$ is a closed positive current on Y°.

Let X^r denote the r-times fiber product of X over Y and let

$$f^r : X^r \to Y \tag{4.1.4}$$

be the natural morphism. Let δ_0 be the positive number as in Section 2 (cf. (2.6.3)) and let ε be a positive rational number such that $\varepsilon < \delta_0$. Let $H_{m,\varepsilon}$ be the singular hermitian metric on

$$(4.1.5) \qquad (1 + m\varepsilon)K_{X^r/Y} - \varepsilon f^{r*} \det h_m$$

constructed as in Section 2 (cf. (2.6.5)). We define the singular hermitian metric $H^+_{m,\varepsilon}$ on $K_{X^r/Y}$ by

$$(4.1.6) \qquad H^+_{m,\varepsilon} := (H_{m,\varepsilon} \cdot f^{r*}(\det h_m)^{\varepsilon})^{\frac{1}{1+m\varepsilon}}.$$

Since

$$(4.1.7) \qquad \Theta_{H^+_{m,\varepsilon}} = \frac{1}{1+m\varepsilon}\left(\Theta_{H_{m,\varepsilon}} + \varepsilon \cdot f^{r*}\Theta_{\det h_m}\right)$$

and $\sqrt{-1}\,\Theta_{\det h_m}$ is semipositive current on Y, we have the following lemma.

Lemma 4.1

$$(4.1.8) \qquad \sqrt{-1}\Theta_{H^+_{m,\varepsilon}} \geqq \frac{\varepsilon}{1+m\varepsilon}\sqrt{-1}f^{r*}\Theta_{h_{\det h_m}}$$

holds on X. □

We set

$$(4.1.9) \qquad \omega_{m,\varepsilon} := \sqrt{-1}\,\Theta_{H^+_{m,\varepsilon}}.$$

Let $d\mu_{can,X^r/Y}$ be the relative canonical measure on the algebraic fiber space $f^r : X^r \longrightarrow Y$. We set

$$(4.1.10) \qquad \omega_{X^r/Y} := \sqrt{-1}\,\partial\bar{\partial}\log\, d\mu_{can,X^r/Y}$$

$d\mu_{can,X^r/Y}$ is C^∞ on a nonempty Zariski open subset U of X^r by Theorem 6.1 below. Then we see that

$$(4.1.11) \qquad \mathcal{F} := \{\xi \in TX^r|U\ ;\ \omega_{X^r/Y}(\xi, \bar{\xi}) = 0\}$$

defines a singular foliation on an open subset V of U defined by

$$(4.1.12) \qquad U_0 := \{x \in U|\omega_{X^r/Y} \text{ is of maximal rank at } x\},$$

i.e., $\mathcal{F}$ is a Monge-Ampère foliation associated with the semi Kähler form $\omega_{X^r/Y}|U_0$. Hence $\mathcal{F}$ has complex analytic leaves on U_0 ([B-K]). But at this moment it is not clear $\mathcal{F}|U_0$ is a complex analytic foliation. By Lemma 4.1, we have the following lemma.

Lemma 4.2 $f^{r*}\Theta_{\det h_m}|\mathcal{F} \equiv 0$ *holds.* □

Proof. Since $d\mu^{-1}_{can,X^r/Y}$ is an AZD of $K_{X^r/Y}$, we see that for every leaf F of $\mathcal{F}$,

$$(4.1.13) \qquad \Theta_{H_{m,\varepsilon}}|F \equiv 0$$

holds.

In fact otherwise, we have a singular hermitian metric:

$$H_{m,\varepsilon}^{1/2} \cdot d\mu_{can,X^r/Y}^{-1/2} \tag{4.1.14}$$

on $K_{X^r/Y}$ with semipositive curvature and strictly bigger numerical dimension than $d\mu_{can,X^r/Y}^{-1}$. This contradicts the fact that $d\mu_{can,X^r/Y}^{-1}$ is an AZD of $K_{X^r/Y}$.

Hence combining (4.1.13) and Lemma 4.1, we see that

$$f^{r*}\Theta_{\det h_m}|F \equiv 0 \tag{4.1.15}$$

holds. This completes the proof of Lemma 4.2. □

4.2 Trivialization along the leaves on Y

Let (E_m, h_m) be as above. Then for any local holomorphic section ξ of E_m on some open subset V of Y,

$$h_m(\sqrt{-1}\Theta_{h_m}(\xi), \xi) \tag{4.2.1}$$

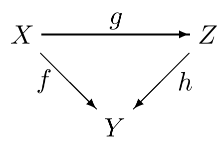
is a semipositive $(1,1)$-current on V by [T4] or [B-P]. Then the curvature $\det E_m$ is computed as:

$$\Theta_{\det h_m}(y) = \sum_{\alpha} h_m(\Theta_{h_m}(\mathbf{e}_\alpha), \mathbf{e}_\alpha), \tag{4.2.2}$$

where $\{\mathbf{e}_\alpha\}$ is an orthonormal basis of $E_{m,y}$ with respect to h_m. Hence $\sqrt{-1}\Theta_{\det h_m}$ is a closed semipositive current on Y. Since

$$f^{r*}\Theta_{\det h_m}|\mathcal{F} \equiv 0 \tag{4.2.3}$$

holds by Lemma 4.2, $(f^{r*}E_m, f^{r*}h_m)$ is flat along every leaf of $\mathcal{F}$. Hence this implies that for every $x \in U_0$ and an orthonormal basis $\{\mathbf{e}_{\alpha,x}\}$ of $(f^{r*}E_m)_x$ with respect to $f^{r*}h_m$, the parallel transport of $\{\mathbf{e}_{\alpha,x}\}$ along the leaf F of $\mathcal{F}$ containing x trivialize $(f^{r*}E_m)|F$ locally. Let



be the relative Iitaka fibration such that Z is the family of relative canonical models and let (L, h_L) be the Hodge $\mathbb{Q}$-line bundle on Z.

