# On a lexical tree for the middle-levels graph problem

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Abstract. A conjecture of I. Hável asserts that all middle-levels graphs $M_k$ of the $(2k+1)$-cubes possess Hamilton cycles. In this work, a tree containing all vertices of certain reduced graphs of the $M_k$ s is introduced and indicated via the Kierstead-Trotter lexical matchings, easing the algorithmic aspects of determining such Hamilton cycles, and here applied to the explicit presentation of some of them, that leads additionally to lower bounds on their numbers.

## 1. Introduction.

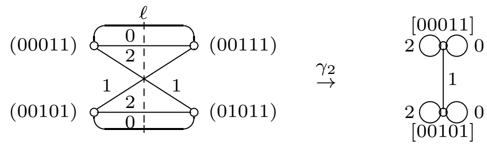
If $1 < n \in \mathbf{Z}$, then the $n$-cube graph $H_n$ is defined as the Hasse diagram of the Boolean lattice on the $n$-element set $[n] = \{0, 1, \dots, n-1\}$. Vertices of $H_n$ will be indicated in three different ways: **(a)** as the subsets $A = \{a_0, a_1, \dots, a_{r-1}\} = a_0a_1 \dots a_{r-1}$ of $[n]$ they stand for, where $0 \leq r \leq n$; **(b)** as the characteristic $n$-vectors $B_A = (b_0, b_1, \dots, b_{n-1}) = b_0b_1 \dots b_{n-1}$ over the field $F_2 = \{0, 1\}$ the subsets $A$ of item (a) represent, given by $b_i = 1$ if and only if $i \in A$, $(i = 0, 1, \dots, n-1)$; **(c)** as the polynomials $\beta_A(x) = b_0 + b_1x + \dots + b_{n-1}x^{n-1}$ associated to the vectors $B_A$ of item (b). We use these three representations interchangeably. An $A$ as above is said to be the *support* of the vector $B_A$. For each $k \in [n]$, the $k$-level $L_k$ of $H_n$ is the vertex subset of $H_n$ formed by those $A \subseteq [n]$ with $|A| = k$.

The middle-levels graph $M_k$ is defined as the subgraph of $H_{2k+1}$ induced by the union of its $k$-level and its $(k+1)$-level, denoted $L_k$ and $L_{k+1}$, and formed by the vertices of weights $k$ and $k+1$, respectively. Thus, $M_k$ is a bipartite graph with vertex parts $L_k$ and $L_{k+1}$ and adjacency given by inclusion or containment.

$M_k$ was conjectured to be Hamiltonian by I. Havel [**3**], for every $1 < k \in \mathbf{Z}$. We discuss a sufficient condition for the existence of a Hamilton cycle in $M_k$ via a reduced graph $R_k$ of $M_k$, for every $k > 1$ [**1, 2**] and define a rooted binary tree $T$ whose vertices count once each vertex of such graphs $R_k$, for all $k > 1$, easing the algorithmic aspects of determining such Hamilton cycles.

### 1.1. Reduced graphs $R_k$.

We consider the following relation $J$ between the elements of $M_k$, seen as polynomials as in (c) above: $\beta_A(x) J \beta_{A'}(x)$ if and only if there exists $i \in \mathbf{Z}$ such that $\beta_{A'}(x) \equiv x^i \beta_A(x) \pmod{1 + x^n}$. It is easy to see that $J$ is an equivalence relation and that there exists a well-defined quotient graph $M_k/J$. For example, $M_2/J$ is the domain of the graph map $\gamma_2$ depicted here and explained below:




where the vertices of $M_k/J$ are denoted $(00011), (00111), (00101), (01011)$, (from left to right and then from top to bottom), representing respectively the $J$-classes

{00011,10001,11000,01100,00110}, {00111,10011,11001,11100,01110},
{00101,10010,01001,10100,01010}, {01011,10101,11010,01101,10110}.

Likewise, we will set both, the vertices of the parts $L_k$ and $L_{k+1}$ of $M_k$ and the vertices of the parts $L_k/J$ and $L_{k+1}/J$ of $M_k/J$, into horizontal lines as pairs symmetrically disposed with respect to an imaginary middle vertical line $\ell$, like the dashed line in the representation of $M_2/J$ above.

