

On tangential cohomology attached to a function on complex foliations

Cristian Ida

Abstract

In this note we study a new cohomology attached to a function along the leaves of complex foliations. We also explain how this cohomology depends on the function and we study a relative cohomology and a Mayer-Vietoris sequence related to this cohomology.

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1 Introduction and preliminaries

The study of geometry and cohomology of holomorphic foliations was initiated by I. Vaisman in [12, 14]. In a different way, a Dolbeault cohomology along the leaves of complex foliations was intensively studied by A. El Kacimi-Alaoui in some recent papers (see [2, 3]). On the other hand, P. Monnier in [7] introduce a new cohomology of smooth manifolds, so called *cohomology attached to a function*. This cohomology was considered for the first time in [6] in the context of Poisson geometry, and more generally, Nambu-Poisson geometry. The main goal of this note is to extend some notions from [2, 3] concerning to [7], giving a similar cohomology attached to a function for foliated forms of type (p, q) on complex foliations. In this sense, in the first section following [2, 3], we briefly recall some preliminaries notions about complex foliations and $\bar{\partial}$ -cohomology along the leaves. Next, we define a Dolbeault cohomology attached to a function for foliated forms of type (p, q) , we define an associated Bott-Chern cohomology, we explain how this cohomology depends on the function, we study a relative cohomology and a Mayer-Vietoris sequence related to this cohomology. In particular, we show that if the function does not vanish, then our cohomology is isomorphic with the Dolbeault cohomology along the leaves. The methods used here are similarly to those used by [7] and are closely related to those used by [1].

1.1 Complex foliations

Let \mathcal{M} be a differentiable manifold of dimension $2m + n$ endowed with a codimension n foliation \mathcal{F} (then the dimension of \mathcal{F} is $2m$).

Definition 1.1. ([3]). The foliation \mathcal{F} is said to be **complex** if it can be defined by an open cover $\{U_i\}$, $i \in I$, of \mathcal{M} and diffeomorphisms $\phi_i : \Omega_i \times \mathcal{O}_i \rightarrow U_i$ (where Ω_i is an open polydisc in \mathbb{C}^m and \mathcal{O}_i is an open ball in \mathbb{R}^n) such that, for every pair $(i, j) \in I \times I$ with $U_i \cap U_j \neq \emptyset$, the coordinate change

$$\phi_{ij} = \phi_j^{-1} \circ \phi_i : \phi_i^{-1}(U_i \cap U_j) \rightarrow \phi_j^{-1}(U_i \cap U_j)$$

is of the form $(z', x') = (\phi_{ij}^1(z, x), \phi_{ij}^2(x))$ with $\phi_{ij}^1(z, x)$ holomorphic in z for x fixed.

An open set U of \mathcal{M} like one of the cover \mathcal{U} is called adapted to the foliation. Any leaf of \mathcal{F} is a complex manifold of dimension m . The notion of complex foliation is a natural generalization of the notion of holomorphic foliation on a complex manifold (see [14]). A manifold \mathcal{M} with a complex foliation \mathcal{F} will be denoted $(\mathcal{M}, \mathcal{F})$. With respect to local coordinates (z, x) , adapted to the complex foliation \mathcal{F} , the *complex structure along the leaves* $J_{\mathcal{F}} : T_{\mathbb{C}}\mathcal{F} \rightarrow T_{\mathbb{C}}\mathcal{F}$, is given by

$$J_{\mathcal{F}}\left(\frac{\partial}{\partial z^a}\right) = i\frac{\partial}{\partial z^a}, \quad J_{\mathcal{F}}\left(\frac{\partial}{\partial \bar{z}^a}\right) = -i\frac{\partial}{\partial \bar{z}^a}, \quad a = 1, \dots, m,$$

where $T_{\mathbb{C}}\mathcal{F} = T\mathcal{F} \otimes_{\mathbb{R}} \mathbb{C}$ is the complexified of the tangent distribution $T\mathcal{F}$.

