SUPER DUALITY AND HOMOLOGY OF UNITARIZABLE MODULES OF LIE ALGEBRAS

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Abstract. The $\mathfrak{u}$-homology formulas for unitarizable modules at negative levels over classical Lie algebras of infinite rank of types $\mathfrak{gl}(n)$, $\mathfrak{sp}(2n)$ and $\mathfrak{so}(2n)$ are obtained. As a consequence, we recover the Enright's formulas for three Hermitian symmetric pairs of classical types $(SU(p,q), SU(p) \times SU(q))$, $(Sp(2n), U(n))$ and $(SO^*(2n), U(n))$.

1. Introduction

In analogy to Kostant's $\mathfrak{u}$-cohomology formulas [Ko], Enright establishes similar formulas [E] for unitarizable highest weight modules of Hermitian symmetric pairs in term of certain complicated subsets of the Weyl groups. The argument there is intricate and involves several equivalences of categories and non-trivial combinatorics of the Weyl groups. Kostant's formula can be rephrased by saying the Kazhdan-Lusztig polynomials associated to finite-dimensional module are monomials. The same statement is true by Enright's formulas for unitarizable highest weight modules. Except for the resemblance of the formulas, there was no obvious connection between Enright's formula and Kostant's formula.

However, the modules appearing in the Howe duality at negative levels [W, H1, H2] over classical Lie algebras of infinite rank are unitarizable modules (cf. [EHW], see also Proposition 2.6 and Remark 2.7 below) and the character formulas for these modules can be obtained by applying the involution of the ring of symmetric functions with infinite variables, which sends the elementary symmetric functions to the complete symmetric functions, to the characters for the corresponding integrable modules over the respective Lie algebras (cf. [CK, CKW]). Remarkably, the $\mathfrak{u}$-homology groups of these modules are also dictated by those of the corresponding integrable modules [CK, CKW]. Recently, the correspondence between $\mathfrak{u}$-homology groups of integrable modules at positive levels and $\mathfrak{u}$-homology groups of unitarizable modules (at negative levels) over the respective Lie algebras can be elucidated in terms of the so called super duality [CWZ, CW], established in [BrS, CL, CLW]. So far there is no explanation of the similarity of these two different $\mathfrak{u}$-homology groups. Super duality gives a first conceptual explanation of this similarity [CLW, Theorem 4.13].

To the best of our knowledge, there is no other proof of Enright's formulas. In this paper, we give a proof of Enright's homology formulas for unitarizable modules by using Kostant's formulas and super duality. The $\mathfrak{u}$-homology formulas (see Theorem 4.4 below) for unitarizable modules over classical Lie algebras of infinite rank of types $\mathfrak{gl}(n)$, $\mathfrak{sp}(2n)$ and $\mathfrak{so}(2n)$ are obtained by combinatorial method. The proof involves relating the combinatorial data of Kostant's formulas for integrable modules over corresponding Lie algebras, that are determined by the super duality, to the data of the Lie algebras under consideration. By applying the truncation functors (cf. [CLW, Section 3.4] to the $\mathfrak{u}$-homology formulas, see also Section 2.4 below), we recover the Enright's formula for three Hermitian symmetric pairs of classical types $(SU(p,q), SU(p)\times SU(q))$, $(Sp(2n), U(n))$ and $(SO^*(2n), U(n))$. However, for $\mathfrak{so}(2n)$, our method can only recover partially Enright's formula for some unitarizable highest weight cases.

The paper is organized as follows. In Section 2, we review and set up notations for the classical Lie algebras of finite and infinite rank. We describe the unitarizable highest weight modules considered in this paper. Combinatorial description of Weyl groups are also given in this section. In Section 3, we compare the actions of certain subsets of Weyl groups on certain numerical data associated with the highest weights. In Section 4, homology formulas for unitarizable modules over Lie algebras of infinite rank are proved. In Section 5, Enright's homology formulas are proved.

We shall use the following notations throughout this article. The symbols $\mathbb{Z}$, $\mathbb{N}$, and $\mathbb{Z}_+$ stand for the sets of all, positive and non-negative integers, respectively. We set $\mathbb{Z}^* := \mathbb{Z}\backslash\{0\}$. For a partition λ, we denote by λ' the transpose partition of λ. Finally all vector spaces, algebras, tensor products, et cetera, are over the field of complex numbers $\mathbb{C}$.

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2. Preliminaries

2.1. **Classical Lie algebras of infinite rank.** In this subsection we review and fix notations on Lie algebras of interest in this paper. For details we refer to the references [K, W, CK, CLW].

2.1.1. *The Lie algebra $\mathfrak{a}_\infty$.* Let $\mathbb{C}^\infty$ be the vector space over $\mathbb{C}$ with an ordered basis $\{\, e_i \,|\, i \in \mathbb{Z}\,\}$ so that an element in $\mathrm{End}(\mathbb{C}^\infty)$ may be identified with a matrix (a_{ij}) $(i,j \in \mathbb{Z})$. Let E_{ij} be the matrix with 1 at the i-th row and j-th column and zero elsewhere. Let $\mathring{\mathfrak{a}}_\infty$ denote the subalgebra of the Lie algebra $\mathrm{End}(\mathbb{C}^\infty)$ spanned by E_{ij} with $i,j \in \mathbb{Z}$. Denote by $\mathfrak{a}_\infty := \mathring{\mathfrak{a}}_\infty \oplus \mathbb{C}K$ the central extension of $\mathring{\mathfrak{a}}_\infty$ by the one-dimensional center $\mathbb{C}K$ given by the 2-cocycle

$$\tau(A,B) := \mathrm{Tr}([J,A]B), \tag{2.1}$$

where $J = \sum_{i\le 0} E_{ii}$ and $\mathrm{Tr}(C)$ is the trace of the matrix C. Observe that the cocycle τ is a coboundary. Indeed, there is embedding $\iota_{\mathring{\mathfrak{a}}}$ from $\mathring{\mathfrak{a}}_\infty$ to $\mathfrak{a}_\infty$ defined by $A \in \mathring{\mathfrak{a}}_\infty$ sending to $A + \mathrm{Tr}(JA)K$ (cf. [CLW, Section 2.5]). It is clear that $\iota_{\mathring{\mathfrak{a}}}(\mathring{\mathfrak{a}}_\infty)$ is an ideal of $\mathfrak{a}_\infty$ and $\mathfrak{a}_\infty$ is a direct sum of the ideals $\iota_{\mathring{\mathfrak{a}}}(\mathring{\mathfrak{a}}_\infty)$ and $\mathbb{C}K$. Note that $\iota_{\mathring{\mathfrak{a}}}(E_{ii}) = E_{ii} + K$ (resp. E_{ii}) for $i \le 0$ (resp. $i \ge 1$).

The Cartan subalgebra $\sum_{i\in\mathbb{Z}}\mathbb{C}E_{ii}\oplus\mathbb{C}K$ is denoted by $\mathfrak{h}_{\mathfrak{a}}$. By assigning degree 0 to the Cartan subalgebra and setting $\deg E_{ij}=j-i$, $\mathfrak{a}_\infty$ is equipped with a $\mathbb{Z}$-gradation $\mathfrak{a}_\infty=\bigoplus_{k\in\mathbb{Z}}(\mathfrak{a}_\infty)_k$. This leads to the following triangular decomposition:

$$\mathfrak{a}_\infty=(\mathfrak{a}_\infty)_+\oplus(\mathfrak{a}_\infty)_0\oplus(\mathfrak{a}_\infty)_-,$$

where $(\mathfrak{a}_\infty)_\pm=\bigoplus_{k\in\pm\mathbb{N}}(\mathfrak{a}_\infty)_k$ and $(\mathfrak{a}_\infty)_0=\mathfrak{h}_{\mathfrak{a}}$.

The set of simple coroots, simple roots and positive roots of $\mathfrak{a}_\infty$ are respectively

$$\begin{aligned}\Pi^\vee_{\mathfrak{a}}&=\{\,\beta_i^\vee:=E_{ii}-E_{i+1,i+1}+\delta_{i0}K\mid i\in\mathbb{Z}\,\},\\ \Pi_{\mathfrak{a}}&=\{\,\beta_i:=\epsilon_i-\epsilon_{i+1}\mid i\in\mathbb{Z}\,\},\\ \Delta^+_{\mathfrak{a}}&=\{\,\epsilon_i-\epsilon_j\mid i<j,\, i,j\in\mathbb{Z}\,\},\end{aligned}$$

where $\epsilon_i\in\mathfrak{h}_{\mathfrak{a}}^*$ is determined by $\langle\epsilon_i,E_{jj}\rangle=\delta_{ij}$ and $\langle\epsilon_i,K\rangle=0$. We also let $\vartheta_{\mathfrak{a}}\in\mathfrak{h}_{\mathfrak{a}}^*$ be defined by $\langle\vartheta_{\mathfrak{a}},K\rangle=1$ and $\langle\vartheta_{\mathfrak{a}},E_{jj}\rangle=0$, for all $j\in\mathbb{Z}$. Let $\rho_{\mathfrak{a}}\in\mathfrak{h}_{\mathfrak{a}}^*$ be determined by $\langle\rho_{\mathfrak{a}},E_{jj}\rangle=-j$, for all $j\in\mathbb{Z}$, and $\langle\rho_{\mathfrak{a}},K\rangle=0$, so that we have $\langle\rho_{\mathfrak{a}},\alpha_i^\vee\rangle=1$, for all $i\in\mathbb{Z}$.

2.1.2. *The Lie algebras $\mathfrak{c}_\infty$ and $\mathfrak{d}_\infty$.* For $\mathfrak{g}=\mathfrak{c},\mathfrak{d}$, let $\mathring{\mathfrak{g}}_\infty$ be the subalgebra of $\mathring{\mathfrak{a}}_\infty$ preserving the following bilinear form on $\mathbb{C}^\infty$:

$$(e_i|e_j)=\begin{cases}(-1)^i\delta_{i,1-j}, & \text{if } \mathfrak{g}=\mathfrak{c},\\ \delta_{i,1-j}, & \text{if } \mathfrak{g}=\mathfrak{d},\end{cases}\qquad i,j\in\mathbb{Z}.$$

Let $\mathfrak{g}_\infty=\mathring{\mathfrak{g}}_\infty\oplus\mathbb{C}K$ be the central extension of $\mathring{\mathfrak{g}}_\infty$ determined by the restriction of the two-cocycle (2.1). Then $\mathfrak{g}_\infty$ has a natural $\mathbb{Z}$-gradation and a triangular decomposition induced from $\mathfrak{a}_\infty$ with $(\mathfrak{g}_\infty)_n=\mathfrak{g}_\infty\cap(\mathfrak{a}_\infty)_n$, for $n\in\mathbb{Z}$. Similar to the $\mathfrak{a}_\infty$ case, the cocycle is a coboundary. Indeed, there are embeddings $\iota_{\mathring{\mathfrak{g}}}$ from $\mathring{\mathfrak{g}}_\infty$ to $\mathfrak{g}_\infty$ defined by $A\in\mathring{\mathfrak{g}}_\infty$ sending to $A+\mathrm{Tr}(JA)K$ [CLW, Section 2.5]. It is clear that $\iota_{\mathring{\mathfrak{g}}}(\mathring{\mathfrak{g}}_\infty)$ is an ideal of $\mathfrak{g}_\infty$ and $\mathfrak{g}_\infty$ is a direct sum of the ideals $\iota_{\mathring{\mathfrak{g}}}(\mathring{\mathfrak{g}}_\infty)$ and $\mathbb{C}K$. Note that $\iota_{\mathring{\mathfrak{g}}}(\widetilde{E}_i)=\widetilde{E}_i-K$ for $i\in\mathbb{N}$ where

$$\widetilde{E}_i=E_{ii}-E_{1-i,1-i}.$$

Note that $(\mathfrak{g}_\infty)_0=\sum_{i\in\mathbb{N}}\mathbb{C}\widetilde{E}_i\oplus\mathbb{C}K$ are Cartan subalgebras, which will be denoted by $\mathfrak{h}_{\mathfrak{g}}$. We let $\epsilon_i\in\mathfrak{h}_{\mathfrak{g}}^*$ be defined by $\langle\epsilon_i,\widetilde{E}_j\rangle=\delta_{ij}$ for $i,j\in\mathbb{N}$ and $\langle\epsilon_i,K\rangle=0$. Then the set of positive roots of $\mathfrak{c}_\infty$ and $\mathfrak{d}_\infty$ are respectively

$$\begin{aligned}\Delta^+_{\mathfrak{c}}&=\{\,\pm\epsilon_i-\epsilon_j\ ,\ -2\epsilon_i\ (i,j\in\mathbb{N},i<j)\,\},\\ \Delta^+_{\mathfrak{d}}&=\{\,\pm\epsilon_i-\epsilon_j\ (i,j\in\mathbb{N},i<j)\,\}.\end{aligned}$$

