A sufficient condition for the existence of an anti-directed 2-factor in a directed graph

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Abstract

Let D be a directed graph with vertex set V and order n. An *anti-directed (hamiltonian) cycle* H in D is a (hamiltonian) cycle in the graph underlying D such that no pair of consecutive arcs in H form a directed path in D. An *anti-directed 2-factor* in D is a vertex-disjoint collection of anti-directed cycles in D

that span V. It was proved in [3] that if the indegree and the outdegree of each vertex of D is greater than $\frac{9}{16}n$ then D contains an anti-directed hamiltonian cycle. In this paper we prove that given a directed graph D, the problem of determining whether D has an anti-directed 2-factor is NP-complete, and we use a proof technique similar to the one used in [3] to prove that if the indegree and the outdegree of each vertex of D is greater than $\frac{25}{48}n$ then D contains an anti-directed 2-factor.

1 Introduction

Let G be a multigraph with vertex set $V(G)$ and edge set $E(G)$. For a vertex $v \in V(G)$, the degree of v in G, denoted by $\deg(v, G)$ is the number of edges of G incident on v. Let $\delta(G) = \min_{v \in V(G)}\{\deg(v, G)\}$. The simple graph underlying G denoted by simp(G) is the graph obtained from G by replacing all multiple edges by single edges. A *2-factor* in G is a collection of vertex-disjoint cycles that span $V(G)$. Let D be a directed graph with vertex set $V(D)$ and arc set $A(D)$. For a vertex $v \in V(D)$, the *outdegree* (respectively, *indegree*) of v in D denoted by $d^+(v, D)$ (respectively, $d^-(v, D)$) is the number of arcs of D directed out of v (respectively, directed into v). Let $\delta(D) = \min_{v \in V(D)}\{\min\{d^+(v, D), d^-(v, D)\}\}$. The *multigraph underlying* D is the multigraph obtained from D by ignoring the directions of the arcs of D. A *directed (Hamilton) cycle* C in D is a (Hamilton) cycle in the multigraph underlying D such that all pairs of consecutive arcs in C form a directed path in D. An *anti-directed (Hamilton) cycle* C in D is a (Hamilton) cycle in the multigraph underlying D such that no pair of consecutive arcs in C form a directed path in D. A *directed 2-factor* in D is a collection of vertex-disjoint directed cycles in D that span $V(D)$. An *anti-directed 2-factor* in D is a collection of vertex-disjoint anti-directed cycles in D that span $V(D)$. Note that every anti-directed cycle in D must have an even number of vertices. We refer the reader to ([1,7]) for all terminology and notation that is not defined in this paper.

The following classical theorems by Dirac [5] and Ghouila-Houri [6] give sufficient conditions for the existence of a Hamilton cycle in a graph G and for the existence of a directed Hamilton cycle in a directed graph D respectively.

Theorem 1 [5] *If G is a graph of order $n \geq 3$ and $\delta(G) \geq \frac{n}{2}$, then G contains a Hamilton cycle.*

Theorem 2 [6] *If D is a directed graph of order n and $\delta(D) \geq \frac{n}{2}$, then D contains a directed Hamilton cycle.*

Note that if D is a directed graph of even order n and $\delta(D) \geq \frac{3}{4}n$ then D contains an anti-directed Hamilton cycle. To see this, let G be the multigraph underlying D and let G' be the subgraph of G consisting of the parallel edges of G. Now, $\delta(D) \geq \frac{3}{4}n$ implies that $\delta(\text{simp}(G')) \geq \frac{n}{2}$ and hence Theorem 1 implies that simp(G') contains a Hamilton cycle which in turn implies that D contains an anti-directed Hamilton cycle.

The following theorem by Grant [7] gives a sufficient condition for the existence of an anti-directed Hamilton cycle in a directed graph D.

Theorem 3 [7] *If D is a directed graph with even order n and if $\delta(D) \geq \frac{2}{3}n + \sqrt{n\log(n)}$ then D contains an anti-directed Hamilton cycle.*

In his paper Grant [7] conjectured that the theorem above can be strengthened to assert that if D is a directed graph with even order n and if $\delta(D) \geq \frac{1}{2}n$ then D contains an anti-directed Hamilton cycle. Mao-cheng Cai [11] gave a counter-example to this conjecture. In [3] the following sufficient condition for the existence of an anti-directed Hamilton cycle in a directed graph was proved.

Theorem 4 [3] *Let D be a directed graph of even order n and suppose that $\frac{1}{2} < p < \frac{3}{4}$. If $\delta(D) \geq pn$ and $n > \frac{\ln(4)}{(p-\frac{1}{2})\ln\left(\frac{p+\frac{1}{2}}{\frac{3}{2}-p}\right)}$, then D contains an anti-directed Hamilton cycle.*

It was shown in [3] that Theorem 4 implies the following corollary that is an improvement on the result in Theorem 3.

Corollary 1 [3] *If D is a directed graph of even order n and $\delta(D) > \frac{9}{16}n$ then D contains an anti-directed Hamilton cycle.*

The following theorem (see [1]) gives a necessary and sufficient condition for the existence of a directed 2-factor in a digraph D.

Theorem 5 *A directed graph $D = (V, A)$ has a directed 2-factor if and only if $|\bigcup_{v \in X} N^+(v)| \geq |X|$ for all $X \subseteq V$.*

We note here that given a directed graph D the problem of determining whether D has a directed Hamilton cycle is known to be NP-complete, whereas, there exists an $O(\sqrt{n}m)$ algorithm (see [1]) to check if a directed graph D of order n and size m has a directed 2-factor. On the other hand, the following theorem proves that given a directed graph D, the problem of determining whether D has a directed 2-factor is NP-complete. We are indebted to Sundar Vishwanath for pointing out the short proof of Theorem 6 given below.