Then we have the following lemma :

Lemma 4.3 *For every leaf F of $\mathcal{F}$, the restriction*

$$(Z, (L, h_L))|f^r(F) \longrightarrow f^r(F) \tag{4.2.4}$$

is locally trivial. □

Proof of Lemma 4.3. By the flatness of (E_m, h_m) along $f^r(F)$, we see that the parallel transport in $(E_m, h_m)|f^r(F)$ locally trivialize E_m as above. This implies that $Z|f^r(F)$ is also trivialized by the parallel transport, since it is the (log) canonical image. Hence $(L, h_L)|f^r(F)$ is also locally trivial (as a metrized family of complex lines). □

4.3 Closedness of leaves

Let $\mathcal{M}$ be the moduli space which parametrizes the equivalence classes of

$$\{(Z_y, (L, h_L)|Z_y)|y \in Y^\circ\} \tag{4.3.1}$$

constructed as in Section 3. Now we consider the moduli map

$$\mu : Y^\circ \longrightarrow \mathcal{M} \tag{4.3.2}$$

defined by

$$\mu(y) := [(Z_y, (L, h_L)|Z_y))], \tag{4.3.3}$$

where $[(Z_y, (L, h_L)|Z_y))]$ denotes the equivalence class in $\mathcal{M}$. Then by Lemma 4.3 for every leaf F of $\mathcal{F}$, $f^r(F)$ is contained in the fiber of $\mu : Y^\circ \longrightarrow \mathcal{M}$. But by the construction, conversely, we see that for every $P \in \mathcal{M}$, $(f^r)^{-1}(\mu^{-1}(P))$ is contained in a leaf of $\mathcal{F}$.

Hence we conclude that for every leaf F of $\mathcal{F}$, $f^r(F)$ is an open subset of the fiber of μ and $f_*\mathcal{F}$ descends to the foliation defined by the moduli map μ. Hence we may take U_0 defined as (4.1.12) to be a nonempty Zariski open subset of X. By the above argument we have the following lemma.

Lemma 4.4 *In the above notations, we have the followings:*

(1) $\mathcal{F}$ decends to the foliation $df(\mathcal{F})$ on Y°,

(2) Every leaf of $df(\mathcal{F})$ is closed in Y° and is a fiber of the moduli map $\mu : Y^\circ \longrightarrow \mathcal{M}$,

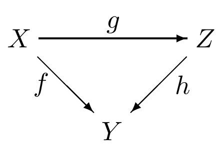
(3) $\mathcal{F}$ is a singular analytic foliation on X. □

Proof. The assertions (1) and (2) have already been proven. The assertion (3) follows from (2) and Lemma 4.3. □

5 Completion of the proof of Theorems 1.9,1.11 and 3.2

In this section we complete the proof of the proof of Theorems 1.9,1.11 and 3.2. But we shall omit the proof of Theorem 1.11, since the proof is essentially the same as the one of Theorem 1.9.

Let $f : X \longrightarrow Y$ be an algebraic fiber space. Suppose that for a general fiber F of f, $\mathrm{Kod}(F) \geqq 0$ holds. Then we have the relative Iitaka fibration:



such that Z is a family of relative canonical models. By taking a suitable modification of X, we may assume that g is a morphism.

Let (L, h_L) be the Hodge line bundle on Z as in Section 2.3. Then we have that

$$f_*\mathcal{O}_X(m!K_{X/Y}) = h_*\mathcal{O}_Y(m!(K_{Z/Y} + L)) \tag{5.0.4}$$

holds for every sufficiently large m. Let Y° be the complement of the discriminant locus of $f : X \to Y$. Let

$$\mu : Y^\circ \to \mathcal{M}. \tag{5.0.5}$$

be the moduli map (4.3.2) as above. Then by the quasi-unipotence of the monodromy ([La]), we see that there exists a positive integer b such that for every $m > 0$

$$\left(\det f_*\mathcal{O}_X(mK_{X/Y})\right)^{\otimes b} \tag{5.0.6}$$

and

$$\left(f_*\mathcal{O}_X(mK_{X/Y})\right)^{\otimes b} \tag{5.0.7}$$

decend to vector bundles on $\mathcal{M}$. Then the relative canonical measure $d\mu_{can,X/Y}$ defines a L^2 metric h_m on $f_*\mathcal{O}_X(mK_{X/Y})$ as in (4.1.3) and then h_m defines a singular hermitian metric $\det h_m$ on $\det f_*\mathcal{O}_X(mK_{X/Y})$. The metric h_m is an invariant metric by Theorem 2.1. In the above notations, we have the following lemma.

Lemma 5.1 *Let a be the minimal positive integer such that $f_*\mathcal{O}_X(aK_{X/Y}) \neq 0$. Let m_0 be a sufficiently large positive integer. Then*

$$F := \mu_* \left(\det f_*\mathcal{O}_X(m_0 a K_{X/Y})\right)^{\otimes b} \tag{5.0.8}$$

is a line bundle on $\mathcal{M}$ with the hermitian metric h_F such that

(1) $\mu^* h_F = h_{am_0}$,

(2) *For every subvariety V in $\mathcal{M}$, $(F|_V, h_F|_V)$ is big on V (cf. Definition 7.4).*

□

Proof of Lemma 5.1. The first assertion (1) is trivial by the construction and the birational invariance of the canonical measures.

By Lemma 4.4 we have the followings:

(1) The foliation $\mathcal{F}$ decends to a foliation $df(\mathcal{F})$ on Y.

(2) $\omega_{Z/Y}$ is generically strictly positive in the transverse direction with respect to $\mathcal{F}$.

(3) μ contracts the leaf of $df(\mathcal{F})$.

Then the second assertion (2) holds, if $V = \mathcal{M}$ by the construction,

For a general V, the assertion (2) follows from the functoriality. □

By Proposition 3.8, we see that $\mathcal{M}$ has a structure of a separable algebraic space. Then by Lemma 5.1 and the quasiprojectivity criterion Theorem 7.3 below, we see that $\mathcal{M}$ is quasiprojective. This completes the proof of Theorem 3.2.

To complete the proof of Theorem 1.9 we use the weak semipositivity (cf. (1.4.1) or (3)(b)) in Theorem 1.9. Then we see that $\mu_*\left(f_*\mathcal{O}_X(mK_{X/Y})\right)^{\otimes b}$ is globally generated on $\mathcal{M}$ for every sufficiently large and divisible m. Then since

$$\mu^*\left(\mu_*\left(f_*\mathcal{O}_X(mK_{X/Y})\right)^{\otimes b}\right) = \left(f_*\mathcal{O}_X(mK_{X/Y})\right)^{\otimes b} \tag{5.0.9}$$

holds by the construcion, we see that $\left(f_*\mathcal{O}_X(mK_{X/Y})\right)^{\otimes b}$ is globally generated on Y° for every sufficiently large m. Then by the finite generation of canonical rings ([B-C-H-M]), this implies that there exists a positive integer m_0 such that $f_*(mK_{X/Y})$ is globally generated over Y°, if $b|m$ and $m \geqq m_0$. This completes the proof of Theorem 1.9. The proof of Theorem 1.11 is similar. □

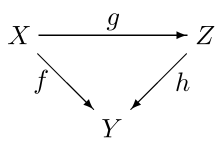
6 Parameter dependence of canonical measures

In this section we prove the following regularity theorem for the relative canonical measure $d\mu_{can,X/Y}$ constructed as in Theorem 2.5.