A convenient rule to set these pairs is given by a bijection $\aleph : L_k \to L_{k+1}$ given by $\aleph(b_0 b_1 \dots b_{n-1}) = \bar{b}_{n-1} \dots \bar{b}_1 \bar{b}_0$, where $\bar{1} = 0$ and $\bar{0} = 1$. Then, each horizontal pair as above is of the form $(B_A, \aleph(B_A))$. Moreover, a skew (that is: non-horizontal) edge $B_A B_{A'}$ of $M_k/J$ (in our adopted representation), where $|A| = k$ and $|A'| = k+1$, is accompanied by another skew edge $\aleph(B_A)\aleph(B_{A'})$, which is obtained from $B_A B_{A'}$ by its reflection on $\ell$, (i.e. as its specular image). Thus, the skew edges of $M_k$ appear in pairs of edges having their end-vertices forming pairs of horizontal vertices. The horizontal edges of $M_k/J$ in our adopted representation have multiplicity $\leq 2$.

By denoting each horizontal pair $(B_A, \aleph(B_A))$ in $M_k/J$ by means of $[B_A]$, where $|A| = k$, we obtain a quotient graph $R_k$ of $M_k/J$, (the quotient graph of $M_k$ claimed in the Introduction), whose vertices are the pairs $[B_A]$ and having an edge $[B_A][B_{A'}]$ representing each pair of skew edges $\{B_A \aleph(B_{A'}), B_{A'} \aleph(B_A)\}$, and a loop at $[B_A]$ representing each horizontal edge $B_A \aleph(B_A)$. Let $\gamma_k : M_k/J \to R_k$ be the corresponding quotient graph map. For example, $R_2$ is represented as the image of the graph map $\gamma_2$ depicted above. Observe that $R_2$ contains two loops per vertex and just one (vertical) edge.

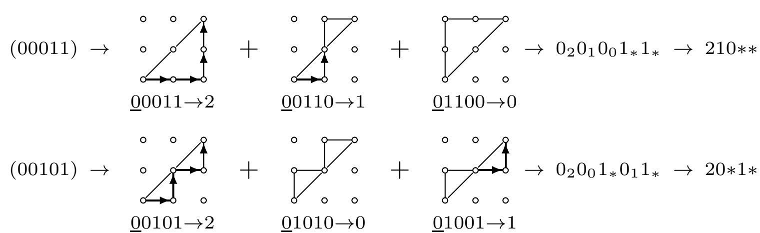
The representation of $M_2/J$ above has its edges indicated with colors 0,1,2, as shown near the edges in the figure. In general, each vertex $v$ of $L_k/J$ will have its incident edges painted with colors $0, 1, \dots, k$ by means of the following procedure, so that $L_k/J$ admits a $(k+1)$-edge-coloring with *color palette* $[k+1]$.

**1.2. Lexical Procedure** [4]**.** For each $v \in L_k/J$, consider the $k+1$ $n$-vectors of the form $b_0 b_1 \dots b_{n-1}$ representing $v$ with $b_0 = 0$. For each such an $n$-vector of the form $b_0 b_1 \dots b_{n-1}$, take a grid $\Gamma = P_{k+1} \square P_{k+1}$, where $P_{k+1}$ is the graph induced by $[k+1]$ in the unit-distance graph of $\mathbf{Z}$. Trace the diagonal $\Delta$ of $\Gamma$ from vertex $(0,0)$ to vertex $(k,k)$. Consider a stepwise increasing index $i \in \mathbf{Z}$ and an accompanying traveling vertex $w$ in $\Gamma$ initialized respectively at $i = 1$ and at $w = (0,0)$. Proceed with an arc marking in $\Gamma$ as follows:

(1) if $b_i = 0$, then mark the arc $(w, w') = (w, w + (1, 0))$ ; otherwise, mark the arc $(w, w') = (w, w + (0, 1))$; let $i := i + 1$ and $w := w'$;

(2) repeat step (1) until $w' = (k, k)$ is fulfilled.

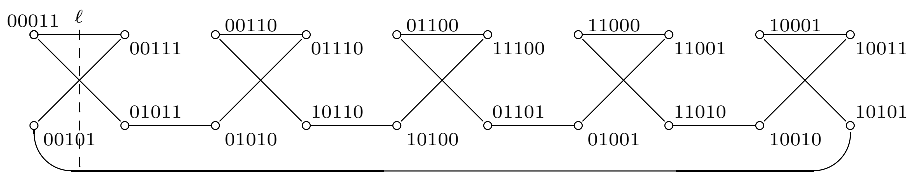
The edge of $L_k/J$ departing from $v$ and obtained from a representative $n$-vector $b_0b_1 \ldots b_{n-1}$ of $v$ as above by the sole complementation of $b_0$, that is by replacing $b_0$ by $\bar{b}_0$, (keeping $b_i$ unchanged, for all $i > 0$), is indicated by the color defined as the number of marked horizontal arcs below $\Delta$. This color is unique among those colors of edges incident to $v$. Moreover, this defines a 1-1 correspondence between $[k + 1]$ and the set of edges incident to $v$ in $L_k/J$. Since this argument can be done also by departing from each vertex $w$ of $L_{k+1}/J$, a 1-factorization of $M_k/J$ is obtained that can be lifted to another one in $M_k$, as in [**4**], and collapsed to another one in $R_k$. This color assignment can be represented as follows, for the case of $K_2/J$:




where the Lexical Procedure is indicated by arrows and plus signs from left to right, departing from each of the two vertices $v = (00011)$ and $v = (00101)$ of $L_2/J$, then going to the right by considering the three different representatives $b_0b_1 \ldots b_{n-1}$ that have- $b_0 = 0$ (where $b_0$ is shown underlined), subsequently marking arcs in $\Gamma$ as indicated, and finally counting the number of marked horizontal arcs that lie below $\Delta$ in each case. (Only marked arcs are traced over $V(\Gamma)$: those below $\Delta$ are indicated by means of arrows, the remaining ones by segments). In each of the cases of a vertex $v$ of $L_2/J$ considered, to the right of the marked representations of $\Gamma$, we have written a modification of the first representation $b_0b_1 \ldots b_{n-1}$ of $v$, obtained by setting as a subindex of each entry 0 the color obtained for it, and a star $*$ for each entry 1. Still to the right of this modification, we have written the string of length $n$ formed by those subindexes in the order they appeared from left to right. We will indicate this final notation by $\delta(v)$.

Each pair of skew edges $B_A\aleph(B_{A'})$ and $B_{A'}\aleph(B_A)$ in $M_k/J$ has the same color from the color palette $[k + 1]$. Thus, the $[k + 1]$-coloring of $M_k/J$ induces a well-defined $[k + 1]$-coloring of $R_k$, as exemplified for $k = 2$ in the first figure above.

**1.3. Sufficient Condition for Hamilton cycles in the $M_k$s [1, 2].** A Hamilton cycle $\eta_k$ in $M_k$ can be constructed from a Hamilton path $\xi_k$ in $R_k$ whose terminal vertices are incident jointly to at least three loops; however, if $2k+1$ is prime then one loop per terminal vertex in $\xi_k$ is enough to insure such Hamilton cycle. (Such a path $\xi_2$ in $R_2$ has two vertices, namely the terminal vertices [00011] and [00101], with four loops altogether). First, we pull back $\xi_k$ in $R_k$ together with a loop at each one of its terminal vertices, via $\gamma_k^{-1}$, onto a Hamilton cycle $\zeta_k$ in $M_k/J$. Second, $\zeta_k$ is pulled back onto a Hamilton cycle $\eta_k$ in $M_k$ by means of the freedom of selection between the two parallel horizontal edges in $M_k$ corresponding to the two loops of one of the terminal vertices of $\xi_k$. In the case $k=2$, a resulting Hamilton cycle $\eta_2$ of $M_2$ is represented here:




where the reflection about $\ell$ is used to transform $\xi_2$ first into a Hamilton cycle $\zeta_2$ of $M_2/J$ (not shown) and then into a path of length $2|V(R_2)|=4$ starting at $00101 = x^2+x^4$ and ending at $01010 = x+x^3$, in the same class mod $1+x^5$, that can be repeated five times to close the desired $\eta_2$, as shown.

## 2. Lexical tree and Hamilton cycles in the $M_k$s.

From now on, we use the notation $\delta(v)$ for the vertices of $R_k$, established at the end of Subsection 1.2. In particular, a Hamilton cycle in $M_k$ is insured by the determination of a Hamilton path $\xi_k$ in $R_k$ from vertex $\delta(000\dots11) = k(k-1)\dots21*\dots*$ to vertex $\delta(010\dots010) = k0*1*2*\dots*(k-1)*$. (For $k=2$, these are the only two vertices of $R_2$, joined by an edge that realizes $\xi_2$). Observe that these two vertices in $R_k$ are incident to two loops each, so that in general a Hamilton cycle $\eta_k$ in $M_k$ would follow by the previous remarks.

**2.1. Lexical Tree.** Notice that $R_1$ is formed by the only vertex $\delta(001) = 10*$. We contend that this vertex is the root of a rooted binary tree $T$, which is the tree claimed in the Introduction before Subsection 1.1, that has as its nodes the vertices of all the graphs $R_k$, for $1 \le k \in \mathbf{Z}$. Such a $T$ is defined as follows, where the concatenation of two strings $X$ and $Y$ is indicated $X|Y$ and $||X||$ = length of $X$:

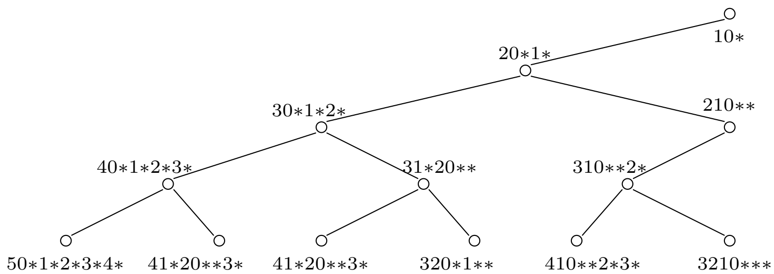
(1) the root of $T$ is $10*$;

(2) the left child of a node $\delta(v) = k|X$ in $T$ with $||X|| = 2k$ is $k+1|X|k|*$;

(3) the right child of a node $\delta(v) = k|X|Y|*$, where $X$ and $Y$ are strings respectively starting with $j < k-1$ and $j+1$, is $k|Y|X|*$;

(4) if $\delta(v) = k|k-1|X$, then $\delta(v)$ does not have a right child.

The restriction of $T$ to its five initial levels looks like:



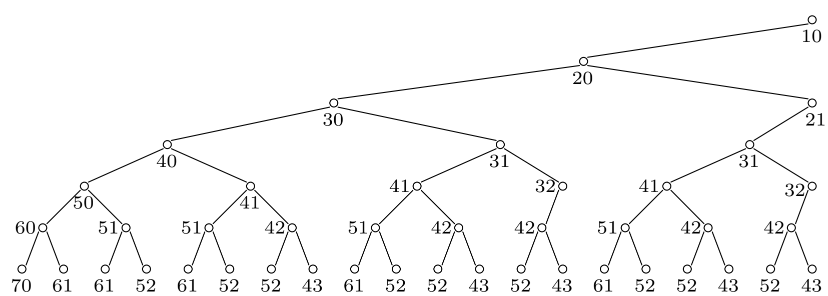

**2.2. Catalan Triangle.** To count the nodes of $T$, we recall the Catalan triangle $\mathcal{T}$, which is a triangular arrangement of positive integers, starting with:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | | | | | | | |
| 1 | 1 | | | | | | |
| 1 | 2 | 2 | | | | | |
| 1 | 3 | 5 | 5 | | | | |
| 1 | 4 | 9 | 14 | 14 | | | |
| 1 | 5 | 14 | 28 | 42 | 42 | | |
| 1 | 6 | 20 | 48 | 90 | 132 | 132 | |
| 1 | 7 | 27 | 75 | 165 | 297 | 429 | 429 |

The numbers in the $j$-th row $\mathcal{T}_j$ of $\mathcal{T}$, where $0 \leq j \in \mathbf{Z}$, are denoted $\tau_0^j, \tau_1^j, \ldots, \tau_j^j$ They satisfy the following properties: **(a)** $\tau_0^j = 1$, for every $j \geq 0$; **(b)** $\tau_1^j = j$ and $\tau_j^j = \tau_{j-1}^j$, for every $j \geq 1$; **(c)** $\tau_i^j = \tau_i^{j-1} + \tau_{i-1}^j$, for every $j \geq 2$ and $i = 1, \ldots, j-2$; **(d)** $\sum_{i=0}^j \tau_i^j = \tau_j^{j+1} = \tau_{j+1}^{j+1}$, for every $j \geq 2$. The following counting properties of $T$ are elementary.

**Theorem 1.** *Level* $0$ *of* $T$ *contains just the root* $10*$. *The number of nodes at a level* $j > 0$ *of* $T$ *is* $\binom{2k+1}{k}$ *if* $j = 2k+1$, *and* $2\binom{2k+1}{k}$ *if* $j = 2k+2$, *where* $k \geq 0$. *On the other hand, for every* $k \geq 1$, *the number of vertices* $v$ *of* $R_k$ *with* $\delta(v) = kjX$ *is equal to* $\tau_j^k$, *where* $j = 0, 1, \ldots, k-1$. *Moreover* $|V(R_k)| = \tau_k^{k+1} = \tau_{k+1}^{k+1}$, *which is the Catalan number* $\frac{1}{2k+1}\binom{2k+1}{k}$. *This number is odd if and only if* $k = 2^r - 1$, *for some integer* $r \geq 0$.

