Two phase Stefan-type problem: Regularization near initial data by phase dynamics.

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Abstract

In this paper we investigate the regularizing behavior of two-phase Stefan problem near initial data. The main step in the analysis is to establish that in any given scale, the scaled solution is very close to a Lipschitz profile in space-time. We introduce a new decomposition argument to generalize the preceding ones ([CJK1]-[CJK2] and [CK]) on one-phase free boundary problems.

1 Introduction

Consider $u_0(x) : B_R(0) \to I\!R$ with $R >> 1$ and $u_0 \geq -1$, $|\{u_0 = 0\}| = 0$ and $u_0(x) = -1$ on $\partial B_R(0)$. (See Figure 1.) The two-phase Stefan problem can be formally written as

$$(ST2) \qquad \begin{cases} u_t - \Delta u = 0 & \text{in} \quad \{u > 0\} \cup \{u < 0\} \\ \dfrac{u_t}{|Du^+|} = |Du^+| - |Du^-| & \text{on} \quad \partial\{u > 0\} \\ u(\cdot, 0) = u_0 & \\ u = -1 & \text{on} \quad \partial B_R(0) \end{cases}$$

where u^+ and u^- respectively denotes the positive and negative parts of u, i.e,

$$u^+ := \max(u, 0) \text{ and } u^- := -\min(u, 0).$$

The classical Stefan problem describes the phase transition between solid/liquid or liquid/liquid interface (see [M] and also [OPR].) In our setting, we consider a bounded domain $\Omega_0 \subset B_R(0)$ and the initial data $u_0(x)$ such that $\{u_0 > 0\} = \Omega_0$

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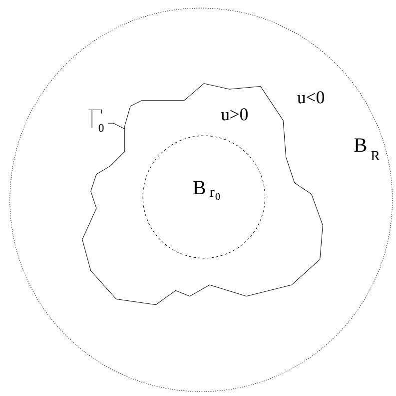

Figure 1: Initial setting of the problem

and $\{u_0 < 0\} = I\!R^n - \Omega_0$. To avoid complicity at the infinity, we consider the problem in the domain $Q = B_R(0) \times [0, \infty)$, with Dirichlet condition

$$u = f(x,t) < 0 \text{ on } \partial B_R(0),$$

where $f(x,t)$ is smooth. In (ST2) we have set $f = -1$ for simplicity. Since our initial data will be only locally Hölder continuous, we employ the notion of viscosity solutions to discuss the evolution of the problem. Viscosity solutions for (ST2) is originally introduced by [ACS1] (also see [CS]). As for existence and uniqueness of viscosity solutions, we refer to [KP].

Note that the second condition of (ST) states that the normal velocity $V_{x,t}$ at each free boundary point $(x,t) \in \partial\{u > 0\}$ is given by

$$V_{x,t} = |Du^+| - |Du^-|(x,t) = (Du^+(x,t) - Du^-(x,t)) \cdot \nu_{x,t},$$

where $\nu_{x,t}$ denotes the spatial unit normal vector of $\partial\{u > 0\}$ at (x,t), pointing inward with respect to the *positive phase* $\{u > 0\}$.

In this paper we investigate the regularizing behavior of the *free boundary* $\partial\{u > 0\}$. Our main result states that when Γ_0 has no sharp corner, then the free boundary immediately regularizes after $t = 0$, and stays regular for a small amount of time. Note that, in general, after some time the free boundary may move away from its initial profile and develop singularities by topological changes, such as merging of two boundary parts. Whether this happens with star-shaped initial data is an open question (see Remark 3.1.)

The well-known results of [ACS1]-[ACS2] states that if a solution as well as its free boundary of $(ST2)$ stays close to a locally Lipschitz profile in a unit space-time neighborhood, then the solution is indeed smooth in a smaller neighborhood. Hence the main step in our analysis is to prove that the free boundary $\partial\{u > 0\}$ stays close to a locally Lipschitz profile over a unit time interval. Indeed proving this step has been the main challenge in the previous work of the authors and Jerison ([CJK1], [CJK2], [CK]) on the study of one-phase free boundary problems. Once this step is established, using the fact that u is a caloric function in *almost Lipschitz* domain, we will have some control over the behavior of re-scaled solutions following the arguments in [CJK1]. Then the appropriate modification of iteration arguments taken in [ACS1]-[ACS2] applies to derive further regularity results (see section 5). In extension of the ideas from one-phase case to our setting, the main obstacle lies in the competition between fluxes of positive and negative phase: to overcome this, we introduce a new decomposition procedure which we explain below.

Before discussing our result in detail, let us introduce precise conditions on the initial data.

(I-a) Ω_0 and u_0 are star-shaped with respect to a ball $B_{r_0}(0) \subset \Omega_0$.

Observe that then the *Lipschitz constant* L of $\partial\Omega_0$ is determined by r_0 and d_0, where $d_0 := \sup\{d(x, B_{r_0}(0) : x \in \partial\Omega_0\}$: i.e., there exist $h = h(r_0)$ and $L = L(\frac{r_0}{d_0})$ such that for any $x_0 \in \partial\Omega_0$, after rotation of coordinates one can represent

$$B_h(x_0) \cap \Omega_0 = \{(x', x_n) : x' \in I\!R^{n-1}, x_n \leq f(x)\} \text{ with } Lipf \leq L. \tag{1.1}$$

For simplicity of the presentation we set $h = 1$.

For a locally Lipschitz domain such as Ω_0, there exist growth rates $0 < \beta < 1 < \alpha$ such that the following holds: Let H be a positive harmonic function in $\Omega_0 \cap B_2(x)$, $x \in \partial\Omega_0$, with Dirichlet condition on $\partial\Omega_0 \cap B_2(x)$, and with value 1 at $x - e_n$. (Here let e_n be the direction of the axis for the Lipschitz graph near x.) Then for $x - se_n \in \Omega_0 \cap B_1(x)$

$$s^\alpha \leq H(x - se_n) \leq s^\beta. \tag{1.2}$$

Now we precisely describe the range of the Lipschitz constant L of Ω_0.

(I-b) $L < L_n$ for a sufficiently small dimensional constant L_n so that

$$5/6 \leq \beta < \alpha \leq 7/6.$$

The remaining conditions are on the regularity of u_0.

(I-c) $-N_0 \leq \Delta u_0 \leq N_0$ in $\Omega_0 \cup (B_R(0) - \Omega_0)$,

(I-d) For $x \in \partial\Omega_0$, $e_n = x/|x|$ and small $s > 0$ (for $0 < s < 1/10$),

$$|Du_0(x \pm se_n)| \geq Cs^{\alpha-1}.$$

Note that (I-c) and (I-d) holds for u_0 which is smooth in its positive and negative phases and is harmonic near the initial free boundary: i.e., $-\Delta u_0 = 0$ in the set $(\{u_0 > 0\} \cup \{u_0 < 0\}) \cap \{x : d(x, \partial\Omega_0) \leq 1\}$.

For a function $u(x,t) : I\!R^n \times [0,\infty) \to I\!R$, let us denote

$$\Omega(u) := \{u > 0\}, \quad \Omega_t(u) := \{u(\cdot, t) > 0\}$$

and

$$\Gamma(u) := \partial\{u > 0\}, \quad \Gamma_t(u) := \partial\{u(\cdot, t) > 0\}.$$

Since $\Gamma_0 = \partial\{u(\cdot, 0) < 0\}$ in our setting, the property is preserved for later times, i.e.,

$$\Gamma_t(u) = \partial\{u(\cdot, t) > 0\} = \partial\{u(\cdot, t) < 0\} \text{ for all } t > 0$$

(see [RB], [GZ], and [KP]).

In the one-phase case the problem can be written as follows:

$$(ST1) \qquad \begin{cases} u_t - \Delta u = 0 & \text{in} \quad \{u > 0\} \\ \dfrac{u_t}{|Du|} = |Du| & \text{on} \quad \partial\{u > 0\} \\ u(\cdot, 0) = u_0^+. \end{cases}$$

In [CK] the following has been proved for (ST1).

Theorem 1.1. *([CK] Theorem 0.1.) Suppose u is a solution of $(ST1)$ in $B_2(0) \times [0,1]$, $0 \in \Gamma_0(u)$, with the initial data $u_0 \geq 0$ satisfying (I-b), (I-c) and (I-d) in $B_2(0)$. Suppose $u_0(-e_n) = 1$. If $\sup_{B_2(0)\times[0,1]} u \leq M_0$, then there exists a small $s > 0$ depending on N_0, M_0 and n such that the free boundary $\Gamma_t(u)$ becomes smooth and averages out in $B_s(0)$. More precisely,*

(a) The free boundary $\Gamma_t(u)$ is C^1 and is a Lipschitz graph with respect to e_n with Lipschitz constant $L' < L_n$ in $B_s(0) \subset I\!R^n$.

(b) The spatial normal of $\Gamma_t(u)$ is continuous in space and time, in $B_s(0)$.

(c) If $x \in \Gamma_0(u) \cap B_s(0)$ and $x + de_n \in \Gamma_t(u) \cap B_s(0)$, then

$$C^{-1}\frac{u(x - de_n, 0)}{d} \leq |Du(x + de_n, t)| = V_{x+de_n,t} \leq C\frac{u(x - de_n, 0)}{d}$$

where C depends on n and M_0. Hence

$$\frac{d}{t} \sim |Du(x + de_n, t)| \sim \frac{u(x - de_n, 0)}{d}.$$

Theorem 1.1 states that the free boundary regularizes in space, in a scale proportional to the distance it has traveled. Note that the regularity results hold up to the initial time and all the regularity assumptions are imposed only on the initial data.

Our aim in this paper is to extend the above theorem to the two-phase case. Here the intuition is rather straightforward, based on the previous results. There are two cases:

(a) One of the phases has much bigger flux than the other: in this case one-phase like phenomena (regularization by the dominant phase proportional to the distance the free oundary traveled) is expected.

(b) Both phases are in balance: in this case one expects regularization due to competition between two phases, resulting in Lipschitz-like behavior over time.

The difficulty in making above heuristics rigorous lies in introducing a proper "sorting" procedure which divides the cases (a) and (b) in a given scale. To enable such procedure, it is essential to show Harnack-type inequalities for solutions of (ST2) in both cases, ensuring that the behavior of solutions can be localized in a proper time-space scale.

To state the main result, we introduce one more notation.