Theorem 6.1 *Let $f : X \longrightarrow Y$ be an algebraic fiber space with* $\mathrm{Kod}(X/Y) \geqq 0$. *Let $d\mu_{can,X/Y}$ be the relative canonical measure on X constructed as in Theorem 2.5. Then $d\mu_{can,X/Y}$ is C^∞ on a nonempty Zariski open subset of X.* □

Remark 6.2 *Essentially the same regularity result holds for the relative log canonical measure $d\mu_{can,(X,D)/Y}$ for the family of KLT pairs $f : (X, D) \longrightarrow Y$ (cf. Theorem 2.5). The proof requires the dynamical construction of (log) canonical measures as in [T8], but otherwise the proof is the same as the one of Theorem 6.1.* □

Here I would like to explain the scheme of the proof of Theorem 6.1. Let



be the relative canonical model. Then the relative canonical Kähler current $\omega_{Z/Y}$ satisfies a partial differential equation on each fiber. Hence the regularity of $\omega_{Z/Y}$ (hence also the regularity of $d\mu_{can,X/Y}$) may be deduced by the parameter dependence of the solution of Monge-Ampère equations.

But after some time, I realized that this approach is extremely difficult to implement. The reason is as follows. Usually since the canonical Kähler current is unique on each fiber of $h : Z \longrightarrow Y$, it is natural to consider the variation of the canonical Kähler current satisfies a partial differential equation on each fiber which is (as is easily seen) essentially the Laplace equation with respect to the cacnonical Kähler current. So far there is no difficulty. The next step is to apply the implicit function theorem. Here the major difficulty arises. Because although the canonical Kähler current is C^∞ on a nonempty Zariski open subset of each fiber, it is singular on a proper analytic subset of the fibers. Hence it seems to be extremely difficult to fix the appropriate function space to apply the implicit function theorem. Also it seems to be very difficult to know the precise asymptotic behavior of the canonical Kähler current near the singularities.

Hence I decided to use the dynamical construction of canonical Kähler currents to deduce the (generic) horizontal smoothness of the relative canonical Kähler current.

The advantage of this approach is that we can deduce the smoothness in terms of Hörmander's L^2-estimate for $\bar{\partial}$-operators. Because in each step, we only need to consider the variation of Bergman projections which is essentially a linear problem. In this way, we can deduce the regularity of the relative canonical Kähler current by the inductive estimates of Bergman projections.

This inductive estimate is very smilar to the construction of Kuranishi family.

6.1 Dynamical construction of the canonical Kähler currents

The canonical Kähler current in Theorem 1.9 can be constructed as the limit of a dynamical system as in ([T4]).

Let X be a smooth projective n-fold with $\mathrm{Kod}(X) \geqq 0$. And let

$$f : X - \cdots \to Y \tag{6.1.1}$$

be the Iitaka fibration associated with the complete linear system $|m_0!K_X|$ for some sufficiently large positive integer m_0. By taking a suitable modifications, we assume the followings:

(1) Y is smooth and f is a morphism.

(2) $f_*\mathcal{O}_X(m_0!K_{X/Y})^{**}$ is a line bundle on Y, where $**$ denotes the double dual.

We set

$$L_{X/Y} := \frac{1}{m_0!} f_*\mathcal{O}_X(m_0!K_{X/Y})^{**}. \tag{6.1.2}$$

In [F-M] this $L_{X/Y}$ is denoted by $L_{X/Y}$. Let a be positive integer such that $f_*\mathcal{O}_X(aK_{X/Y}) \neq 0$. Then we see that

$$H^0(X, \mathcal{O}_X(maK_X)) \simeq H^0(Y, \mathcal{O}_Y(ma(K_Y + L_{X/Y}))) \tag{6.1.3}$$

holds for every $m \geqq 0$. In particular $\mathrm{Kod}(X) = \dim Y$ holds. Hence by (6.1.3), we see that $K_Y + L_{X/Y}$ is big. Let A be an ample line bundle on Y such that for every pseudoeffective singular hermitian line bundle (F, h_F) on Y, $\mathcal{O}_Y(jK_Y + A + F) \otimes \mathcal{I}(h_F)$ is globally generated for every $0 \leqq j \leqq a$.

The existence of such an ample line bundle A follows from Nadel's vanishing theorem ([N, p.561]). Let h_A be a C^∞ hermitian metric on A with strictly positive curvature. We construct a sequence of singular hermitian metrics $\{h_m\}_{m \geqq 1}$ and a sequence of Bergman kernels $\{K_m\}_{m \geqq 1}$ as follows.

We set

$$K_1 := \begin{cases} K(Y, K_Y + A, h_A), & \text{if } a > 1 \\ \\ K(Y, K_Y + L_{X/Y} + A, h_{L_{X/Y}} \cdot h_A), & \text{if } a = 1 \end{cases} \tag{6.1.4}$$

where for a singular hermitian line bundle (F, h_F) $K(Y, K_Y + F, h_F)$ is (the diagonal part of) the Bergman kernel of $H^0(Y, \mathcal{O}_Y(K_Y + F) \otimes \mathcal{I}(h_F))$ as (1.5.9). Then we set

$$h_1 := (K_1)^{-1}. \tag{6.1.5}$$

We continue this process. Suppose that we have constructed K_{m-1} and the singular hermitian metric h_{m-1} on $(m-1)K_Y + \lfloor \frac{m-1}{a} \rfloor a L_{X/Y} + A$. Then we define
(6.1.6)

$$K_m := \begin{cases} K(Y, mK_Y + \lfloor \frac{m}{a} \rfloor a L_{X/Y} + A, h_{m-1}) & \text{if } m \not\equiv 0 \bmod a \\ \\ K(Y, m(K_Y + L_{X/Y}) + A, h^a_{L_{X/Y}} \otimes h_{m-1}) & \text{if } m \equiv 0 \bmod a \end{cases}$$

and

$$h_m := (K_m)^{-1}. \tag{6.1.7}$$

Thus inductively we construct the sequences $\{h_m\}_{m \geqq 1}$ and $\{K_m\}_{m \geqq 1}$. This inductive construction is essentially the same one originated by the author in [T3]. The following theorem asserts that the above dynamical system yields the canonical Kähler current on Y.