The tree $T$ has a simplified equivalent form $T'$ obtained by indicating each one of its nodes by just the first two symbols of its notation $\delta(v)$. A portion of $T'$ larger than the one of $T$ shown above looks like:




Moreover, an alternative equivalent form $T''$ of $T$ is obtained by denoting each one of its nodes $v$ by means of the finite sequence obtained by concatenating the second symbols of the nodes of a path from the root node 10 of $T'$ to $v$. This way, we have that the correspondence from de the nodes of $T''$ to the nodes of $T$ for all nodes of $T''$ of length $\leq 3$ is:

| | | |
|---|---|---|
| $00 \to 20{*}1{*}$ | $000 \to 30{*}1{*}2{*}$ | $0000 \to 40{*}1{*}2{*}3{*}$ |
| | | $0001 \to 41{*}2{*}30{*}{*}$ |
| | | $0002 \to 42{*}30{*}1{*}{*}$ |
| | | $0003 \to 430{*}1{*}2{*}{*}$ |
| | | $0011 \to 41{*}20{*}{*}3{*}$ |
| | $001 \to 31{*}20{*}{*}$ | $0012 \to 420{*}{*}31{*}{*}$ |
| | | $0013 \to 431{*}20{*}{*}{*}$ |
| | $002 \to 320{*}1{*}{*}$ | $0022 \to 420{*}1{*}{*}3{*}$ |
| | | $0023 \to 4320{*}1{*}{*}{*}$ |
| $01 \to 210{*}{*}$ | $011 \to 310{*}{*}2{*}$ | $0111 \to 410{*}{*}2{*}3{*}$ |
| | | $0112 \to 42{*}310{*}{*}{*}$ |
| | | $0113 \to 4310{*}{*}2{*}{*}$ |
| | $012 \to 3210{*}{*}{*}$ | $0122 \to 4210{*}{*}{*}3{*}$ |
| | | $0123 \to 43210{*}{*}{*}{*}$ |

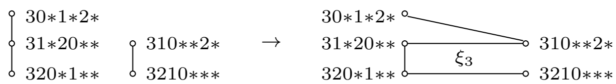
Furthermore, this provides an alternative notation for the vertices of the graphs $R_k$, replacing their notation as in the tree $T$ with the alternative equivalent notation provided by tree $T''$. Thus, it can be observed the following fact.

**Theorem 2.** *There exists a one-to-one correspondence* $\Phi$ *from the set of nondecreasing integer* $k$*-sequences* $a_0a_1\dots a_{k-1}$ *with* $a_0=0$ *and* $0 \leq a_i \leq i$ $(0 \leq i < k)$ *onto* $V(R_k)$*, in which each new position of* $a_0a_1\dots a_{k-1}$ *from left to right indicates a new branch of* $T$ *descending to the left in a path that starts at the root* $10*$ *of* $T$*, followed by a right path in* $T$ *whose length equals the number filling that position. The final vertex of* $T$ *in the resulting path obtained from* $a_0a_1\dots a_{k-1}$ *is its image in* $V(R_k)$ *through* $\Phi$*.*

Each nondecreasing integer $k$-sequence (NDI$k$S) $S$ as in Theorem 2 indicates a path $P$ from the root of $T$ to a specific node $v$ of $T$. For example in $R_2$ and $R_3$:

$$\begin{aligned}
&00 \to (10{*}, 20{*}1{*});\\
&01 \to (10{*}, 20{*}1{*}, 210{*}{*});\\
&000 \to (10{*}, 20{*}1{*}, 30{*}1{*}2{*});\\
&001 \to (10{*}, 20{*}1{*}, 30{*}1{*}2{*}, 31{*}20{*}{*});\\
&002 \to (10{*}, 20{*}1{*}, 30{*}1{*}2{*}, 31{*}20{*}{*}, 320{*}1{*}{*});\\
&011 \to (10{*}, 20{*}1{*}, 210{*}{*}, 310{*}{*}2{*});\\
&012 \to (10{*}, 20{*}1{*}, 210{*}{*}, 310{*}{*}2{*}, 3210{*}{*}{*}).
\end{aligned}$$

Each NDI$k$S $S$ represents a path $P$ in $T$ departing from its root and obtained by advancing in $S$ from left to right, starting from the first entry, 0, in $S$, with each new entry attained in $S$ indicating a left child $w$ of the previously attained node in $P$ and with the number filling that entry indicating the number of right children in $P$ up to the next left child $w'$ in $P$, if at least one such $w'$ remains, or until $v$.




**2.3. Some Hamilton cycles [1, 2]: Case. $\mathbf{k=3}$**. For a fixed $k$, consider the induced graph $T_k = T[V(R_k)]$. Its edges descend to the right in $T$. In representing $T_k$, we trace those edges vertically, keeping the height of the levels as in $T$. For $k = 3$, this looks like:

on the left, while on the right we have traced, joining the vertices of $R_3$, a Hamilton path $\xi_3$ with its terminal vertices incident to two loops each.