- For $x_0 \in \Gamma_0 = \Gamma_0(u)$ and $e_n := x_0/|x_0|$, define

$$R^+(x_0,d) := \frac{u^+(x_0 - de_n, 0)}{u^-(x_0 + de_n, 0)}, \quad R^-(x_0,d) := \frac{u^-(x_0 + de_n, 0)}{u^+(x_0 - de_n, 0)}$$

and

$$t(x_0,d) := \min[\frac{d^2}{u^+(x_0 - de_n, 0)}, \frac{d^2}{u^-(x_0 + de_n, 0)}].$$

Theorem 1.2 (Main Theorem)**.** *Suppose u is a solution of (ST2) with initial data u_0 satisfying (Ia)-(Id) with $\Omega_0(u) \subset B_2(0)$. Then there exists a constant d_0 depending on the dimension n and N_0 such that the following holds. If $x_0 \in \Gamma_0(u)$ and $d \leq d_0$, then $\Gamma(u)$ is a Lipschitz graph in space-time in the region $B_{2d}(x_0) \times [t(x_0,d)/2, t(x_0,d)]$ with $\Gamma(u)$ intersecting with $B_d(x_0) \times [t(x_0,d)/2, t(x_0,d)]$. Further, there exists a positive dimensional constant M such that the following holds.*

(a) If $R^+(x_0,d) \geq M$, then

$$|Du^+|(x,t) \sim \frac{u^+(x_0 - de_n, 0)}{d}$$

in $B_{2d}(x_0) \times [\frac{t(x_0,d)}{2}, t(x_0,d)]$ and

$$V_{x,t} \sim \frac{u^+(x_0 - de_n, 0)}{d}$$

on $\Gamma(u) \cap (B_{2d}(x_0) \times [\frac{t(x_0,d)}{2}, t(x_0,d)])$*.*

The parallel statements hold for u^- *if* $R^-(x_0,d) \geq M$*.*

(b) *If* $R^+(d), R^-(d) \leq M$*, then*

$$|Du^{\pm}|(x,t) \sim \frac{u^+(x_0 - de_n, 0)}{d} \sim \frac{u^-(x_0 + de_n, 0)}{d}$$

in $B_{2d}(x_0) \times [t(x_0,d)/2, t(x_0,d)]$*.*

Remark 1.3. *1. Note that in the first case,* $t(x_0,d)$ *indeed is comparable to the time that* $\Gamma(u)$ *has traveled from* x_0 *to* $x_0 \pm de_n$*, and thus we can say that the free boundary regularizes in a scale proportional to the distance it has traveled.*

2. Our result extends to the case where the star-shaped condition (I-a) is replaced by

(I-a)' Ω_0 *is locally Lipschitz with a sufficiently small Lipschitz constant.*

We discuss the difference in the proof in this case, in section 6.

Let us finish this section with an outline of the paper. In section 2 we introduce preliminary results and notations, to be used in the paper. In section 3 we prove some properties on the evolution of solutions of (ST2) with star-shaped data. In addition to Harnack inequalities, we show that the solution stays near the star-shaped profile for a unit time (Lemma 3.1), which in turn yields that the solution stays very close to harmonic functions (Lemma 3.6). Making use of the results in section 3, in section 4 we perform a decomposition procedure to show that for a unit time all parts of the free boundary stay close to Lipschitz profiles, regardless of the local dynamics between the phases. This completes our main step in the analysis. In section 5 we describe the procedure leading to further regularization, pointing out the main difference between the previous results. In section 6 we discuss a generalized proof for the corresponding regularization result (Theorem6.1) when the star-shapedness of the initial data (I-a) is replaced by a local version (I-a)'.

2 Preliminary lemmas and notations

We introduce some notations.

• For $x \in I\!R^n$, denote $x = (x', x_n) \in I\!R^{n-1} \times I\!R$ where $x_n = x \cdot e_n$.

• Let $B_r(x)$ be the space ball of radius r, centered at x.

• Let $Q_r := B_r(0) \times [-r^2, r^2]$ be the parabolic cube and let $K_r := B_r(0) \times [-r, r]$ be the hyperbolic cube.

• A caloric function in $\Omega \cap Q_r$ will denote a nonnegative solution of the heat equation, vanishing along the lateral boundary of Ω.

• For $x_0 \in \Gamma_0$ and $e_n = x_0/|x_0|$, define

$$t(x_0, d) := \min[\frac{d^2}{u^+(x_0 - de_n, 0)}, \frac{d^2}{u^-(x_0 + de_n, 0)}].$$

• Given $\epsilon > 0$, a function w is called ϵ-monotone in the direction τ if

$$u(p + \lambda\tau) \geq u(p) \text{ for any } \lambda \geq \epsilon.$$

• $W_x(\theta^x, e)$ and $W_t(\theta^t, \nu)$ with $e \in I\!R^n$ and $\nu \in span(e_n, e_t)$ respectively denote a spatial circular cone of aperture $2\theta^x$ and axis in the direction of e, and a two-dimensional space-time cone in (e_n, e_t) plane of aperture $2\theta^t$ and axis in the direction of ν.

• w is ϵ-monotone in a cone of directions if w is ϵ-monotone in every direction in the cone.

• C is called an universal constant if it depends only on the dimension n and the regularity constant N_0 of u_0.

The first lemma is a direct consequence of the interior Harnack inequalities proved in [C-C].

Lemma 2.1 ([C-C]). *Suppose $w(x) : I\!R^n \to I\!R$ has bounded Laplacian. Then w is Hölder continuous with its constant depending on the Laplacian bound.*

Lemma 2.2 ([FGS1], Theorem 3). *Let Ω be a domain in $I\!R^n \times I\!R$ such that $(0,0)$ is on its lateral boundary. Suppose Ω is a $Lip^{1,1/2}$ domain, i.e.,*

$$\Omega = \{(x', x_n, t) : |x'| < 1, |x_n| < 2L, |t| < 1, x_n \leq f(x', t)\},$$

where f satisfies $|f(x', t) - f(y', s)| \leq L(|x' - y'| + |t - s|^{1/2}$.) If u is a caloric function in Ω, then there exists $C = C(n, L)$, where L is the Lipschitz constant for Ω, such that

$$\frac{u(x,t)}{v(x,t)} \leq C \frac{u(-Le_n, 1/2)}{v(-Le_n, -1/2)}.$$

for $(x,t) \in Q_{1/2}$.

Lemma 2.3 ([ACS1], Theorem 1). *Let Ω be a Lipschitz domain in $I\!R^n \times I\!R$, i.e.,*

$$Q_1 \cap \Omega = Q_1 \cap \{(x,t) : x_n \leq f(x', t)\},$$

where f satisfies $|f(x,t) - f(y,s)| \leq L(|x - y| + |t - s|)$. Let u be a caloric function in $Q_1 \cap \Omega$ with $(0,0) \in \partial\Omega$ and $u(-e_n, 0) = m > 0$ and $\sup_{Q_1} u = M$. Then there exists a constant C, depending only on n, L, $\frac{m}{M}$ such that

$$u(x, t + \rho^2) \leq C u(x, t - \rho^2)$$

for all $(x,t) \in Q_{1/2} \cap \Omega$ and for $0 \leq \rho \leq d_{x,t}$.

Lemma 2.4 ([ACS1], Lemma 5)**.** *Let u and Ω be as in Lemma 2.3. Then there exist $a, \delta > 0$ depending only on n, L, $\frac{m}{M}$ such that*

$$w_+ := u + u^{1+a} \text{ and } w_- := u - u^{1+a}$$

are subharmonic and superharmonic, respectively, in $Q_\delta \cap \Omega \cap \{t = 0\}$.

Next we state several properties of harmonic functions:

Lemma 2.5 ([D])**.** *Let u_1, u_2 be two nonnegative harmonic functions in a domain D of $I\!R^n$ of the form*

$$D = \{(x', x_n) \in I\!R^{n-1} \times I\!R : |x'| < 2, |x_n| < 2L, x_n > f(x')\}$$

with f a Lipschitz function with constant less than L and $f(0) = 0$. Assume further that $u_1 = u_2 = 0$ along the graph of f. Then in

$$D_{1/2} = \{|x'| < 1, |x_n| < L, x_n > f(x')\}$$

we have

$$0 < C_1 \leq \frac{u_1(x', x_n)}{u_2(x', x_n)} \cdot \frac{u_2(0, L)}{u_1(0, L)} \leq C_2$$

with C_1, C_2 depending only on L.

Lemma 2.6 ([JK])**.** *Let D, u_1 and u_2 be as in Lemma 2.5. Assume further that*

$$\frac{u_1(0, L/2)}{u_2(0, L/2)} = 1.$$

Then, $u_1(x', x_n)/u_2(x', x_n)$ is Hölder continuous in $\bar{D}_{1/2}$ for some coefficient α, both α and the C^α norm of u_1/u_2 depending only on L.

Lemma 2.7 ([C2])**.** *Let u be as in Lemma 2.5. Then there exists $c > 0$ depending only on L such that for $0 < d < c$, $\frac{\partial}{\partial x_n} u(0, d) \geq 0$ and*

$$C_1 \frac{u(0, d)}{d} \leq \frac{\partial u}{\partial x_n}(0, d) \leq C_2 \frac{u(0, d)}{d}$$

where $C_i = C_i(M)$.

Lemma 2.8 ([JK], Lemma 4.1)**.** *Let Ω be Lipschitz domain contained in $B_{10}(0)$. There exists a dimensional constant $\beta_n > 0$ such that for any $\zeta \in \partial\Omega$, $0 < 2r < 1$ and positive harmonic function u in $\Omega \cap B_{2r}(\zeta)$, if u vanishes continuously on $B_{2r}(\zeta) \cap \partial\Omega$, then for $x \in \Omega \cap B_r(\zeta)$,*

$$u(x) \leq C\left(\frac{|x - \zeta|}{r}\right)^{\beta_n} \sup\{u(y) : y \in \partial B_{2r}(\zeta) \cap \Omega\}$$

where C depends only on the Lipchitz *constants of Ω.*

Next, we point out that we use the notion of viscosity solutions for our investigation. When $\{u_0 = 0\}$ is of zero Lebesgue measure, it was proved in [KP] that the viscosity solution of $(ST2)$ is unique and coincides with the usual weak solutions. (See [KP] for the definition as well as other properties of viscosity solutions.) Below we state important properties of viscosity solutions.

Lemma 2.9. *Suppose u is a viscosity solution of (ST2). Then*

(a) u is caloric in its positive and negative phases.

(b) $-u$ is also a viscosity solution of (ST2) with boundary data $-g$.

(c) $u^+ = \max(u, 0)$ (or $u^- = -\min(u, 0)$) is a viscosity subsolution (or supersolution) of (ST2) with initial data u_0^+ (or u_0^-).

Lemma 2.10 (Comparison principle, [KP])**.** *Let u, v be respectively viscosity sub- and supersolutions of (ST2) in $D \times (0, T) \subset Q$ with initial data $u_0 \prec v_0$ in D. If $u \le v$ on ∂D and $u < v$ on $\partial D \cap \bar{\Omega}(u)$ for $0 \le t < T$, then $u(\cdot, t) \prec v(\cdot, t)$ in D for $t \in [0, T)$.*

Below we state a distance estimate for the free boundary and Harnack inequality for the one-phase solution u of (ST1).