Theorem 6.3 *([T7]) Let X be a smooth projective variety of nonnegative Kodaira dimension and let $f : X \longrightarrow Y$ be the Iitaka fibration as above. Let m_0 and $\{h_m\}_{m \geq 1}$ be the sequence of hermitian metrics as above and let n denote* $\dim Y$. *Then*

$$h_\infty := \liminf_{m \to \infty} \sqrt[m]{(m!)^n \cdot h_m} \tag{6.1.8}$$

is a singular hermitian metric on $K_Y + L_{X/Y}$ such that

$$\omega_Y = \sqrt{-1}\,\Theta_{h_\infty} \tag{6.1.9}$$

holds, where ω_Y is the canonical Kähler current on Y as in Theorem 1.9 and $n = \dim Y$.

More precisely

$$K_\infty := \lim_{m \longrightarrow \infty} h_A^{\frac{1}{am}} \cdot K_{am}^{\frac{1}{am}}$$

exists in L^1-topology (as a limit of bounded volume forms on Y) and $h_\infty = K_\infty^{-1}$ holds. In particular $\omega_Y = \sqrt{-1}\Theta_{h_\infty}$ (in fact h_∞) is unique and is independent of the choice of A and h_A. □

Remark 6.4 *Similar theorem holds for a KLT pair with nonnegative (log) Kodaira dimension. See [T8]. But the corresponding dynamical system is not a single dynamical system, but is an infinite sequence of dynamical systems.* □

6.2 Family of dynamical systems

In this subsection, we shall consider the dynamical systems in Section 6.1 on the relative canonical models. Let

$$\begin{array}{ccc} X & \xrightarrow{\quad g \quad} & Z \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle h} & \\ & Y & \end{array}$$

be the relative canonical model as in Theorem 6.1.

Now we consider the relative version of the construction in Section 6.1. Let A be a sufficiently ample line bundle on Z and let h_A be a C^∞-hermitian metric on A. We set

$$Y^\circ := \{y \in Y | f : X \longrightarrow Y \text{ is smooth over } y\}. \tag{6.2.1}$$

For every $y \in Y^\circ$ we construct a sequence of singular hermitian metrics $\{h_{m,y}\}_{m \geqq 1}$ and a sequence of Bergman kernels $\{K_{m,y}\}$ as follows.

We set

$$K_{1,y} := \begin{cases} K(Z_y, K_{Z_y} + A|Z_y, h_A|Z_y), & \text{if } a > 1 \\ \\ K(Z_y, K_{Z_y} + L_{X/Z}|Z_y + A|Z_y, h_{L_{X/Z}} \cdot h_A|Z_y), & \text{if } a = 1 \end{cases} \tag{6.2.2}$$

Then we set

$$h_{1,y} := (K_{1,y})^{-1}. \tag{6.2.3}$$

We continue this process. Suppose that we have constructed K_{m-1} and the singular hermitian metric h_{m-1} on $(m-1)K_Z + \lfloor \frac{m-1}{a} \rfloor a L_{X/Z} + A$. Then we define

(6.2.4)

$$K_{m,y} := \begin{cases} K(Z_y, mK_{Z_y} + \lfloor \frac{m}{a} \rfloor a L_{X/Z}|Z_y + A, h_{m-1,y}) & \text{if } m \not\equiv 0 \bmod a \\ \\ K(Y, m(K_{Z_y} + L_{X/Z}|Z_y) + A, h^a_{L_{X/Z}}|Z_y \otimes h_{m-1,y}) & \text{if } m \equiv 0 \bmod a \end{cases}$$

and

$$h_{m,y} := (K_{m,y})^{-1}. \tag{6.2.5}$$

Thus inductively we construct the sequences $\{h_{m,y}\}_{m \geqq 1}$ and $\{K_{m,y}\}_{m \geqq 1}$ for every $y \in Y^\circ$.

By [B-P], we see that $\sqrt{-1}\partial\bar{\partial} \log K_m$ extends to a closed positive current on Y. Hence by Theorem 6.3, we see that the relative canonical Kähler current:

$$\omega_{Z/Y} := \lim_{m \longrightarrow \infty} \frac{\sqrt{-1}}{m} \partial\bar{\partial} \log K_m \tag{6.2.6}$$

extends to a closed positive current on Y. We denote the extended current again by $\omega_{Z/Y}$. We shall prove Theorem 6.1 by estimating the variation of $K_{m,y}$ with respect to the parameter $y \in Y^\circ$.

6.3 Variation of Bergman projections

Let U be an open subset of Y° such that U is biholomorphic to the unit polydisk Δ^k in $\mathbb{C}^k$ with ceneter O via a local coordinate $(y_1, \cdots, y_k)$. Let $Z_U := h^{-1}(U)$ and let

(6.3.1) $$h_U : Z_U \longrightarrow U$$

be the restriction of h. Let us trivialize $h_U : Z_U \longrightarrow U$ differentiably as

$$\Phi : Z_U \longrightarrow Z_0 \times U,$$

where Z_0 denotes the central fiber $h_U^{-1}(O)$. Let

(6.3.2)
$$P_{m,y} : L^2(Z_y, A_y + mK_{Z_y} + \lfloor \frac{m}{a} \rfloor aL_{X/Z,y}) \rightarrow H^0(Z_y, \mathcal{O}_{Z_y}(A_y + mK_{Z_y} + \lfloor \frac{m}{a} \rfloor aL_{X/Z,y}))$$

the Bergman projection, i.e., the orthogonal projection with respect to the L^2-inner product g_m defined by

(6.3.3) $$g_m(\sigma_y, \sigma'_y) := \int_{Z_y} \sigma \cdot \overline{\sigma'} \cdot h_{m-1,y},$$

if $a \not| m$ and

(6.3.4) $$g_m(\sigma_y, \sigma'_y) := \int_{Z_y} \sigma \cdot \overline{\sigma'} \cdot h_{m-1,y} \cdot h^a_{X/Z,y},$$

if $a|m$. Hearafter we shall omit g_m, if without fear of confusion. Then the above trivialization gives a trivialization:

(6.3.5)
$$\mathcal{L}^2(Z_U, A|U + mK_{Z/Y}|U + \lfloor \frac{m}{a} \rfloor aL_{X/Z}|U) \rightarrow L^2(Z_0, A_0 + mK_{Z_0} + \lfloor \frac{m}{a} \rfloor aL_{X/Z,0}) \times U,$$

where $\mathcal{L}^2(Z_U, A|U + mK_{Z/Y}|U + \lfloor \frac{m}{a} \rfloor aL_{X/Z}|U)$ denotes the Hilbert space bundle

(6.3.6) $$\pi_{L^2} : \mathcal{L}^2(Z_U, A|U + mK_{Z/Y}|U + \lfloor \frac{m}{a} \rfloor aL_{X/Z}|U) \rightarrow U$$

such that
$$\pi_{L^2}^{-1}(y) := L^2(Z_y, A_y + mK_{Z_y} + \lfloor \frac{m}{a} \rfloor aL_{X/Z}|Z_y).$$

Let
$$\bar{\partial}_y : C^\infty(Z_y, A_y + mK_{Z_y} + \lfloor \frac{m}{a} \rfloor aL_{X/Z}|Z_y) \rightarrow A^{0,1}(Z_y, A_y + mK_{Z_y} + \lfloor \frac{m}{a} \rfloor aL_{X/Z}|Z_y)$$

denote the $\bar{\partial}$-operator. We set

(6.3.7) $$\mathbb{H}_{m,y} := H^0(Z_y, \mathcal{O}_{Z_y}(A_y + mK_{Z_y} + \lfloor \frac{m}{a} \rfloor aL_{X/Z,y})).$$

Let $\sigma_y \in C^\infty(Z_y, A_y + mK_{Z_y} + \lfloor \frac{m}{a} \rfloor aL_{X/Z}|Z_y)$ be an arbitrary element. Let us consider the $\bar{\partial}$-equation:

(6.3.8) $$\bar{\partial}_y(Q_{m,y}(\sigma_y)) = \bar{\partial}_y \sigma_y$$
(6.3.9) $$Q_{m,y}(\sigma_y) \perp \mathbb{H}_{m,y}.$$

Then

$$Q_{m,y} : \mathbb{L}^2_m \longrightarrow \mathbb{H}^{\perp}_{m,y} \tag{6.3.10}$$

is the orthogonal projection. Hence the Bergman projection is given by

$$P_{m,y}(\sigma_y) = \sigma_y - Q_{m,y}(\sigma_y). \tag{6.3.11}$$

This implies that

$$D_y P_{m,y} = -D_y Q_{m,y} \tag{6.3.12}$$

holds, where D_y denotes the hermitian connection with respect to g_m (cf. (6.3.4)).

Let us calculate the variation of $P_{m,y}$ at $y = 0$. Let $\sigma \in \mathbb{H}_{m,0}$ and let us extend σ as a section $\tilde{\sigma}$ of the Hilbert space bundle (6.3.6) by the parallel displacement with respect to the hermitian connection with respect to g_m along a smooth curve on Y. We note that since the connection may not be flat, the parallel displacement depends on the choice of the smooth curve. Hereafter we shall fix a differential curve to fix the extension $\tilde{\sigma}$.

Then differentiating the equation:

$$\bar{\partial}_y \tilde{\sigma}(y) = \bar{\partial}_y Q_{m,y}(\tilde{\sigma}(y))$$

with respect to y at $y = 0$, we obtain the equation:

$$\theta_{m,0}(\sigma) = \bar{\partial}_0(D_y Q_{m,y}(\sigma)) \tag{6.3.13}$$

where $\theta_{m,0}$ represents the Kodaira-Spencer class.

We shall decompose $D_y Q_{m,y}(\tilde{\sigma})$ as

$$D_y Q_{m,y}(\tilde{\sigma}) = D_y Q_{m,y}(\tilde{\sigma})_{\mathbb{H}} + D_y Q_{m,y}(\tilde{\sigma})_{\mathbb{H}^{\perp}} \tag{6.3.14}$$

corresponding to the orthogonal decomposition

$$\mathbb{L}_{m,y} = \mathbb{H}_{m,y} \oplus \mathbb{H}^{\perp}_{m,y}.$$

Then we have that

$$\theta_{m,0}(\sigma) = \bar{\partial}_0(D_y Q_{m,y}(\sigma))_{\mathbb{H}^{\perp}} \tag{6.3.15}$$

holds, i.e., $D_y Q_{m,y}(\sigma)_{\mathbb{H}^{\perp}}$ is the minimal solution of (6.3.15). Now we shall fix the standard Kähler metric on $Y \sim \Delta$ induced by the standard Kähler metric on $\mathbb{C}$. Let us estimate the operator norm of

$$(D_y Q_{m,y})_{\mathbb{H}^{\perp}} : \mathbb{H}_{m,0} \longrightarrow \mathbb{H}^{\perp}_{m,0}. \tag{6.3.16}$$

The norm is estimated by Hörmander's L^2-estimate for $\bar{\partial}$-operators.

First we see that $\theta_{m,0}$ consists of the Kodaira-Spencer class of the deformation of $Z_{m,0}$ and the Kodaira-Spencer class of the bundle $\lfloor m/a \rfloor a L_{X/Z,0}$. Then there exists a positive constant C_0 independent of m such that

$$\parallel \theta_{m,0} \parallel_{L^\infty} \leqq C_0 \tag{6.3.17}$$

holds. On the other hand by Hörmander's L^2-estimate, we see that

Lemma 6.5 *There exists a positive constant C_1 such that*

(6.3.18) $$\| (D_y Q_{m,y})_{\mathbb{H}^\perp} \| \leqq C_1$$

holds for every $m \geqq 1$. □

For $k = 2$, differentiating (6.3.15), we have the equation

(6.3.19) $$(D_y \theta_{m,y})(\sigma)(0) = \bar{\partial}_0 (D_y^2 Q_{m,y}(\sigma)_{\mathbb{H}^\perp}) + \theta_{m,0}(D_y Q_{m,y}(\sigma)_{\mathbb{H}^\perp}).$$

Hence we may estimate $D_y^2 Q_{m,y}(\sigma)_{\mathbb{H}^\perp}$ as

$$\| D_y^2 Q_{m,y}(\sigma)_{\mathbb{H}^\perp} \| \leqq C_2$$

for some positive constant C_2 independent of m. For $k \geqq 2$, inductively we have:

Lemma 6.6 *For every $k \geqq 1$, there exists a positive constant C_k such that*