Set

$$\alpha_0^\vee=\begin{cases}-\widetilde{E}_1+K, & \text{for } \mathfrak{c}_\infty,\\ -\widetilde{E}_1-\widetilde{E}_2+2K, & \text{for } \mathfrak{d}_\infty,\end{cases}\qquad \alpha_0=\begin{cases}-2\epsilon_1, & \text{for } \mathfrak{c}_\infty,\\ -\epsilon_1-\epsilon_2, & \text{for } \mathfrak{d}_\infty.\end{cases}$$

The set of simple coroots and simple roots of $\mathfrak{g}_\infty$ are respectively

$$\begin{aligned}\Pi^\vee_{\mathfrak{g}}&=\{\,\alpha_0^\vee,\,\alpha_i^\vee=\widetilde{E}_i-\widetilde{E}_{i+1}\ (i\in\mathbb{N})\,\},\\ \Pi_{\mathfrak{g}}&=\{\,\alpha_0,\,\alpha_i=\epsilon_i-\epsilon_{i+1}\ (i\in\mathbb{N})\,\}.\end{aligned}$$

Let $\vartheta_{\mathfrak{g}} \in \mathfrak{h}_{\mathfrak{g}}^*$ defined by $\langle \vartheta_{\mathfrak{g}}, \widetilde{E}_i \rangle = 0$ for $i \in \mathbb{N}$ and $\langle \vartheta_{\mathfrak{g}}, K \rangle = r$ with $r = 1$ (resp. $\frac{1}{2}$) for $\mathfrak{g} = \mathfrak{c}$ (resp. $\mathfrak{d}$). We also let $\rho_{\mathfrak{g}} \in \mathfrak{h}_{\mathfrak{g}}^*$ be determined by

$$\langle \rho_{\mathfrak{g}}, \widetilde{E}_j \rangle = \begin{cases} -j, & \text{for } \mathfrak{g} = \mathfrak{c}, \\ -j+1, & \text{for } \mathfrak{g} = \mathfrak{d}, \end{cases} j \in \mathbb{N}, \qquad \text{and} \qquad \langle \rho_{\mathfrak{g}}, K \rangle = 0.$$

We have $\langle \rho_{\mathfrak{g}}, \alpha_i^\vee \rangle = 1$ for $i \in \mathbb{N}$ and $\mathfrak{g} = \mathfrak{c}, \mathfrak{d}$.

2.1.3. *Levi subalgebras.* For $\mathfrak{g} = \mathfrak{a}, \mathfrak{c}, \mathfrak{d}$, let $\Delta_{\mathfrak{g}} := \Delta_{\mathfrak{g}}^+ \cup \Delta_{\mathfrak{g}}^-$, where $\Delta_{\mathfrak{g}}^- = -\Delta_{\mathfrak{g}}^+$. Then $\Delta_{\mathfrak{g}}$ is the set of roots of $\mathfrak{g}_\infty$. Let $\Delta_{\mathfrak{g},c}^{\pm} := \Delta_{\mathfrak{g}}^{\pm} \cap \left(\sum_{j \neq 0} \mathbb{Z}\alpha_j \right)$ and $\Delta_{\mathfrak{g},n}^{\pm} := \Delta_{\mathfrak{g}}^{\pm} \setminus \Delta_{\mathfrak{g},c}^{\pm}$. Denote by $\mathfrak{g}_\alpha$ the root space corresponding to $\alpha \in \Delta_{\mathfrak{g}}$. Set

$$\mathfrak{u}_{\mathfrak{g}}^{\pm} := \sum_{\alpha \in \Delta_{\mathfrak{g},n}^{\pm}} \mathfrak{g}_\alpha, \quad \mathfrak{l}_{\mathfrak{g}} := \sum_{\alpha \in \Delta_{\mathfrak{g},c}^{\pm}} \mathfrak{g}_\alpha \oplus \mathfrak{h}_{\mathfrak{g}}. \tag{2.2}$$

Then we have $\mathfrak{g}_\infty = \mathfrak{u}_{\mathfrak{g}}^+ \oplus \mathfrak{l}_{\mathfrak{g}} \oplus \mathfrak{u}_{\mathfrak{g}}^-$. The Lie algebras $\mathfrak{l}_{\mathfrak{g}}$ and $\mathfrak{g}_\infty$ share the same Cartan subalgebra $\mathfrak{h}_{\mathfrak{g}}$. Moreover, $\mathfrak{l}_{\mathfrak{g}}$ has a triangular decomposition induced from $\mathfrak{g}_\infty$. For $\mu \in \mathfrak{h}_{\mathfrak{g}}^*$, we denote respectively by $L(\mathfrak{g}_\infty, \mu)$ and $L(\mathfrak{l}_{\mathfrak{g}}, \mu)$ the irreducible highest weight $\mathfrak{g}_\infty$-module and $\mathfrak{l}_{\mathfrak{g}}$-module with highest weight μ with respect to the triangular decompositions.

For a root $\alpha \in \Delta_{\mathfrak{g}}$, $\mathfrak{g} = \mathfrak{a}, \mathfrak{c}, \mathfrak{d}$, define the reflection σ_α by

$$\sigma_\alpha(\mu) := \mu - \langle \mu, \alpha^\vee \rangle \alpha, \quad \mu \in \mathfrak{h}_{\mathfrak{g}}^*.$$

Here and after, $\alpha^\vee$ denote the coroot of the root α. Let $I_{\mathfrak{a}} = \mathbb{Z}$ and $I_{\mathfrak{g}} = \mathbb{N}$ for $\mathfrak{g} = \mathfrak{c}, \mathfrak{d}$. For $j \in I_{\mathfrak{g}} \cup \{0\}$, let $\sigma_j = \sigma_{\alpha_j}$. Let $W_{\mathfrak{g}}$ be the subgroup of $\mathrm{Aut}(\mathfrak{h}_{\mathfrak{g}}^*)$ generated by the reflections σ_j with $j \in I_{\mathfrak{g}} \cup \{0\}$, i.e. $W_{\mathfrak{g}}$ is the Weyl group of $\mathfrak{g}_\infty$. For each $w \in W_{\mathfrak{g}}$, $\ell_{\mathfrak{g}}(w)$ denote the length of w. We also define

$$w \circ \mu := w(\mu + \rho_{\mathfrak{g}}) - \rho_{\mathfrak{g}}, \quad \mu \in \mathfrak{h}_{\mathfrak{g}}^*, w \in W_{\mathfrak{g}}.$$

Consider $W_{\mathfrak{g},0}$ the subgroup of $W_{\mathfrak{g}}$ generated by σ_j with $j \neq 0$. Let $W_{\mathfrak{g}}^0$ denote the set of the minimal length left coset representatives of $W_{\mathfrak{g}}/W_{\mathfrak{g},0}$ (cf. [V, Liu, Ku]). We have $W_{\mathfrak{g}} = W_{\mathfrak{g}}^0 W_{\mathfrak{g},0}$. For $k \in \mathbb{Z}_+$, set

$$W_{\mathfrak{g},k}^0 := \{ w \in W_{\mathfrak{g}}^0 \mid \ell_{\mathfrak{g}}(w) = k \}.$$

Finally, for $\mathfrak{g} = \mathfrak{a}, \mathfrak{c}, \mathfrak{d}$, let $(\cdot|\cdot)$ be a bilinear form defined on subspace of $\mathfrak{h}_{\mathfrak{g}}^*$ satisfying

$$(\epsilon_i|\epsilon_j) = \delta_{ij}, \quad (\vartheta_{\mathfrak{g}}|\epsilon_i) = (\epsilon_i|\vartheta_{\mathfrak{g}}) = (\vartheta_{\mathfrak{g}}|\vartheta_{\mathfrak{g}}) = 0 \quad \text{for } i, j \in I_{\mathfrak{g}}.$$

Recall that $I_{\mathfrak{a}} = \mathbb{Z}$ and $I_{\mathfrak{g}} = \mathbb{N}$ for $\mathfrak{g} = \mathfrak{c}, \mathfrak{d}$.

2.2. **Finite dimensional Lie algebras.** For the rest of the paper, let $\mathfrak{g}$ stand for $\mathfrak{a}, \mathfrak{c}, \mathfrak{d}$. We shall fix the following notations:

$$\overline{\mathfrak{a}} := \mathfrak{a}, \ \overline{\mathfrak{c}} := \mathfrak{d}, \ \overline{\mathfrak{d}} := \mathfrak{c}.$$

Remark 2.1. For $\mathfrak{x} = \mathfrak{c}, \mathfrak{d}$, let $\mathfrak{g}^{\mathfrak{x}}$ and $\overline{\mathfrak{g}}^{\mathfrak{x}}$ be the Lie algebras defined in [CLW, Section 2] with $m = 0$. Then $\mathfrak{c}_\infty = \mathfrak{g}^{\mathfrak{c}}$, $\mathfrak{d}_\infty = \mathfrak{g}^{\mathfrak{d}}$, $\overline{\mathfrak{c}}_\infty \cong \overline{\mathfrak{g}}^{\mathfrak{c}}$ and $\overline{\mathfrak{d}}_\infty \cong \overline{\mathfrak{g}}^{\mathfrak{d}}$. Note that K send to $-K$ for the isomorphisms $\overline{\mathfrak{c}}_\infty \cong \overline{\mathfrak{g}}^{\mathfrak{c}}$ and $\overline{\mathfrak{d}}_\infty \cong \overline{\mathfrak{g}}^{\mathfrak{d}}$.

For $m, n \in \mathbb{N}$, the subalgebra of $\mathring{\mathfrak{a}}_\infty$ spanned by E_{ij} with $1-m \le i, j \le n$, denoted by $\mathfrak{t}_{m,n}\mathfrak{a}$, is isomorphic to the general linear algebra $\mathfrak{gl}(m+n)$. The subalgebras $(\mathfrak{t}_{n,n}\mathfrak{a}) \cap \mathring{\mathfrak{c}}_\infty$ and $(\mathfrak{t}_{n,n}\mathfrak{a}) \cap \mathring{\mathfrak{d}}_\infty$ are isomorphic to the symplectic Lie algebra $\mathfrak{sp}(2n)$ and orthogonal Lie algebra $\mathfrak{so}(2n)$, denoted by $\mathfrak{t}_n\mathfrak{c}$ and $\mathfrak{t}_n\mathfrak{d}$ respectively. We shall drop the subscript of $\mathfrak{t}$ if there has no ambiguity.

For $\overline{\mathfrak{g}} = \mathfrak{a}, \mathfrak{c}, \mathfrak{d}$, the embeddings $\iota_{\mathring{\overline{\mathfrak{g}}}}$ restricted to $\mathfrak{t}\overline{\mathfrak{g}}$ are also denoted by $\iota_{\mathring{\overline{\mathfrak{g}}}}$. Let $\Delta^+_{\mathfrak{t}\overline{\mathfrak{g}}}$ denote the set of positive roots of $\mathfrak{t}\overline{\mathfrak{g}}$ with respect to the triangular decomposition induced from $\overline{\mathfrak{g}}_\infty$. We also let $\Delta_{\mathfrak{t}\overline{\mathfrak{g}}} = \Delta^+_{\mathfrak{t}\overline{\mathfrak{g}}} \cup -\Delta^+_{\mathfrak{t}\overline{\mathfrak{g}}}$ and $\Delta^+_{\mathfrak{t}\overline{\mathfrak{g}},n} = \Delta^+_{\overline{\mathfrak{g}},n} \cap \Delta^+_{\mathfrak{t}\overline{\mathfrak{g}}}$. Set $\mathfrak{h}_{\mathfrak{t}\overline{\mathfrak{g}}} = \mathfrak{h}_{\overline{\mathfrak{g}}} \cap \mathfrak{t}\overline{\mathfrak{g}}$, $\mathfrak{u}^\pm_{\mathfrak{t}\overline{\mathfrak{g}}} = \mathfrak{u}^\pm_{\overline{\mathfrak{g}}} \cap \mathfrak{t}\overline{\mathfrak{g}}$ and $\mathfrak{l}_{\mathfrak{t}\overline{\mathfrak{g}}} = \mathfrak{l}_{\overline{\mathfrak{g}}} \cap \mathfrak{t}\overline{\mathfrak{g}}$. Note that $\mathfrak{t}\overline{\mathfrak{g}}$ and $\mathfrak{l}_{\mathfrak{t}\overline{\mathfrak{g}}}$ share the same a Cartan subalgebra $\mathfrak{h}_{\mathfrak{t}\overline{\mathfrak{g}}}$. Moreover, $\mathfrak{l}_{\mathfrak{t}\overline{\mathfrak{g}}}$ has a triangular decomposition induced from $\mathfrak{t}\overline{\mathfrak{g}}$. For $\mu \in \mathfrak{h}^*_{\mathfrak{t}\overline{\mathfrak{g}}}$, we denote respectively by $L(\mathfrak{t}\overline{\mathfrak{g}}, \mu)$ and $L(\mathfrak{l}_{\mathfrak{t}\overline{\mathfrak{g}}}, \mu)$ the irreducible highest weight $\mathfrak{t}\overline{\mathfrak{g}}$-module and $\mathfrak{l}_{\mathfrak{t}\overline{\mathfrak{g}}}$-module with highest weight μ with respect to the triangular decompositions. For $\mu \in \mathfrak{h}^*_{\mathfrak{t}\overline{\mathfrak{g}}}$, $L(\mathfrak{l}_{\mathfrak{t}\overline{\mathfrak{g}}}, \mu)$ is extended to an $(\mathfrak{l}_{\mathfrak{t}\overline{\mathfrak{g}}} + \mathfrak{u}^+_{\mathfrak{t}\overline{\mathfrak{g}}})$-module by letting $\mathfrak{u}^+_{\mathfrak{t}\overline{\mathfrak{g}}}$ act trivially. Let $\mathfrak{p}_{\mathfrak{t}\overline{\mathfrak{g}}} = \mathfrak{l}_{\mathfrak{t}\overline{\mathfrak{g}}} + \mathfrak{u}^+_{\mathfrak{t}\overline{\mathfrak{g}}}$. Define as usual the parabolic Verma module with highest weight μ by

$$N(\mathfrak{t}\overline{\mathfrak{g}}, \mu) = \mathrm{Ind}^{\mathfrak{t}\overline{\mathfrak{g}}}_{\mathfrak{p}_{\mathfrak{t}\overline{\mathfrak{g}}}} L(\mathfrak{l}_{\mathfrak{t}\overline{\mathfrak{g}}}, \mu).$$