Theorem 6 [14] *Given a directed graph D, the problem of determining whether D has an anti-directed 2-factor. is NP-complete.*

Proof. Clearly the the problem of determining whether D has an anti-directed 2-factor is in NP. A graph G is said to be k-edge colorable if the edges of G can be colored with k colors in such a way that no two adjacent edges receive the same color. It is well known that given a cubic graph G, it is NP-complete to determine if G is 3-edge colorable. Now, given a cubic graph $G = (V, E)$, construct a directed graph $D = (V, A)$, where for each $\{u, v\} \in E$, we have the oppositely directed arcs (u, v) and (v, u) in A. It is clear that G is 3-edge colorable if and only if D contains an anti-directed 2-factor. This proves that the the problem of determining whether a directed graph D has an anti-directed 2-factor is NP-complete. ■

In Section 1 of this paper we prove the following theorem that gives a sufficient condition for the existence of an anti-directed 2-factor in a directed graph.

Theorem 7 *Let D be a directed graph of even order n and suppose that $\frac{1}{2} < p < \frac{3}{4}$. If $\delta(D) \geq pn$ and $n > \frac{\ln(4)}{(p-\frac{1}{2})\ln\left(\frac{p+\frac{1}{2}}{\frac{3}{2}-p}\right)}$(??), then D contains an anti-directed 2-factor.*

In Section 1 we will show that Theorem 7 implies the following corollary.

Corollary 2 [3] *If D is a directed graph of even order n and $\delta(D) > \frac{25}{48}n$ then D contains an anti-directed 2-factor.*

2 Proof of Theorem 7 and its Corollary

A partition of a set S with $|S|$ being even into $S = X \cup Y$ is an *equipartition* of S if $|X| = |Y| = \frac{|S|}{2}$. The proof of Theorem 4 mentioned in the introduction made extensive use of the following theorem by Chvátal [4].

Theorem 8 [4] *Let G be a bipartite graph of even order n and with equipartition $V(G) = X \cup Y$. Let $(d_1, d_2, \ldots, d_n)$ be the degree sequence of G with $d_1 \leq d_2 \leq \ldots \leq d_n$. If G does not contain a Hamilton cycle, then for some $i \leq \frac{n}{4}$ we have that $d_i \leq i$ and $d_{\frac{n}{2}} \leq \frac{n}{2} - i$.*

We prepare for the proof of Theorem 7 by proving Theorems 10 and 11 which give necessary degree conditions (similar to those in Theorem 8) for the non-existence of a 2-factor in a bipartite graph G of even order n with equipartition $V(G) = X \cup Y$. Let $G = (V, E)$ be a bipartite graph of even order n and with equipartition $V(G) =$

$X \cup Y$. For $U \subseteq X$ (respectively $U \subseteq Y$) define $N^{(2)}(U)$ as being the multiset of vertices $v \in Y$ (respectively $v \in X$) such that $(u, v) \in E$ for some $u \in U$ and with v appearing twice in $N^{(2)}(U)$ if there are two or more vertices $u \in U$ with $(u, v) \in E$ and v appearing once in $N^{(2)}(U)$ if there is exactly one $u \in U$ with $(u, v) \in E$. We will use the following theorem by Ore [12] that gives a necessary and sufficient condition for the non-existence of a 2-factor in a bipartite graph of even order n with equipartition $V(G) = X \cup Y$.

Theorem 9 *Let $G = (V, E)$ be a bipartite graph of even order n and with equipartition $V(G) = X \cup Y$. G contains no 2-factor if and only if there exists some $U \subseteq X$ such that $|N^{(2)}(U)| < 2|U|$.*

For a bipartite graph $G = (V, E)$ of even order n and with equipartition $V(G) = X \cup Y$, a set $U \subseteq X$ or $U \subseteq Y$ is defined to be a *deficient* set of vertices in G if $|N^{(2)}(U)| < 2|U|$.

We now prove four Lemmas that will be used in the proof of Theorems 10 and 11.

Lemma 1 *Let G be a bipartite graph of even order n and with equipartition $V(G) = X \cup Y$. If U is a minimal deficient set of vertices in G then $2|U| - 2 \leq |N^{(2)}(U)|$.*

Proof. Clear by the minimality of U. ■

Lemma 2 *Let G be a bipartite graph of even order n and with equipartition $V(G) = X \cup Y$, and let U be a minimal deficient set of vertices in G. Let $M \subseteq N(U)$ be the set of vertices in $N(U)$ that are adjacent to exactly one vertex in U. Then, no vertex of U is adjacent to more than one vertex of M.*

Proof. If a vertex $u \in U$ is adjacent to two vertices of M, since U is a deficient set of vertices in G, we have that $|N^{(2)}(U - u)| \leq |N^{(2)}(U)| - 2 < 2|U| - 2 = 2|U - u|$. This implies that $U - u$ is a deficient set of vertices in G, which in turn contradicts the minimality of U. ■

Lemma 3 *Let G be a bipartite graph of even order n and with equipartition $V(G) = X \cup Y$, and suppose that G does not contain a 2-factor. If U is a minimal deficient set in G with $|U| = k$, then $\deg(u) \leq k$ for each $u \in U$ and $|\{u \in U : \deg(u) \leq k - 1\}| \geq k - 1$.*

Proof. Suppose that $\deg(u) \geq k+1$ for some $u \in U$ and let $M \subseteq N(U)$ be the set of vertices in $N(U)$ that are adjacent to exactly one vertex in U. Then Lemma 2 implies that u is adjacent to at most one vertex in M which implies that u is adjacent to at least k vertices in $N(U) - M$. This implies that $|N^{(2)}(U)| \geq 2k$, which contradicts the assumption that U is a deficient set. This proves that $\deg(u) \leq k$ for each $u \in U$. If two vertices in U have degree k then similarly Lemma 2 implies that $|N^{(2)}(U)| \geq 2k$, which contradicts the assumption that U is a deficient set. This proves the second part of the Lemma. ■