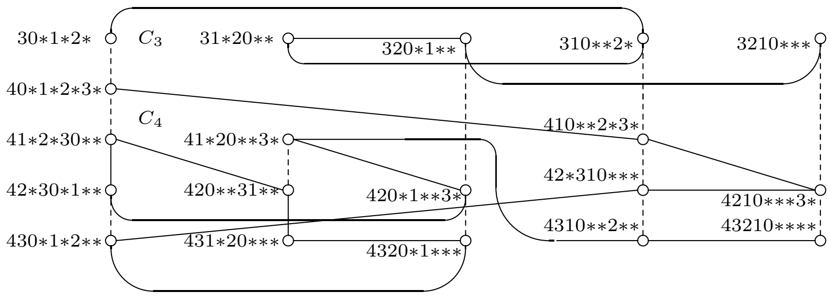
Let us analyze a little further the Hamilton path $\xi_3$ depicted on $R_3$. By translating adequately the vertices of $\xi_3$ mod $1+x^7$, shown vertically on the left below, we can see to their right a corresponding representative path $\xi'$ in $M_3$ separated by double arrows (indicative of the bijection $\aleph$) from its image $\aleph(\xi')$. All entries 0, 1 here bear subindexes as agreed, and extensively for the images of vertices through $\aleph$, in its corresponding backward form. Corresponding notation for a loop is included for each of the two terminal vertices of $\xi_3$ before and after the data corresponding to $\xi_3$ and $\xi'$. The 6-path resulting from $\xi_3$ and the two terminal loops are presented in the penultimate column, by combining the non-$*$ symbols of both vertices incident to each edge, with a hat over the coordinate in which a 0-1 switch took place, accompanied to the right by their images through $\aleph$:

| | | | | | | |
|---|---|---|---|---|---|---|
| | $1_20_*1_10_*1_01_30_*$ | $\leftrightarrow$ | $1_*0_30_01_*0_11_*0_2$ | | | |
| $30{*}1{*}2{*}$ | $1_*0_21_*0_30_01_*0_1$ | $\leftrightarrow$ | $1_10_*1_01_30_*1_20_*$ | $2213\hat{0}31$ | $\leftrightarrow$ | $13\hat{0}3122$ |
| $310{*}{*}2{*}$ | $1_30_*1_20_*0_*1_01_1$ | $\leftrightarrow$ | $0_10_01_*1_*0_21_*0_3$ | $322300\hat{1}$ | $\leftrightarrow$ | $\hat{1}003223$ |
| $31{*}20{*}{*}$ | $0_30_11_*0_20_01_*1_*$ | $\leftrightarrow$ | $0_*0_*1_01_20_*1_11_3$ | $\hat{3}122001$ | $\leftrightarrow$ | $100221\hat{3}$ |
| $320{*}1{*}{*}$ | $0_*0_*1_10_*1_01_21_3$ | $\leftrightarrow$ | $0_30_20_01_*0_11_*1_*$ | $3112\hat{0}23$ | $\leftrightarrow$ | $32\hat{0}2113$ |
| $3210{*}{*}{*}$ | $0_30_20_10_01_*1_*1_*$ | $\leftrightarrow$ | $0_*0_*0_*1_01_11_21_3$ | $32\hat{1}0023$ | $\leftrightarrow$ | $3200\hat{1}23$ |
| | $0_*0_*0_*1_01_11_21_3$ | $\leftrightarrow$ | $0_30_20_10_01_*1_*1_*$ | $321\hat{0}123$ | $\leftrightarrow$ | $321\hat{0}123$ |

We just extended the idea of the initial fifth (reflected about $\ell$) of the Hamilton cycle $\eta_2$ in $M_2$ depicted previously, to the case of an initial seventh, (also reflected about $\ell$), of a Hamilton cycle $\eta_3$ in $M_3$. Continuing in the same fashion six more times, translating adequately mod $1+x^7$, a Hamilton cycle in $M_3$ is obtained. The six edges indicated on the penultimate column could be presented also with the hat positions as the leftmost ones: $\hat{0}312213, \hat{1}322300, \hat{3}122001, \hat{0}233112, \hat{1}002332, \hat{0}123321$. Every edge of $R_k$ can be presented in this way. The Hamilton path $\xi_3$ can also be given by the sequence of hat positions: 1301, (to which 0 is prefixed and postfixed for the terminal loops). In the example for $k=2$ above, a similar sequence for $\xi_2$ reduces to 1.

**2.4. Case. $\mathbf{k=4}$**. In the same way, for $k=4$, the following sequence (of hat positions) works for a Hamilton path $\xi_4$ in $R_4$: 1241201234032, representable as follows, where $\xi_3$ is also included, on top, just for comparison:

with the edges of the resulting $\xi_4$ in $R_4$ drawn fully and the remaining edges of $T_4$ dashed, as are the edges from $V(R_3)$ to $V(R_4)$ in $T$. In general, for each vertex




$v \in V(R_{k-1})$, there is path descending from the left child of $v$ and continuing to the right on vertices of $V(R_k)$, for each $k > 0$, and this procedure covers all the vertices of $R_k$.