We also notice that the *Nijenhuis tensor along the leaves* associated to $J_{\mathcal{F}}$, defined by

$$N_{\mathcal{F}}(X, Y) = 2\{[J_{\mathcal{F}}X, J_{\mathcal{F}}Y] - [X, Y] - J_{\mathcal{F}}[J_{\mathcal{F}}X, Y] - J_{\mathcal{F}}[X, J_{\mathcal{F}}Y]\},$$

vanish for every $X, Y \in \Gamma(T_{\mathbb{C}}\mathcal{F})$.

Let $(\mathcal{M}, \mathcal{F})$ and $(\mathcal{M}', \mathcal{F}')$ be two complex foliations. A *morphism* from $(\mathcal{M}, \mathcal{F})$ to $(\mathcal{M}', \mathcal{F}')$ is a differentiable mapping $f : \mathcal{M} \rightarrow \mathcal{M}'$ which sends every leaf F of \mathcal{F} into a leaf F' of \mathcal{F}' such that the restriction map $f : F \rightarrow F'$ is holomorphic. We say that a morphism $f : (\mathcal{M}, \mathcal{F}) \rightarrow (\mathcal{M}', \mathcal{F}')$ is an *isomorphism* of complex foliations (automorphism of $(\mathcal{M}, \mathcal{F})$ if $(\mathcal{M}', \mathcal{F}') = (\mathcal{M}, \mathcal{F})$) if f is a diffeomorphism whose restriction to any leaf $F \rightarrow F'$ (where $F' = f(F)$) is a biholomorphism. For some examples of complex foliations, see for instance [2].

1.2 $\bar{\partial}_{\mathcal{F}}$ -cohomology

Let us consider $\Omega^{p,q}(\mathcal{F})$ be the space of foliated differential forms of type (p, q) that is, differential forms on \mathcal{M} which can be written in local coordinates $(z^1, \dots, z^m, x^1, \dots, x^n)$, adapted to the foliation by

$$\varphi = \sum \varphi_{A_p \bar{B}_q}(z, x) dz^{A_p} \wedge d\bar{z}^{B_q}, \quad (1.1)$$

where $A_p = (a_1 \dots a_p)$, $B_q = (b_1 \dots b_q)$, and the sum is after the indices $a_1 \leq \dots \leq a_p$; $b_1 \leq \dots \leq b_q$. Also, the coefficient functions $\varphi_{a_1 \dots a_p \bar{b}_1 \dots \bar{b}_q}$ are skew symmetric in the indices (a_1, \dots, a_p) and (b_1, \dots, b_q) , respectively.

The Cauchy-Riemann operators along the leaves, are locally defined by

$$\partial_{\mathcal{F}}\varphi = \sum_{a=1}^m \frac{\partial\varphi_{A_p\bar{B}_q}}{\partial z^a} dz^a \wedge dz^{A_p} \wedge d\bar{z}^{B_q}, \quad \bar{\partial}_{\mathcal{F}}\varphi = \sum_{a=1}^m \frac{\partial\varphi_{A_p\bar{B}_q}}{\partial \bar{z}^a} d\bar{z}^a \wedge dz^{A_p} \wedge d\bar{z}^{B_q}. \quad (1.2)$$

These operators have the properties $\partial_{\mathcal{F}}^2 = \bar{\partial}_{\mathcal{F}}^2 = 0$ and $\partial_{\mathcal{F}}\bar{\partial}_{\mathcal{F}} + \bar{\partial}_{\mathcal{F}}\partial_{\mathcal{F}} = 0$.

The differential complex

$$0 \longrightarrow \Omega^{p,0}(\mathcal{F}) \xrightarrow{\bar{\partial}_{\mathcal{F}}} \Omega^{p,1}(\mathcal{F}) \xrightarrow{\bar{\partial}_{\mathcal{F}}} \dots \xrightarrow{\bar{\partial}_{\mathcal{F}}} \Omega^{p,m}(\mathcal{F}) \longrightarrow 0$$

is called the $\bar{\partial}_{\mathcal{F}}$ -complex of $(\mathcal{M}, \mathcal{F})$; its cohomology $H^{p,q}(\mathcal{F})$ is called the *foliated Dolbeault cohomology* of the complex foliation $(\mathcal{M}, \mathcal{F})$. Locally, the operator $\bar{\partial}_{\mathcal{F}}$ satisfies a Dolbeault-Grothendieck Lemma, (see [2]).