Lemma 4 *Let $G = (V, E)$ be a bipartite graph of even order n and with equipartition $V(G) = X \cup Y$ and suppose that $U \subseteq X$ is a minimal deficient set in G. Let $Y_0 = \{v \in Y : v \notin N(U)\}$, $Y_1 = \{v \in Y : |U \cap N(v)| = 1\}$, and $Y_2 = \{v \in Y : |U \cap N(v)| \geq 2\}$. Let $U^* = Y_0 \cup Y_1$. Then U^* is a deficient set in G.*

Proof. Let $X_0 = X - U, X_1 = \{u \in U : (u, v) \in E$ for some $v \in Y_1\}$, and $X_2 = U - X_1$. Note that $|X| = |Y|$ implies that $|X_0|+|X_1|+|X_2| = |Y_0|+|Y_1|+|Y_2|$. Now, since by Lemma 2 we have that $|X_1| = |Y_1|$, this implies that $|X_0|+|X_2| = |Y_0|+|Y_2|$. Since U is a deficient set we have that $|N^{(2)}(U)| = |Y_0|+2|Y_2| < 2|U| = 2(|X_1|+|X_2|$. Hence, $|Y_1| + 2(|X_0| + |X_2| - |Y_0|) < 2(|X_1| + |X_2|)$, which in turn implies that $2|X_0| + |X_1| < 2(|Y_0| + |Y_1|)$. This proves that U^* is a deficient set in G. ■

We are now ready to prove two theorems which give necessary degree conditions (similar to those in Theorem 8) for the non-existence of a 2-factor in a bipartite graph G of even order n with equipartition $V(G) = X \cup Y$.

Theorem 10 *Let G be a bipartite graph of even order $n = 4s \geq 12$ and with equipartition $V(G) = X \cup Y$. Let $(d_1, d_2, \ldots, d_n)$ be the degree sequence of G with $d_1 \leq d_2 \leq \ldots \leq d_n$. If G does not contain a 2-factor, then either*

(1) for some $k \leq \frac{n}{4}$ we have that $d_k \leq k$ and $d_{k-1} \leq k - 1$, or,

(2) $d_{\frac{n}{4}-1} \leq \frac{n}{4} - 1$.

Proof. We will prove that for some $k \leq \frac{n}{4}$, G contains k vertices with degree at most k, and that of these k vertices, $(k - 1)$ vertices have degree at most $(k - 1)$, or, that G contains at least $\frac{n}{4} - 1$ vertices of degree at most $\frac{n}{4} - 1$.
Since G does not contain a 2-factor, Theorem 9 implies that G contains a deficient set of vertices. Let $U \subseteq X$ be a minimal deficient set of vertices in G. If $|U| \leq \frac{n}{4}$, then Lemma 3 implies that statement (1) is true and the result holds.
Now suppose that $|U| > \frac{n}{4}$. As in the statement of Lemma 4, let $Y_0 = \{v \in Y : v \notin N(U)\}$, $Y_1 = \{v \in Y : |U \cap N(v)| = 1\}$, and $Y_2 = \{v \in Y : |U \cap N(v)| \geq 2\}$. Let $U^* = Y_0 \cup Y_1$. Then Lemma 4 implies that U^* is a deficient set in G. If $|U^*| \leq \frac{n}{4}$

then again statement (1) is true and the result holds.
Now suppose that $|U^*| > \frac{n}{4}$, and as in the proof of Lemma 4, let $X_0 = X - U, X_1 = \{u \in U : (u,v) \in E$ for some $v \in Y_1\}$, and $X_2 = U - X_1$. We have that $\deg(u) \leq 1 + |Y_2|$ for each $u \in U$, and hence we may assume that $|Y_2| \geq \frac{n}{4} - 1$, else the result holds. Similarly, since $\deg(u) \leq 1 + |X_0|$ for each $u \in U^*$, we may assume that $|X_0| \geq \frac{n}{4} - 1$. Note that $|U| > \frac{n}{4}$ and $|X_0| \geq \frac{n}{4} - 1$ implies that $|U| = \frac{n}{4} + 1$, and that $|U^*| > \frac{n}{4}$ and $|Y_2| \geq \frac{n}{4} - 1$ implies that $|U^*| = \frac{n}{4} + 1$. Now, since U is a minimal deficient set of vertices in G, Lemma 1 implies that $|X_1| = 2$ or $X_1 = 3$. If $|X_1| = 2$ then at least $\frac{n}{4} - 1$ of the vertices in U must have degree at most $\frac{n}{4} - 1$, and statement (2) of the theorem is true. Finally, if $|X_1| = 3$ then at least $\frac{n}{2} - 4$ (and hence at least $\frac{n}{4} - 1$ because $n \geq 12$) of the vertices in each of U and U^* must have degree at most $\frac{n}{4} - 1$, and statement (2) of the theorem is true. ■

Theorem 11 *Let G be a bipartite graph of even order $n = 4s + 2 \geq 14$ and with equipartition $V(G) = X \cup Y$. Let $(d_1, d_2, \ldots, d_n)$ be the degree sequence of G with $d_1 \leq d_2 \leq \ldots \leq d_n$. If G does not contain a 2-factor, then either*

(1) for some $k \leq \frac{(n-2)}{4}$ we have that $d_k \leq k$ and $d_{k-1} \leq k - 1$, or,

(2) $d_{\frac{(n-2)}{2}} \leq \frac{(n-2)}{4}$.