The space $\mathfrak{h}^*_{\mathfrak{t}\overline{\mathfrak{g}}}$ is spanned by ϵ_i with $1 \le i \le n$ (resp. $1-m \le i \le n-1$) for $\overline{\mathfrak{g}} = \mathfrak{c}, \mathfrak{d}$ (resp. $\mathfrak{a}$) and therefore $\mathfrak{h}^*_{\mathfrak{t}\overline{\mathfrak{g}}}$ can be regarded as a subspace of $\mathfrak{h}^*_{\overline{\mathfrak{g}}}$. Note that $\mathfrak{h}^*_{\mathfrak{t}\overline{\mathfrak{g}}}$ is an invariant subspace of σ_i for $1 \le i \le n$ (resp. $1-m \le i \le n-1$) for $\overline{\mathfrak{g}} = \mathfrak{c}$ or $\mathfrak{d}$ (resp. $\mathfrak{a}$). The restriction of these σ_i to $\mathfrak{h}^*_{\mathfrak{t}\overline{\mathfrak{g}}}$ are also denoted by σ_i. Let $W_{\mathfrak{t}\overline{\mathfrak{g}}}$ be the subgroup of $\mathrm{Aut}(\mathfrak{h}^*_{\mathfrak{t}\overline{\mathfrak{g}}})$ generated by these σ_is. Then $W_{\mathfrak{t}\overline{\mathfrak{g}}}$ is the Weyl group of $\mathfrak{t}\overline{\mathfrak{g}}$. For each $w \in W_{\mathfrak{t}\overline{\mathfrak{g}}}$ we let $\ell_{\mathfrak{t}\overline{\mathfrak{g}}}(w)$ denote the length of w. Consider $W_{\mathfrak{t}\overline{\mathfrak{g}},0}$ the subgroup of $W_{\mathfrak{t}\overline{\mathfrak{g}}}$ generated by σ_j with $j \neq 0$. Let $W^0_{\mathfrak{t}\overline{\mathfrak{g}}}$ denote the set of the minimal length representatives of the left coset space $W_{\mathfrak{t}\overline{\mathfrak{g}}}/W_{\mathfrak{t}\overline{\mathfrak{g}},0}$ (cf. [Liu, Ku]). For $k \in \mathbb{Z}_+$, set $W^0_{\mathfrak{t}\overline{\mathfrak{g}},k} := \{\, w \in W^0_{\mathfrak{t}\overline{\mathfrak{g}}} \,|\, \ell_{\mathfrak{t}\overline{\mathfrak{g}}}(w) = k \,\}$. We also define

$$w \circ \mu := w(\mu + \rho_{\mathfrak{t}\overline{\mathfrak{g}}}) - \rho_{\mathfrak{t}\overline{\mathfrak{g}}}, \quad \mu \in \mathfrak{h}^*_{\mathfrak{t}\overline{\mathfrak{g}}}, w \in W_{\mathfrak{t}\overline{\mathfrak{g}}}.$$

Finally, let $\rho_{\mathfrak{t}\overline{\mathfrak{g}}}$ denote the half sum of the positive roots. Then $\rho_{\mathfrak{t}\overline{\mathfrak{g}}}(h) = \rho_{\overline{\mathfrak{g}}}(h)$ (resp. $\rho_{\mathfrak{a}}(h) + \frac{1}{2}(n-m+1)$) for $h \in \mathfrak{h}_{\mathfrak{t}\overline{\mathfrak{g}}}$ with $\overline{\mathfrak{g}} = \mathfrak{c}, \mathfrak{d}$ (resp. $\overline{\mathfrak{g}} = \mathfrak{a}$).

2.3. **Combinatorial descriptions of Weyl groups.** In this section, we present combinatorial descriptions of certain aspects of infinite Weyl groups $W_{\mathfrak{g}}$ (cf. [BB]). Recall that $\mathbb{Z}^* := \mathbb{Z} \backslash \{0\}$.

Define $\phi_{\mathfrak{g}} \in \mathfrak{h}^*_{\mathfrak{g}}$ by

$$\phi_{\mathfrak{g}} = \begin{cases} -\sum_{i \le 0} \epsilon_i, & \text{if } \mathfrak{g} = \mathfrak{a}; \\ \sum_{i \in \mathbb{N}} \epsilon_i, & \text{if } \mathfrak{g} = \mathfrak{c}, \mathfrak{d}. \end{cases}$$

Every element $\sigma \in \mathfrak{h}^*_{\mathfrak{g}}$ can be uniquely represented by $\sum_{i \in I_{\mathfrak{g}}} \xi_i \epsilon_i + q\vartheta_{\mathfrak{g}}$ with $\xi_i, q \in \mathbb{C}$. For $\mathfrak{g} = \mathfrak{c}, \mathfrak{d}$ and $i \in \mathbb{N}$, we define $\epsilon_{-i} = -\epsilon_i$. It is easy to see by computing the actions of σ_i that the actions of $W_{\mathfrak{g}}$ on $\mathfrak{h}_{\mathfrak{g}}^*$ is given by

$$\sigma(\sum_{i\in\mathbb{Z}}\xi_i\epsilon_i+q\vartheta_{\mathfrak{a}})=\sum_{i\leq 0}(\xi_i+q)\epsilon_{\tilde{\sigma}(i)}+\sum_{i>0}\xi_i\epsilon_{\tilde{\sigma}(i)}+q\phi_{\mathfrak{a}}+q\vartheta_{\mathfrak{a}},\qquad \text{if } \mathfrak{g}=\mathfrak{a}; \tag{2.3}$$

$$\sigma(\sum_{i\in\mathbb{N}}\xi_i\epsilon_i+q\vartheta_{\mathfrak{g}})=\sum_{i\in\mathbb{N}}(\xi_i-q\langle\vartheta_{\mathfrak{g}},K\rangle)\epsilon_{\tilde{\sigma}(i)}+q\langle\vartheta_{\mathfrak{g}},K\rangle\phi_{\mathfrak{g}}+q\vartheta_{\mathfrak{g}},\quad \text{if } \mathfrak{g}=\mathfrak{c},\mathfrak{d}, \tag{2.4}$$

where $\tilde{\sigma}$ is a permutation of $\mathbb{Z}$ (i.e. $\tilde{\sigma}$ is a bijection on $\mathbb{Z}$ satisfying $\tilde{\sigma}(j)=j$ for $|j|\gg 0$) for $\mathfrak{g}=\mathfrak{a}$ and $\tilde{\sigma}$ is a signed permutation of $\mathbb{Z}^*$ (i.e. $\tilde{\sigma}$ is a bijection on $\mathbb{Z}^*$ satisfying $\tilde{\sigma}(j)=j$ for $|j|\gg 0$ and $\tilde{\sigma}(-i)=-\tilde{\sigma}(i)$ for $i\in\mathbb{Z}^*$) for $\mathfrak{g}=\mathfrak{c},\mathfrak{d}$. Therefore $\sigma\mapsto\tilde{\sigma}$ is a representation on $\mathbb{Z}$ and $\mathbb{Z}^*$ for $\mathfrak{g}=\mathfrak{a}$ and $\mathfrak{g}=\mathfrak{c},\mathfrak{d}$, respectively. Moreover, they are faithful representations. It is clear that the image of $W_{\mathfrak{a}}$ in $\mathrm{Aut}(\mathbb{Z})$ is the set of permutations of $\mathbb{Z}$ and the image of $W_{\mathfrak{c}}$ (resp. $W_{\mathfrak{d}}$) in $\mathrm{Aut}(\mathbb{Z}^*)$ is the set of a signed (resp. even signed) permutations of $\mathbb{Z}^*$. A signed permutation $\tilde{\sigma}$ of $\mathbb{Z}^*$ is called even signed permutation if $|\{i\in\mathbb{N}\mid\tilde{\sigma}(i)<0\}|$ is a even number. We shall identify $W_{\mathfrak{g}}$ with the image of $W_{\mathfrak{g}}$ in $\mathrm{Aut}(\mathbb{Z})$ (resp. $\mathrm{Aut}(\mathbb{Z}^*)$) for $\mathfrak{g}=\mathfrak{a}$ (resp. $\mathfrak{c},\mathfrak{d}$) for the rest of the paper. Note that for $i\in\mathbb{Z}$, $\tilde{\sigma}_i(i)=i+1$, $\tilde{\sigma}_i(i+1)=i$ and $\tilde{\sigma}_i(j)=j$ for all $j\neq i,i+1$. Also for $\mathfrak{g}=\mathfrak{c},\mathfrak{d}$ and $i\in\mathbb{N}$, $\tilde{\sigma}_i(i)=i+1$, $\tilde{\sigma}_i(i+1)=i$ and $\tilde{\sigma}_i(j)=j$ for all $j\neq i,i+1$ while $\tilde{\sigma}_0(1)=-1$ (resp. -2), $\tilde{\sigma}_0(2)=2$ (resp. -1), and $\tilde{\sigma}_0(j)=j$ for all $j\geq 3$ for $\mathfrak{g}=\mathfrak{c}$ (resp. $\mathfrak{d}$). We shall use these representations for the rest of the paper and we shall simply write $\sigma(j)$ instead of $\tilde{\sigma}(j)$.

Recall that $\ell_{\mathfrak{g}}$ denote the length function on $W_{\mathfrak{g}}$ and $W_{\mathfrak{g}}^0$ denote the set of the minimal length left coset representatives of $W_{\mathfrak{g}}/W_{\mathfrak{g},0}$. We have

$$W_{\mathfrak{g}}^0=\begin{cases}\{\sigma\in W_{\mathfrak{a}}\,|\,\sigma(i)<\sigma(j) \text{ for } i<j\leq 0 \text{ and } \ 0<i<j\}, & \text{if } \mathfrak{g}=\mathfrak{a};\\ \{\sigma\in W_{\mathfrak{g}}|\sigma(i)<\sigma(j), \text{ for } 1\leq i<j\}, & \text{if } \mathfrak{g}=\mathfrak{c},\mathfrak{d}\end{cases} \tag{2.5}$$

(see, e.g. [BB, Lemma 2.4.7, Proposition 8.1.4 and Proposition 8.2.4]) and for $\sigma\in W_{\mathfrak{g}}^0$,

$$\ell_{\mathfrak{g}}(\sigma)=\begin{cases}|\{(i,j)\in\mathbb{Z}\times\mathbb{Z}\mid i<j,\sigma(i)>\sigma(j)\}|, & \text{if } \mathfrak{g}=\mathfrak{a};\\ |\{(i,j)\in\mathbb{N}\times\mathbb{N}\mid i\leq j,\sigma(-i)>\sigma(j)\}|, & \text{if } \mathfrak{g}=\mathfrak{c};\\ |\{(i,j)\in\mathbb{N}\times\mathbb{N}\mid i<j,\sigma(-i)>\sigma(j)\}|, & \text{if } \mathfrak{g}=\mathfrak{d}\end{cases} \tag{2.6}$$

(see, e.g. [BB, Corollary 1.5.2, Corollary 8.1.1 and Corollary 8.2.1]).

Lemma 2.2. *For $\sigma\in W_{\mathfrak{c}}^0$ with $\sigma(i)<0$ for $i\leq j$, and $\sigma(i)>0$ for $i>j$, define $\overline{\sigma}\in W_{\mathfrak{d}}^0$ by*

$$\overline{\sigma}(i)=\begin{cases}\sigma(i)-1, & \text{if } \ i\leq j;\\ 1, & \text{if } \ i=j+1 \text{ and } j \text{ is even};\\ -1, & \text{if } \ i=j+1 \text{ and } j \text{ is odd};\\ \sigma(i-1)+1, & \text{if } \ i\geq j+2.\end{cases}$$

For each $k\geq 0$, the map from $W_{\mathfrak{c},k}^0$ to $W_{\mathfrak{d},k}^0$ sending σ to $\overline{\sigma}$ is a bijection.