Lemma 2.11 ([CK], Lemma 2.2)**.** *Let u be given as in Theorem 1.1. There exists $t_0 = t_0(N_0, M_0, n) > 0$ such that if $x_0 \in \Gamma_0$ and $t \le t_0$, then*

$$\frac{1}{C} t^{1/(2-\alpha)} \le d(x_0, t) \le C t^{1/(2-\beta)} \tag{2.1}$$

where α and β are given in (1.2), C depends on N_0, M_0 and n, and $d(x_0, t)$ denotes the distance that Γ moved from the point x_0 during the time t, i.e.,

$$d(x_0, t) := \sup\{d : u(x_0 + de_n, t) > 0\}.$$

Lemma 2.12 ([CK], Lemma 2.3)**.** *Let u be given as in Theorem 1.1. There exists d_0 depending on N_0, M_0 and n such that if $x_0 \in \Gamma_0$ and $d \le d_0$, then*

$$u(x_0 - de_n, t) \le Cu(x_0 - de_n, 0) \text{ for } 0 \le t \le t(x_0, d)$$

where C depends on N_0, M_0 and n.

The following monotonicity formula by Alt-Caffarelli-Friedman prevents the scenario that both phases compete with large pressure in our problem.

Lemma 2.13 ([ACF])**.** *Let h_+ and h_- be nonnegative continuous functions in $B_1(0)$ such that $\Delta h_\pm \ge 0$ and $h_+ \cdot h_- = 0$ in $B_1(0)$. Then the functional*

$$\phi(r) = \frac{1}{r^4} \int_{B_r(0)} \frac{|\nabla h_+|^2}{|x|^{n-2}} dx \int_{B_r(0)} \frac{|\nabla h_-|^2}{|x|^{n-2}} dx$$

is monotone increasing in r, $0 < r < 1$.

Corollary 2.14. *Let $\partial\Omega_0 \subset I\!R^n$ be star-shaped with respect to $B_1(0) \subset \Omega_0$ and suppose $B_{4/3}(0) \subset \Omega_0 \subset B_{5/3}(0)$. Let h_+ be the harmonic function in $\Omega_0 - B_1(0)$ with boundary values $h_+ = 0$ on $\partial\Omega_0$, and $h_+ = 1$ on $\partial B_1(0)$. Let h_- be the harmonic function in $B_2(0) - \Omega_0$ with boundary values $h_- = 0$ on $\partial\Omega_0$, and $h_- = 1$ on $\partial B_2(0)$. Then there exists a sufficiently large dimensional constant $M > 0$ such that*

$$\frac{h_+(x_0 - re_n)}{r} \geq M \text{ implies } \frac{h_-(x_0 + re_n)}{r} \leq 1$$

for $x_0 \in \partial\Omega_0$, $e_n = x/|x|$ and $0 \leq r \leq 1/6$.

Proof. It follows from Lemma 2.13 since

$$\begin{aligned}
&\left(\frac{h_+(x_0 - re_n)}{r} \cdot \frac{h_-(x_0 + re_n)}{r}\right)^2 \\
\sim\quad & \frac{1}{(2r)^4}\int_{B_{r/2}(x_0 - re_n)} \frac{|\nabla h_+|^2}{|x - x_0|^{n-2}}dx \cdot \int_{B_{r/2}(x_0 + re_n)} \frac{|\nabla h_-|^2}{|x - x_0|^{n-2}}dx \\
\leq\quad & \frac{1}{(2r)^4}\int_{B_{2r}(x_0)} \frac{|\nabla h_+|^2}{|x - x_0|^{n-2}}dx \cdot \int_{B_{2r}(x_0)} \frac{|\nabla h_-|^2}{|x - x_0|^{n-2}}dx \\
=\quad & \phi(2r) \leq \phi(1/3) \leq C_n.
\end{aligned}$$

□

3 Properties of solutions with star-shaped initial data

Lemma 3.1. *If Ω_0 and u_0 are star-shaped with respect to the ball $B_{r_0}(0) \subset \Omega_0$, then $\Omega_t(u)$ and $u(\cdot, t)$ stays σ-close to star-shaped for all $0 \leq t \leq \frac{1}{3}\sigma^{1/5}$. (See Figure 2)*

Proof. 1. Observe that, for any $a > 0$, the parabolic scaling $(x, t) \to (ax, a^2t)$ preserves both the heat operator and the boundary motion law in $(ST2)$. Therefore, for any $\sigma > 0$ the function

$$u_1(x, t) := u((1 + \sigma)(x - x_0) + x_0, (1 + \sigma)^2 t)$$

is also a viscosity solution of (ST2) with corresponding initial data.

2. Choose $x_0 \in B_{r_0}(0)$. Take a small $c_0 > 0$ such that $B_{r_0+c_0}(0) \subset \Omega_0$. We claim that for $0 \leq \delta \leq \sigma^{6/5}$,

$$u_1(x, 0) \leq u(x, \delta) \text{ in } B_R(0) - B_{r_0+c_0}(0) \tag{3.1}$$

if σ is small enough. To show (3.1), let us introduce another function

$$\tilde{u}(x,0) := u((1+\frac{\sigma}{2})(x-x_0)+x_0,0).$$

Also let v^- be the solution of (ST1) with initial data u_0^-, and with $v^- = 1$ on $\partial B_R(0)$. Then by comparison, $-v^- \leq u$ and $B_R(0) - \Omega_t(-v^-) \subset \Omega_t(u)$. Hence by Lemma 2.11 applied for $-v^-$,

$$\Omega_0(\tilde{u}) \subset \Omega_t(u) \text{ for } 0 \leq t \leq \sigma^{7/6}.$$

Moreover, due to our assumption,

$$\tilde{u}(x,0) \leq u_0(x).$$

Therefore, the maximum principle for caloric functions implies

$$w(x,t) \leq u(x,t)$$

where w solves the heat equation in the cylindrical domain $D = \Omega_0(\tilde{u}) \times [0,\sigma^{7/6}]$ with initial data $\tilde{u}(x,0)$ and zero boundary data on $\partial\Omega_0(\tilde{u}) \times [0,\sigma^{7/6}]$.

Now w_t solves the heat equation in D,

$$w_t = \Delta w \geq -C \text{ at } t=0, \text{ and } w_t = 0 \text{ on } \partial\Omega_0(\tilde{u}).$$

Therefore we conclude that $w_t \geq -C$ in D. In particular

$$w(x,\delta) \geq \tilde{u}(x,0) - C\delta. \tag{3.2}$$

Next we compare $u_1(x,0)$ with $w(x,\delta)$. Observe that for $x \in B_R(0) - B_{r_0+c_0}(0)$,

$$\begin{aligned} u_1(x,0) &= \tilde{u}(x,0) + \textstyle\int_{\sigma/2}^{\sigma}((x-x_0)\cdot Du((1+s)(x-x_0)+x_0,0))ds \\ &\leq \tilde{u}(x,0) - c_0\sigma^{7/6} \\ &\leq \tilde{u}(x,0) - C\sigma^{6/5} \\ &\leq w(x,\delta) \leq u(x,\delta) \end{aligned}$$

for $0 \leq \delta \leq \sigma^{6/5}$, where the first inequality follows from our assumption (I-d) on u_0, the second inequality follows if σ is sufficiently small, and the third inequality follows from (3.2). Hence we conclude (3.1).

3. Our goal is to prove that for $0 \leq \delta \leq \sigma^{6/5}$,

$$u_1(x,t) \leq u_2(x,t) := u(x,t+\delta) \tag{3.3}$$

in $(B_R(0) - B_{r_0+c_0}(0)) \times [0,\sigma^{1/5}]$. Note that the inequality holds at $t=0$ by step 2. However, we needs a bit more arguments since we do not know yet whether the lateral boundary data on $\partial B_{r_0+c_0}(0)$ is properly ordered.

Suppose

$$\Omega(u_1) \subset \Omega(u) \text{ for } 0 \leq t \leq t_0$$

and $\Omega(u_1)$ contacts $\partial\Omega(u)$ for the first time at $t = t_0$. Observe then that

$$f(x,t) := u(x, t+\delta) - u_1(x,t)$$

solves the heat equation in $\Omega(u_1)$ with nonnegative boundary data for $0 \leq t \leq t_0$, with

$$f(x,0) \geq 0 \text{ in } B_R(0) - B_{r_0+c_0}(0).$$

Indeed following the computation given above, it follows that

$$f(x,0) \geq c_0\sigma \text{ in } B_{r_0+c_0}(0) - B_{r_0+\frac{c_0}{2}}(0).$$

On the other hand, due to the fact that $w_t \geq -C$ and $\delta \leq \sigma^{6/5}$, we have

$$f(x,0) \geq (w(x,\delta) - w(x,0)) + (w(x,0) - u_1(x,0)) \geq -C\sigma^{6/5} \text{ in } B_{r_0+\frac{c_0}{2}}(0).$$

Therefore we have

$$f(x,t) > 0 \text{ on } \partial B_{r_0+c_0}(0) \times [0, t_0]$$

if $t_0 << 1$. But then this contradicts Theorem 2.10 applied to the region $(B_R(0) - B_{r_0+c_0}(0)) \times [0, t_0]$.

4. From (3.3) of step 3, we obtain

$$u((1+\sigma)(x - x_0) + x_0, (1+\sigma)^2 t) \leq u(x, t+\delta) \tag{3.4}$$

in $(B_R(0) - B_{r_0+c_0}(0)) \times [0, \sigma^{1/5}]$ for any $x_0 \in B_{r_0}(0)$, as long as σ and δ are sufficiently small and satisfy $0 \leq \delta \leq \sigma^{6/5}$. As a result, for $0 \leq t \leq \frac{1}{3}\sigma^{1/5}$, we can choose $\delta = \sigma(2+\sigma)t \leq \sigma^{6/5}$ such that

$$(1+\sigma)^2 t = t + \delta.$$

It follows then from (3.4) that the function $u(\cdot, t)$ is σ-monotone with respect to the cone of directions W_x in $(B_R(0) - B_{r_0+c_0}(0))$ for $t \in [0, \frac{1}{3}\sigma^{1/5}]$.

$$\text{(Here} \quad W_x = \{\nu \in S^n : \nu = \frac{x - x_0}{|x - x_0|} \text{ for some } x_0 \in B_{r_0}(0)\}.)$$

□

Remark 3.2. *Observe that, due to $(I - b)$, we have for $x \in \Gamma_0$*

$$t(x,d) := \min[\frac{d^2}{u^+(x - de_n, 0)}, \frac{d^2}{u^-(x + de_n, 0)}] \in [d^{7/6}, d^{5/6}] << d^{4/5} \tag{3.5}$$

where $t(x,d)$ is the time it takes for the free boundary to regularize in $B_d(0)$. Therefore, $u(\cdot, t)$ is at least d^4-monotone with respect to W_x in $(B_R(0) - B_{r_0+c_0}(0))$ for $0 \leq t \leq t(x_0, d)$. This property will serve as a basis for our regularization argument in section 3.