**2.5. Case. $\mathbf{k = 5}$ : A lower bound.** Let $\Phi_k^0$ and $\Phi_k^1$ be respectively the images, through the correspondence $\Phi$ of Theorem 2, of the smallest and largest $k$-sequences in the domain of $\Phi$. (The Hamilton paths $\xi_k$ obtained above for $k = 2, 3, 4$ started and ended respectively at $\Phi_k^0$ and $\Phi_k^1$). Two different Hamilton paths in $R_5$ playing the role of $\xi_5$ in the previous considerations about $\xi_k$ are given by the following sequences of hat positions, where the initial and final vertices are respectively $\Phi_5^0$ and $\Phi_5^1$:

151520315150523234251535452515013135313 53;
404035240405032321304020103040542420242 02;

so they generate corresponding Hamilton cycles in $M_6$, by the previous discussion. Cconsequently, a lower bound for the number of Hamilton cycles in $M_5$ is 2.

**2.6. Case. $\mathbf{k = 6}$ : A lower bound.** Here is how to obtain 29 different Hamilton cycles in $M_6$. They all arise from the Hamilton cycle in $R_6$ determined by the following cycle of hat positions, departing from $\Phi_1^6$ and shown in a three-line display:

(53464103016153032023143043236025201010425 31
530201013403410643405040126525360315010405 20
412340615016560510502320616135342030636304521)

By removing the first (final) edge of this cycle, with hat position 5 (1), we obtain a Hamilton path in $R_6$ with final (initial) vertex $\Phi_6^1$ incident to two loops and initial (final) vertex incident to one loop, enough to insure a Hamilton cycle in $M_6$ in each case. The same holds if we represent the same cycle, but starting in the second line of the display, which departs from $\Phi_6^0$ and accounts for another pair of Hamilton cycles in $M_6$. A fifth Hamilton cycle arises if we start in the third line of the display, where the first hat position corresponds to an edge with hat position 4, preceding and succeeding vertices with one and two loops, respectively.

By removing an edge with one of the following order numbers in the cycle of hat positions displayed above:

1,28,41,42,43,44,45,60,62,100,101,107,108,96,104,105,114,122,127,128,129,130,131,132,

a Hamilton path in $R_6$ is obtained that has a loop at each one of its two terminal vertices, thus insuring a Hamilton cycle in $M_6$ in each case (since $2k + 1 = 13$ is prime), which yields a total of 29 Hamilton cycles in $M_6$. This was a list of 24 hat positions, but three of the intervening terminal vertices had two loops each, yielding a total of five new loops, which were considered above, yielding the claimed lower bound on the number of Hamilton cycles of $M_6$, namely 29.

**2.7. Adjacency table.** An adjacency table for the vertices of $R_k$ can be obtained by writing backwards their lexical expressions, via an interpretation of the function $\aleph$ in terms of the lexical symbols of each vertex heading an adjacency column, as in the following table for $k = 3$, where each lexical expression of a vertex is accompanied by its order of presentation in $T[V(R_k)]$:

| $30{*}1{*}2{*}$ | 1 | $31{*}20{*}{*}$ | 2 | $320{*}1{*}{*}$ | 3 | $310{*}{*}2{*}$ | 4 | $3210{*}{*}{*}$ | 5 |
|---|---|---|---|---|---|---|---|---|---|
| $\hat{3}{*}2{*}1{*}0$ | 1 | $\hat{3}{*}2{*}{*}01$ | 4 | $\hat{3}{*}{*}1{*}02$ | 3 | $\hat{3}{*}{*}02{*}1$ | 2 | $\hat{3}{*}{*}{*}012$ | 5 |
| ${*}{*}1{*}0\hat{2}3$ | 3 | ${*}{*}0\hat{2}{*}13$ | 2 | ${*}\hat{2}{*}1{*}03$ | 1 | ${*}{*}{*}01\hat{2}3$ | 5 | ${*}\hat{2}{*}{*}013$ | 4 |
| ${*}{*}0\hat{1}3{*}2$ | 4 | ${*}\hat{1}3{*}{*}02$ | 2 | ${*}{*}0\hat{1}{*}23$ | 5 | ${*}\hat{1}{*}03{*}2$ | 1 | ${*}{*}\hat{1}{*}023$ | 3 |
| ${*}\hat{0}3{*}2{*}1$ | 1 | ${*}{*}1{*}\hat{0}23$ | 3 | ${*}{*}\hat{0}2{*}13$ | 2 | ${*}{*}\hat{0}13{*}2$ | 4 | ${*}{*}{*}\hat{0}123$ | 5 |