(6.3.20) $$\| (D_y^k Q_{m,y})_{\mathbb{H}^\perp} \| \leqq C_k$$

holds for every $m \geqq 1$. □

6.4 Estimate of the holomorphic part

Now we shall estimate the holmorphic part of the derivatives of $Q_{m,y}$ at $y = 0$. Let

$$\tau_{hol} \in H^0(Z, \mathcal{O}_Z(mK_{Z/Y} + \lfloor \frac{m}{a} \rfloor aL_{X/Z})),$$

be an arbitrary holomorphic section. Differentiating the trivial identity:

(6.4.1) $$g_m(Q_{m,y}(\tilde{\sigma}), \tau_{hol}) = 0,$$

we obtain that for every positive integer ℓ

(6.4.2) $$\sum_{i+j=\ell} \int_X h_{m-1} \left(D_y^i Q_{m,y}(\tilde{\sigma}), D_y^j \tau_{hol} \right) = 0$$

holds, where g_m denotes the L^2-metric defined by (6.3.3) and (6.3.4). Hence (6.4.2) implies that we can estimate $D_y^k Q_{m,y}$ in terms of the esimate of $\{D_y^\ell Q_{m-1,y}\}_{\ell=0}^{k-1}$.

Lemma 6.7 *There exists a positive constant C'_k independent of m such that*

(6.4.3) $$\| (D_y^k Q_{m,y})_{\mathbb{H}} \| \leqq C'_k$$

holds for every $m \geqq 1$. □

Proof. Let τ_0 be an element of $H^0(Z_0, \mathcal{O}_{Z_0}(mK_{Z/Y} + \lfloor \frac{m}{a} \rfloor aL_{X/Z}))$. We extend τ_0 to the ℓ-th infinitesimal neighbourhood $Z_0^{(\ell)}$ of Z_0 by the successive extension. By the L^2-estimates, we may take the extension $\tau^{(\ell)}$ so that

(6.4.4) $$\| D^\ell \tau^{(\ell)} \|_{(\ell)} \leqq C_{(\ell)}$$

holds for some positive constant $C_{(\ell)}$ independent of m. Then replacing τ_{hol} by $\tau^{(\ell)}$ in (6.4.2), by induction on ℓ, we have the lemma. □

6.5 Variation of Bergman kernels

The parameter dependence of Bergman kernels can be deduced from the variation of the Bergman projections. By the trivial equality:

$$(6.5.1) \qquad (D_y^k P_m)(\tilde{\sigma})(z) = \int_{X(\zeta)} h_{m-1} \cdot D_y^k K_m(z,\zeta) \cdot \tilde{\sigma}(\zeta),$$

(where the integral is taken with respect to the parameter ζ)

$$(6.5.2) \qquad \sum_{i=0}^{k} (D_y^{k-i} K_m(z,w), D_y^i Q_m(\tilde{\sigma}))_{g_m} = 0$$

holds. Then by induction on k and the extremal property of Bergman kernels, there exists a positive constant C independent of m such that

$$(6.5.3) \qquad |(D_y^k P_m)(\tilde{\sigma})(z)|_{h_{m-1}} = \left(D_y^k K_m(z,\zeta), \tilde{\sigma}(\zeta)\right) \leqq C \cdot m^{\frac{n}{2}} \parallel \tilde{\sigma} \parallel$$

holds for every $z \in Z_y$. Combining (6.5.3), this implies that there exists a positive constant $C(k)$ depending only on k such that

$$(6.5.4) \qquad |m^{-n} D_y^k K_m(z,\zeta)|_{h_{m-1}} < C(k)$$

holds on Z_y. Hence by the Sobolev's embedding theorem, we see that there exists a positive constant $\hat{C}_k$ independent of m such that

$$(6.5.5) \qquad |\left((m!)^{-n} K_m\right)^{\frac{1}{m}}|_{C^k} \leqq \hat{C}_k$$

holds on Z_y By Theorem 6.3, this means that the relative canonical measure $d\mu_{can,X/Y}$ is C^∞ on a nonempty Zariski open subset of X. This completes the proof of Theorem 6.1. □

7 Appendix

In this section, we collect several analytic tools used in this article.

7.1 Ampleness criterion for line bundles on quasiprojective varieties

In this section we prove a criterion of quasiprojectivity used in the previous section. The criterion is almost the same as in [Sch-T]. But it is slightly stronger.

Let X be a not necessarily reduced algebraic space with compactification $\overline{X}$ in the sense of algebraic spaces, and let L be a holomorphic line bundle on $\overline{X}$ with a *positive* singular hermitian metric h in the following sense.

Definition 7.1 *Let Z be a reduced complex space and L a holomorphic line bundle. A* singular *hermitian metric h on L is a singular hermitian metric h on $L|Z_{reg}$ with the following property: There exists a desingularization $\pi : \widetilde{Z} \longrightarrow Z$ such that h can be extended from Z_{reg} to a singular hermitian metric $\widetilde{h}$ on π^*L over $\widetilde{Z}$.* □

Condition 7.2 (P) *We say that the positivity condition (P) holds, if*

(i) For all $p \in X$ and any holomorphic curve $C \subset \overline{X}$ through p the (positive, d-closed) current $\sqrt{-1}\Theta_h|C$ is well-defined, and the Lelong number $\nu(\sqrt{-1}\Theta_h|C, p)$ vanishes,

(ii) For any smooth locally closed subspace $Z \subset X$ of $\dim Z > 0$, $h|_Z$ is well defined and $(L|_Z, h|_Z)$ is big (cf. Definition 7.4 below). □

Now we state the criterion.

Theorem 7.3 *Let X be an irreducible, not necessarily reduced algebraic space with a compactification $\overline{X}$. Let L be a holomorphic line bundle on $\overline{X}$. The map*

$$\Phi_{|mL|} : \overline{X} \rightharpoonup \mathbb{P}^N(m),$$

where $N(m) = \dim |mL|$, defines an embedding of X for sufficiently large m, if it satisfies condition (P). □

The proof of Theorem 7.3 is essentially the same as the one of [Sch-T, Theorem 6] except the use of Theorem 7.6 below to perturbe the metric to a metric with strictly positive curvature.

7.2 Kodaira's lemma for big pseudoeffective line bundles

In this subsection, we prove a singular hermitian version of Kodaira's lemma (cf. [K-O, Appendix]).

First we shall define the big singular hermitian line bundle.