Proof. We will prove that for some $k \leq \frac{n}{4}$, G contains k vertices with degree at most k, and that of these k vertices, $(k-1)$ vertices have degree at most $(k-1)$, or, that G contains at least $\frac{(n-2)}{2}$ vertices of degree at most $\frac{(n-2)}{4}$.
Since G does not contain a 2-factor, Theorem 9 implies that G contains a deficient set of vertices. Without loss of generality let $U \subseteq X$ be a minimum cardinality deficient set of vertices in G. If $|U| \leq \frac{(n-2)}{4}$, then Lemma 3 implies that statement (1) is true and the result holds.
Now suppose that $|U| > \frac{(n-2)}{4}$. As in the statement of Lemma 4, let $Y_0 = \{v \in Y : v \notin N(U)\}$, $Y_1 = \{v \in Y : |U \cap N(v)| = 1\}$, and $Y_2 = \{v \in Y : |U \cap N(v)| \geq 2\}$. Let $U^* = Y_0 \cup Y_1$. Then Lemma 4 implies that U^* is a deficient set in G. Since U is a minimum cardinality deficient set of vertices in G, we have that$|U^*| \geq |U| > \frac{(n-2)}{4}$.
Now, as in the proof of Lemma 4, let $X_0 = X - U, X_1 = \{u \in U : (u,v) \in E$ for some $v \in Y_1\}$, and $X_2 = U - X_1$. We have that $\deg(u) \leq 1 + |Y_2|$ for each $u \in U$, and hence we may assume that $|Y_2| \geq \frac{(n-2)}{4} - 1$, else the result holds. Similarly, since $\deg(u) \leq 1 + |X_0|$ for each $u \in U^*$, we may assume that $|X_0| \geq \frac{(n-2)}{4} - 1$. Note that $|U| > \frac{(n-2)}{4}$ and $|X_0| \geq \frac{(n-2)}{4} - 1$ implies that $\frac{(n-2)}{4} + 1 \leq |U| \leq \frac{(n-2)}{4} + 2$. We now examine the two cases: $|U| = \frac{(n-2)}{4} + 1$ and $|U| = \frac{(n-2)}{4} + 2$.

(1) $|U| = \frac{(n-2)}{4} + 1$. In this case we must have that $|X_0| = \frac{(n-2)}{4}$. Note that $|X_1| \leq 3$ because if $|X_1| \geq 4$ then since U is a minimal deficient set of vertices, we would have that $|Y_2| \leq \frac{(n-2)}{4} - 2$, a contradiction to the assumption at this point that $|Y_2| \geq \frac{(n-2)}{4} - 1$. We now examine the following four subcases separately.

(1)a $|X_1| = 0$. In this case we have that $|Y_1| = 0$ and $|X_2| = \frac{(n-2)}{4} + 1$. Since U is a minimal deficient set of vertices, Lemma 1 implies that $|Y_2| = \frac{(n-2)}{4}$ and $|Y_0| = \frac{(n-2)}{4} + 1$. Thus, $X_2 \cup Y_0$ is a set of $\frac{n}{2} + 1$ vertices of degree at most $\frac{(n-2)}{4}$ which meets the requirement of the theorem..

(1)b $|X_1| = 1$. In this case we have that $|Y_1| = 1$ and $|X_2| = \frac{(n-2)}{4}$. Since U is a minimal deficient set of vertices, Lemma 1 implies that $|Y_2| = \frac{(n-2)}{4}$ and $|Y_0| = \frac{(n-2)}{4}$. Thus, $X_2 \cup Y_0$ is a set of $\frac{n}{2} + 1$ vertices of degree at most $\frac{(n-2)}{4}$ each as required by the theorem.

(1)c $|X_1| = 2$. In this case we have that $|Y_1| = 2$ and $|X_2| = \frac{(n-2)}{4} - 1$. Since U is a minimal deficient set of vertices, Lemma 1 implies that $|Y_2| = \frac{(n-2)}{4} - 1$ and $|Y_0| = \frac{(n-2)}{4}$. Thus, $X_2 \cup X_1 \cup Y_0$ is a set of $\frac{n}{2}$ vertices of degree at most $\frac{(n-2)}{4}$ which meets the requirement of the theorem.

(1)d $|X_1| = 3$. In this case we have that $|Y_1| = 3$ and $|X_2| = \frac{(n-2)}{4} - 2$. Since U is a minimal deficient set of vertices, Lemma 1 implies that $|Y_2| = \frac{(n-2)}{4} - 1$ and $|Y_0| = \frac{(n-2)}{4} - 1$. Thus, $X_2 \cup X_1 \cup Y_0$ is a set of $\frac{n}{2} - 1$ vertices of degree at most $\frac{(n-2)}{4}$ as required by the theorem.

(2) $|U| = \frac{(n-2)}{4} + 2$. In this case we have that $|X_0| = \frac{(n-2)}{4} - 1$. Since U is a minimum cardinality deficient set of vertices, we also have that $|U^*| = |U| = \frac{(n-2)}{4} + 2$. Hence we now have that $|Y_2| = |X_0| = \frac{(n-2)}{4} - 1$. Thus, $U \cup U^*$ is a set of $\frac{n}{2} + 3$ vertices of degree at most $\frac{(n-2)}{4}$ which meets the requirement of the theorem.

■

Lemma 5 *Let x, y, r be positive numbers such that $x \geq y$ and $r < y$. Then* $\frac{(x+r)(x-r)}{(y+r)(y-r)} \geq (\frac{x}{y})^2$.