Proof. By (2.5), it is a bijection from $W_{\mathfrak{c}}^0$ to $W_{\mathfrak{d}}^0$. By (2.6), we have $\ell_{\mathfrak{c}}(\sigma)=\ell_{\mathfrak{d}}(\overline{\sigma})$ for $\sigma\in W_{\mathfrak{c}}^0$. The lemma follows. □

Let $\{\xi_i\}_{i\in\mathbb{N}}$ be a sequence of real numbers. Define $\xi_{-i} := -\xi_i$ for $i \in \mathbb{N}$. For any sequence of strictly decreasing negative real numbers $\{\xi_i\}_{i\in\mathbb{N}}$ and $\sigma \in W^0_{\mathfrak{g}}$ with $\mathfrak{g} = \mathfrak{c}, \mathfrak{d}$, it is easy to see that $\{\xi_{\sigma(i)}\}_{i\in\mathbb{N}}$ is a sequence of strictly decreasing real numbers. The following lemma follows from the definition of $\overline{\sigma}$.

Lemma 2.3. *Let $\{\xi_i\}_{i\in\mathbb{N}}$ be a sequence of strictly decreasing negative real numbers. Define $\overline{\xi}_{i+1} = \xi_i$ for $i \in \mathbb{N}$ and $\overline{\xi}_1 = 0$. Then for all $\sigma \in W^0_{\mathfrak{c}}$, we have*

$$\{\xi_{\sigma(i)} \mid i \in \mathbb{N}\} \cup \{0\} = \{\overline{\xi}_{\overline{\sigma}(i)} \mid i \in \mathbb{N}\},$$

where $\overline{\sigma}$ is defined in Lemma 2.2.

2.4. **Unitarizable highest weight modules.** Recall that $\mathfrak{g}$ stand for $\mathfrak{a}, \mathfrak{c}, \mathfrak{d}$, and $\overline{\mathfrak{a}} = \mathfrak{a}$, $\overline{\mathfrak{c}} = \mathfrak{d}$ and $\overline{\mathfrak{d}} = \mathfrak{c}$. In this subsection we classify the highest weights of irreducible unitarizable quasi-finite highest weight $\overline{\mathfrak{g}}_\infty$-modules with respect to the anti-linear anti-involution ω defined below.

For a partition $\lambda = (\lambda_1, \lambda_2, \cdots)$, the transpose partition of λ is denoted by $\lambda' = (\lambda'_1, \lambda'_2, \cdots)$. For $\mathfrak{g} = \mathfrak{c}, \mathfrak{d}$, a partition λ and $d \in \mathbb{C}$, define

$$\Lambda^{\mathfrak{g}}(\lambda, d) := \sum_{i\in\mathbb{N}} \lambda'_i \epsilon_i + d\vartheta_{\mathfrak{g}} \in \mathfrak{h}^*_{\mathfrak{g}}, \quad \overline{\Lambda}^{\mathfrak{g}}(\lambda, d) = \sum_{i\in\mathbb{N}} \lambda_i \epsilon_i - \frac{d\langle\vartheta_{\mathfrak{g}}, K\rangle}{\langle\vartheta_{\overline{\mathfrak{g}}}, K\rangle} \vartheta_{\overline{\mathfrak{g}}} \in \mathfrak{h}^*_{\overline{\mathfrak{g}}}. \tag{2.7}$$

Let $\mathcal{D}(\mathfrak{g})$ denote the set of pairs (λ, d) with $d \in \mathbb{Z}_+$ satisfying $\lambda'_1 \leq d$ if $\mathfrak{g} = \mathfrak{c}$; and $\lambda'_1 + \lambda'_2 \leq d$ if $\mathfrak{g} = \mathfrak{d}$. For a pair of partitions $\lambda = (\lambda^-, \lambda^+)$ and $d \in \mathbb{C}$, define $\Lambda^{\mathfrak{a}}(\lambda, d), \overline{\Lambda}^{\mathfrak{a}}(\lambda, d) \in \mathfrak{h}^*_{\mathfrak{a}}$ by

$$\Lambda^{\mathfrak{a}}(\lambda, d) = -\sum_{i\in\mathbb{Z}_+} (\lambda^-)'_{i+1} \epsilon_{-i} + \sum_{i\in\mathbb{N}} (\lambda^+)'_i \epsilon_i + d\vartheta_{\mathfrak{a}},$$
$$\overline{\Lambda}^{\mathfrak{a}}(\lambda, d) = -\sum_{i\in\mathbb{Z}_+} \lambda^-_{i+1} \epsilon_{-i} + \sum_{i\in\mathbb{N}} \lambda^+_i \epsilon_i - d\vartheta_{\mathfrak{a}}.$$

Let $\mathcal{D}(\mathfrak{a})$ denote the set of pairs (λ, d) satisfying $d \in \mathbb{Z}_+$ and $(\lambda^-)'_1 + (\lambda^+)'_1 \leq d$.

Let $\mathfrak{k}$ be a Lie algebra equipped with an anti-linear anti-involution ω, and let V be a $\mathfrak{k}$-module. A Hermitian form $\langle \cdot \mid \cdot \rangle$ on V is said to be contravariant if $\langle av|v'\rangle = \langle v|\omega(a)v'\rangle$, for all $a \in \mathfrak{k}$, $v, v' \in V$. A $\mathfrak{k}$-module equipped with a positive definite contravariant Hermitian form is called a unitarizable $\mathfrak{k}$-module. Assume that $\mathfrak{k} = \oplus_{j\in\mathbb{Z}} \mathfrak{k}_j$ (possibly $\dim \mathfrak{k}_j = \infty$) is a $\mathbb{Z}$-graded Lie algebra and $\mathfrak{k}_0$ is abelian. A graded $\mathfrak{k}$-module $M = \oplus_{j\in\mathbb{Z}} M_j$ is called quasi-finite if $\dim M_j < \infty$ for all $j \in \mathbb{Z}$ [KR].

Remark 2.4. Let V be a highest weight $\mathfrak{g}_\infty$-module with highest weight ξ. Using the arguments as in [LZ, Section 4], we have V is quasi-finite if and only if ξ satisfies $\xi(E_{ii}) = 0$ (resp. $\xi(\widetilde{E}_{ii}) = 0$) for $|i| \gg 0$ (resp. $i \gg 0$) for $\mathfrak{g} = \mathfrak{a}$ (resp. $\mathfrak{c}, \mathfrak{d}$). Therefore every quasi-finite integrable highest weight $\mathfrak{g}_\infty$-module is of the form $L(\mathfrak{g}_\infty, \Lambda^{\mathfrak{g}}(\lambda, d))$ for some $(\lambda, d) \in \mathcal{D}(\mathfrak{g})$.

Now we consider the anti-linear anti-involution ω on $\mathfrak{a}_\infty$ defined by

$$\omega(E_{ij}) = \begin{cases} E_{ji}, & \text{for } i, j \leq 0 \text{ or } i, j > 0; \\ -E_{ji}, & \text{for } i > 0, j \leq 0 \text{ or } i \leq 0, j > 0, \end{cases} \quad \text{and} \quad \omega(K) = K.$$

For $\mathfrak{g} = \mathfrak{c}, \mathfrak{d}$, the restriction of the anti-linear anti-involution ω on $\mathfrak{a}_\infty$ to $\mathfrak{g}_\infty$ gives an anti-linear anti-involution on $\mathfrak{g}_\infty$, which will also be denoted by ω.

For $d \in \mathbb{C}$ and a pair of partitions $\lambda = (\lambda^-, \lambda^+)$ with $\lambda^+_{n+1} = \lambda^-_{m+1} = 0$, let $\Gamma_{\mathfrak{t}\overline{\mathfrak{a}}}(\lambda, d)$ be the element in $\mathfrak{h}^*_{\mathfrak{t}\overline{\mathfrak{a}}}$ determined by

$$\Gamma_{\mathfrak{t}\overline{\mathfrak{a}}}(\lambda, d) = \sum_{i=1}^{m} (-d - \lambda^-_i)\epsilon_{-i+1} + \sum_{i=1}^{n} \lambda^+_i \epsilon_i.$$

For $d \in \mathbb{C}$ and a partition λ satisfying $\lambda_{n+1} = 0$, let $\Gamma_{\mathfrak{t}\overline{\mathfrak{g}}}(\lambda, d)$ be the element in $h^*_{\mathfrak{t}\overline{\mathfrak{g}}}$ determined by

$$\Gamma_{\mathfrak{t}\overline{\mathfrak{g}}}(\lambda, d) = \begin{cases} \sum_{i=1}^{n} (\lambda_i + \frac{d}{2})\epsilon_i, & \text{for } \overline{\mathfrak{g}} = \mathfrak{c}, \\ \sum_{i=1}^{n} (\lambda_i + d)\epsilon_i, & \text{for } \overline{\mathfrak{g}} = \mathfrak{d}. \end{cases}$$

Let $\mathcal{D}_{\mathfrak{t}}(\mathfrak{g})$ denote the subset of $\mathcal{D}(\mathfrak{g})$ consisting of elements in (λ, d) satisfying $\lambda_{n+1} = 0$ for $\overline{\mathfrak{g}} = \mathfrak{c}, \mathfrak{d}$ (resp. $\lambda^+_{n+1} = 0$ and $\lambda^-_{m+1} = 0$ for $\overline{\mathfrak{g}} = \mathfrak{a}$).

Now we introduce the truncation functors [CLW, Section 3.4]. Let $M = \bigoplus_\beta M_\beta$ be a semisimple $\mathfrak{h}_{\overline{\mathfrak{g}}}$-module such that M_β is the weight space of M with weight β. The truncation functor $\mathfrak{tr}_{\mathfrak{t}\overline{\mathfrak{h}}}$ is defined by sending M to $\bigoplus_\nu M_\nu$, summed over $\sum_{i=1-m}^{n} \mathbb{C}\epsilon_i + \mathbb{C}\vartheta_{\overline{\mathfrak{g}}}$ (resp. $\sum_{i=1}^{n} \mathbb{C}\epsilon_i + \mathbb{C}\vartheta_{\overline{\mathfrak{g}}}$) for $\overline{\mathfrak{g}} = \mathfrak{a}$ (resp. $\mathfrak{c}, \mathfrak{d}$). For $(\lambda, d) \in \mathcal{D}(\mathfrak{g})$, $L(\overline{\mathfrak{g}}_\infty, \overline{\Lambda}^{\mathfrak{g}}(\lambda, d))$ is a $\mathfrak{t}\overline{\mathfrak{g}}$-module through the embedding $\iota_{\tilde{\overline{\mathfrak{g}}}}$ defined in Section 2.2. $\mathfrak{tr}_{\mathfrak{t}\overline{\mathfrak{h}}}(L(\overline{\mathfrak{g}}_\infty, \overline{\Lambda}^{\mathfrak{g}}(\lambda, d)))$ is an irreducible $\mathfrak{t}\overline{\mathfrak{g}}$-module and

$$\mathfrak{tr}_{\mathfrak{t}\overline{\mathfrak{h}}}(L(\overline{\mathfrak{g}}_\infty, \overline{\Lambda}^{\mathfrak{g}}(\lambda, d))) = L(\mathfrak{t}\overline{\mathfrak{g}}, \Gamma_{\mathfrak{t}\overline{\mathfrak{g}}}(\lambda, d)) \tag{2.8}$$

for any partition λ with $\lambda_{n+1} = 0$ and $\overline{\mathfrak{g}} = \mathfrak{c}, \mathfrak{d}$ [CLW, Lemma 3.2]. The same result is also true for $\overline{\mathfrak{g}} = \mathfrak{a}$ and pair of partitions $\lambda = (\lambda^-, \lambda^+)$ with $\lambda^+_{n+1} = \lambda^-_{m+1} = 0$ by using the same arguments as in [CLW]. The anti-linear anti-involution ω on $\overline{\mathfrak{g}}_\infty$ induces an anti-linear anti-involution on $\mathfrak{t}\overline{\mathfrak{g}}$, which will also be denoted by ω.

By cumbersome but straight forward computations, the following theorem is reformulated the Theorem 2.4 and some results of sections $7, 8, 9$ in [EHW] in terms of partitions.