Definition 7.4 (L, h_L) *be a pseudoeffective singular hermitian line bundle on a projective manifold X. We set*
(7.2.1)
$\nu_{num}(L, h_L) := \sup\{\dim V \mid V$ is a subvariety of X such that $h_L|_V$ is well defined

(7.2.2) and $(L, h_L)^{\dim V} \cdot V > 0\}$.

We call $\nu_{num}(L, h_L)$ the **numerical Kodaira dimension** *of (L, h_L). If $\nu_{num}(L, h_L) =$* $\dim X$ *we say that (L, h_L) is big.* □

Lemma 7.5 *Let X be a smooth projective variety and let $|H|$ be a very ample linear system. Then there exists a smooth member $H' \in |H|$, such that*

(7.2.3) $$\mathcal{I}(h_L^m) \otimes \mathcal{O}_{H'} = \mathcal{I}(h_L^m \mid_{H'})$$

holds for every $m \geqq 1$. □

Proof of Lemma 7.5. Let A be a sufficiently ample line bundle such that $\mathcal{O}_X(A + mL)) \otimes \mathcal{I}(h_L^m)$ is globally generated for all $m \geqq 1$. Let $\{\sigma_j^{(m)}\}_{j=1}^{N_m}$ be a (complete) basis of $H^0(X, \mathcal{O}_X(A + mL)) \otimes \mathcal{I}(h_L^m))$. We consider the subset

(7.2.4) $$U := \{F \in |H|; F \text{ is smooth}, \int_F \mid \sigma_j^{(m)} \mid^2 \cdot h_L^m \cdot h_A \cdot dV_F < +\infty$$

(7.2.5) for every m and $1 \leqq j \leqq N_m.\}$

of $|H|$, where dV_F denotes the volume form on F induced by the Kähler form ω. We claim that such U is the complement of at most a countable union of proper subvarieties of $|H|$. Let us fix a positive integer m.

$$E_m := \{F \in |H|; F \text{ is smooth}, \int_F |\sigma_j^{(m)}|^2 \cdot h_L^m \cdot h_A \, dV_F = +\infty\} \tag{7.2.6}$$

is of measure 0 by Fubini's theorem. Then since $U = |H| - \cup_{m=1}^{\infty} E_m$, we complete the proof of Lemma 7.5. □

Theorem 7.6 *Let X be a projective manifold and let (L, h_L) be a big psedoeffective singular hermitian line bundle. Then there exists a singular hermitian metric h_L^+ on L such that*

(1) $\sqrt{-1}\Theta_{h_L^+}$ is strictly positive everywhere on X,

(2) $h_L^+ \geqq h_L$ holds on X.

□

Let us explain the relation between Theorem7.6 and the original Kodaira's lemma . Let D be an ample divisor on a smooth projective variety X. Let us identify divisors with line bundles. By Kodaira's lemma, there exists a C^∞ hermitian metrics h_D, h_E on D, E respectively (the notion of hermitian metrics naturally extends to the case of $\mathbb{Q}$-line bundles) such that the curvature of $h_D \cdot h_E^{-1}$ is stricly positive. Let σ_E be a multivalued holomorphic section of E with divisor E such that $h_E(\sigma_E, \sigma_E) \leqq 1$ on X. Then

$$h_D^+ := \frac{h_D}{h_E(\sigma_E, \sigma_E)} \tag{7.2.7}$$

is a singular hermitian metric on D such that

(1) $\sqrt{-1}\Theta_{h_D^+}$ is strictly positive everywhere on X.

(2) $h_D \leqq h_D^+$ holds on X.

In this way Theorem 7.6 can be viewed as an analogue of the usual Kodaira's lemma to the case of big pseudoeffective singular hermitian line bundles.

7.3 Proof of Theorem 7.6

The proof of Theorem 7.6 presented here is not very much different from the original proof of Kodaira's lemma (cf. [Ka2] or [K-O, Appendix]). But it requires estimates of Bergman kernels and additional care for the multiplier ideal sheaves.

Let X be a smooth projective variety of dimension n and let (L, h_L) be a big pseudoeffective singular hermitian line bundle on X. Let ω be a Kähler form on X and let dV be the associated volume form on X. Let H be a smooth very ample divisor on X.

Lemma 7.7 *There exists a positive integer m_0 such that $m_0(L,h_L)-H$ is big, i.e.,*

$$\limsup_{\ell\to\infty} \ell^{-n}\cdot \dim H^0(X,\mathcal{O}_X(\ell(m_0L-H)\otimes\mathcal{I}(h_L^{m_0\ell}))>0 \tag{7.3.1}$$

holds. □

Proof of Lemma 7.7. Replacing H by a suitable member of $|H|$, by Lemma 7.5, we may assume that

$$\mathcal{I}(h_L^m)\mid_H=\mathcal{I}(h_L^m\mid_H) \tag{7.3.2}$$

holds for every $m \geqq 1$. Let us consider the exact sequence

$$0\to H^0(X,\mathcal{O}_X(mL-H)\otimes\mathcal{I}(h_L^m))\to H^0(X,\mathcal{O}_X(mL)\otimes\mathcal{I}(h_L^m)) \tag{7.3.3}$$

$$\to H^0(H,\mathcal{O}_H(mL)\otimes\mathcal{I}(h_L^m\mid_H)). \tag{7.3.4}$$

Then since $\mu(L,h_L)>0$ and

$$\dim H^0(H,\mathcal{O}_H(mL)\otimes\mathcal{I}(h_L^m\mid_H))=O(m^{n-1}) \tag{7.3.5}$$

we see that for every sufficiently large m,

$$H^0(X,\mathcal{O}_X(mL-H)\otimes\mathcal{I}(h_L^m))\neq 0 \tag{7.3.6}$$

holds. □

To prove Lemma 7.7, we need to refine the above argument a little bit. Let m_0 be a positive integer such that

$$m_0>n\cdot\frac{(L,h_L)^{n-1}\cdot H}{(L,h_L)^n} \tag{7.3.7}$$

holds. For very general $H_1^{(\ell)},\cdots H_\ell^{(\ell)}\in|H|$, by Lemma 7.5, replacing m by $m_0\ell$ and H by ℓH, we have the exact sequence

(7.3.8)

$$0\to H^0(X,\mathcal{O}_X(\ell(m_0L-H))\otimes\mathcal{I}(h_L^{m_0\ell}))\to H^0(X,\mathcal{O}_X(m_0\ell L)\otimes\mathcal{I}(h_L^{m_0\ell})).$$