Proof. $y^2(x^2 - r^2) \geq (y^2 - r^2)x^2$, so the result follows. ■

Proof of Theorem 7. For an equipartition of $V(D)$ into $V(D) = X \cup Y$, let $B(X \to Y)$ be the bipartite directed graph with vertex set $V(D)$, equipartition $V(D) = X \cup Y$, and with $(x, y) \in A(B(X \to Y))$ if and only if $x \in X$, $y \in Y$, and, $(x, y) \in A(D)$. Let $B(X, Y)$ denote the bipartite graph underlying $B(X \to Y)$. It is clear that $B(X, Y)$ contains a Hamilton cycle if and only if $B(X \to Y)$ contains an anti-directed Hamilton cycle. We will prove that there exists an equipartition of $V(D)$ into $V(D) = X \cup Y$ such that $B(X, Y)$ contains a Hamilton cycle.

In the argument below, we make the simplifying assumption that $d^+(v) = d^-(v) = \delta(D)$ for each $v \in V(D)$. It is straightforward (see the remark at the end of the proof) to see that the argument extends to the case in which some indegrees or outdegrees are greater than $\delta(D)$.

Let $v \in V(D)$. Let n_k denote the number of equipartitions of $V(D)$ into $V(D) = X \cup Y$ for which $\deg(v, B(X, Y)) = k$. Since $v \in X$ or $v \in Y$ and since $d^+(v) = d^-(v) = \delta(D)$, we have that $n_k = 2\binom{\delta}{k}\binom{n-\delta-1}{\frac{n}{2}-k}$. Note that if $k > \frac{n}{2}$ or if $k < \delta - \frac{n}{2} + 1$ then $n_k = 0$. Thus the total number of equipartitions of $V(D)$ into $V(D) = X \cup Y$ is

$$T = \sum_{k=\delta-\frac{n}{2}+1}^{\frac{n}{2}} n_k = \sum_{k=\delta-\frac{n}{2}+1}^{\frac{n}{2}} 2\binom{\delta}{k}\binom{n-\delta-1}{\frac{n}{2}-k} = \binom{n}{\frac{n}{2}}. \tag{1}$$

Denote by $N = \binom{n}{\frac{n}{2}}$ the total number of equipartitions of $V(D)$. For a particular equipartition of $V(D)$ into $V(D) = X_i \cup Y_i$, let $(d_1^{(i)}, d_2^{(i)}, \ldots, d_n^{(i)})$ be the degree sequence of $B(X_i, Y_i)$ with $d_1^{(i)} \le d_2^{(i)} \le \ldots \le d_n^{(i)}$, $i = 1, 2, \ldots, N$, and, let $P_i = \{j : d_j^i \le \frac{n}{4}\}$. If $B(X_i, Y_i)$ does not contain a Hamilton cycle then Theorem 8 implies that there exists $k \le \frac{n}{4}$ such that $d_k^i \le k$ and hence, $|\{d_j^i : d_j^i \le k, j = 1, 2, \ldots, n\}| \ge k$. This in turn implies that $\sum_{j \in P_i} \frac{1}{d_j^i} \ge 1$. Hence, the number of equipartitions of $V(D)$ into $V(D) = X \cup Y$ for which $B(X, Y)$ does not contain a Hamilton cycle is at most

$$S = n\left(\frac{n_2}{2} + \frac{n_3}{3} + \ldots + \frac{n_{\lfloor \frac{n}{4} \rfloor}}{\lfloor \frac{n}{4} \rfloor}\right) \tag{2}$$

Thus, to show that there exists an equipartition of $V(D)$ into $V(D) = X \cup Y$ such that $B(X, Y)$ contains a Hamilton cycle, it suffices to show that $T > S$, i.e.,

$$\sum_{k=\delta-\frac{n}{2}+1}^{\frac{n}{2}} 2\binom{\delta}{k}\binom{n-\delta-1}{\frac{n}{2}-k} > n \sum_{k=2}^{\lfloor \frac{n}{4} \rfloor} \frac{2\binom{\delta}{k}\binom{n-\delta-1}{\frac{n}{2}-k}}{k} \tag{3}$$

We break the proof of (3) into three cases.

Case 1: $n = 4m$ and $\delta = 2d$ for some positive integers m and d.

For $i = 0, 1, \ldots, \frac{n}{4} - 2$, let $A_i = n_{(d+i)} = 2\binom{\delta}{d+i}\binom{n-\delta-1}{2m-d-i}$, and let $B_i = n_{(\frac{n}{4}-i)} = 2\binom{\delta}{m-i}\binom{n-\delta-1}{m+i}$. Clearly, (3) is satisfied if we can show that

$$A_i > \frac{nB_i}{\frac{n}{4} - i}, \text{ for each } i = 0, 1, \ldots, \frac{n}{4} - 2. \tag{4}$$

We prove (4) by recursion on i. We first show that $A_0 > \frac{nB_0}{\frac{n}{4}}$, i.e. $n_{\frac{\delta}{2}} > n\left(\frac{n_{\frac{n}{4}}}{\frac{n}{4}}\right) = 4n_{\frac{n}{4}}$. Let $\delta = \frac{n}{2} + s$. We have that

$$\begin{aligned}
\frac{A_0}{B_0} &= \frac{(\frac{n}{4})!(\delta - \frac{n}{4})!(\frac{n}{4})!(\frac{3n}{4} - \delta - 1)!}{\frac{\delta}{2}!\frac{\delta}{2}!(\frac{n}{2} - \frac{\delta}{2})!(\frac{n}{2} - \frac{\delta}{2} - 1)!} \\
&= \frac{(\frac{n}{4})!(\frac{n}{4} + s)!(\frac{n}{4})!(\frac{n}{4} - s - 1)!}{(\frac{n}{4} + \frac{s}{2})!(\frac{n}{4} + \frac{s}{2})!(\frac{n}{4} - \frac{s}{2})!(\frac{n}{4} - \frac{s}{2} - 1)!} \\
&= \frac{(\frac{n}{4} + s)(\frac{n}{4} + s - 1)\ldots(\frac{n}{4} + \frac{s}{2} + 1)(\frac{n}{4})(\frac{n}{4} - 1)\ldots(\frac{n}{4} - \frac{s}{2} + 1)}{(\frac{n}{4} + 1)(\frac{n}{4} + 2)\ldots(\frac{n}{4} + \frac{s}{2})(\frac{n}{4} - \frac{s}{2} - 1)(\frac{n}{4} - \frac{s}{2} - 2)\ldots(\frac{n}{4} - s)}
\end{aligned}$$