Theorem 2.5. *For $\mathfrak{g} = \mathfrak{a}, \mathfrak{c}, \mathfrak{d}$, let $\xi \in \mathfrak{h}^*_{\mathfrak{t}\overline{\mathfrak{g}}}$.*

i. *$L(\mathfrak{t}\overline{\mathfrak{a}}, \xi)$ is unitarizable with respect to ω if and only if $\xi = \Gamma_{\mathfrak{t}\overline{\mathfrak{a}}}(\lambda, d) + k \sum_{i=-m+1}^{n} \epsilon_i$ for some pair of partitions $\lambda = (\lambda^+, \lambda^-)$ with $\lambda^-_m = \lambda^+_n = 0$ and $d, k \in \mathbb{R}$ satisfying $d \geq \min\{(\lambda^-)'_1 + n - 1, (\lambda^+)'_1 + m - 1\}$, or $d \in \mathbb{Z}$ and $d \geq (\lambda^-)'_1 + (\lambda^+)'_1$. Moreover, $N(\mathfrak{t}\overline{\mathfrak{a}}, \Gamma_{\mathfrak{t}\overline{\mathfrak{a}}}(\lambda, d) + k \sum_{i=-m+1}^{n} \epsilon_i)$ are irreducible for pair of partitions $\lambda = (\lambda^+, \lambda^-)$ with $\lambda^-_m = \lambda^+_n = 0$ and $d, k \in \mathbb{R}$ satisfying $d > \min\{(\lambda^-)'_1 + n - 1, (\lambda^+)'_1 + m - 1\}$.*

ii. *$L(\mathfrak{t}\overline{\mathfrak{d}}, \xi)$ is unitarizable with respect to ω if and only if $\xi = \Gamma_{\mathfrak{t}\overline{\mathfrak{d}}}(\lambda, d)$ for some partition λ with $\lambda_n = 0$ and $d \in \mathbb{R}$ satisfying $d \geq n - 1 + \lambda'_2$, or $d \in \mathbb{Z}$ and $d \geq \lambda'_1 + \lambda'_2$. Moreover, $N(\mathfrak{t}\overline{\mathfrak{d}}, \Gamma_{\mathfrak{t}\overline{\mathfrak{d}}}(\lambda, d))$ are irreducible for partition λ with $\lambda_n = 0$ and $d > n - 1 + \lambda'_2$.*

iii. *Assume that $\xi \in \mathfrak{h}^*_{\mathfrak{t}\overline{\mathfrak{c}}}$ with $\xi(\widetilde{E}_{n-1}) = \xi(\widetilde{E}_n)$. $L(\mathfrak{t}\overline{\mathfrak{c}}, \xi)$ is unitarizable with respect to ω if and only if $\xi = \Gamma_{\mathfrak{t}\overline{\mathfrak{c}}}(\lambda, d)$ for some partition λ with $\lambda_{n-1} = \lambda_n = 0$ and $d \in \mathbb{R}$ satisfying $d \geq \frac{1}{2}(\lambda'_1 + n) - 1$ if $n - \lambda'_1$ is even; $d \geq \frac{1}{2}(\lambda'_1 + n - 1) - 1$ if*

$n-\lambda_1'$ is odd, or $d\in\mathbb{Z}$ and $d\geq\lambda_1'$. Moreover, $N(\mathfrak{t}\overline{\mathfrak{c}},\Gamma_{\mathfrak{t}\overline{\mathfrak{c}}}(\lambda,d))$ are irreducible for partition λ with $\lambda_{n-1}=\lambda_n=0$ and $d\in\mathbb{R}$ satisfying $d>\frac{1}{2}(\lambda_1'+n)-1$ if $n-\lambda_1'$ is even; $d>\frac{1}{2}(\lambda_1'+n-1)-1$ if $n-\lambda_1'$ is odd.

Proposition 2.6. *For $\mathfrak{g}=\mathfrak{a},\mathfrak{c},\mathfrak{d}$, let $L(\overline{\mathfrak{g}}_\infty,\xi)$ be an irreducible quasi-finite highest weight $\overline{\mathfrak{g}}_\infty$-module with highest weight ξ. Then $L(\overline{\mathfrak{g}}_\infty,\xi)$ is unitarizable with respect to the anti-linear anti-involution ω if and only if $\xi=\overline{\Lambda}^{\mathfrak{g}}(\lambda,d)$ for some $(\lambda,d)\in\mathcal{D}(\mathfrak{g})$.*

Proof. Let $L(\overline{\mathfrak{g}}_\infty,\xi)$ be a unitarizable irreducible quasi-finite highest weight $\overline{\mathfrak{g}}_\infty$-module. By Remark 2.4, ξ satisfies $\xi(E_{ii})=0$ (resp. $\xi(\widetilde{E}_{ii})=0$) for $|i|\gg 0$ (resp. $i\gg 0$) for $\overline{\mathfrak{g}}=\overline{\mathfrak{a}}$ (resp. $\overline{\mathfrak{c}},\overline{\mathfrak{d}}$). It is easy to see that $d\in\mathbb{R}$ and $\xi(\widetilde{E}_{ii})-\xi(\widetilde{E}_{i+1,i+1})\in\mathbb{Z}_+$ (resp. $\xi(E_{ii})-\xi(E_{i+1,i+1})\in\mathbb{Z}_+$) for all i (resp. $i\neq 0$) for $\overline{\mathfrak{g}}=\overline{\mathfrak{c}},\overline{\mathfrak{d}}$ (resp. $\overline{\mathfrak{a}}$). This implies $\xi=\overline{\Lambda}^{\mathfrak{g}}(\lambda,d)$ for some partition λ (resp. pair of partitions $\lambda=(\lambda^+,\lambda^-)$) and $d\in\mathbb{R}$ for $\overline{\mathfrak{g}}=\overline{\mathfrak{c}},\overline{\mathfrak{d}}$ (resp. $\overline{\mathfrak{a}}$). Now applying truncation functor to $L(\overline{\mathfrak{g}}_\infty,\xi)$ with $n\gg d$ (resp. $m,n\gg d$) for $\overline{\mathfrak{g}}=\overline{\mathfrak{c}},\overline{\mathfrak{d}}$ (resp. $\overline{\mathfrak{a}}$), $\mathfrak{tr}_{\mathfrak{t}\overline{\mathfrak{h}}}(L(\overline{\mathfrak{g}}_\infty,\xi))$ is a unitarizable $\mathfrak{t}\overline{\mathfrak{g}}$-module with respect to ω. By Theorem 2.5 and (2.8), we have $d\in\mathbb{Z}$ and $(\lambda,d)\in\mathcal{D}_{\mathfrak{t}}(\mathfrak{g})$. Hence $\xi=\overline{\Lambda}^{\mathfrak{g}}(\lambda,d)$ for some $(\lambda,d)\in\mathcal{D}(\mathfrak{g})$. Conversely, the irreducible highest weight $\overline{\mathfrak{g}}_\infty$-modules $L(\overline{\mathfrak{g}}_\infty,\overline{\Lambda}^{\mathfrak{g}}(\lambda,d))$ are modules appearing in the Howe dualities at negative levels described in [W]. These modules are unitarizable and quasi-finite. The proof is completed. □

Remark 2.7. The modules described in the proposition are modules appearing in the Howe dualities at negative levels described in [W] (cf. [LZ, Theorem 5.6, 5.8, 5.9]).

3. Numerical data of the highest weights

In this section, we shall provide combinatorial descriptions of $\overline{\Lambda}^{\mathfrak{g}}(\lambda,d)$ in terms of $\Lambda^{\mathfrak{g}}(\lambda,d)$.

Definition 3.1. Let $\{a_i\}_{i\in\mathbb{N}}$ and $\{b_i\}_{i\in\mathbb{N}}$ be two strictly decreasing sequences of integers (resp. half integers). Then the sequences $\{a_i\}_{i\in\mathbb{N}}$ and $\{b_i\}_{i\in\mathbb{N}}$ are said to form a dual pair if $\mathbb{Z}$ (resp. $\frac{1}{2}+\mathbb{Z}$) is the disjoint union of the two sequences $\{a_i\}_{i\in\mathbb{N}}$ and $\{-b_i\}_{i\in\mathbb{N}}$.

Define the function ρ on $\mathbb{N}$ by $\rho(i)=-i$ for all $i\in\mathbb{N}$. The following lemma is well known (see e.g. [M, (1.7)]).

Lemma 3.2. *For any partition λ, the sequences $\{\lambda_i+\rho(i)\}_{i\in\mathbb{N}}$ and $\{\lambda_i'+\rho(i)+1\}_{i\in\mathbb{N}}$ form a dual pair.*

Recall that $\phi_{\mathfrak{g}}=\sum_{i\in\mathbb{N}}\epsilon_i$ for $\mathfrak{g}=\mathfrak{c},\mathfrak{d}$.

Lemma 3.3. *For $\mathfrak{g}=\mathfrak{c},\mathfrak{d}$ and $(\lambda,d)\in\mathcal{D}(\mathfrak{g})$, let $\{\zeta_i\}_{i\in\mathbb{N}}$ and $\{\overline{\zeta}_i\}_{i\in\mathbb{N}}$ be two sequences determined by*

$$\Lambda^{\mathfrak{g}}(\lambda,d)+\rho_{\mathfrak{g}}-d\langle\vartheta_{\mathfrak{g}},K\rangle\phi_{\mathfrak{g}}=\sum_{i\in I_{\mathfrak{g}}}\zeta_i\epsilon_i+d\vartheta_{\mathfrak{g}},$$

$$\overline{\Lambda}^{\mathfrak{g}}(\lambda,d)+\rho_{\overline{\mathfrak{g}}}+d\langle\vartheta_{\mathfrak{g}},K\rangle\phi_{\overline{\mathfrak{g}}}=\sum_{i\in I_{\mathfrak{g}}}\overline{\zeta}_i\epsilon_i-\frac{d\langle\vartheta_{\mathfrak{g}},K\rangle}{\langle\vartheta_{\overline{\mathfrak{g}}},K\rangle}\vartheta_{\overline{\mathfrak{g}}}.$$

Then $\{\zeta_i\}_{i\in\mathbb{N}}$ and $\{\overline{\zeta}_i\}_{i\in\mathbb{N}}$ form a dual pair. Moreover, $\zeta_i<0$ for $i\in\mathbb{N}$ and $\mathfrak{g}\neq\mathfrak{d}$. In the case $\mathfrak{g}=\mathfrak{d}$, $\zeta_i<0$ for $i\geq 2$, and $\zeta_1<0$ (resp. $=0$ and >0) for $\lambda_1'<\frac{d}{2}$ (resp. $=\frac{d}{2}$ and $>\frac{d}{2}$).

Proof. By Lemma 3.2, $\{\zeta_i\}_{i\in\mathbb{N}}$ and $\{\overline{\zeta}_i\}_{i\in\mathbb{N}}$ form a dual pair. It is clear that $\zeta_i<0$ for $i\in\mathbb{N}$ and $\mathfrak{g}=\mathfrak{c}$. For $\mathfrak{g}=\mathfrak{d}$, we have $\lambda_2'\leq\frac{d}{2}$ and hence $\zeta_i<0$ for $i\geq 2$. Also, $\zeta_1=\lambda_1'-\frac{d}{2}<0$ (resp. $=0$ and >0) for $\lambda_1'<\frac{d}{2}$ (resp. $=\frac{d}{2}$ and $>\frac{d}{2}$). □

Lemma 3.4. *For $\mathfrak{g}=\mathfrak{c},\mathfrak{d}$ and $(\lambda,d)\in\mathcal{D}(\mathfrak{g})$, let $\{\zeta_i\}_{i\in\mathbb{N}}$ and $\{\overline{\zeta}_i\}_{i\in\mathbb{N}}$ be two sequences defined in Lemma 3.3. Define $N(\lambda,d)=\{(i,j)\in\mathbb{N}\times\mathbb{N}\mid \overline{\zeta}_i+\overline{\zeta}_j=0, i,j\in\mathbb{N}\}$, $J=\{j\in\mathbb{N}\mid (j,k)\notin N(\lambda,d),\forall k\in\mathbb{N}\}$, $\mathcal{S}=\{\zeta_i\mid i\geq 1\}$ and $\overline{\mathcal{S}}=\{\overline{\zeta}_i\mid i\in J\}$.*

i. *For $\mathfrak{g}=\mathfrak{c}$, we have $\overline{\mathcal{S}}=\mathcal{S}$ and $\overline{\zeta}_{d+1}=0$.*
ii. *For $\mathfrak{g}=\mathfrak{d}$, we have*

$$\begin{aligned}
\overline{\mathcal{S}}=\mathcal{S} \text{ and } \zeta_i\neq 0\neq\overline{\zeta}_i \text{ for all } i, &\qquad \text{if } d \text{ is odd;}\\
\overline{\mathcal{S}}\cup\{0\}=\mathcal{S} \text{ and } \zeta_1=0, &\qquad \text{if } d \text{ is even and } \lambda_1'=\frac{d}{2};\\
\overline{\mathcal{S}}=\mathcal{S} \text{ and } \overline{\zeta}_i=0 \text{ for some } i, &\qquad \text{if } d \text{ is even and } \lambda_1'\neq\frac{d}{2}.
\end{aligned}$$

Proof. We shall only prove the case $\mathfrak{g}=\mathfrak{d}$. The proof of the other cases are similar and easier. For $j\geq 2$, we have $\zeta_1+\zeta_j\leq\lambda_1'+\lambda_2'-d-j+1\leq -1$ and hence $\zeta_1\neq-\zeta_j$ for $j\geq 2$. Since $\{\zeta_i\}_{i\in\mathbb{N}}$ and $\{\overline{\zeta}_i\}_{i\in\mathbb{N}}$ form a dual pair, $\zeta_1\neq\pm\zeta_j$ for $j\geq 2$ and ζ_i are negative for $i\geq 2$, we have $\zeta_i\in\overline{\mathcal{S}}$ for $i\geq 2$, and $\zeta_1\in\overline{\mathcal{S}}$ for $\zeta_1\neq 0$. This implies $\overline{\mathcal{S}}\supseteq\mathcal{S}\backslash\{0\}$. For $x\in\overline{\mathcal{S}}$, we have $-x\notin\overline{\mathcal{S}}$ and hence $-x\in-\mathcal{S}$. Therefore $\overline{\mathcal{S}}=\mathcal{S}\backslash\{0\}$. By Lemma 3.2, $\mathcal{S}$ (resp. $\overline{\mathcal{S}}$) contains 0 if and only if d is even and $\lambda_1'=\frac{d}{2}$ (resp. $\lambda_1'\neq\frac{d}{2}$). The proof is completed. □

Recall that $\phi_{\mathfrak{a}}=-\sum_{i\leq 0}\epsilon_i$.