2 $\bar{\partial}_{\mathcal{F}}$ -cohomology attached to a function

In this section, we consider a new $\bar{\partial}_{\mathcal{F}}$ -cohomology associated to a function on the foliated manifold $(\mathcal{M}, \mathcal{F})$. This new cohomology is also defined in terms of foliated forms of type (p, q) . More precisely, if $(\mathcal{M}, \mathcal{F})$ is a complex foliation and f is a function on \mathcal{M} , we define the foliated coboundary operators

$$\partial_{\mathcal{F},f} : \Omega^{p,q}(\mathcal{F}) \rightarrow \Omega^{p+1,q}(\mathcal{F}), \quad \partial_{\mathcal{F},f}\varphi = f\partial_{\mathcal{F}}\varphi - (p+q)\partial_{\mathcal{F}}f \wedge \varphi, \quad (2.1)$$

$$\bar{\partial}_{\mathcal{F},f} : \Omega^{p,q}(\mathcal{F}) \rightarrow \Omega^{p,q+1}(\mathcal{F}), \quad \bar{\partial}_{\mathcal{F},f}\varphi = f\bar{\partial}_{\mathcal{F}}\varphi - (p+q)\bar{\partial}_{\mathcal{F}}f \wedge \varphi. \quad (2.2)$$

It is easy to check that $\partial_{\mathcal{F},f}^2 = \bar{\partial}_{\mathcal{F},f}^2 = 0$ and $\partial_{\mathcal{F},f}\bar{\partial}_{\mathcal{F},f} + \bar{\partial}_{\mathcal{F},f}\partial_{\mathcal{F},f} = 0$. So, we obtain a differential complex

$$0 \longrightarrow \Omega^{p,0}(\mathcal{F}) \xrightarrow{\bar{\partial}_{\mathcal{F},f}} \Omega^{p,1}(\mathcal{F}) \xrightarrow{\bar{\partial}_{\mathcal{F},f}} \dots \xrightarrow{\bar{\partial}_{\mathcal{F},f}} \Omega^{p,m}(\mathcal{F}) \longrightarrow 0 \quad (2.3)$$

called the *Dolbeault complex associated to the function f* of \mathcal{F} ; its cohomology $H_f^{p,q}(\mathcal{F})$ is called the *Dolbeault cohomology associated to the function f* of complex foliation \mathcal{F} .

More generally, for any integer k , we define the coboundary operator

$$\bar{\partial}_{\mathcal{F},f}^k : \Omega^{p,q}(\mathcal{F}) \rightarrow \Omega^{p,q+1}(\mathcal{F}), \quad \bar{\partial}_{\mathcal{F},f}^k\varphi = f\bar{\partial}_{\mathcal{F}}^k\varphi - (p+q-k)\bar{\partial}_{\mathcal{F}}f \wedge \varphi. \quad (2.4)$$

We still have $(\bar{\partial}_{\mathcal{F},f}^k)^2 = 0$ and we denote by $H_{f,k}^{p,q}(\mathcal{F})$ the cohomology of this complex. We shall restrict our attention to the cohomology $H_f^{\bullet,\bullet}(\mathcal{F})$ but most results readily generalize to the cohomology $H_{f,k}^{\bullet,\bullet}(\mathcal{F})$.