Lemma 3.3. **(Harnack at** $t = 0$**)** *Let* $x \in \Gamma_0$*, then for all* $s > 0$ *and for* $0 \leq t \leq t(x, s)$ *we have*

$$u^+(x - se_n, t) \leq C_1 u^+(x - se_n, 0)$$

and

$$u^-(x + se_n, t) \leq C_1 u^-(x + se_n, 0)$$

where $e_n = x/|x|$*.*

Proof. Let v^+ solve the one-phase Stefan problem (ST1) with initial data $v_0^+(x) = u_0^+(x)$. Then v^+ is also a solution of (ST2) with $u_0(x) \leq v_0^+(x)$, and thus by Theorem 2.10 we have

$$u(x,t) \leq v^+(x,t).$$

Therefore it follows from one-phase Harnack inequality applied for $v^+(x,t)$ that

$$u^+(x - se_n, t) \leq v^+(x - se_n, t) \leq C_1 v^+(x - se_n, 0) = C_1 u(x - e_n, 0)$$

for $0 \leq t \leq t_0$ where $t_0 = s^2/u(x - se_n, 0) \geq t(x, s)$.

As for $u^-(x,t)$, we compare u^- with the solution v^- of (ST1) with initial data $v_0^-(x) = u_0^-(x)$ and with boundary data $v^- = 1$ on $\partial B_R(0)$. The rest of the argument is parallel to above.

□

Lemma 3.4. **(Backward Harnack at** $t = 0$**)** *Let* $x \in \Gamma_0$ *and let* $s > 0$*. Then for* $0 \leq t \leq t(x, s)$

$$u^+(x - se_n, 0) \leq C_1 u^+(x - se_n, t)$$

and

$$u^-(x + se_n, 0) \leq C_1 u^-(x + se_n, t)$$

Proof. We will only show the lemma for u^+. The other part follows by a parallel argument. Let v^- solve (ST1) with initial data u_0^- and with boundary data 1 on $\partial B_R(0)$. Then $-v^-$ is also a solution of (ST2) with $-v_0^- \leq u_0$, and thus by Theorem 2.10, $-v^- \leq u$ and

$$\{v^- = 0\} \subset \{u \geq 0\}.$$

Note that $\Omega(v^-)$ moves according to the one-phase dynamics, which has been studied in detail by [CK2]. In particular we know that $\Omega(v^-)$ will be Lipschitz at each time. Moreover, for a boundary point $(x,t) \in \Gamma(v^-)$ and $d := \text{dist}(x, \Gamma_0(v^-))$, the normal velocity $V_{x,t}$ satisfies

$$V_{x,t} = |Dv^-(x,t)| \sim \frac{v^-(x + 2de_n, 0)}{2d} \leq d^{\beta-1} \leq t^{\frac{\beta-1}{2-\alpha}} \tag{3.6}$$

where the last inequality follows from Lemma 2.11. Let $v^*(x,t)$ solve the heat equation in $\{v^- = 0\}$ with initial data $u_0(x)$ and boundary data 0 on $\partial\{v^- = 0\}$. Since

$$\Omega(v^*) = \{v^- = 0\} \subset \{u \geq 0\},$$

we have $v^*(x,t)\leq u(x,t)$. Moreover, for any given $t>0$, $\tilde{v}^-(x,s):=v^-(\sqrt{t}x,ts)$ satisfies the assumptions of Lemma 2.4. Thus it follows that $v^-(\cdot,t)$ is t^a-close to a harmonic function in $B_{\sqrt{t}}(x)$ for some $a>0$, where $x\in\Gamma_0$. Moreover, due to the assumption on the initial data, $(v^*)_t=\Delta v^*\geq -C$ at $t=0$. Also on $\Gamma(v^*)$,

$$(v^*)_t/|Dv^*|=-(v^-)_t/|Dv^-|=-|Dv^-|\geq -t^{\frac{\beta-1}{2-\alpha}}$$

where the last inequality follows from (3.6). Since $\Omega(v^*)$ is Lipschitz and $\Gamma_t(v^*)=\Gamma_t(v^-)$ is regularized in time (Theorem 1.1), (3.6) also holds for $|Dv^*|$.

Hence on $\Gamma(v^*)$,

$$(v^*)_t=-|Dv^-||Dv^*|\geq -t^{\frac{2(\beta-1)}{2-\alpha}}>-t^{-2/5}.$$

Since $(v^*)_t$ solves a heat equation in $\Omega(v^*)$, it follows that for $x\in\Gamma_0$,

$$(v^*)_t\geq -t^{-2/5}\text{ in } B_{\sqrt{t}/2}(x-\sqrt{t}e_n)\times[0,t]. \tag{3.7}$$

Then since $v^*(x-\sqrt{t}e_n,0)\geq(\sqrt{t})^\alpha\geq(\sqrt{t})^{7/6}=t^{7/12}$,

$$\begin{aligned} u^+(x-\sqrt{t}e_n,0)=v^*(x-\sqrt{t}e_n,0) &\leq C_1v^*(x-\sqrt{t}e_n,t)\\ &\leq C_1u^+(x-\sqrt{t}e_n,t)\end{aligned}$$

where the first inequality follows from (3.7). Since $\Gamma(v^*)=\Gamma(v^-)$ is Lipschitz in a parabolic scaling, v^* is almost harmonic. Hence $v^*(\cdot,t)$ is bigger than the harmonic function $\omega^t(x)$ in $\Omega_t(v^*)\cap B_{\sqrt{t}}(x)$ with its value

$$\omega^t(x-\sqrt{t}e_n)=(C_1)^{-1}u^+(x-\sqrt{t}e_n,0).$$

Note that if $0\leq t\leq t(x,s)$, then $s<\sqrt{t}$. Hence for $0\leq t\leq t(x,s)$,

$$C_1u^+(x-se_n,t)\geq C_1v^*(x-se_n,t)\geq C_1\omega^t(x-se_n)\geq Cu^+(x-se_n,0),$$

where the last inequality follows since the one-phase result implies a power law on the movement of $\Gamma(v^-)=\Gamma(v^*)$ (see Lemma 2.5 of [CJK1]), and this yields a bound on $u^+(x-se_n,0)/\omega^t(x-se_n)$.

Similar arguments apply to u^-, if we consider the function v^+ solving (ST1) with initial data u_0^+, and the function $v^\star$ solving the heat equation in $\{v^+=0\}$ with initial data u_0 and with boundary data 0 on $\Gamma(v^+)$ and -1 on $\partial B_R(0)$.

$\square$

Lemma 3.5. **(Distance estimate at** $t=0$**)** *Let $x\in\Gamma_0$ and let s be a sufficiently small positive constant. If*

$$\frac{|u^+(x-se_n,0)|}{s}\leq m\quad\textit{and}\quad\frac{|u^-(x+se_n,0)|}{s}\leq m,$$

then for $t\in[0,\frac{s}{m}]$,

$$d(x,t)=\sup\{r: x+re_n \textit{ or } x-re_n\in\Gamma_t(u)\}\leq s.$$

Proof. Let v^+ solve (ST1) with initial data u_0^+, and let v^- solve (ST1) with initial data u_0^- and with $v^- = 1$ on $\partial B_R(0)$. Then by comparison, $-v^- \leq u \leq v^+$ and the lemma follows from the one-phase result Theorem 1.1. $\square$

In the following lemma, we approximate our solution by harmonic functions.

Lemma 3.6. (Spatial regularity in the whole domain) *For $x_0 \in \Gamma_0$ and $r > 0$, there exists a function $\omega(x,t) := \omega^+(x,t) - \omega^-(x,t)$ such that*

(a) $\omega(\cdot,t)$ is harmonic in its positive and negative phase in $(1+r)\Omega_t(u) - (1-r)\Omega_t(u)$, and $\Omega(\omega^+)$, $\Omega(\omega^-)$ are star-shaped;

(b) For a dimensional constant $C > 0$, we have

$$\omega^+(x,t) \leq u^+(x,t) \leq C\omega^+((1-r^{5/4})x,t)$$

and

$$\omega^-(x,t) \leq u^-(x,t) \leq C\omega^-((1+r^{5/4})x,t)$$

in $B_r(x_0) \times [r^2, t(x_0,r)]$.

Remark 3.7. *Note that we do not know yet whether the solution is close to a Lipschitz graph in time. Also, note that $t(x_0,r) \geq r^{7/6} \gg r^2$, and $\partial\{\omega^+ > 0\}$ need not be $\partial\{\omega^- > 0\}$.*

Proof. 1. We will only show the lemma for u^+. Let $\Gamma^\star$ be the free boundary obtained from the one-phase problem (ST1) with the initial data u_0^+, and let Ω^* be the region bounded by $\Gamma^\star$. Let v_1 solve the heat equation in Ω^* and in $B_R(0)\times[0,1]-\Omega^*$, with initial data u_0 and with $v_1 = -1$ on $\partial B_R(0)$. Similarly, we define v_2, whose free boundary is obtained from the one-phase solution with initial data u_0^-. Then by comparison,

$$v_2 \leq u \leq v_1.$$

Hence the free boundary of u is trapped between the free boundaries of v_1 and v_2. Also, since one-phase versions v_1 and v_2 behave nicely, we have those functions almost harmonic up to r-neighborhood of their free boundaries for $r^2/2 \leq t \leq r^2$. Next note that the range of t is $0 \leq t \leq t(x_0,r)$, and thus both of the sets $\Gamma_t(v_1)$ and $\Gamma_t(v_2)$ are within distance r of $\Gamma_0(u)$ in $B_r(x_0)$. In particular, using the one-phase result, i.e., arguing as in Lemmas 2.1 and 2.3 of [CK], we obtain

$$v_2(x_0 - 2re_n, t) \sim u_0(x_0 - 2re_n, 0) \sim v_1(x_0 - 2re_n, t) \tag{3.8}$$

for $0 \leq t \leq t(x_0,r)$.

2. Observe that

$$t(x_0,r) \leq r^{2-\alpha} \leq r^{5/6} := \tau.$$

Due to Lemma 3.1, we know that at each time, $\Omega_t(u)$ is τ^5 -close to a star-shaped domain D_t up to the time $t = \tau$, i.e.,

$$D_t \subset \Omega_t(u) \subset (1+\tau^5)D_t \subset (1+r^4)D_t \tag{3.9}$$

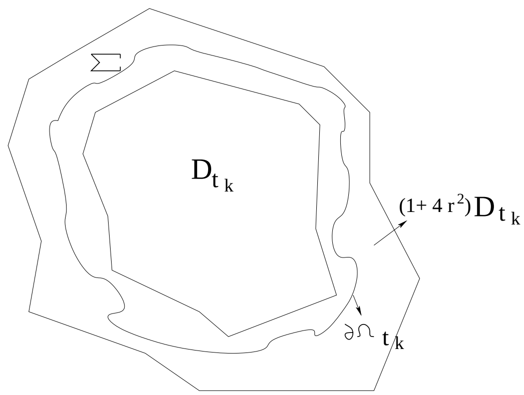

Figure 2: Approximation of the positive phase by a star-shaped domain

for $0 \leq t \leq \tau$.