Lemma 3.5. *For $(\lambda,d)\in\mathcal{D}(\mathfrak{a})$, let $\{\zeta_i\}_{i\in\mathbb{Z}}$ and $\{\overline{\zeta}_i\}_{i\in\mathbb{Z}}$ be two sequences determined by*

$$\begin{aligned}
\Lambda^{\mathfrak{a}}(\lambda,d)+\rho_{\mathfrak{a}}-d\phi_{\mathfrak{a}}&=\sum_{i\in\mathbb{Z}}(\zeta_i-1)\epsilon_i+d\vartheta_{\mathfrak{a}},\\
\overline{\Lambda}^{\mathfrak{a}}(\lambda,d)+\rho_{\mathfrak{a}}+d\phi_{\mathfrak{a}}&=\sum_{i\in\mathbb{Z}}\overline{\zeta}_i\epsilon_i-d\vartheta_{\mathfrak{a}}.
\end{aligned}$$

Define $N(\lambda,d)=\{(i,j)\in I_{\mathfrak{a}}\times I_{\mathfrak{a}}\mid \overline{\zeta}_i=\overline{\zeta}_j, i\leq 0<j\}$, $J_+=\{j\in\mathbb{N}\mid (i,j)\notin N(\lambda,d),\forall i\leq 0\}$, $J_-=\{i\in\mathbb{Z}\mid (i,j)\notin N(\lambda,d),\forall j\in\mathbb{N}\}$, $\mathcal{S}_+=\{\zeta_i\mid i\geq 1\}$, $\mathcal{S}_-=\{\zeta_i\mid i\leq 0\}$, $\overline{\mathcal{S}}_+=\{\overline{\zeta}_i\mid i\in J_+\}$ and $\overline{\mathcal{S}}_-=\{\overline{\zeta}_i\mid i\in J_-\}$. Then we have $\overline{\mathcal{S}}_+=-\mathcal{S}_-$ and $\overline{\mathcal{S}}_-=-\mathcal{S}_+$.

Proof. Let $\mathcal{B}_+=\{\overline{\zeta}_i\mid i\in\mathbb{N}\}$ and $\mathcal{B}_-=\{\overline{\zeta}_i\mid i\leq 0\}$. By Lemma 3.2, we have

$$(-\mathcal{S}_+)\sqcup\mathcal{B}_+=\mathbb{Z}\quad\text{and}\quad(-\mathcal{S}_-)\sqcup\mathcal{B}_-=\mathbb{Z}.$$

For $x\in\overline{\mathcal{S}}_+$, we have $x\notin\mathcal{B}_-$ by the definition of $\overline{\mathcal{S}}_+$ and hence $x\in-\mathcal{S}_-$. Therefore $\overline{\mathcal{S}}_+\subseteq-\mathcal{S}_-$. Now assume $x\in-\mathcal{S}_-$. We have $x\notin\mathcal{B}_-$. Since $\{\zeta_i\}_{i\in\mathbb{Z}}$ is strictly increasing, we have $x \notin -\mathcal{S}_+$ and hence $x \in \mathcal{B}_+$. Thus $x \in \mathcal{B}_+ \backslash \mathcal{B}_- = \overline{\mathcal{S}}_+$ and therefore $-\mathcal{S}_- \subseteq \overline{\mathcal{S}}_+$. Similarly, we have $-\mathcal{S}_+ = \overline{\mathcal{S}}_-$. The proof is completed. □

We shall use the notations defined in Lemma 3.4 and Lemma 3.5 for the rest of the paper. By (2.3) and Lemma 3.5, we have (for $(\lambda, d) \in \mathcal{D}(\mathfrak{a})$, $\sigma \in W_{\mathfrak{a}}$)

$$\sigma^{-1}(\Lambda^{\mathfrak{a}}(\lambda,d)+\rho_{\mathfrak{a}}) = \sum_{i\in\mathbb{Z}}(\zeta_i-1)\epsilon_{\sigma^{-1}(i)}+d\phi_{\mathfrak{a}}+d\vartheta_{\mathfrak{a}} = \sum_{i\in\mathbb{Z}}\zeta_{\sigma(i)}\epsilon_i-\sum_{i\in\mathbb{Z}}\epsilon_i+d\phi_{\mathfrak{a}}+d\vartheta_{\mathfrak{a}}. \tag{3.1}$$

By Lemma 3.4 and (2.4), we have (for $(\lambda, d) \in \mathcal{D}(\mathfrak{g})$, $\sigma \in W_{\mathfrak{g}}$ and $\mathfrak{g} = \mathfrak{c}, \mathfrak{d}$)

$$\sigma^{-1}(\Lambda^{\mathfrak{g}}(\lambda,d) + \rho_{\mathfrak{g}}) = \sum_{i\in\mathbb{N}}\zeta_{\sigma(i)}\epsilon_i + d\langle\vartheta_{\mathfrak{g}}, K\rangle\phi_{\mathfrak{g}} + d\vartheta_{\mathfrak{g}} + d\vartheta_{\mathfrak{g}}. \tag{3.2}$$

For η belonging to the subspace of $\mathfrak{h}^*_{\overline{\mathfrak{g}}}$ spanned by ϵ_js and $\vartheta_{\overline{\mathfrak{g}}}$, let $[\eta]^+$ denote the unique $\Delta^+_{\overline{\mathfrak{g}},c}$-dominant element in $W_{\overline{\mathfrak{g}},0}$-orbit of $\eta \in \mathfrak{h}^*_{\overline{\mathfrak{g}}}$. The following two propositions are important for proving the main theorem in the next section.

Proposition 3.6. *Let $\{j_i\}_{i\in\mathbb{N}}$ be the strictly increasing sequence with $J = \{j_i \mid i \in \mathbb{N}\}$. For $(\lambda, d) \in \mathcal{D}(\mathfrak{g})$ with $\mathfrak{g} = \mathfrak{c}, \mathfrak{d}$ and a partition μ with $\Lambda^{\mathfrak{g}}(\mu, d) = \sigma^{-1} \circ \Lambda^{\mathfrak{g}}(\lambda, d)$ for some $\sigma \in W^0_{\mathfrak{g},k}$, we have*

$$\begin{aligned}&\overline{\Lambda}^{\mathfrak{g}}(\mu,d) + \rho_{\overline{\mathfrak{g}}} + d\langle\vartheta_{\mathfrak{g}},K\rangle\phi_{\overline{\mathfrak{g}}}\\ &= \begin{cases}\big[\sum_{i\in\mathbb{N}\backslash J}\overline{\zeta}_i\epsilon_i + \sum_{i\in\mathbb{N}}\overline{\zeta}_{j_{\sigma(i)}}\epsilon_{j_i} - \frac{d\langle\vartheta_{\mathfrak{g}},K\rangle}{\langle\vartheta_{\overline{\mathfrak{g}}},K\rangle}\vartheta_{\overline{\mathfrak{g}}}\big]^+, & \text{if } 0 \notin \mathcal{S};\\ \big[\sum_{i\in\mathbb{N}\backslash J}\overline{\zeta}_i\epsilon_i + \sum_{i\in\mathbb{N}}\overline{\zeta}_{j_{\sigma^0(i)}}\epsilon_{j_i} - \frac{d\langle\vartheta_{\mathfrak{g}},K\rangle}{\langle\vartheta_{\overline{\mathfrak{g}}},K\rangle}\vartheta_{\overline{\mathfrak{g}}}\big]^+, & \text{if } 0 \in \mathcal{S}.\end{cases}\end{aligned}$$

Here σ^0 appears only in the case $\mathfrak{g} = \mathfrak{d}$ and it is determined by $\overline{\sigma^0} = \sigma$ (see Lemma 2.2 and Lemma 2.3).

Proof. In the proof, union means disjoint union. Let $\{\xi_i\}_{i\in\mathbb{N}}$ and $\{\overline{\xi}_i\}_{i\in\mathbb{N}}$ be two sequences determined by

$$\Lambda^{\mathfrak{g}}(\mu,d) + \rho_{\mathfrak{g}} - d\langle\vartheta_{\mathfrak{g}},K\rangle\phi_{\mathfrak{g}} = \sum_{i\in I_{\mathfrak{g}}}\xi_i\epsilon_i + d\vartheta_{\mathfrak{g}},$$

$$\overline{\Lambda}^{\mathfrak{g}}(\mu,d) + \rho_{\overline{\mathfrak{g}}} + d\langle\vartheta_{\mathfrak{g}},K\rangle\phi_{\overline{\mathfrak{g}}} = \sum_{i\in I_{\mathfrak{g}}}\overline{\xi}_i\epsilon_i - \frac{d\langle\vartheta_{\mathfrak{g}},K\rangle}{\langle\vartheta_{\overline{\mathfrak{g}}},K\rangle}\vartheta_{\overline{\mathfrak{g}}}.$$

Assume $0 \notin \mathcal{S}$. We have $\{\overline{\zeta}_{j_i}\}_{i\in\mathbb{N}} = \{\zeta_i\}_{i\in\mathbb{N}}$ by Lemma 3.4. By Lemma 3.3, Lemma 3.4, and the fact that σ acts on $\mathbb{Z}^*$ as a signed permutation, we have

$$\{-\zeta_{\sigma(i)} \mid i \in \mathbb{N}\} \sqcup \{\overline{\zeta}_{j_{\sigma(i)}} \mid i \in \mathbb{N}\} \sqcup \{\overline{\zeta}_i \mid i \in \mathbb{N}\backslash J\} = \mathbb{Z} \text{ (or } \frac{1}{2} + \mathbb{Z}).$$

Since $\{\xi_i\}_{i\in\mathbb{N}}$ and $\{\overline{\xi}_i\}_{i\in\mathbb{N}}$ form a dual pair, and $\{\xi_i \mid i \in \mathbb{N}\} = \{\zeta_{\sigma(i)} \mid i \in \mathbb{N}\}$ by (3.2), we have $\{\overline{\xi}_i \mid i \in \mathbb{N}\} = \{\overline{\zeta}_{j_{\sigma(i)}} \mid i \in \mathbb{N}\} \sqcup \{\overline{\zeta}_i \mid i \in \mathbb{N}\backslash J\}$. Therefore the proposition holds for this case since $\{\overline{\xi}_i\}_{i\in\mathbb{N}}$ is a decreasing sequence.

The case of $0 \in \mathcal{S}$ only occurs when $\mathfrak{g} = \mathfrak{d}$ with $\zeta_1 = 0$. We have $\{\overline{\zeta}_{j_i} \mid i \in \mathbb{N}\} = \{\zeta_i \mid i \in \mathbb{N}\}\backslash\{0\}$. Since σ acts on $\mathbb{Z}^*$ as a signed permutation, by Lemma 2.3, we have

$$\{-\zeta_{\sigma(i)} \mid i \in \mathbb{N}\} \sqcup \{\overline{\zeta}_{j_{\sigma^0(i)}} \mid i \in \mathbb{N}\} \sqcup \{\overline{\zeta}_i \mid i \in \mathbb{N}\backslash J\} = \mathbb{Z}.$$

Now the proposition also follows in this case using the arguments above. □

Proposition 3.7. *Let $\{j_i\}_{i\in\mathbb{Z}}$ be the strictly decreasing sequence such that $J_+ = \{j_i \mid i \le 0\}$ and $J_- = \{k_i \mid i \in \mathbb{N}\}$, and let $J = J_- \sqcup J_+$. For $(\lambda, d) \in \mathcal{D}(\mathfrak{a})$ and a partition μ with $\Lambda^{\mathfrak{a}}(\mu, d) = \sigma^{-1} \circ \Lambda^{\mathfrak{a}}(\lambda, d)$ for some $\sigma \in W^0_{\mathfrak{a},k}$, we have*

$$\overline{\Lambda}^{\mathfrak{a}}(\mu, d) + \rho_{\mathfrak{a}} + d\phi_{\mathfrak{a}} = \Big[\sum_{i\in\mathbb{Z}\backslash J} \overline{\zeta}_i \epsilon_i + \sum_{i\in\mathbb{Z}} \overline{\zeta}_{j_{\sigma(i)}} \epsilon_{j_i} - d\vartheta_{\mathfrak{a}}\Big]^+.$$