**2.8. Summation sequences.** For each $k > 1$, consider the sequence $S_1$ whose terms are the lengths of the paths obtained by restricting $T$ to $V(R_k)$ taken from left to right, followed, if $k > 2$, by the sequence $S_2$ of summations of maximum decreasing subsequences of $S_1$, also taken from left to right, followed, if $k > 3$, by the sequence $S_3$ of summations of maximum decreasing subsequences of $S_2$, and so on, in order to obtain $k - 1$ sequences $S_1, \dots, S_{k-1}$, where $S_{k-1}$ has just one number. For example:

| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $k$=2 | $S_1$ | 2; | | | | | | | | | | | | | |
| $k$=3 | $S_1$ | 3, | 2; | | | | | | | | | | | | |
| | $S_2$ | – | 5; | | | | | | | | | | | | |
| $k$=4 | $S_1$ | 4, | 3, | 2; | 3, | 2; | | | | | | | | | |
| | $S_2$ | – | – | 9, | – | 5; | | | | | | | | | |
| | $S_3$ | – | – | – | – | 14; | | | | | | | | | |
| $k$=5 | $S_1$ | 5, | 4, | 3, | 2; | 4, | 3, | 2; | 3, | 2; | 4, | 3, | 2; | 3, | 2; |
| | $S_2$ | – | – | – | 14, | – | – | 9, | – | 5; | – | – | 9, | – | 5; |
| | $S_3$ | – | – | – | – | – | – | – | – | 28, | – | – | – | – | 14; |
| | $S_4$ | – | – | – | – | – | – | – | – | – | – | – | – | – | 42; |
| ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... |

showing that the components of $T[V(R_k)]$, taken from left to right, are paths whose lengths form $S_1$, which can be recovered via backtracking in $\mathcal{T}$ from the single element of $S_{k-1}$, namely $\tau_k^k$, using the Catalan triangle according to the structure of the partial sums, where some commas separating the terms of the sequences are replaced by semicolons in order to indicate where each partial sum ends up.

**2.9. Counting nodes of $R_k$ in the levels of $\mathcal{T}$.** Also, $\mathcal{T}$ allows to determine the number of elements of $R_k$ at each level of $T$. In fact, we may rewrite $\mathcal{T}$ with its elements inside parentheses preceded by the number denoting a level of $T$, meaning that $R_k$ just contains at that level the number enclosed in parentheses:

| | | | | | | |
|---|---|---|---|---|---|---|
| $k$=2 | 1(1) | 2(1) | | | | |
| $k$=3 | 2(1) | 3(2) | 4(2) | | | |
| $k$=4 | 3(1) | 4(3) | 5(5) | 6(5) | | |
| $k$=5 | 4(1) | 5(4) | 6(9) | 7(14) | 8(14) | |
| ... | ... | ... | ... | ... | ... | ... |

As mentioned in Subsection 1.2, the Lexical Procedure yields 1-factorizations of $R_k$, $M_k/J$ and $M_k$ by means of the edge colors $0, 1, \ldots, k$. This yields the lexical matchings of [**4**]. This lexical approach and the quotient graphs $M_k/J$ and $R_k$ are compatible, because each edge $e$ of $M_k$ has the same lexical color in $[k+1]$ for both arcs composing $e$, (not the case for the modular approach of [**5**]).

**2.10. Ordered partitions of positive integers.** To each NDI$k$S

$$A = a_0 a_1 \ldots a_{k-1}$$

we associate an integer $(k-1)$-sequence $\phi(A) = b_1 \ldots b_{k-1}$ with $\sum_{i=1}^{k-1} = k-1$ and $b_i \leq i$, for $i = 1, \ldots, k-1$, as follows: Let $b_1 = a_1$ and let $b_i = a_i - a_{i-1}$, for $i = 2, \ldots, k-1$. Then, the path $P$ from the root of $T$ to the node $v$ of $T$ represented by $A$ can be traced via $\phi(A)$ by inspecting it from left to right: each new entry inspected (starting with the first one) represents a left child, in the order they appear in $P$, from which a right path whose length is the integer occupying that entry. Then, we can restate Theorem 2 as follows.

**Theorem 3.** *There exists a one-to-one correspondence from the set of integer $k$-sequences $b_1 \ldots b_{k-1}$ with $\sum_{i=1}^{k-1} b_i = k-1$ and $b_i \leq i$, for $i = 1, \ldots, k-1$, onto $V(R_k)$.*

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