Then by Lemma 3.3 and (3.9) with $\beta \geq 5/6$,

$$u(x,t) \leq r^{(13/20)(5/6)} = r^{13/24} \text{ on } \partial(1 - r^{13/20})D_0$$

for $0 \leq t \leq \tau$. (Here note that we can apply Lemma 3.3 up to the time τ since

$$t(z, r^{13/20}) \geq r^{13(2-\beta)/20} > \tau \text{ for any } z \in \Gamma_0.)$$

Then by the τ^5-monotonicity of u,

$$u(x,t) \leq r^{13/24} \text{ on } B_R(0) - (1 - r^{13/20} + r^4)D_0 \tag{3.10}$$

for $0 \leq t \leq \tau$. Since $\Gamma_t(u)$ stays in the $\tau^{5/6}$-neighborhood of $\Gamma_0(u)$ up to τ, we obtain that ∂D_t stays in the $r^{25/36}$-neighborhood of ∂D_0 up to the time τ. Since $r^{25/36} < r^{13/20}$, (3.10) implies

$$u(x,t) \leq r^{13/24} \text{ on } B_R(0) - D_s \tag{3.11}$$

for any $0 \leq s, t \leq \tau$.

3. Let

$$t_0 = 0 \leq t_1 = r^2 \leq t_2 = 2r^2 \leq ... \leq t_{k_0} = k_0 r^2 \leq \tau$$

and fix a number b such that

$$5/4 \leq b < 61/48.$$

We will construct a supersolution of (ST2) in

$$(B_R(0) - (1 + r^b)D_{t_k}) \times [t_k, t_{k+1}].$$

Let $w^k(x)$ be the harmonic function in

$$\Sigma := (1 + 4r^b)D_{t_k} - D_{t_k}$$

with boundary data zero on $\partial(1+4r^b)D_{t_k}$ and $C_n r^{13/24}$ on ∂D_{t_k}, where C_n is a sufficiently large dimensional constant. Extend $w(x)=0$ in $I\!R^n-\Sigma$. Next define

$$\Phi(x,t) := \inf\{\omega(y) : |x-y| \leq r^b - (t-t_k)\frac{r^{b-2}}{2}\}$$

in $(B_R(0)-(1+r^b)D_{t_k})\times[t_k,t_{k+1}]$. We claim that ω is a supersolution of (ST2) since our constant b satisfies

$$r^{b-2} > r^{\frac{13}{24}-b}. \tag{3.12}$$

To check this, first note that $\Phi(\cdot,t)$ is superharmonic in its positive set and $\Phi_t \geq 0$. Hence we only need to show that

$$\frac{\Phi_t}{|D\Phi|} \geq |D\Phi| \text{ on } \Gamma(\Phi). \tag{3.13}$$

Due to the definition of Φ, $\Gamma_t(\Phi)$ has an interior ball of radius at least $r^b/2$ for $t_k \leq t \leq t_{k+1}$. This and the superharmonicity of Φ in the positive set yields that

$$|D\Phi| \leq \frac{Cr^{13/24}}{r^b} \text{ on } \Gamma(\Phi)$$

for a dimensional constant $C>0$. Moreover $\Gamma(\Phi)$ evolves with normal velocity $\frac{1}{2}r^{b-2}$. Since (3.12) holds for our choice of b (i.e., for $5/4 \leq b < 61/48$), we conclude (3.13) for r smaller than a dimensional constant $r(n)$. Now we compare u with Φ in

$$(B_R(0)-(1+r^b)D_{t_k})\times[t_k,t_{k+1}].$$

Note that by (3.11),

$$u^+ \leq \Phi \text{ on } \partial(1+r^b)D_{t_k}$$

if C_n is chosen sufficiently large. Also at $t=t_k$, (3.9) implies

$$u(\cdot,t_k) \leq 0 \leq \Phi(\cdot,t_k) \text{ on } B_R(0)-(1+r^b)D_{t_k}.$$

Hence we get $u \leq \Phi$ in $(I\!R^n-(1+r^b)D_{t_k})\times[t_k,t_{k+1}]$. This implies

$$\Omega(u) \subset \Omega(\Phi) \cup ((1+r^b)D_{t_k}\times[t_k,t_{k+1}]) := \tilde{\Omega}(\Phi) \tag{3.14}$$

for $t_k \leq t \leq t_{k+1}$.

4. Next we let $v(x,t)$ solve the heat equation in

$$\tilde{\Omega}(\Phi) - ((1-3r)\Omega_0(u)\times[t_k,t_{k+1}])$$

with initial data $v(\cdot,t_k)=u(\cdot,t_k)$ and boundary data zero on $\Gamma(\Phi)$ and $v=u$ on $(1-3r)\Gamma_0(u)$. Observe that, due to (3.14), we have $u^+ \leq v$ for $t_k \leq t \leq t_{k+1}$.

Since $\tilde{\Omega}(\Phi)$ is star-shaped and expands with its normal velocity $< r^{b-2}$ which is less than r^{-1}, Lemma 2.4 applies to $\tilde{v}(x,t) := v(rx,r^2t)$. In particular there exists a constant $C>0$ such that

$$(1/C)v(x,t) \leq h_1(x,t) \leq Cv(x,t)$$

for $(t_k + t_{k+1})/2 \leq t \leq t_{k+1}$, where $h_1(\cdot, t)$ is the harmonic function in $\Omega_t(v) - (1-2r)\Omega_0(u)$ with boundary data zero on $\Gamma_t(v)$ and v on $(1-2r)\Gamma_0(u)$.

Hence we conclude that

$$u^+ \leq v \leq Ch_1$$

in $(B_R(0) - (1-2r)\Omega_0(u)) \times [(t_k + t_{k+1})/2, t_{k+1}]$.

5. Similar arguments, now pushing the boundary purely by the minus phase given by the harmonic function yields that

$$B_R(0) - \tilde{\Omega}_t(\Psi) := \Pi_t \subset \Omega_t(u)$$

for $t_k \leq t \leq t_{k+1}$, where

$$\Pi_t = \{x \in D_{t_k} : \mathrm{dist}(x, \partial D_{t_k}) \geq 3r^b + \frac{r^{b-2}}{2}(t - t_k)\}.$$

Let $w(x,t)$ solve the heat equation in

$$\Pi - ((1-3r)\Omega_0(u) \times [t_k, t_{k+1}])$$

with initial data $u(\cdot, t_k)$ and boundary data zero on $\partial\Pi$, and u on $(1-3r)\Gamma_0(u)$. Then $u \geq w(x,t)$.

Since Π is star-shaped and shrinks with its normal velocity $< r^{b-2}$ which is less than r^{-1}, Lemma 2.4 applies to $\tilde{w}(x,t) := w(rx, r^2 t)$. In particular there exists $C > 0$ such that

$$u^+ \geq w \geq (1/C)h_2$$

for $(t_k + t_{k+1})/2 \leq t \leq t_{k+1}$, where $h_2(\cdot, t)$ is the harmonic function in $\Pi_t - (1-2r)\Omega_0(u)$ with boundary data coinciding with that of w.

6. Lastly we will show that h_1 and h_2 are not too far away, i.e.

$$h_1(x,t) \leq Ch_2(x - 8r^b e_n, t)$$

with a dimensional constant $C > 0$. Since u is between $(1/C)h_2$ and Ch_1, this will conclude our lemma for $(t_k + t_{k+1})/2 \leq t \leq t_{k+1}$. Then by changing the time intervals $[t_k, t_{k+1}]$ to $[t_k + r^2/2, t_{k+1} + r^2/2]$, we obtain lemma for $r^2 \leq r \leq t(x_0, r)$.

To prove the claim, observe that

$$\Omega_t(w) \subset \Omega_t(v) \subset (1 + 8r^b)\Omega_t(w)$$

Moreover, observe that

$$\begin{aligned} v_2((1+r^b)x, (1+r^b)^2(t-t_k) + t_k) &\leq v(x,t), w(x,t) \\ &\leq v_1((1-r^b)x, (1-r^b)^2(t-t_k) + t_k) \end{aligned}$$

for $t_k \leq t \leq t_{k+1}$. This and (3.8) yield

$$v(x_0 - 2re_n, t) \sim w(x_0 - 2re_n, t) \sim u(x_0 - 2re_n, 0).$$

It follows that

$$w(x,t) \leq v(x,t) \leq Cw(x - 8r^b e_n, t) \text{ on } (1-2r)\Gamma_0 \times [t_k, t_{k+1}].$$

Hence due to Dahlberg's lemma, we conclude that

$$h_1(x,t) \leq C_1 v(x,t) \leq C_2 w(x - 8r^b e_n, t) \leq C_3 h_2(x - 8r^b e_n, t)$$

in $B_r(x_0) \times [(t_k + t_{k+1})/2, t_{k+1}]$. Since the inequality holds for any $5/4 \leq b < 61/48$, we can conclude the lemma.

□

Proposition 3.8. (Regularization in bad balls) *For a fixed $x_0 \in \Gamma_0(u)$, suppose that either*

$$u^+(x_0 - re_n, t_0) \geq Mu^-(x_0 + re_n, t_0)$$

or

$$u^-(x_0 + re_n, t_0) \geq Mu^+(x_0 - re_n, t_0)$$

for $M > M_n$, where M_n is a sufficiently large dimensional constant. Then for $r \leq 1/M_n$, there exists a dimensional constant $C > 0$ such that

$$|\nabla u^+(x,t)| \leq C\frac{u^+(x_0 - re_n, t_0)}{r} \quad \textit{and} \quad |\nabla u^-(x,t)| \leq C\frac{u^-(x_0 + re_n, t_0)}{r}$$

in $B_r(x_0) \times [t(x_0, r)/2, t(x_0, r)]$.

Remark 3.9. *1. In the next section, we will extend this Lemma for later times, i.e., for $x_0 \in \Gamma_{t_0}$. (See Lemma 4.7.)*

2. Note that the situation given in Proposition 3.8 is essentially a perturbation of the one-phase case in [CK]. The main step in the proof is in verification of this observation: i.e., by barrier arguments we will show that our solution is very close to a re-scaled version of the one-phase solution of (ST), for which the regularity of solutions are well-understood (see Theorem 1.1).