Now, applications of Lemma 1 give

$$\begin{aligned}
\frac{A_0}{B_0} &\geq \frac{(\frac{n}{4} + \frac{3s}{4} + \frac{1}{2})^{\frac{s}{2}}}{(\frac{n}{4} + \frac{s}{4} + \frac{1}{2})^{\frac{s}{2}}} \frac{(\frac{n}{4} - \frac{s}{4} + \frac{1}{2})^{\frac{s}{2}}}{(\frac{n}{4} - \frac{3s}{4} - \frac{1}{2})^{\frac{s}{2}}} \\
&\geq \frac{(\frac{n}{4} + \frac{s}{4} + \frac{1}{2})^{s}}{(\frac{n}{4} - \frac{s}{4})^{s}}
\end{aligned} \tag{5}$$

Since $\delta \geq pn$, we have that $s = \delta - \frac{n}{2} \geq (p - \frac{1}{2})n$. Thus, (5) gives

$$\frac{A_0}{B_0} \geq \left(\frac{\frac{n}{4} + \frac{(p-\frac{1}{2})n}{4}}{\frac{n}{4} - \frac{(p-\frac{1}{2})n}{4}}\right)^{(p-\frac{1}{2})n} = \left(\frac{p + \frac{1}{2}}{\frac{3}{2} - p}\right)^{(p-\frac{1}{2})n} \tag{6}$$

Because $n > \frac{\ln(4)}{(p-\frac{1}{2})\ln\left(\frac{p+\frac{1}{2}}{\frac{3}{2}-p}\right)}$, (6) implies that $\frac{A_0}{B_0} > 4$, thus proving (4) for $i = 0$.

We now turn to the recursive step in proving (4) and assume that $A_k > \frac{nB_k}{\frac{n}{4}-k}$, for $0 < k < \frac{n}{4} - 2$. We will show that

$$\frac{A_{k+1}}{A_k} \geq \left(\frac{\frac{n}{4} - k}{\frac{n}{4} - k - 1}\right) \frac{B_{k+1}}{B_k} \tag{7}$$

This will suffice because (7) together with the recursive hypothesis implies that $A_{k+1} \geq \left(\frac{\frac{n}{4}-k}{\frac{n}{4}-k-1}\right)\frac{A_k}{B_k}B_{k+1} > \left(\frac{\frac{n}{4}-k}{\frac{n}{4}-k-1}\right)\frac{n}{\frac{n}{4}-k}B_{k+1} = \frac{n}{\frac{n}{4}-k-1}B_{k+1}$. We have that

$$\frac{A_{k+1}}{A_k} = \frac{\binom{\delta}{\frac{\delta}{2}+k+1}\binom{n-\delta-1}{\frac{n}{2}-\frac{\delta}{2}-k-1}}{\binom{\delta}{\frac{\delta}{2}+k}\binom{n-\delta-1}{\frac{n}{2}-\frac{\delta}{2}-k}} = \frac{\left(\frac{\delta}{2}-k\right)\left(\frac{n}{2}-\frac{\delta}{2}-k\right)}{\left(\frac{\delta}{2}+k+1\right)\left(\frac{n}{2}-\frac{\delta}{2}+k\right)},$$

$$\text{and, } \frac{B_{k+1}}{B_k} = \frac{\binom{\delta}{\frac{n}{4}-k-1}\binom{n-\delta-1}{\frac{n}{4}+k+1}}{\binom{\delta}{\frac{n}{4}-k}\binom{n-\delta-1}{\frac{n}{4}+k}} = \frac{\left(\frac{n}{4}-k\right)\left(\frac{3n}{4}-\delta-k-1\right)}{\left(\delta-\frac{n}{4}+k+1\right)\left(\frac{n}{4}+k+1\right)}.$$

Hence, letting $\delta = \frac{n}{2}+s$, we have that

$$\begin{aligned}
\frac{\left(\frac{A_{k+1}}{A_k}\right)}{\left(\frac{B_{k+1}}{B_k}\right)} &= \frac{\left(\frac{\delta}{2}-k\right)\left(\frac{n}{2}-\frac{\delta}{2}-k\right)\left(\delta-\frac{n}{4}+k+1\right)\left(\frac{n}{4}+k+1\right)}{\left(\frac{n}{4}-k\right)\left(\frac{3n}{4}-\delta-k-1\right)\left(\frac{\delta}{2}+k+1\right)\left(\frac{n}{2}-\frac{\delta}{2}+k\right)} \\
&= \frac{\left(\frac{n}{4}+\frac{s}{2}-k\right)\left(\frac{n}{4}-\frac{s}{2}-k\right)\left(\frac{n}{4}+s+k+1\right)\left(\frac{n}{4}+k+1\right)}{\left(\frac{n}{4}-k\right)\left(\frac{n}{4}-s-k-1\right)\left(\frac{n}{4}+\frac{s}{2}+k+1\right)\left(\frac{n}{4}-\frac{s}{2}+k\right)} \qquad (8)
\end{aligned}$$

Note that in equation (8) we have, $\frac{\left(\frac{n}{4}+\frac{s}{2}-k\right)}{\left(\frac{n}{4}-k\right)} > 1$, $\frac{\left(\frac{n}{4}+s+k+1\right)}{\left(\frac{n}{4}+\frac{s}{2}+k+1\right)} > 1$, $\frac{\left(\frac{n}{4}+k+1\right)}{\left(\frac{n}{4}-\frac{s}{2}+k\right)} > 1$, and in addition because $k < \frac{n}{4}$, it is easy to verify that $\frac{\left(\frac{n}{4}-\frac{s}{2}-k\right)}{\left(\frac{n}{4}-s-k-1\right)} > \frac{\left(\frac{n}{4}-k\right)}{\left(\frac{n}{4}-k-1\right)}$. Now (8) implies (7) which in turn proves (4). This completes the proof of Case 1.