Using (2.2), by direct calculus we obtain

Proposition 2.1. *If $f, g \in \mathcal{F}(\mathcal{M})$ then*

- (i) $\bar{\partial}_{\mathcal{F},f+g} = \bar{\partial}_{\mathcal{F},f} + \bar{\partial}_{\mathcal{F},g}$, $\bar{\partial}_{\mathcal{F},0} = 0$, $\bar{\partial}_{\mathcal{F},-f} = -\bar{\partial}_{\mathcal{F},f}$;
- (ii) $\bar{\partial}_{\mathcal{F},fg} = f\bar{\partial}_{\mathcal{F},g} + g\bar{\partial}_{\mathcal{F},f} - fg\bar{\partial}_{\mathcal{F}}$, $\bar{\partial}_{\mathcal{F},1} = \bar{\partial}_{\mathcal{F}}$, $\bar{\partial}_{\mathcal{F}} = \frac{1}{2}(f\bar{\partial}_{\mathcal{F},\frac{1}{f}} + \frac{1}{f}\bar{\partial}_{\mathcal{F},f})$, and
- (iii) $\bar{\partial}_{\mathcal{F},f}(\varphi \wedge \psi) = \bar{\partial}_{\mathcal{F},f}\varphi \wedge \psi + (-1)^{\deg \varphi} \varphi \wedge \bar{\partial}_{\mathcal{F},f}\psi$.

2.1 Bott-Chern cohomology

Definition 2.1. The differential complex

$$\dots \Omega^{p-1,q-1}(\mathcal{F}) \xrightarrow{\partial_{\mathcal{F},f} \bar{\partial}_{\mathcal{F},f}} \Omega^{p,q}(\mathcal{F}) \xrightarrow{\partial_{\mathcal{F},f} \oplus \bar{\partial}_{\mathcal{F},f}} \Omega^{p+1,q}(\mathcal{F}) \oplus \Omega^{p,q+1}(\mathcal{F}) \dots \quad (2.5)$$

is called the *Bott-Chern complex associated to the function f* of \mathcal{F} and the corresponding Bott-Chern cohomology groups are given by

$$H_{f,BC}^{p,q}(\mathcal{F}) = \frac{\ker\{\partial_{\mathcal{F},f} : \Omega^{p,q} \rightarrow \Omega^{p+1,q}\} \cap \ker\{\bar{\partial}_{\mathcal{F},f} : \Omega^{p,q} \rightarrow \Omega^{p,q+1}\}}{\text{im}\{\partial_{\mathcal{F},f} \bar{\partial}_{\mathcal{F},f} : \Omega^{p-1,q-1} \rightarrow \Omega^{p,q}\}}. \quad (2.6)$$

It is easy to see that $\bigoplus_{p,q} H_{f,BC}^{p,q}(\mathcal{F})$ inherits a bigraded algebra structure induced by the exterior product of these forms. The above definition imply the canonical map

$$H_{f,BC}^{p,q}(\mathcal{F}) \rightarrow H_f^{p,q}(\mathcal{F}). \quad (2.7)$$

Now, let us consider the dual of the Bott-Chern cohomology groups associated to the function f , given by

$$H_{f,A}^{p,q}(\mathcal{F}) = \frac{\ker\{\partial_{\mathcal{F},f} \bar{\partial}_{\mathcal{F},f} : \Omega^{p,q} \rightarrow \Omega^{p+1,q+1}\}}{\text{im}\{\partial_{\mathcal{F},f} : \Omega^{p-1,q} \rightarrow \Omega^{p,q}\} + \text{im}\{\bar{\partial}_{\mathcal{F},f} : \Omega^{p,q-1} \rightarrow \Omega^{p,q}\}}$$

called the *Aeppli cohomology groups associated to the function f* of complex foliation \mathcal{F} .