Proof. Without loss of generality, we may assume that

$$u^+(x_0 - re_n, 0) \geq Mu^-(x_0 + re_n, 0).$$

1. First we show that after a small amount of time u become almost harmonic near the free bounadry. By Lemmas 3.3 and 3.4 imply that for $0 \leq t \leq t(x_0, r)$,

$$u^+(x_0 - re_n, t) \sim u^+(x_0 - re_n, 0), \quad u^-(x_0 + re_n, t) \sim u^-(x_0 + re_n, 0) \quad (3.15)$$

Also note that, by the assumption on the initial data u_0, Lemma 3.6 holds at $t = 0$. In other words, there exists a function $\omega(x, 0) = \omega_0(x)$ such that

(a) ω_0 is harmonic in its positive and negative phases in $(1 + r)\Omega_0(u) - (1 - r)\Omega_0(u)$;

(b) $\Omega(\omega_0^+)$ and $\Omega(\omega_0^-)$ are star-shaped;

(c) In $B_r(x_0)$, we have

$$\omega_0^+(x) \leq u_0^+(x) \leq C\omega_0^+((1-r^{5/4})x) \tag{3.16}$$

and

$$\omega_0^-(x) \leq u_0^-(x) \leq C\omega_0^-((1+r^{5/4})x). \tag{3.17}$$

Next we improve (3.16) and (3.17) for later times, and obtained the inequalities with $C = (1+r^a)$ for $t \geq r^{3/2}$. By the distance estimate-Lemma 2.11, the free boundary of u moves less that $r^{9/7} < r^{5/4}$ during the time $t = r^{3/2}$. Then we let v_1 solve the heat equation in cylindrical domains

$$(1+2r^{5/4})\Omega_0(\omega^+) \times [0, r^{3/2}] \cup (B_2(0) - (1+2r^{5/4})\Omega_0(\omega^+)) \times [0, r^{3/2}],$$

with initial data u_0 and lateral boundary data zero on $(1+2r^{5/4})\Gamma_0(\omega^+) \times [0, r^{3/2}]$, and -1 on $\partial B_2(0) \times [0, r^{3/2}]$. Similarly, we let v_2 solve the heat equation in cylindrical domains

$$(1-2r^{5/4})\Omega_0(\omega^+) \times [0, r^{3/2}] \cup (B_2(0) - (1-2r^{5/4})\Omega_0(\omega^+)) \times [0, r^{3/2}]$$

with initial data u_0 and lateral boundary data zero on $(1-2r^{5/4})\Gamma_0(\omega^+) \times [0, r^{3/2}]$, and -1 on $\partial B_2(0) \times [0, r^{3/2}]$. Then by comparison, $v_2 < u < v_1$. Also by Lemma 2.4 and $\beta \geq 5/6$,

$$|v_1 - v_2| \leq r^{\frac{5}{4} \times \frac{5}{6}} = r^{25/24}$$

in the domain. Note that on $(1-r^{6/7})\Gamma_0(\omega^+)$, $|v_1| \geq r^{\frac{6}{7} \times \frac{7}{6}} = r$ and thus

$$|v_1 - v_2| \leq r^{a_1}|v_1| \text{ on } (1-r^{6/7})\Gamma_0(\omega^+) \text{ for } a_1 = 1/24.$$

Similarly,

$$|v_1 - v_2| \leq r^{a_1}|v_2| \text{ on } (1+r^{6/7})\Gamma_0(\omega^+).$$

Then since v_1 and v_2 are almost harmonic in the $r^{3/4}$-neighborhood of their boundaries for $\frac{1}{2}r^{3/2} \leq t \leq r^{3/2}$, the above inequalities on $|v_1 - v_2|$ imply the following: for $\frac{1}{2}r^{3/2} \leq t \leq r^{3/2}$, there exist positive harmonic functions $\tilde{\omega}^+(\cdot, t)$ and $\tilde{\omega}^-(\cdot, t)$ defined respectively in

$$\Omega_t(v_2^+) \cap (B_R(0) - (1-r^{1-b})\Omega_0(\omega^+)) \text{ and } \Omega_t(v_1^-) \cap (1+r^{1-b})\Omega_0(\omega^+))$$

where $b = 1/7$, such that for some $a > 0$

$$\tilde{\omega}^+(x,t) \leq u^+(x,t) \leq (1+r^a)\tilde{\omega}^+((1-4r^{5/4})x, t) \tag{3.18}$$

and

$$\tilde{\omega}^-(x,t) \leq u^-(x,t) \leq (1+r^a)\tilde{\omega}^-((1+4r^{5/4})x, t). \tag{3.19}$$

Now on the time interval $[0, r^{3/2}] + \frac{k}{2}r^{3/2}$, $1 \le k \le m$, we construct v_1 and v_2 so that they solve the heat equation in the cylindrical domains with

$$\Gamma(v_1) = (1 + 2r^{5/4})\Gamma_{\frac{k}{2}r^{3/2}}(\omega^+) \times [\frac{k}{2}r^{3/2}, (1 + \frac{k}{2})r^{3/2}]$$

and

$$\Gamma(v_2) = (1 - 2r^{5/4})\Gamma_{\frac{k}{2}r^{3/2}}(\omega^+) \times [\frac{k}{2}r^{3/2}, (1 + \frac{k}{2})r^{3/2}].$$

Then by a similar argument as above, we obtain harmonic functions $\tilde{\omega}^{\pm}(\cdot, t)$ satisfying (3.18) and (3.19) for

$$\frac{1+k}{2}r^{3/2} \le t \le (1 + \frac{k}{2})r^{3/2}.$$

Hence we conclude (3.18) and (3.19) for $r^{3/2} \le t \le t(x_0, r)$.

2. Next we re-scale $u(x, t)$ as follows:

$$\tilde{u}(x, t) := \alpha^{-1}u(rx + x_0, r^2\alpha^{-1}t) \text{ in } 2Q_{x_0},$$

where $\alpha := u^+(x_0 - re_n, t_0) < 1$. Then $\tilde{u}(x, t)$ solves

$$\begin{cases} (\alpha\partial_t - \Delta)\tilde{u} = 0 & \text{in} \quad \Omega(\tilde{u}) \\ V = |D\tilde{u}^+| - |D\tilde{u}^-| & \text{on} \quad \Gamma(\tilde{u}) \\ \tilde{u}(-e_n, 0) = 1 & \\ \tilde{u}(e_n, 0) = -1/N & \text{where} \quad N \ge M. \end{cases}$$

Furthermore, (3.15) implies that for $0 \le t \le 1$,

$$\tilde{u}^+(-e_n, t) \sim 1, \quad \tilde{u}^-(e_n, t) \sim \frac{1}{N}.$$

Let $\tilde{w}$ be the corresponding re-scaled version of $\tilde{\omega}$ given in (3.18) and (3.19), then in $B_{r^{-b}}(0) \cap \Omega_0(\tilde{u})$ we have

$$(1 - r^a)\tilde{w}^+((1 + 4r^{5/4})x, \alpha r^{-1/2}) \le \tilde{u}^+(x, \alpha r^{-1/2}) \le \tilde{w}^+(x, \alpha r^{-1/2}) \tag{3.20}$$

and

$$(1 - r^a)\tilde{w}^-(x, \alpha r^{-1/2}) \le \tilde{u}^-(x, \alpha r^{-1/2}) \le \tilde{w}^-((1 + 4r^{5/4})x, \alpha r^{-1/2}) \tag{3.21}$$

Here note that

$$\alpha r^{-1/2} = \sqrt{r} \cdot \frac{u^+(x_0 - re_n, t_0)}{r} \le r^{1/3}.$$

Lastly, for given $x_0 \in \Gamma(\tilde{u}) \cap B_1(0)$, a similar argument as in (3.7) implies that

$$\tilde{u}(x, t) \le (1 + r^b)\tilde{u}(x, 0) \text{ in } \partial B_{\frac{1}{2}r^{-b}}(r^{-b}e_n) \times [0, 1]. \tag{3.22}$$

3. We claim that we can construct a supersolution U_1 and a subsolution U_2 of (ST2) such that

$$U_2(x,t) \le \tilde{u}(x,t) \le U_1(x,t) \le U_2(x - \sqrt{\epsilon} e_n, t) \text{ in } B_1(0) \times [\alpha r^{-1/2}, 1]$$

and that U_2 is a smooth solution with uniformly Lipschitz boundary in space and time. Then for sufficiently small $r > 0$ the lemma will follow from analysis parallel to that of [ACS2].

To illustrate the main ideas, let us first assume that

(a) (3.20) and (3.21) hold in the entire ring domain $R \times [0,1]$, where

$$R = \{x : d(x, \Gamma_0(\tilde{u})) \le r^{-b}\};$$

(b) $\tilde{u}(x,t) \le (1 + r^b)\tilde{u}(x,0)$ on $\partial R \times [0,1]$.

Let U_1^+ be the solution of (HS) in $\Sigma = (I\!R^n - (\Omega_0 - R)) \times [0,1]$ with initial data $\tilde{w}(x,t)$ and boundary data $(1+r^b)\tilde{u}(x,0)$, and let

$$U_1 = U_1^+ - U_1^- \text{ in } R \times [0,1],$$

where $U_1^-(\cdot,t)$ is the harmonic function in $R - \Omega(U_1^+)$ with fixed boundary data zero on $\Gamma(U_1^+)$ and C/N on $\partial R - \Omega(U_1^+)$. Then U_1 is a supersolution of (ST2) in Σ, and thus by Theorem 2.10 and the assumptions (a)-(b) we have $\tilde{u} \le U_1$ in Σ.

4. The construction of the subsolution U_2 is a bit less straightforward. We use

$$U_2^+(x,t) := (1-\epsilon) \sup_{|y-x| \le \sqrt{\epsilon}(1-c(t))} U_1^+((1+\sqrt{\epsilon})y, t),$$

where $\epsilon = 1/N$ and $c(t) := t^{4/5}$. Then we define

$$U_2 = U_2^+ - U_2^- \text{ in } R \times [0,1],$$

where R is the ring domain as given above and $U_2^-(\cdot,t)$ is the harmonic function in $R - \Omega(U_2^+)$ with fixed boundary data zero on $\Gamma(U_2^+)$ and C/N on $\partial R - \Omega(U_2^+)$. Then U_2 satisfies the free boundary condition

$$V_{U_2} \le (1+\epsilon)|DU_2^+| - \sqrt{\epsilon} c'(t).$$

Therefore, U_2 is a subsolution of (ST2) if we can show that

$$\sqrt{\epsilon} c'(t) \ge \epsilon |DU_2^+| + |DU_2^-| \text{ on } \Gamma(U_2) \tag{3.23}$$

and $\int_0^1 c'(s) ds \le 1$.

The analysis performed in [CK], as in the proof of (c) of Theorem 1.1, yields the following: at a fixed time t, $\Gamma(U_1)$ regularizes in the scale of $d := d(t)$ which solves

$$t = \frac{d^2}{U_1(-de_n, 0)}.$$

Therefore,

$$|DU_2^+| \sim \frac{U_2^+(-de_n,0)}{d} \text{ and } |DU_2^-| \sim \frac{U_2^-(de_n,0)}{d}$$

on

$$\Gamma(U_2)\times[t/2,t].$$

Observe that since $\beta \geq 5/6$,

$$U_2^+(-de_n,0) \leq d^{5/6} \text{ and } U_2^-(de_n,0) \leq \epsilon d^{5/6},$$

then we have

$$\epsilon\frac{U_2^+(-de_n,0)}{d} + \frac{U_2^-(de_n,0)}{d} \leq \epsilon d^{-1/6} \leq \sqrt{\epsilon}t^{-1/5}.$$

where the last inequality follows from

$$t = d^2/U_1(-de_n,0) \leq d^2/d^\alpha \leq d^{5/6}.$$

Hence $c(t) = t^{4/5}$ satisfies (3.23), and we conclude that U_2 is a subsolution of (ST2).