Proof. In the proof, union means disjoint union. Let $\{\xi_i\}_{i\in\mathbb{Z}}$ and $\{\overline{\xi}_i\}_{i\in\mathbb{Z}}$ be two sequences determined by

$$\Lambda^{\mathfrak{a}}(\mu, d) + \rho_{\mathfrak{a}} + \sum_{i\in\mathbb{Z}} \epsilon_i - d\phi_{\mathfrak{a}} = \sum_{i\in\mathbb{Z}} \xi_i \epsilon_i + d\vartheta_{\mathfrak{a}},$$
$$\overline{\Lambda}^{\mathfrak{a}}(\mu, d) + \rho_{\mathfrak{a}} + d\phi_{\mathfrak{a}} = \sum_{i\in\mathbb{Z}} \overline{\xi}_i \epsilon_i - d\vartheta_{\mathfrak{a}}.$$

By Lemma 3.5, we have

$$\mathbb{Z} = (-\mathcal{S}_+) \sqcup (\overline{\mathcal{S}}_+) \sqcup \{\overline{\zeta}_i \mid i \in \mathbb{N}\backslash J_+\} = (-\mathcal{S}_+) \sqcup (-\mathcal{S}_-) \sqcup \{\overline{\zeta}_i \mid i \in \mathbb{N}\backslash J_+\}.$$

Therefore we have $\mathbb{Z} = \{-\zeta_{\sigma(i)} \mid i \in \mathbb{Z}\} \sqcup \{\overline{\zeta}_i \mid i \in \mathbb{N}\backslash J_+\}$ because σ acts as a permutation on $\mathbb{Z}$. Since $\xi_i = \zeta_{\sigma(i)}$ for $i \in \mathbb{Z}$ by (3.1) and $\zeta_{\sigma(i)} = -\overline{\zeta}_{j_{\sigma(i)}}$ for $i \in \mathbb{Z}$ by Lemma 3.5, we have

$$\mathbb{Z} = \{-\zeta_{\sigma(i)} \mid i \in \mathbb{N}\} \sqcup \{-\zeta_{\sigma(i)} \mid i \le 0\} \sqcup \{\overline{\zeta}_i \mid i \in \mathbb{N}\backslash J_+\}$$
$$= \{-\xi_i \mid i \in \mathbb{N}\} \sqcup \{\overline{\zeta}_{j_{\sigma(i)}} \mid i \le 0\} \sqcup \{\overline{\zeta}_i \mid i \in \mathbb{N}\backslash J_+\}.$$

Since $\{\xi_i\}_{i\in\mathbb{N}}$ and $\{\overline{\xi}_i\}_{i\in\mathbb{N}}$ form a dual pair, we have $\{\overline{\xi}_i \mid i \in \mathbb{N}\} = \{\overline{\zeta}_{j_{\sigma(i)}} \mid i \le 0\} \sqcup \{\overline{\zeta}_i \mid i \in \mathbb{N}\backslash J_+\}$. Similarly, we have $\{\overline{\xi}_i \mid i \le 0\} = \{\overline{\zeta}_{j_{\sigma(i)}} \mid i \in \mathbb{N}\} \sqcup \{\overline{\zeta}_i \mid i \in (-\mathbb{Z}_+)\backslash J_-\}$. Therefore the proposition holds since $\{\overline{\xi}_i\}_{i\in\mathbb{N}}$ is a decreasing sequence and $\{\overline{\xi}_{-i}\}_{i\in\mathbb{Z}_+}$ is an increasing sequence. □

4. $\mathfrak{u}^-_{\overline{\mathfrak{g}}}$-homology formulas for $\overline{\mathfrak{g}}_\infty$-modules

In this section we give a combinatorial proof of Enright's $\mathfrak{u}^-_{\overline{\mathfrak{g}}}$-homology formula [E] for the unitarizable highest weight $\overline{\mathfrak{g}}_\infty$-modules with highest weight $\overline{\Lambda}^{\mathfrak{g}}(\lambda, d)$.

For a module V over Lie algebra $\mathcal{G}$, let $\mathrm{H}_k(\mathcal{G}; V)$ denote k-th homology group of $\mathcal{G}$ with coefficients in V. It is well known that the homology groups $\mathrm{H}_k(\mathfrak{u}^-_{\overline{\mathfrak{g}}}; V)$ are $\mathfrak{l}_{\overline{\mathfrak{g}}}$-modules. The $\mathfrak{u}^-_{\overline{\mathfrak{g}}}$-homology of unitarizable highest weight modules are described by the following theorem which was obtained in [CK, Theorem 7.2] for $\overline{\mathfrak{g}}_\infty = \mathfrak{a}_\infty$ and in [CKW, Theorem 6.5] for $\overline{\mathfrak{g}}_\infty = \mathfrak{c}_\infty, \mathfrak{d}_\infty$. The following theorem holds for more general situation by using the correspondence of homology group in the sense of super duality [CLW, Theorem 4.10] together with Kostant's formulas for integrable $\mathfrak{g}_\infty$-modules (cf. [J, Ko, V, CK]).

Theorem 4.1. *For $(\lambda, d) \in \mathcal{D}(\mathfrak{g})$ with $\mathfrak{g} = \mathfrak{c}, \mathfrak{d}$ (resp. $\mathfrak{g} = \mathfrak{a}$), we have, as $\mathfrak{l}_{\overline{\mathfrak{g}}}$-modules,*

$$\mathrm{H}_k\big(\mathfrak{u}^-_{\overline{\mathfrak{g}}}; L(\overline{\mathfrak{g}}_\infty, \overline{\Lambda}^{\mathfrak{g}}(\lambda, d))\big) \cong \bigoplus_{\mu} L(\mathfrak{l}_{\overline{\mathfrak{g}}}, \overline{\Lambda}^{\mathfrak{g}}(\mu, d)),$$

where μ is summed over all partitions (resp. pairs of partitions) such that $\Lambda^{\mathfrak{g}}(\mu, d) = w^{-1} \circ \Lambda^{\mathfrak{g}}(\lambda, d)$ for some $w \in W^0_{\mathfrak{g},k}$.

For ξ belonging the subspace of $\mathfrak{h}^*_{\overline{\mathfrak{g}}}$ spanned by ϵ_js and $\vartheta_{\overline{\mathfrak{g}}}$, let $\Psi(\xi) = \{\alpha \in \Delta^+_{\overline{\mathfrak{g}},n} \,|\, (\xi + \rho_{\overline{\mathfrak{g}}} \,|\, \alpha) = 0\}$ and define $\Phi(\xi)$ to be the subset of $\Delta^+_{\overline{\mathfrak{g}},n}$ consisting of roots β satisfying the following conditions [E, DES]:

i. $\langle \xi + \rho_{\overline{\mathfrak{g}}}, \beta^\vee \rangle \in \mathbb{N}$;
ii. $(\beta \,|\, \alpha) = 0$ for all $\alpha \in \Psi(\xi)$;
iii. β is short if $\Psi(\xi)$ contains a long root.

Let $W_{\overline{\mathfrak{g}}}(\xi)$ be the subgroup of $W_{\overline{\mathfrak{g}}}$ that is generated by the reflections s_β with $\beta \in \Phi(\xi)$. Define $\Delta_{\overline{\mathfrak{g}}}(\xi)$ to be the subset of $\Delta_{\overline{\mathfrak{g}}}$ consisting of the roots $\vartheta \in \Delta_{\overline{\mathfrak{g}}}$ such that s_ϑ lies in $W_{\overline{\mathfrak{g}}}(\xi)$.

For $(\lambda, d) \in \mathcal{D}(\mathfrak{g})$, let $\Delta_{\overline{\mathfrak{g}}}(\lambda, d) = \Delta_{\overline{\mathfrak{g}}}(\overline{\Lambda}^{\mathfrak{g}}(\lambda, d))$ and $W_{\overline{\mathfrak{g}}}(\lambda, d) = W_{\overline{\mathfrak{g}}}(\overline{\Lambda}^{\mathfrak{g}}(\lambda, d))$. Then $\Delta_{\overline{\mathfrak{g}}}(\lambda, d)$ is an abstract root system and $W_{\overline{\mathfrak{g}}}(\lambda, d)$ is the Weyl group of $\Delta_{\overline{\mathfrak{g}}}(\lambda, d)$ [E, EHW] (see also Lemma 4.2 below). Let $\Delta^+_{\overline{\mathfrak{g}}}(\lambda, d) = \Delta_{\overline{\mathfrak{g}}}(\lambda, d) \cap \Delta^+_{\overline{\mathfrak{g}}}$ be the set of positive roots of $\Delta_{\overline{\mathfrak{g}}}(\lambda, d)$. Set $W_{\overline{\mathfrak{g}},0}(\lambda, d) = W_{\overline{\mathfrak{g}}}(\lambda, d) \cap W_{\overline{\mathfrak{g}},0}$. Let $W^0_{\overline{\mathfrak{g}}}(\lambda, d)$ denote the set of the minimal length representatives of the left coset space $W_{\overline{\mathfrak{g}}}(\lambda, d)/W_{\overline{\mathfrak{g}},0}(\lambda, d)$ and let $W^0_{\overline{\mathfrak{g}},k}(\lambda, d)$ be the subset of $W^0_{\overline{\mathfrak{g}}}(\lambda, d)$ consisting of elements σ with $\ell_{(\lambda,d)}(\sigma) = k$, where $\ell_{(\lambda,d)}$ is the length function on $W_{\overline{\mathfrak{g}}}(\lambda, d)$.

For $(\lambda, d) \in \mathcal{D}(\mathfrak{g})$, let $J^0 = J \sqcup \{j \mid \overline{\zeta}_j = 0\}$ for $\mathfrak{g} = \mathfrak{c}, \mathfrak{d}$ and define

$$\Upsilon(\lambda, d) = \begin{cases} \{\epsilon_i - \epsilon_j \in \Delta^+_{\overline{\mathfrak{g}}};\ (i \in J_-,\ j \in J_+)\}, \text{ for } \mathfrak{g} = \mathfrak{a}; \\ \{-\epsilon_i - \epsilon_j \in \Delta^+_{\overline{\mathfrak{g}}};\ (i < j,\ i, j \in J^0)\}, \text{ for } \mathfrak{g} = \mathfrak{c}; \\ \{-\epsilon_i - \epsilon_j \in \Delta^+_{\overline{\mathfrak{g}}};\ (i < j,\ i, j \in J)\}, \text{ if } J^0 \neq J \text{ or } \frac{d}{2} \notin \mathbb{Z}, \text{for } \mathfrak{g} = \mathfrak{d}; \\ \{-\epsilon_i - \epsilon_j, -2\epsilon_i \in \Delta^+_{\overline{\mathfrak{g}}};\ (i < j,\ i, j \in J)\}, \text{ if } J^0 = J \text{ and } \frac{d}{2} \in \mathbb{Z}, \text{for } \mathfrak{g} = \mathfrak{d}. \end{cases}$$

Lemma 4.2. *For $(\lambda, d) \in \mathcal{D}(\mathfrak{g})$, we have*

$$\Delta_{\overline{\mathfrak{g}}}(\lambda, d) = \begin{cases} \{\epsilon_i - \epsilon_j \in \Delta_{\overline{\mathfrak{g}}};\ (i \neq j,\ i, j \in J_- \sqcup J_+)\}, \text{ for } \mathfrak{g} = \mathfrak{a}; \\ \{\pm(\pm\epsilon_i - \epsilon_j) \in \Delta_{\overline{\mathfrak{g}}};\ (i < j,\ i, j \in J^0)\}, \text{ for } \mathfrak{g} = \mathfrak{c}; \\ \{\pm(\pm\epsilon_i - \epsilon_j) \in \Delta_{\overline{\mathfrak{g}}};\ (i < j,\ i, j \in J)\}, \text{ if } J^0 \neq J \text{ or } \frac{d}{2} \notin \mathbb{Z}, \text{for } \mathfrak{g} = \mathfrak{d}; \\ \{\pm(\pm\epsilon_i - \epsilon_j), \pm 2\epsilon_i \in \Delta_{\overline{\mathfrak{g}}};\ (i < j,\ i, j \in J)\}, \text{ if } J^0 = J \text{ and } \frac{d}{2} \in \mathbb{Z}, \text{for } \mathfrak{g} = \mathfrak{d}. \end{cases}$$

Proof. For $(\lambda, d) \in \mathcal{D}(\mathfrak{g})$, we have $\Phi(\overline{\Lambda}^{\mathfrak{g}}(\lambda, d)) \subseteq \Upsilon(\lambda, d)$ by Lemma 3.4 and Lemma 3.5. Using the relations of the Weyl groups, it is easy to observe that $\Upsilon(\lambda, d) \subseteq \Delta_{\overline{\mathfrak{g}}}(\lambda, d)$. Now the lemma follows by using the relations of the Weyl groups again. □

Lemma 4.3. *For $(\lambda, d) \in \mathcal{D}(\mathfrak{g})$, there is a bijection from $W^0_{\mathfrak{g},k}$ to $W^0_{\overline{\mathfrak{g}},k}(\lambda, d)$.*

Proof. By Lemma 4.2, it is clear that $W^0_{\mathfrak{g},k} = W^0_{\overline{\mathfrak{g}},k}(\lambda, d)$ for the cases $\mathfrak{g} = \mathfrak{a}$ and $\mathfrak{g} = \mathfrak{d}$ with $J^0 \neq J$ or $\frac{d}{2} \notin \mathbb{Z}$. For the cases $\mathfrak{g} = \mathfrak{c}$ and $\mathfrak{g} = \mathfrak{d}$ with $J^0 = J$ and $\frac{d}{2} \in \mathbb{Z}$, the lemma follows from Lemma 4.2 and Lemma 2.2. □

Using Theorem 4.1, Proposition 3.6, Proposition 3.7, Lemma 4.3 and Lemma 2.3 together with (2.3) and (2.4), we have the following theorem.