Proposition 2.2. *The exterior product induces a bilinear map*

$$\wedge : H_{f,BC}^{p,q}(\mathcal{F}) \times H_{f,A}^{r,s}(\mathcal{F}) \rightarrow H_{f,A}^{p+r,q+s}(\mathcal{F}). \quad (2.8)$$

Proof. Let $\varphi, \psi \in \Omega^{p,q}(\mathcal{F})$. If φ is $(\partial_{\mathcal{F},f} + \bar{\partial}_{\mathcal{F},f})$ -closed and ψ is $\partial_{\mathcal{F},f} \bar{\partial}_{\mathcal{F},f}$ -closed then $\varphi \wedge \psi$ is $\partial_{\mathcal{F},f} \bar{\partial}_{\mathcal{F},f}$ -closed. Also, if φ is $(\partial_{\mathcal{F},f} + \bar{\partial}_{\mathcal{F},f})$ -closed and ψ is $(\partial_{\mathcal{F},f} + \bar{\partial}_{\mathcal{F},f})$ -exact then $\varphi \wedge \psi$ is $(\partial_{\mathcal{F},f} + \bar{\partial}_{\mathcal{F},f})$ -exact and if φ is $\partial_{\mathcal{F},f} \bar{\partial}_{\mathcal{F},f}$ -exact and ψ is $\partial_{\mathcal{F},f} \bar{\partial}_{\mathcal{F},f}$ -closed then $\varphi \wedge \psi$ is $(\partial_{\mathcal{F},f} + \bar{\partial}_{\mathcal{F},f})$ -exact.

For the last assertion, we have

$$\begin{aligned} \varphi \wedge \psi &= \partial_{\mathcal{F},f} \bar{\partial}_{\mathcal{F},f} \theta \wedge \psi \\ &= \frac{1}{2}(\partial_{\mathcal{F},f} + \bar{\partial}_{\mathcal{F},f})[(\bar{\partial}_{\mathcal{F},f} - \partial_{\mathcal{F},f})\theta \wedge \psi + (-1)^{p+q}\theta \wedge (\partial_{\mathcal{F},f} - \bar{\partial}_{\mathcal{F},f})\psi]. \end{aligned}$$

□

In particular, we have

$$H_{f,BC}^{p,q}(\mathcal{F}) \times H_{f,A}^{n-p,n-q}(\mathcal{F}) \rightarrow H_{f,A}^{n,n}(\mathcal{F}).$$

2.2 Dependence on the function

A natural question to ask about the cohomology $H_f^{p,q}(\mathcal{F})$ is how it depends on the function f . Similar with the proposition 3.2. from [7], we explain this fact for our foliated cohomology. We have

Proposition 2.3. *If $h \in \mathcal{F}(\mathcal{M})$ does not vanish, then the cohomologies $H_f^{\bullet,\bullet}(\mathcal{F})$ and $H_{fh}^{\bullet,\bullet}(\mathcal{F})$ are isomorphic.*

Proof. For each $p, q \in \mathbb{N}$, consider the linear isomorphism

$$\phi^{p,q} : \Omega^{p,q}(\mathcal{F}) \rightarrow \Omega^{p,q}(\mathcal{F}), \quad \phi^{p,q}(\varphi) = \frac{\varphi}{h^{p+q}}. \quad (2.9)$$

If φ is a foliated (p, q) -form on $(\mathcal{M}, \mathcal{F})$, one checks easily that

$$\phi^{p,q+1}(\bar{\partial}_{\mathcal{F}, fh}\varphi) = \bar{\partial}_{\mathcal{F}, f}(\phi^{p,q}(\varphi)), \quad (2.10)$$

so ϕ induces an isomorphism between the cohomologies $H_f^{\bullet,\bullet}(\mathcal{F})$ and $H_{fh}^{\bullet,\bullet}(\mathcal{F})$. \square

Corollary 2.1. *If the foliated function f does not vanish, then $H_f^{\bullet,\bullet}(\mathcal{F})$ is isomorphic to the foliated Dolbeault cohomology $H^{\bullet,\bullet}(\mathcal{F})$.*

Proof. We take $h = \frac{1}{f}$ in the above proposition. \square

2.3 A Mayer-Vietoris sequence and a homotopy morphism

Since the differential $\bar{\partial}_{\mathcal{F}, f}$ commutes with the restrictions to open subsets, one can construct, in the same way as for the de Rham cohomology (see [1]), a Mayer-Vietoris exact sequence, namely:

Theorem 2.1. *If $\mathcal{U} = \{U, V\}$ is an open cover of \mathcal{M} , we have the long exact sequence*

$$\begin{aligned} \dots \rightarrow H_f^{p,q-1}(\mathcal{F}|_{U \cap V}) \rightarrow H_f^{p,q}(\mathcal{F}) \xrightarrow{A} H_f^{p,q}(\mathcal{F}|_U) \oplus H_f^{p,q}(\mathcal{F}|_V) \xrightarrow{B} \\ \xrightarrow{B} H_f^{p,q}(\mathcal{F}|_{U \cap V}) \rightarrow \dots, \end{aligned}$$

where for $[\varphi] \in H_f^{p,q}(\mathcal{F})$ and $([\sigma_U], [\tau_V]) \in H_f^{p,q}(\mathcal{F}|_U) \oplus H_f^{p,q}(\mathcal{F}|_V)$, we define

$$A([\varphi]) = ([\sigma_U], [\tau_V]) \text{ and } B([\sigma_U], [\tau_V]) = [\sigma|_{U \cap V} - \tau|_{U \cap V}].$$

Following [7], we give

Definition 2.2. Let $(\mathcal{M}, \mathcal{F})$ and $(\mathcal{M}', \mathcal{F}')$ two complex foliations and $f \in \mathcal{F}(\mathcal{M})$ and $f' \in \mathcal{F}(\mathcal{M}')$. A *morphism* from the pair $(\mathcal{M}, \mathcal{F}, f)$ to the pair $(\mathcal{M}', \mathcal{F}', f')$ is a pair (ϕ, α) formed by a morphism $\phi : (\mathcal{M}, \mathcal{F}) \rightarrow (\mathcal{M}', \mathcal{F}')$ and a real valued function $\alpha : \mathcal{M} \rightarrow \mathbb{R}$, such that α does not vanish on \mathcal{M} and $f' \circ \phi = \alpha f$.

If (ϕ, α) is a morphism from the pair $(\mathcal{M}, \mathcal{F}, f)$ to the pair $(\mathcal{M}', \mathcal{F}', f')$ then the map $\Omega^{p,q}(\mathcal{F}') \mapsto \Omega^{p,q}(\mathcal{F})$ defined by $\varphi \mapsto \frac{\phi^* \varphi}{\alpha^{p+q}}$ induces an homomorphism in cohomology $H_{f'}^{p,q}(\mathcal{F}') \mapsto H_f^{p,q}(\mathcal{F})$. We also notice that if ϕ is diffeomorphism then $H_{f'}^{p,q}(\mathcal{F}')$ and $H_f^{p,q}(\mathcal{F})$ are isomorphic.

2.4 A relative cohomology

The relative de Rham cohomology was first defined in [1] p. 78. In this subsection we construct a similar version for our foliated cohomology.

Let $\mu : \mathcal{M} \rightarrow \mathcal{M}'$ be a morphism of two complex foliations. If F is a leaf of $(\mathcal{M}, \mathcal{F})$ then under inclusion $F \xrightarrow{i} \mathcal{M}$ we have $\bar{\partial}_{\mathcal{F}} = i^* \bar{\partial}$. We also notice that the following diagram

$$\begin{array}{ccc} F & \xrightarrow{i} & \mathcal{M} \\ \mu \downarrow & & \downarrow \mu \\ F' & \xrightarrow{i'} & \mathcal{M}' \end{array}$$

is commutative. Then by above discussion one gets

$$\bar{\partial}_{\mathcal{F}} \mu^* = \mu^* \bar{\partial}'_{\mathcal{F}'}. \quad (2.11)$$

Indeed, for $\varphi \in \Omega^{p,q}(\mathcal{F}')$ we have

$$\begin{aligned} \bar{\partial}_{\mathcal{F}}(\mu^* \varphi) &= i^* \bar{\partial}(\mu^* \varphi) = i^* \mu^*(\bar{\partial}' \varphi) = (\mu i)^*(\bar{\partial}' \varphi) \\ &= (i' \mu)^*(\bar{\partial}' \varphi) = \mu^* i'^* \bar{\partial}' \varphi = \mu^* \bar{\partial}'_{\mathcal{F}'} \varphi, \end{aligned}$$

where $\bar{\partial}'$ denotes the corresponding operator on \mathcal{M}' .