Case 2: $n = 4m$ and $\delta = 2j+1$ for some positive integers m and j.
For $i = 0, 1, \ldots, \frac{n}{4}-2$, let $A_i = n_{(j+i)} = 2\binom{\delta}{j+i}\binom{n-\delta-1}{2m-j-i}$, and as in Case 1, let $B_i = n_{(\frac{n}{4}-i)} = 2\binom{\delta}{m-i}\binom{n-\delta-1}{m+i}$. As in Case 1, we prove by recursion on i that inequality (4) is satisfied for A_i and B_i defined here. Towards this end, let $\delta = \frac{n}{2}+s$ where s is odd. We have that,

$$\begin{aligned}
\frac{A_0}{B_0} &= \frac{(\frac{n}{4})!(\delta-\frac{n}{4})!(\frac{n}{4})!(\frac{3n}{4}-\delta-1)!}{j!(\delta-j)!(\frac{n}{2}-j)!(\frac{n}{2}-\delta+j-1)!} \\
&= \frac{(\frac{n}{4})!(\frac{n}{4}+s)!(\frac{n}{4})!(\frac{n}{4}-s-1)!}{(\frac{n}{4}+\frac{s}{2}-\frac{1}{2})!(\frac{n}{4}+\frac{s}{2}+\frac{1}{2})!(\frac{n}{4}-\frac{s}{2}+\frac{1}{2})!(\frac{n}{4}-\frac{s}{2}-\frac{3}{2})!} \\
&= \frac{(\frac{n}{4}+s)(\frac{n}{4}+s-1)\ldots(\frac{n}{4}+\frac{s}{2}+\frac{3}{2})(\frac{n}{4})(\frac{n}{4}-1)\ldots(\frac{n}{4}-\frac{s}{2}+\frac{3}{2})}{(\frac{n}{4}+\frac{s}{2}-\frac{1}{2})(\frac{n}{4}+\frac{s}{2}-\frac{3}{2})\ldots(\frac{n}{4}+1)(\frac{n}{4}-\frac{s}{2}-\frac{3}{2})(\frac{n}{4}-\frac{s}{2}-\frac{5}{2})\ldots(\frac{n}{4}-s)} \\
&\geq \frac{(\frac{n}{4}+s)(\frac{n}{4}+s-1)\ldots(\frac{n}{4}+\frac{s}{2}+\frac{3}{2})(\frac{n}{4}-1)\ldots(\frac{n}{4}-\frac{s}{2}+\frac{3}{2})}{(\frac{n}{4}+\frac{s}{2}-\frac{1}{2})(\frac{n}{4}+\frac{s}{2}-\frac{3}{2})\ldots(\frac{n}{4}+1)(\frac{n}{4}-\frac{s}{2}-\frac{3}{2})\ldots(\frac{n}{4}-s+1)}\frac{\frac{n}{4}}{(\frac{n}{4}-s)}
\end{aligned}$$

Now, applications of Lemma 1 give

$$\begin{aligned}
\frac{A_0}{B_0} &\geq \frac{(\frac{n}{4}+\frac{3s}{4}+\frac{3}{4})^{(\frac{s}{2}-\frac{1}{2})}}{(\frac{n}{4}+\frac{s}{4}+\frac{1}{4})^{(\frac{s}{2}-\frac{1}{2})}} \frac{(\frac{n}{4}-\frac{s}{4}+\frac{1}{4})^{(\frac{s}{2}-\frac{1}{2})}}{(\frac{n}{4}-\frac{3s}{4}-\frac{1}{4})^{(\frac{s}{2}-\frac{1}{2})}} \frac{\frac{n}{4}}{(\frac{n}{4}-s)} \\
&\geq \frac{(\frac{n}{4}+\frac{s}{4}+\frac{1}{2})^{s-1}}{(\frac{n}{4}-\frac{s}{4})^{s-1}} \frac{\frac{n}{4}}{(\frac{n}{4}-s)} \\
&\geq \frac{(\frac{n}{4}+\frac{s}{4}+\frac{1}{2})^{s}}{(\frac{n}{4}-\frac{s}{4})^{s}}
\end{aligned}$$

This is exactly inequality (5) obtained in proving Case 1. The rest of the proof for Case 2 is similar to that of Case 1 and we omit it.

Case 3: $n \equiv 2 \pmod 4$.
In this case we point out that a proof similar to that in cases 1 and 2 above verifies the result.