Now we can use the fact

$$U_2 \leq \tilde{u} \leq U_1 \text{ in } B_c(0)\times[0,c]$$

to conclude that $\tilde{u}$ is $\sqrt{\epsilon}$- close to U_1: a Lipschitz (and smooth) solution in $B_1(0)\times[1/2,1]$. Once we can confirm this, everything else follows from analysis parallel to that of [ACS2] with the choice of a sufficiently small ϵ.

5. Now we proceed to the general proof without the simplified assumptions (a) and (b) in step 3, which are replaced with local inequalities (3.20)-(3.21) and (3.22). For this we need to perturb the initial data outside of $B_1(0)$ (see section 4, p 2781-2783 of [CJK2]), to obtain functions $W_1(x)$ and $W_2(x)$ which satisfies the followings:

(a) $\{W_k > 0\}$ with $k = 1,2$ is star-shaped and coincides with $\Omega_{\alpha r^{-1/2}}(\tilde{w})$ in $B_{r^{-b}}(0)$;

(b) $\{W_2 > 0\} \subset \Omega_{\alpha r^{-1/2}}(\tilde{w}) \subset \{W_1 > 0\}$;

(c) $d(x,\{W_k > 0\}) \geq r^{-b}$ with $k = 1,2$ for $x \in \Gamma_{\alpha r^{-1/2}}(\tilde{w}) \cap (I\!\!R^n - B_{2r^{-b}}(0))$;

(d) W_k is harmonic in $\{W_k > 0\} - K$ with boundary data zero on $\Gamma(W_k)$ and $(1+r^b)\tilde{w}(x,\alpha r^{-1/2})$ on ∂K, where

$$K = \{x : d(x,\Gamma(W_k)) \geq r^{-b}\}.$$

Let U_k be the solution of Hele-Shaw problem in

$$I\!\!R^n - \frac{1}{2}\{W_k > 0\}\times[\alpha r^{-1/2},1]$$

with initial data W_1 and with lateral boundary data $(1+r^b)\tilde{w}(x,\alpha r^{-1/2})$. Due to Proposition 4.1 of [CJK2], for sufficiently small $r > 0$, the level sets of U_1 is then ϵc-close to those of U_2 in $B_1(0)\times[0,1]$. Hence we can use U_2 instead of U_1 in step 4. and proceed as in step 4 to conclude. □

4 Decomposition based on local phase dynamics

Throughout the rest of the paper, we fix $x_0 \in \Gamma_0$ and a sufficiently small constant $r > 0$, and will prove the regularization of the solution in $B_r(x_0) \times [t(x_0, r)/2, t(x_0, r)]$. We also fix a constant $M \geq M_n$, where M_n is a sufficiently large dimensional constant. If the ratio between $u^+(x_0 - re_n, 0)$ and $u^-(x_0 + re_n, 0)$ is bigger than M, then we can directly apply Proposition 3.8 to prove the main theorem. Therefore we assume that

$$M^{-1}u^-(x_0 + re_n, 0) \leq u^+(x_0 - re_n, 0) \leq Mu^-(x_0 + re_n, 0). \tag{4.1}$$

Let

$$C_0 := \max[\frac{u^+(x_0 - re_n, 0)}{r}, \frac{u^-(x_0 + re_n, 0)}{r}].$$

Then since u_0^+ and u_0^- are comparable with harmonic functions, C_0 is less than a constant depending on n and M (See Corollary 2.14). Also note that

$$C_0 \geq r^{\alpha-1} \geq r^{1/6}.$$

Let

$$A^+ = \{x \in \Gamma_0 \cap B_{2r}(x_0) : \frac{u^+(x - se_n, 0)}{s} \geq MC_0 \text{ for some } r^{5/4} \leq s \leq r\}$$

and

$$A^- = \{x \in \Gamma_0 \cap B_{2r}(x_0) : \frac{u^-(x + se_n, 0)}{s} \geq MC_0 \text{ for some } r^{5/4} \leq s \leq r\}.$$

Denote

$$A = A^+ \cup A^-.$$

Lemma 4.1. *If*

$$\frac{u^\pm(x \mp se_n, 0)}{s} \geq MC_0 \textit{ for some } s \leq r,$$

then

$$\frac{u^\mp(x \pm se_n, 0)}{s} \leq C_0.$$

Proof. Since $u_0^\pm$ are comparable with harmonic functions $h^\pm$, we can argue similarly as in Corollary 2.14. Observe

$$\begin{aligned} \frac{u_0^+(x - se_n)}{s} \cdot \frac{u_0^-(x + se_n)}{s} \quad &\sim \quad \frac{h^+(x - se_n)}{s} \cdot \frac{h^-(x + se_n)}{s} \\ &\lesssim \quad \sqrt{\phi(r)} \lesssim C_0^2. \end{aligned}$$

□

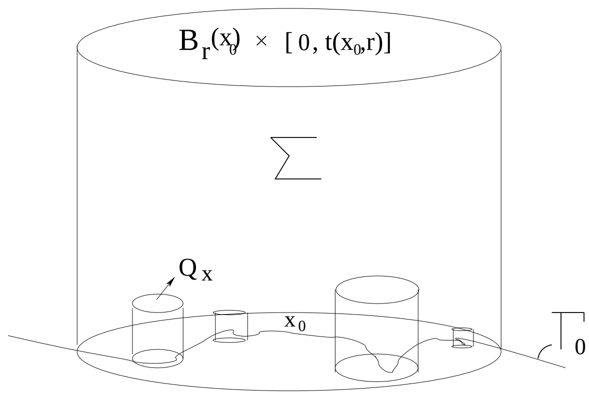

Figure 3: Decomposition of the domain

Now for $x \in A^+$, we can find the largest constant $r_x < r$ such that

$$\frac{u^+(x - r_x e_n, 0)}{r_x} = MC_0$$

then let

$$Q_x = B_{r_x}(x) \times [0, \frac{r_x}{MC_0}].$$

Also for $x \in A^-$, we can similarly define r_x and Q_x. Let

$$\Sigma := B_r(x_0) \times [0, t(x_0, r)] - \bigcup_{x \in A} Q_x. \tag{4.2}$$

(See Figure 3)

The following statement is a direct consequence of the definition (4.2).

Lemma 4.2. *If $x \in \Gamma_0 \cap \Sigma_0$, then for all $r^{5/4} \le s \le r$*

$$\frac{u^+(x - se_n, 0)}{s}, \frac{u^-(x + se_n, 0)}{s} \le MC_0.$$

The next proposition is the main result in this section, which states that the solution is "well-behaved" in Σ.

Proposition 4.3. *There exists a dimensional constant $K > 0$ such that for all $(x, t) \in \Gamma \cap \Sigma$*

$$(A) \qquad \frac{u^+(x - se_n, t)}{s}, \frac{u^-(x + se_n, t)}{s} < KMC_0 \quad \textit{for} \quad r^{5/4} \le s \le r.$$

Before proving Proposition 4.3, we show an immediate consequence of the proposition: we are ready to show that $\Gamma(u)$ is close to a Lipschitz graph in time as well as in space.

Corollary 4.4. *for* $(x,t)\in\Gamma\cap\Sigma$, *suppose* $(x+ke_n,t+\tau)\in\Gamma$. *Then*

$$|k|\leq r^{5/4}\quad \textit{if}\quad \tau\in[0,\frac{r^{5/4}}{K_1MC_0}].$$

where K_1 *is a dimensional constant.*

Proof. Due to Lemma 3.6, at any time $0\leq t\leq t(x_0,r)$, we have

$$h^{\pm}(x,t)\leq u^{\pm}(x,t)\leq C_1h^{\pm}(x\mp r^{5/4}e_n,t). \tag{4.3}$$

in $B_r(x_0)$, where $h:=h^+(\cdot,t)-h^-(\cdot,t)$ is harmonic in its positive and negative phase in $(1+r)\Omega_t(u)-(1-r)\Omega_t(u)$, and the domains $\Omega(h^+)$ and $\Omega(h^-)$ are both star-shaped with respect to $B_{r_0}(0)$. Let us pick $(y_0,t_0)\in\Gamma\cap\Sigma$. Due to Proposition 4.3, (4.3) and the Harnack inequality for harmonic functions, we have

$$\sup_{y\in B_{10r^{5/4}}(y_0)} u(y,t_0)\leq CC_1KMC_0r^{5/4} \tag{4.4}$$

where C is a dimensional constant. On the other hand, due to Lemma 3.1 and $t_0^5\leq r^{25/6}$, we have

$$u(\cdot,t_0)\leq 0 \text{ in } B_{\frac{1}{2}r^{5/4}}(y_0+r^{5/4}e_n). \tag{4.5}$$

Let

$$y_1:=y_0+r^{5/4}e_n,\ C_2:=CC_1KMC_0,\ r(t):=\frac{1}{2}r^{5/4}-C_3(t-t_0)$$

where $C_3=CC_2$. Next we define $\phi(x,t)$ in the domain

$$\Pi:=B_{2r^{5/4}}(y_1)\times[t_0,t_0+\frac{r^{5/4}}{C_3}]$$

such that

$$\begin{cases} -\Delta\phi(\cdot,t)=0 & \text{in} \quad B_{2r^{5/4}}(y_1)-B_{r(t)}(y_1)\\ \phi=2C_2r^{5/4} & \text{on} \quad \partial B_{2r^{5/4}}(y_1)\\ \phi=0 & \text{in} \quad B_{r(t)}(y_1). \end{cases}$$

Then by (4.3), (4.4) and (4.5), $u\prec\phi$ at $t=t_0$ in Π. Let T_0 be the first time where u hits ϕ from below in Π. Since (4.4) also holds for any $(x,t)\in\Gamma\cap\Sigma$ in place of (y_0,t_0), we have $u<\phi$ on the parabolic boundary of $\Pi\cap\{t_0\leq t\leq T_0\}$. On the other hand, if C is chosen sufficiently large, then

$$\frac{\phi_t}{|D\phi|}=C_3\geq|D\phi| \text{ on } \partial B_{r(t)}(y_1)\times[t_0,t_1:=t_0+\frac{r^{5/4}}{4C_3}],$$

and thus ϕ is a supersolution of (ST). This and Theorem 2.10 applied to u and ϕ in Π yields a contradiction, and we conclude that $\Gamma(u)$ lies outside of

$B_{\frac{1}{4}r^{5/4}}(y_0 + r^{5/4}e_n)$ for $t_0 \leq t \leq t_1$. Similarly, by constructing a negative radial barrier and comparing it with u, one can show that $\Gamma(u)$ lies outside of $B_{\frac{1}{4}r^{5/4}}(y_0 - r^{5/4}e_n)$ for $t_0 \leq t \leq t_1$. Hence we conclude.

