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M-SHELLABILITY OF DISCRETE POLYMATROIDS

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Abstract. In this note we show that every discrete polymatroid is M-shellable. This gives, in a partial case, a positive answer to a conjecture of Chari and improves a recent result of Schweig where he proved that the h-vector of a lattice path matroid satisfies a conjecture of Stanley.

1. Introduction and Preliminaries

A *matroid* M is a pair $(E(M), \mathcal{B}(M))$ consisting of a finite set $E(M)$ and a collection $\mathcal{B}(M)$ of subsets of $E(M)$, called *bases* of M, that satisfy the following two conditions:

(B1) $\mathcal{B}(M) \neq \emptyset$, and
(B2) for each pair of distinct sets B, B' in $\mathcal{B}(M)$ and for each element $x \in B \setminus B'$, there is an element $y \in B' \setminus B$ such that $(B - x) \cup y$ is in $\mathcal{B}(M)$.

Subsets of bases are called *independent sets*. The collection of independent sets of a matroid form an abstract simplicial complex, called *matroid complex*.

For a $(d-1)$-dimensional simplicial complex Δ, let f_i be the number of $(i-1)$-dimensional faces of Δ (i.e. the faces of cardinal i), and $f(\Delta) = (f_0, f_1, \ldots, f_d)$ its f-vector. The h-vector $h(\Delta) = (h_0, h_1, \ldots, h_d)$ is defined by $H(y) = F(y-1)$, where $H(y) = \sum_{i=0}^{d} h_i y^{d-i}$ and $F(y) = \sum_{i=0}^{d} f_i y^{d-i}$.

A *monomial order ideal* Γ on a set $V = \{x_1, \ldots, x_n\}$ of variables is a set of monomials $x_1^{a_1} \ldots x_n^{a_n}$ such that $u \in \Gamma$ and $v|u$ imply that $v \in \Gamma$. The *degree sequence* of Γ is $h(\Gamma) = (h_0, h_1, \ldots)$, where $h_i = \#\{u \in \Gamma | \deg u = i\}$. We will not distinguish between a monomial order ideal and its poset (ordered by divisibility).

A *pure M-vector* is the degree sequence of an order ideal of monomials, whose maximal elements have the same degree.

The following conjecture of Stanley [5] is one of the most important conjectures on h-vector of matroid complexes.

Conjecture 1.1. (*Stanley*) *The h-vector of a matroid complex is a pure M-vector.*

A poset Q is an *M-poset* if there exists a monomial M on a finite set E of indeterminates (variables) such that Q is isomorphic to the poset (ordered by divisibility) on the set of monomials on E that divide M. Equivalently, an M-poset is a direct product of chains.

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Given two elements $x \leq y$ of a poset P, the interval $[x, y]$ is called an M-*interval* if it is an M-poset. A pure poset P is called M-*partitionable* if P can be partitioned into M-intervals $[x_1, y_1], \dots, [x_n, y_n]$ such that for each $1 \leq i \leq n$, y_i is a maximal element of the poset P. Such a partition is called an M-*partition* of the poset P.

Definition 1.2. An M-*shelling* of a poset P is an M-partition of P along with an ordering of the M-intervals such that the union of the elements in any initial subsequence of M-intervals is an order ideal of P. A poset P is M-*shellable*, if it admits an M-shelling.

Chari [2] proposed a stronger version of Stanley's conjecture for h-vectors of matroid complexes based on the concept of M-shellability:

Conjecture 1.3. (*Chari*) *The h-vector of a matroid complex is a shellable M-vector.*

Recall that a pure M-vector is called shellable if it is the degree sequence of an M-shellable order ideal of monomials.

Herzog and Hibi [3] introduced discrete polymatroid, which it is a generalization of matroids. Let Γ be a pure monomial order ideal on the variables $\{x_1, \dots, x_r\}$ and for any $m \in \Gamma$, the degree of x_i in m is denoted by m_i. We say Γ is a discrete polymatroid if, for any two maximal monomials $m, m' \in \Gamma$ and index i with $m_i > m'_i$, there exists an index j such that $m_j < m'_j$ and $\frac{x_j}{x_i} m \in \Gamma$, cf. [4, Definition 4.1.].

The aim of this paper is to show that every discrete polymatroid is M-shellable (Theorem 2.1). We apply this result to show that the h-vector of a lattice path matroid (see Section 2 for definition) satisfies Conjecture 1.3.

2. Main Theorem

Theorem 2.1. *Every discrete polymatroid is M-shellable.*

Proof. Let Γ be a discrete polymatroid on the set $\{x_1, \dots, x_r\}$ of variables and let p be the number of maximal elements of Γ. The proof is by induction on p. If $p = 1$,the basic case, then Γ is an M-poset and the assertion is obvious. So assume that $p > 1$. Then there exist an index j and two maximal elements m and m' in Γ with $m_j \neq m'_j$. With no lose of generality, we assume that $j = r$. Now, put

- $k = \max\{m_r \mid m \in \Gamma\}$;
- $\Gamma_1 = \{m \in \Gamma \mid x_r^k \nmid m\}$;
- $\Gamma_2 = \Gamma - \Gamma_1$; and
- $\Gamma' = \{\frac{m}{x_r^k} \mid m \in \Gamma_2\}$.