Now, if $f' \in \mathcal{F}(\mathcal{M}')$ then by (2.11) one gets

$$\bar{\partial}_{\mathcal{F}, \mu^* f'} \mu^* = \mu^* \bar{\partial}'_{\mathcal{F}', f'}. \quad (2.12)$$

Indeed, for $\varphi \in \Omega^{p,q}(\mathcal{F}')$ by direct calculus we have

$$\begin{aligned} \bar{\partial}_{\mathcal{F}, \mu^* f'}(\mu^* \varphi) &= \mu^* f' \bar{\partial}_{\mathcal{F}}(\mu^* \varphi) - (p+q) \bar{\partial}_{\mathcal{F}}(\mu^* f') \wedge \mu^* \varphi \\ &= \mu^* f' \mu^*(\bar{\partial}'_{\mathcal{F}'} \varphi) - (p+q) \mu^*(\bar{\partial}'_{\mathcal{F}'} f') \wedge \mu^* \varphi \\ &= \mu^*(f' \bar{\partial}'_{\mathcal{F}'} \varphi) - \mu^*((p+q) \bar{\partial}'_{\mathcal{F}'} f' \wedge \varphi) \\ &= \mu^*(\bar{\partial}'_{\mathcal{F}', f'} \varphi). \end{aligned}$$

We define the differential complex

$$\dots \xrightarrow{\tilde{\partial}_{\mathcal{F}}} \Omega_{\mathcal{F}}^{p,q}(\mu) \xrightarrow{\tilde{\partial}_{\mathcal{F}}} \Omega_{\mathcal{F}}^{p,q+1}(\mu) \xrightarrow{\tilde{\partial}_{\mathcal{F}}} \dots$$

where

$$\Omega_{\mathcal{F}}^{p,q}(\mu) = \Omega^{p,q}(\mathcal{F}') \oplus \Omega^{p,q-1}(\mathcal{F}) \text{ and } \tilde{\partial}_{\mathcal{F}}(\varphi, \psi) = (\bar{\partial}'_{\mathcal{F}', f'} \varphi, \mu^* \varphi - \bar{\partial}_{\mathcal{F}, \mu^* f'} \psi).$$

Taking into account $\bar{\partial}'_{\mathcal{F}', f'}{}^2 = \bar{\partial}_{\mathcal{F}, \mu^* f'}^2 = 0$ and (2.12) we easily verify that $\tilde{\partial}_{\mathcal{F}}^2 = 0$. Denote the cohomology groups of this complex by $H_{f'}^{p,q}(\mu)$.

If we regraduate the complex $\Omega^{p,q}(\mathcal{F})$ as $\widetilde{\Omega}^{p,q}(\mathcal{F}) = \Omega^{p,q-1}(\mathcal{F})$, then we obtain an exact sequence of differential complexes

$$0 \longrightarrow \widetilde{\Omega}^{p,q}(\mathcal{F}) \xrightarrow{\alpha} \Omega_{\mathcal{F}}^{p,q}(\mu) \xrightarrow{\beta} \Omega^{p,q}(\mathcal{F}') \longrightarrow 0 \quad (2.13)$$

with the obvious mappings α and β given by $\alpha(\psi) = (0, \psi)$ and $\beta(\varphi, \psi) = \varphi$, respectively. From (2.13) we have an exact sequence in cohomologies,

$$\dots \longrightarrow H_{\mu^* f'}^{p,q-1}(\mathcal{F}) \xrightarrow{\alpha^*} H_{f'}^{p,q}(\mu) \xrightarrow{\beta^*} H_{f'}^{p,q}(\mathcal{F}') \xrightarrow{\delta^*} H_{\mu^* f'}^{p,q}(\mathcal{F}) \longrightarrow \dots$$