Theorem 4.4. *For $(\lambda, d) \in \mathcal{D}(\mathfrak{g})$ and $k \in \mathbb{Z}_+$, we have, as $\mathfrak{l}_{\overline{\mathfrak{g}}}$-modules,*

$$\mathrm{H}_k\big(\mathfrak{u}^-_{\overline{\mathfrak{g}}}; L(\overline{\mathfrak{g}}_\infty, \overline{\Lambda}^{\mathfrak{g}}(\lambda, d))\big) \cong \bigoplus_{w \in W^0_{\overline{\mathfrak{g}},k}(\lambda,d)} L\big(\mathfrak{l}_{\overline{\mathfrak{g}}}, [w^{-1}(\overline{\Lambda}^{\mathfrak{g}}(\lambda, d) + \rho_{\overline{\mathfrak{g}}})]^+ - \rho_{\overline{\mathfrak{g}}}\big).$$

Remark 4.5. There has the counterpart of the Theorem 4.11 in [CLW] for $\mathfrak{u}^+$-cohomology in the sense of [Liu, Section 4]. The analogous statement is also true for $\mathfrak{g} = \mathfrak{a}$. The formulas for $\mathfrak{u}^+$-cohomology can be proved by the same argument as in the proof in [CLW]. Therefore, there is an analogue of Theorem 4.4 for $\mathfrak{u}^+_{\overline{\mathfrak{g}}}$-cohomology in the sense of [Liu]. The formulas for the cohomology can be proved by the same argument as in the proof of the theorem above.

5. Homology formulas for unitarizable modules over finite dimensional Lie algebras

In the section we shall give a new proof of Enright's homology formulas for unitarizable modules over classical Lie algebras corresponding to the three Hermitian symmetric pairs of classical types $(SU(p,q), SU(p) \times SU(q))$, $(Sp(n,\mathbb{R}), U(n))$ and $(SO^*(2n), U(n))$.

For ξ belonging to $\mathfrak{h}^*_{\mathfrak{t}\overline{\mathfrak{g}}}$, let $\Psi(\xi) = \{\alpha \in \Delta^+_{\mathfrak{t}\overline{\mathfrak{g}},n} \,|\, (\xi + \rho_{\mathfrak{t}\overline{\mathfrak{g}}} \,|\, \alpha) = 0\}$ and define $\Phi(\xi)$ to be the subset of $\Delta^+_{\mathfrak{t}\overline{\mathfrak{g}},n}$ consisting of roots β satisfying the following conditions [E, DES]:

i. $\langle \xi + \rho_{\mathfrak{t}\overline{\mathfrak{g}}}, \beta^\vee \rangle \in \mathbb{N}$;
ii. $(\beta \,|\, \alpha) = 0$ for all $\alpha \in \Psi(\xi)$;
iii. β is short if $\Psi(\xi)$ contains a long root.

Let $W_{\mathfrak{t}\overline{\mathfrak{g}}}(\xi)$ be the subgroup of $W_{\mathfrak{t}\overline{\mathfrak{g}}}$ that is generated by the reflections s_β with $\beta \in \Phi(\xi)$. Associated to $W_{\mathfrak{t}\overline{\mathfrak{g}}}(\xi)$, let $\Delta_{\mathfrak{t}\overline{\mathfrak{g}}}(\xi)$ denote the subset of $\Delta_{\mathfrak{t}\overline{\mathfrak{g}}}$ consisting of the roots ϑ such that s_ϑ lies in $W_{\mathfrak{t}\overline{\mathfrak{g}}}(\xi)$. We also let $[\xi]^+$ be the unique $\Delta^+_{\mathfrak{t}\overline{\mathfrak{g}},c}$-dominant element in the $W_{\mathfrak{t}\overline{\mathfrak{g}},0}$-orbit of ξ.

Assume that the irreducible module $L(\mathfrak{t}\overline{\mathfrak{g}}, \xi)$ is unitarizable with highest weight $\xi \in \mathfrak{h}^*_{\mathfrak{t}\overline{\mathfrak{g}}}$. Then $\Delta_{\mathfrak{t}\overline{\mathfrak{g}}}(\xi)$ is an abstract root system and $W_{\mathfrak{t}\overline{\mathfrak{g}}}(\xi)$ is the Weyl group of $\Delta_{\mathfrak{t}\overline{\mathfrak{g}}}(\xi)$ by [E, EHW]. Let $\Delta^+_{\mathfrak{t}\overline{\mathfrak{g}}}(\xi) = \Delta_{\mathfrak{t}\overline{\mathfrak{g}}}(\xi) \cap \Delta^+_{\mathfrak{t}\overline{\mathfrak{g}}}$ be the set of positive roots of $\Delta_{\mathfrak{t}\overline{\mathfrak{g}}}(\xi)$. Set $W_{\mathfrak{t}\overline{\mathfrak{g}},0}(\xi) = W_{\mathfrak{t}\overline{\mathfrak{g}}}(\xi) \cap W_{\mathfrak{t}\overline{\mathfrak{g}},0}$. Let $W^0_{\mathfrak{t}\overline{\mathfrak{g}}}(\xi)$ denotes the set of the minimal length representatives of the left coset space $W_{\mathfrak{t}\overline{\mathfrak{g}}}(\xi)/W_{\mathfrak{t}\overline{\mathfrak{g}},0}(\xi)$ and let $W^0_{\mathfrak{t}\overline{\mathfrak{g}},k}(\xi)$ be the subset of $W^0_{\mathfrak{t}\overline{\mathfrak{g}}}(\xi)$ consisting of elements σ with $\ell_\xi(\sigma) = k$, where ℓ_ξ is the length function on $W_{\mathfrak{t}\overline{\mathfrak{g}}}(\xi)$.

Theorem 5.1. *For $\overline{\mathfrak{g}} = \mathfrak{a}, \mathfrak{c}$ or $\mathfrak{d}$, let $L(\mathfrak{t}\overline{\mathfrak{g}}, \xi)$ be a unitarizable $\mathfrak{t}\overline{\mathfrak{g}}$-module with highest weight $\xi \in \mathfrak{h}^*_{\mathfrak{t}\overline{\mathfrak{g}}}$. Assume that ξ satisfies the assumption of the Case (iii) of Theorem 2.5 (cf. Case (ii) of* [EHW, Theorem 9.4]*) for $\mathfrak{t}\overline{\mathfrak{g}} \cong \mathfrak{so}(2n)$. For $k \in \mathbb{Z}_+$, we have, as*

$\mathfrak{l}_{\mathfrak{t}\overline{\mathfrak{g}}}$-*modules,*

$$H_k\big(\mathfrak{u}^-_{\mathfrak{t}\overline{\mathfrak{g}}}; L(\mathfrak{t}\overline{\mathfrak{g}},\xi)\big) \cong \bigoplus_{w\in W^0_{\mathfrak{t}\overline{\mathfrak{g}},k}(\xi)} L\big(\mathfrak{l}_{\mathfrak{t}\overline{\mathfrak{g}}}\,, [w^{-1}(\xi+\rho_{\mathfrak{t}\overline{\mathfrak{g}}})]^+ - \rho_{\mathfrak{t}\overline{\mathfrak{g}}}\big).$$

Proof. Since $H_k(\mathfrak{u}^-_{\mathfrak{t}\overline{\mathfrak{a}}}; L(\mathfrak{t}\overline{\mathfrak{a}}, \mu + k\sum_{i=-m+1}^n \epsilon_i)) = H_k(\mathfrak{u}^-_{\mathfrak{t}\overline{\mathfrak{a}}}; L(\mathfrak{t}\overline{\mathfrak{a}},\mu))\otimes L(\mathfrak{l}_{\mathfrak{t}\overline{\mathfrak{a}}}, k\sum_{i=-m+1}^n \epsilon_i)$ for all $i\geq 0$ and $\mu\in\mathfrak{h}^*_{\mathfrak{t}\overline{\mathfrak{a}}}$, it is sufficient to show all ξ with $k=0$ appearing in the Case (i) of Theorem 2.5 when $\overline{\mathfrak{g}}=\overline{\mathfrak{a}}$.

First we assume that $\xi = \Gamma_{\mathfrak{t}\overline{\mathfrak{g}}}(\lambda, d)$ with $d\notin\mathbb{Z}$. Then we have $\Delta_{\mathfrak{t}\overline{\mathfrak{g}}}(\xi)=\emptyset$ and $L(\mathfrak{t}\overline{\mathfrak{g}},\xi) = N(\mathfrak{t}\overline{\mathfrak{g}},\xi)$ by Theorem 2.5. Therefore $L(\mathfrak{t}\overline{\mathfrak{g}},\xi)$ is a free $\mathfrak{u}^-_{\mathfrak{t}\overline{\mathfrak{g}}}$-modules and hence $H_k\big(\mathfrak{u}^-_{\mathfrak{t}\overline{\mathfrak{g}}}; L(\mathfrak{t}\overline{\mathfrak{g}},\xi)\big) = L(\mathfrak{l}_{\mathfrak{t}\overline{\mathfrak{g}}}\,,\xi)$ (resp. $=0$) for $k=0$ (resp. $k>0$). Thus the theorem holds for this case.

Now we assume that $\xi = \Gamma_{\mathfrak{t}\overline{\mathfrak{g}}}(\lambda,d)$ for some $(\lambda,d)\in\mathcal{D}_{\mathfrak{t}}(\mathfrak{g})$. By a direct calculation, we have $\Delta_{\mathfrak{t}\overline{\mathfrak{g}}}(\xi) = \Delta_{\overline{\mathfrak{g}}}(\lambda,d)\cap\Delta_{\mathfrak{t}\overline{\mathfrak{g}}}$. Recall that $\mathfrak{tr}_{\mathfrak{t}\overline{\mathfrak{h}}}(L(\overline{\mathfrak{g}}_\infty, \overline{\Lambda}^{\mathfrak{g}}(\lambda,d))) = L(\mathfrak{t}\overline{\mathfrak{g}}, \Gamma_{\mathfrak{t}\overline{\mathfrak{g}}}(\lambda,d))$ for $(\lambda,d)\in\mathcal{D}_{\mathfrak{t}}(\mathfrak{g})$. Since $\mathfrak{tr}_{\mathfrak{t}\overline{\mathfrak{h}}}(L(\overline{\mathfrak{g}}_\infty, \overline{\Lambda}^{\mathfrak{g}}(\lambda,d))) = L(\mathfrak{t}\overline{\mathfrak{g}}, \Gamma_{\mathfrak{t}\overline{\mathfrak{g}}}(\lambda,d))$ and the homology commutes with the truncation functor, we have

$$H_k(\mathfrak{u}^-_{\mathfrak{t}\overline{\mathfrak{g}}}; L(\mathfrak{t}\overline{\mathfrak{g}},\xi)) = \mathfrak{tr}_{\mathfrak{t}\overline{\mathfrak{h}}}(H_k(\mathfrak{u}^-_{\overline{\mathfrak{g}}}; L(\overline{\mathfrak{g}}_\infty, \overline{\Lambda}^{\mathfrak{g}}(\lambda,d)))).$$

Note that $H_k(\mathfrak{u}^-_{\overline{\mathfrak{g}}}; L(\overline{\mathfrak{g}}_\infty, \overline{\Lambda}^{\mathfrak{g}}(\lambda,d)))$ with $k\geq 0$ decompose into the direct sum of irreducible $\mathfrak{l}_{\overline{\mathfrak{g}}}$-modules of the form $L(\mathfrak{l}_{\overline{\mathfrak{g}}}, \overline{\Lambda}^{\mathfrak{g}}(\mu,d))$ for some partition μ (resp. pair of partitions $\mu=(\mu^-,\mu^+)$) if $\mathfrak{g}=\mathfrak{c},\mathfrak{d}$ (resp. $\mathfrak{a}$) and $\mathfrak{tr}_{\mathfrak{t}\overline{\mathfrak{h}}}(L(\mathfrak{l}_{\overline{\mathfrak{g}}}, \overline{\Lambda}^{\mathfrak{g}}(\mu,d))) = L(\mathfrak{l}_{\mathfrak{t}\overline{\mathfrak{g}}}, \Gamma_{\mathfrak{t}\overline{\mathfrak{g}}}(\mu,d))$. Therefore, the theorem also holds for this case by Theorem 4.4. □

Remark 5.2. By Remark 4.5, Enright's cohomology formulas for unitarizable modules over classical Lie algebras with highest weights satisfying the assumption in the theorem above can be proved in the same manner as in above.

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