Claim: Γ_1 and Γ' are discrete polymatroids.

Proof of Claim: We only show that Γ_1 is discrete polynomial. A similar argument works for Γ'. First note that Γ_1 is a monomial order ideal. Since, for $m \in \Gamma_1$ and $u \mid m$ we get that $x_r^k \nmid u$ which implies that $u \in \Gamma_1$.To prove the purity of Γ_1, we assume that this is not the case and get a contradiction. By assumption, there exist a maximal element m in Γ_1 and an element $m' \in \Gamma_2$ such that $m \mid m'$. So $m' = x_r^t m$, for some $t > 0$. Let m'' be a maximal elements in Γ with $m''_r < k$. Then there exists an index j such that $\frac{x_j}{x_i}m' = x_j x_r^{t-1}m \in \Gamma$ which it is contradict m is a maximal element of Γ_1. Thus Γ_1 is pure. To complete the proof we assume that m and m' be two monomials in Γ_1 with $m_i > m'_i$, for some i. Then there exists an index j such that $m_j < m'_j$ and $\frac{x_j}{x_i}m \in \Gamma$, since Γ is a discrete polymatroid. If $j \neq r$, then $x_r^k \nmid \frac{x_j}{x_i}m$, since $x_r^k \nmid m$. For $j = r$ we have $m_r < m'_r < k$ and then $(\frac{x_j}{x_i}m)_r < k$. Therefore Γ_1 is a discrete polymatroid. This complete the proof of the claim.

By induction hypothesis, there exist the following M-shelling orders for Γ_1 and Γ':

$$\Gamma_1 = [a_1, b_1]\dot{\cup} \cdots \dot{\cup}[a_n, b_n] \text{ and } \quad \Gamma' = [c_1, d_1]\dot{\cup} \cdots \dot{\cup}[c_l, d_l].$$

We claim that the following order

$$\Gamma = [a_1, b_1]\dot{\cup} \cdots \dot{\cup}[a_n, b_n]\dot{\cup}[x_r^k c_1, x_r^k d_1]\dot{\cup} \cdots \dot{\cup}[x_r^k c_l, x_r^k d_l]$$

is an M-shelling for Γ. It suffices to show that every initial subsequence $A = \Gamma_1\dot{\cup}[x_r^k c_1, x_r^k d_1]\dot{\cup} \cdots \dot{\cup}[x_r^k c_s, x_r^k d_s]$ $(s < l)$ is an order ideal. Assume the contrary. Then there exist $m \in A - \Gamma_1$ and $u \in \Gamma - A$ with $u \mid m$. Therefore, $\frac{u}{x_r^k} \in [c_1, d_1]\dot{\cup} \cdots \dot{\cup}[c_s, d_s]$. Since $\frac{u}{x_r^k} \mid \frac{m}{x_r^k}$, and Γ' is M-shellable. It contradicts $u \in \Gamma - A$. Now the proof is complete. □

Note that the converse of Theorem 2.1 does not hold. As a counterexample, one can consider the monomial order ideal Σ with maximal elements xy and z^2. It is easy to see that Σ is M-shellable but it is not a discrete polymatroid.

A sequence $(h_0, h_1, \ldots, h_r)$ is called a *PM-vector* if it is the degree sequence of some discrete polymatroid. Clearly, every PM-vector is a pure M-vector. But Theorem 2.1 gives the following generalization of this fact.

Corollary 2.2. *Every PM-vector is a shellable M-vector.*

The h-vector of Σ in the example before Corollary 2.2 is $(1, 3, 2)$. It shows that $(1, 3, 2)$ is a shellable M-vector, but it is indeed a PM-vector (take the discrete polymatroid with maximal elements xy and yz). However we guess these two classes of vectors are very closed.

We end the paper by a result on lattice path matroids.

Fix two lattice paths $P = p_1p_2 \ldots p_{m+r}$ and $Q = q_1q_2 \ldots q_{m+r}$ from $(0, 0)$ to (m, r) with P never going above Q. For every lattice path R between P and Q, let $\mathcal{N}(R)$ be the set of R's north steps.

In [1], the authors showed that $M[P, Q] = \{\mathcal{N}(R) : R \text{ is a path between } Q \text{ and } P\}$ is a matroid. $M[P, Q]$ is called a *lattice path matroid*.

Schweig [4, Theorem 3.6.] showed that lattice path matroids satisfy Conjecture 1.1. Even more, he proved that the h-vector of a lattice path matroid is a PM-vector, [4, Corollary 4.5.]. This result of Schweig and Corollary 2.2 together imply the following result, which says that lattice path matroids satisfy Conjecture 1.3.

Corollary 2.3. *The h-vector of a lattice path matroid is a shellable M-vector.*

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