Remark: We argue that there was no loss of generality in our assumption at the beginning of the proof of Theorem 7 that $d^+(v) = d^-(v) = \delta(D)$ for each $v \in V(D)$. Let $D^* = (V^*, A(D^*)$ be a directed graph with $d^+(v) \geq \delta(D^*)$, and $d^-(v) \geq \delta(D^*)$ for each $v \in V(D^*)$. Let $v \in V(D^*)$, and, let n_k^* denote the number of equipartitions of $V(D^*)$ into $V(D^*) = X \cup Y$ for which $\deg(v, B(X,Y)) = k$. We can delete some arcs pointed into v and some arcs pointed out of v to get a directed graph $D = (V^*, A(D))$ in which $d^+(v) = d^-(v) = \delta(D^*)$. Now as before let n_k denote the number of equipartitions of $V(D)$ into $V(D) = X \cup Y$ for which $\deg(v, B(X,Y)) = k$. It is clear that $\sum_{k=2}^{q} n_k \geq \sum_{k=2}^{q} n_k^*$ for each q, and that $\sum_{k=\delta-\frac{n}{2}+1}^{\frac{n}{2}} n_k = \sum_{k=\delta-\frac{n}{2}+1}^{\frac{n}{2}} {n_k}^*$ = total number of equipartitions of $V(D^*)$. Hence, the proof above that $T > S$ holds with n_k replaced by n_k^*. ■

We now prove the corollaries of Theorem 7 mentioned in the introduction.

Proof of Corollary 1. If $n \leq 10$ then $\delta(D) > \frac{2}{3}n$ and Theorem 6 implies that D has an anti-directed Hamilton cycle. Hence, assume that $n > 10$, and for given n, let p be the unique real number such that $\frac{1}{2} < p < \frac{3}{4}$ and $n = \frac{\ln(4)}{(p-\frac{1}{2})\ln\left(\frac{p+\frac{1}{2}}{\frac{3}{2}-p}\right)}$. The result follows from Theorem 7 if $\delta(D) > pn$ and since $\delta(D) > \frac{1}{2}n + \sqrt{n\ln(2)}$, it suffices to show that $pn \leq \frac{1}{2}n + \sqrt{n\ln(2)}$. Let $x = p - \frac{1}{2}$ and note that

$0 < x < \frac{1}{4}$. Now, $pn \leq \frac{1}{2}n + \sqrt{n\ln(2)}$ if and only if $xn \leq \sqrt{n\ln(2)}$ if and only if $\sqrt{\frac{\ln(4)}{x\ln\left(\frac{1+x}{1-x}\right)}} \leq \frac{\sqrt{\ln(2)}}{x}$ if and only if $2x \leq \ln(1+x) - \ln(1-x)$. Since $0 < x < \frac{1}{4}$, we have that $\ln(1+x) - \ln(1-x) = \sum_{k=0}^{\infty} \frac{2x^{2k+1}}{2k+1}$ and this completes the proof of Corollary 1. ■

Proof of Corollary 2. For $p = \frac{9}{16}$, $177 < \frac{\ln(4)}{(p-\frac{1}{2})\ln\left(\frac{p+\frac{1}{2}}{\frac{3}{2}-p}\right)} < 178$. Hence, Theorem 7 implies that the corollary is true for all $n \geq 178$. If $n < 178$, $\delta(D) > \frac{9}{16}n$, and, $n \not\equiv 0 \pmod 4$, we can verify that inequality (3) is satisfied by direct computation. If $n < 178$, $\delta(D) > \frac{9}{16}n$, and, $n \equiv 0 \pmod 4$, a use of Theorem 8 that is stronger than its use in deriving the bound S in equation (2) yields that the number of equipartitions of $V(D)$ into $V(D) = X \cup Y$ for which $B(X,Y)$ does not contain a Hamilton cycle is at most

$$S' = n\left(\frac{n_2}{2} + \frac{n_3}{3} + \ldots + \frac{n_{\lfloor \frac{n}{4} \rfloor}}{2\lfloor \frac{n}{4} \rfloor}\right). \tag{9}$$

Direct computation now verifies that $T > S'$. ■

Proof of Corollary 3. If $n \leq 14$ is even and $\delta(D) > \frac{1}{2}n$ then we have that $\delta(D) > \frac{9}{16}n$ and Corollary 2 implies Corollary 3. ■

References

[1] Jørgen Bang-Jensen, Gregory Z. Gutin, *Digraphs: Theory, Algorithms and Applications*, Springer Monographs in Mathematics (2008).

[2] J.A. Bondy and U.S.R. Murty, *Graph Theory*, Springer, GTM 244 (2008).

[3] Arthur H. Busch, Michael S. Jacobson, Timothy Morris, Michael J. Plantholt, and Shailesh K. Tipnis, Improved sufficient conditions for the existence of anti-directed Hamilton cycles in digraphs, submitted.

[4] V. Chvátal, On Hamilton's ideals, *J. Comb. Th. B* **12** (1972), 163-168.

[5] Dirac G.A., Some theorems on abstract graphs, *Proc. London Math. Soc.* 2 (1952), 69-81.

[6] A. Ghouila-Houri, Une condition suffisante d'existence d'un circuit Hamiltonien, *C.R. Acad. Sci. Paris* 156 (1960) 495-497.

[7] D.D. Grant, Anti-directed Hamilton cycles in digraphs, *Ars Combinatoria* 10 (1980) 205-209.

[8] R. Häggkvist and A. Thomason, Oriented Hamilton cycles in digraphs, *J. of Graph Theory*, 19, No. 4, (1995) 471-479.

[9] F. Harary, *Graph Theory*, Addison-Wesley, Reading, MA(1969).

[10] R. Häggkvist and A. Thomason, Oriented hamilton cycles in oriented graphs, *Combinatorics, geometry and probability* Cambridge University Press, Cambridge 1997, MR 1476456.

[11] Mao-cheng Cai, A counterexample to a conjecture of Grant, *Discrete Mathematics* 44 (1983), 111.

[12] , O. Ore, Studies in directed graphs I, *Ann. Math.* **63** (1956), 383-406.

[13] Michael Plantholt and Shailesh Tipnis, Vertex-oriented Hamilton cycles in digraphs, submitted to *Electronic Journal of Combinatorics.*

[14] Sundar Vishwanath, personal communication.