It is easily seen that $\delta^* = \mu^*$. Here μ^* denotes the corresponding map between cohomology groups. Let $\varphi \in \Omega_{\mathcal{F}}^{p,q}(\mu)$ be a $\overline{\partial}_{\mathcal{F}', f'}$ -closed form, and $(\varphi, \psi) \in \Omega_{\mathcal{F}}^{p,q}(\mu)$. Then $\widetilde{\partial}_{\mathcal{F}}(\varphi, \psi) = (0, \mu^* \varphi - \overline{\partial}_{\mathcal{F}, \mu^* f'} \psi)$ and by the definition of the operator δ^* we have

$$\delta^*[\varphi] = [\mu^* \varphi - \overline{\partial}_{\mathcal{F}, \mu^* f'} \psi] = [\mu^* \varphi].$$

Hence we finally get a long exact sequence

$$\dots \longrightarrow H_{\mu^* f'}^{p,q-1}(\mathcal{F}) \xrightarrow{\alpha^*} H_{f'}^{p,q}(\mu) \xrightarrow{\beta^*} H_{f'}^{p,q}(\mathcal{F}') \xrightarrow{\mu^*} H_{\mu^* f'}^{p,q}(\mathcal{F}) \xrightarrow{\alpha^*} \dots \quad (2.14)$$

We have

Corollary 2.2. *If the complex foliations $(\mathcal{M}, \mathcal{F})$ and $(\mathcal{M}', \mathcal{F}')$ are of the $(n + 2m)$ -th and $(n' + 2m')$ -th dimension, respectively, then*

- (i) $\beta^* : H_{f'}^{p,m+1}(\mu) \rightarrow H_{f'}^{p,m+1}(\mathcal{F}')$ is an epimorphism,
- (ii) $\alpha^* : H_{\mu^* f'}^{p,m'}(\mathcal{F}) \rightarrow H_{f'}^{p,m'+1}(\mu)$ is an epimorphism,
- (iii) $\beta^* : H_{f'}^{p,q}(\mu) \rightarrow H_{f'}^{p,q}(\mathcal{F}')$ is an isomorphism for $q > m + 1$,
- (iv) $\alpha^* : H_{\mu^* f'}^{p,q}(\mathcal{F}) \rightarrow H_{f'}^{p,q+1}(\mu)$ is an isomorphism for $q > m'$,
- (v) $H_{f'}^{p,q}(\mu) = 0$ for $q > \max\{m + 1, m'\}$.

Finally, we notice that for $f = 1$ the operator $\widetilde{\partial}_{\mathcal{F}}$ defined by $\widetilde{\partial}_{\mathcal{F}}(\varphi, \psi) = (\overline{\partial}_{\mathcal{F}'} \varphi, \mu^* \varphi - \overline{\partial}_{\mathcal{F}} \psi)$ satisfies a Dolbeault type Lemma, namely

Theorem 2.2. ([4]). *Let φ be a foliated differential form of type (p, q) defined on $U' \subset \mathcal{M}'$ and ψ be a foliated differential form of type $(p, q - 1)$ defined on $U \subset \mathcal{M}$ such that $\widetilde{\partial}_{\mathcal{F}}(\varphi, \psi) = (0, 0)$. Then, there exists a foliated differential form φ_1 of type $(p, q - 1)$ defined on $V' \subset U'$ and a foliated differential form θ_1 of type $(p, q - 2)$ defined on $V \subset U$ and such that $(\varphi, \psi) = \widetilde{\partial}_{\mathcal{F}}(\varphi_1, \psi_1)$.*

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Cristian Ida
Department of Algebra, Geometry and Differential Equations
University Transilvania of Braşov, România
Address: Braşov 500091, Str. Iuliu Maniu 50, România
email: *cristian.ida@unitbv.ro*