(7.3.9)

$$\to\oplus_{i=1}^{\ell}H^0(H_i^{(\ell)},\mathcal{O}_{H_i}(m_0\ell L)\otimes\mathcal{I}(h_L^{m_0\ell}\mid_{H_i})).$$

We note that $\{H_i^{(\ell)}\}_{i=1}^{\ell}$ are chosen for each ℓ. If we take $\{H_i^{(\ell)}\}_{i=1}^{\ell}$ very general, we may assume that

$$\dim H^0(H_i^{(\ell)},\mathcal{O}_{H_i}(mL)\otimes\mathcal{I}(h_L^m\mid_{H_i})) \tag{7.3.10}$$

is independent of $1\leqq i\leqq\ell$ for every m. This implies that

$$\limsup_{\ell\to\infty}\ell^{-n}\cdot\dim H^0(X,\mathcal{O}_X(\ell(m_0L-H))\otimes\mathcal{I}(h_L^{m_0\ell})) \tag{7.3.11}$$

$$\geqq\frac{1}{n!}(L,h_L)^n\cdot m_0^n-\frac{1}{(n-1)!}\{(L,h_L)^{n-1}\cdot H\}\cdot m_0^{n-1} \tag{7.3.12}$$

holds. By (7.3.7), we see that

$$\frac{1}{n!}(L,h_L)^n \cdot m_0^n - \frac{1}{(n-1)!}\{(L,h_L)^{n-1} \cdot H\}m_0^{n-1} \tag{7.3.13}$$

is positive. This completes the proof of Lemma 7.7. □

Let A be a sufficiently ample line bundle on X and let h_A be a C^∞ hermitian metric such that the curvature of h_A is everywhere strictly positive on X. Here the meaning of "sufficiently ample" will be specified later. Let m be a positive integer. Let us consider the inner product

$$(\sigma,\sigma') := \int_X h_A \cdot h_L^m \cdot \sigma \cdot \bar{\sigma}'\, dV \tag{7.3.14}$$

on $H^0(X,\mathcal{O}_X(A+mL)\otimes\mathcal{I}(h_L^m))$ and let K_m be the associated (diagonal part of) Bergman kernel. Let us consider the subspace:

$$H^0(X,\mathcal{O}_X(A+\ell(m_0L-H))\otimes\mathcal{I}(h_L^{m_0\ell})) \subset H^0(X,\mathcal{O}_X(A+m_0\ell L)\otimes\mathcal{I}(h_L^{m_0\ell})) \tag{7.3.15}$$

as a Hilbert subspace and let $K^+_{m_0\ell}$ denotes the associated Bergman kernel with respect to the restriction of the inner product on $H^0(X,\mathcal{O}_X(A+m_0\ell L)\otimes\mathcal{I}(h_L^{m_0\ell}))$ to the subspace $H^0(X,\mathcal{O}_X(A+\ell(m_0L-H))\otimes \mathcal{I}(h_L^{m_0\ell}))$. Then by definition, we have the trivial inequality :

$$K^+_{m_0\ell} \leqq K_{m_0\ell} \tag{7.3.16}$$

holds on X for every $\ell \geqq 1$.

The next lemma follows from the same argument as in [Dem]

Lemma 7.8 *([Dem]) If A is sufficiently ample,*

$$h_L := \text{the lower envelope of } (\limsup_{m\to\infty} \sqrt[m]{K_m})^{-1}. \tag{7.3.17}$$

holds. □

Remark 7.9 *In [Dem], Demailly considered the local version of Lemma 7.8, but the same proof works thanks to the sufficiently ample line bundle A.* □

We note that

$$\int_X h_A \cdot h_L^m \cdot K_m \cdot dV = \dim H^0(X,\mathcal{O}_X(A+mL)\otimes\mathcal{I}(h_L^m)) \tag{7.3.18}$$

and

$$\int_X h_A \cdot h_L^{m_0\ell} \cdot K^+_{m_0\ell} \cdot dV = \dim H^0(X,\mathcal{O}_X(A+\ell(m_0L-H))\otimes\mathcal{I}(h_L^{m_0\ell})) \tag{7.3.19}$$

hold. Hence by Lemma 7.7

$$\limsup_{\ell\to\infty}\left((m_0\ell)^{-n}\cdot\int_X h_L^{m_0\ell}\cdot K^+_{m_0\ell}\cdot dV\right) > 0 \tag{7.3.20}$$

holds. Then by Fatou's lemma, we see that

$$\int_X \limsup_{\ell\to\infty} \frac{h_A \cdot h_L^{m_0\ell} \cdot K_{m_0\ell}^+}{(m_0\ell)^n} \geqq \limsup_{\ell\to\infty} \int_X \frac{h_A \cdot h_L^{m_0\ell} \cdot K_{m_0\ell}^+}{(m_0\ell)^n} > 0 \tag{7.3.21}$$

hold. In particular

$$\limsup_{\ell\to\infty} \frac{h_A \cdot h_L^{m_0\ell} \cdot K_{m_0\ell}^+}{(m_0\ell)^n} \tag{7.3.22}$$

is not identically 0[4]. This implies that

$$\limsup_{\ell\to\infty} \sqrt[m_0\ell]{K_{m_0\ell}^+} \tag{7.3.23}$$

is not identically 0 and by Lemma 7.8 and (4.3), it is finite. Let h_H be a C^∞ hermitian metric on H with strictly positive curvature and let τ be a global holomorphic section of $\mathcal{O}_X(H)$ with divisor H such that $h_H(\tau,\tau) \leqq 1$ holds on X. We set

$$h_L^+ := (\limsup_{\ell\to\infty} \sqrt[m_0\ell]{K_{m_0\ell}^+}\ )^{-1} \cdot h_H(\tau,\tau). \tag{7.3.24}$$

Then h_L^+ is a singular hermitian metric on L, since $(\limsup_{\ell\to\infty} \sqrt[m_0\ell]{K_{m_0\ell}^+}\ )^{-1} \cdot \mid \tau \mid^2$ can be viewed as a singular hermitian metric on $L - H$ with semipositive curvature current. By the construction it is clear that the curvature current of h_L^+ is bigger than or equal to the curvature of h_H. In particular the curvature current of h_L^+ is strictly positive. And by the construction

$$h_L \leqq h_L^+ \tag{7.3.25}$$

holds on X. This completes the proof of Theorem 7.6. □

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[4] At this moment, there is a possibility that it is identically $+\infty$.

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