$\square$

We proceed to show our main result, Proposition 4.3. The following lemmas are used in the proof of the proposition.

- For $x_0 \in \Gamma_{t_0}$, define

$$t(x_0, r) := \min[\frac{r^2}{u^+(x_0 - re_n, t_0)}, \frac{r^2}{u^-(x_0 + re_n, t_0)}].$$

Lemma 4.5 (Harnack at later times). *Fix $s \in [r^{5/4}, r]$. If $(y_0, t_0) \in \Gamma \cap \Sigma$, then*

$$u^+(y_0 - se_n, t_0) \geq c_1 u^+(y_0 - se_n, t_0 + \tau)$$

and

$$u^-(y_0 + se_n, t_0) \geq c_1 u^-(y_0 + se_n, t_0 + \tau)$$

for $0 \leq \tau \leq t(y_0, s)/2$ and $c_1 > 0$.

Proof. We will show the lemma for u^+: the statement on u^- follows via parallel arguments.

1. Let $(y_0, t_0) \in \Gamma \cap \Sigma$ and let $s \in [r^{5/4}, r]$. Let h^+ be given as in (4.3). Due to Lemma 3.3 and Lemma 3.4, we have

$$\begin{aligned} h^+(y_0 - 2re_n, t_1) &\leq u^+(y_0 - 2re_n, t_1) \\ &\leq Cu^+(y_0 - 2re_n, t_2) \leq Ch^+(y_0 - (2r + r^{5/4})e_n, t_2) \end{aligned}$$

for $0 \leq t_1, t_2 \leq t_0 + t(y_0, r)/2$. (Here note that $y_0 \in B_r(x_0)$.) In particular

$$u^+(y_0 - 2re_n, t) \leq Ch^+(y_0 - (2r + r^{5/4})e_n, t_0) \leq C_1 h^+(y_0 - 2re_n, t_0) \quad (4.6)$$

for $t \leq t_0 + t(y_0, s)/2$.

2. Now let v^+ solve (ST1) in $(I\!R^n - (1 - 2r)D_{t_0}) \times [t_0, t_0 + t(y_0, s)/2]$ with initial and boundary data $C_2 h^+(x - 2se_n, t)$. Since $s \geq r^{5/4}$, (4.3) implies

$$\Omega_t(u) \subset \Omega_{t_0}(v^+) \subset \Omega_t(v^+) \text{ in } B_{2s}(y_0) \times [t_0, t_0 + t(y_0, s)/2]. \quad (4.7)$$

Then by (4.7), (4.6) and (4.3),

$$u^+ \leq v^+ \text{ in } B_s(y_0) \times [t_0, t_0 + t(y_0, s)/2]$$

if we choose C_2 as a multiple of C_1 by a dimensional constant. Moreover, due to the Harnack inequality for one-phase (ST1), one can conclude that

$$\begin{aligned} u^+(y_0 - se_n, t_0 + \tau) &\leq v^+(y_0 - se_n, t_0 + \tau) \\ &\leq Cv^+(y_0 - se_n, t_0) \\ &= CC_2h^+(y_0 - 3se_n, t_0) \\ &\leq C_3h^+(y_0 - se_n, t_0) \\ &\leq C_3u^+(y_0 - se_n, t_0) \end{aligned}$$

for $0 \leq \tau \leq \dfrac{s^2}{v^+(y_0 - se_n, t_0)} \sim t(y_0, s)/2$. Here the first inequality uses $u^+ \leq v^+$, the second uses the Harnack inequality for v^+, the third one uses the Harnack inequality for harmonic functions and the last one uses (4.3). $\square$

Lemma 4.6 (**Backward harnack).** *Suppose that (A) holds up to time $t = T_0 \leq t(x_0, r)$. If $(y_0, t_0) \in \Gamma$ and $t_0 \leq T_0$, then for $0 \leq \tau \leq t(y_0, s)/2$,*

$$u^+(y_0 - se_n, t_0) \leq Cu^+(y_0 - se_n, t_0 + \tau)$$

and

$$u^-(y_0 + se_n, t_0) \leq Cu^-(y_0 + se_n, t_0 + \tau)$$

where $0 \leq s \leq r$ and C is a universal constant.

Proof. We will show the argument for u^+, due to the symmetric nature of the claim. The argument here will be similar to that of Lemma 3.4, replacing the initial data u_0^+ and u_0^- (used in the construction of barriers) by $h^+(x, t_0)$ and $h^-(x, t_0)$ given in (4.3).

We consider v_1: a one-phase solution of (ST1) in

$$\Pi := (1 + r)\Omega_{t_0} \times [t_0, t_0 + t(y_0, s)/2]$$

with initial and lateral boundary data C_1h^-. Then $v_1 \leq u$ in Π. Now let v_2 solve the heat equation in $\{v_1 = 0\} \times [t_0, t_0 + t(y_0, s)/2]$ with initial data

$$v_2(\cdot, t_0) = \begin{cases} h^+(\cdot, t_0) & \text{in} \quad \{v_1(\cdot, t_0) = 0\} - (1 - r)\{h^+(\cdot, t_0) > 0\} \\ \tilde{h}(\cdot) & \text{in} \quad (1 - r)\{h^+(\cdot, t_0) > 0\}, \end{cases}$$

where $\tilde{h}(\cdot)$ is a C^2 extension function of $h^+(\cdot, t_0)$ chosen so that $\tilde{h}(\cdot) \leq u^+(\cdot, t_0)$. The rest of the proof is the same as that of Lemma 3.4.

$\square$

Lemma 4.7. (Regularization in bad balls) *For a fixed* $(x_0, t_0) \in \Gamma(u)$*, and suppose*

$$u^+(x_0 - re_n, t_0) \geq Mu^-(x_0 + re_n, t_0)$$

or

$$u^-(x_0 + re_n, t_0) \geq Mu^+(x_0 - re_n, t_0)$$

for $M > M_n$*, where* M_n *is a dimensional constant. Then for* $r \leq 1/M_n$*, there exists a dimensional constant* $C > 0$ *such that*

$$|\nabla u^+| \leq C \frac{u^+(x_0 - re_n, t_0)}{r} \quad \textit{and} \quad |\nabla u^-| \leq C \frac{u^-(x_0 + re_n, t_0)}{r}$$

in $B_r(x_0) \times [t_0 + t(x_0, r)/2, t_0 + t(x_0, r)]$.

Proof. The proof of this lemma is parallel to that of Proposition 3.8. We use Harnack and backward Harnack inequalities (Lemmas 4.5 and 4.6) instead of Lemmas 3.3 and 3.4. Also we have Lemma 3.6. □

We are now ready to prove our main result, Proposition 4.3. Observe that (A) holds up to some $T_0 > 0$ by Lemma 4.2 and Lemma 3.3.

Proof of Proposition 4.3. Let K be a sufficiently large dimensional constant such that $K \gg M$. Let us assume that (A) breaks down for u^+ for the first time at $t = T_0$. Then

$$\frac{u^+(z_0 - se_n, T_0)}{s} = KMC_0 \tag{4.8}$$

for some $(z_0, T_0) \in \Gamma \cap \Sigma$ and $r^{5/4} \leq s \leq r$. Let

$$h = \sup\{h : \frac{u^+(z_0 - ke_n, T_0)}{k} \geq M^2 C_0 \text{ for } s \leq k \leq h\}. \tag{4.9}$$

Note that $h < r/2$ due to Lemma 3.3 and the definition of C_0, and $h > 2s$ due to Lemma 3.6. By the definition of h we have

$$\frac{u^+(z_0 - he_n, T_0)}{h} = M^2 C_0. \tag{4.10}$$

Let us find t_0: the closest time before T_0 such that for some $(y_0, t_0) \in \Gamma$

$$T_0 - t_0 = t(y_0, h)/2 \text{ and } y_0/|y_0| = z_0/|z_0|.$$

Then Lemma 4.5 implies

$$\frac{u^+(y_0 - he_n, t_0)}{h} \sim \frac{u^+(y_0 - he_n, T_0)}{h} \sim \frac{u^+(z_0 - he_n, T_0)}{h} = M^2 C_0.$$

Since $u^+(\cdot, t_0)$ and $u^-(\cdot, t_0)$ are comparable to harmonic functions (Lemma 3.6), a similar argument as in Lemma 4.1 implies that

$$\frac{u^-(y_0 + he_n, t_0)}{h} \lesssim C_0 \lesssim \frac{1}{M^2} \frac{u^+(y_0 - he_n, t_0)}{h}.$$

Hence by Lemma 4.7, we have

$$|\nabla u^+(\cdot, T_0)| \sim M^2 C_0 \text{ in } B_h(y_0)$$

Since $B_s(z_0) \subset B_h(y_0)$, this would contradict (4.8) since $K \gg M$.

$\square$

5 Regularization after $t = t(r)$.

Recall that $x_0 \in \Gamma_0$ and $r > 0$ are fixed, and they satisfy (4.1). Our goal is to prove the regularization of the free boundary after the time $t(x_0, r)/2$ in $B_r(x_0)$. Define

$$Q_r(x_0) := B_r(x_0) \times [t(x_0, r)/2, t(x_0, r)] \subset \Sigma.$$

Let us briefly review the information we have on u so far. Due to Lemma 3.6 and Corollary 4.4, our solution u is ϵ-monotone in $Q_r(x_0)$, with respect to a space and time cone, where the space cone $W_x(e_n, \theta_0)$ satisfies

$$|\theta_0 - \pi| = O(L),$$

where L is the Lipschitz constant of the initial domain Ω_0 given by (1.1). Moreover, due to Proposition 4.3, u does not grow too big over time, which along with Lemma 3.8 guarantees that there is no big flux of u coming in from outside of $B_r(x_0)$ to perturb our solution. Therefore the theory developed in [ACS1]-[ACS2], which says localized solutions with flat free bondaries are smooth, applies with appropriate modifications if we have L small enough such that the waiting time phenomena as seen in [CK2] is prevented. More precise description of the situation as well as precise modifications are detailed below.

As a result of Proposition 4.3, (A) holds up to

$$t = t(x_0, r) \leq C r^{2-\alpha} < r^{3/4}.$$

Moreover $Q_r(x_0) \subset \Sigma$, and thus Corollary 4.4 and Lemma 3.1, the free boundary $\Gamma(u)$ is $r^{4/3}$-monotone in $Q_r(x_0)$ with respect to the time cone $W_t(e_n, \tan^{-1}(1/K_1 M C_0))$ and the space cone $W_x(e_n, \theta_0)$. Here θ_0 is the angle corresponding to the Lipschitz constant of Γ_0, and $t(x_0, r) = \frac{r}{C_0}$.

On the other hand, by Lemma 3.3 and the definition of C_0,

$$\frac{u(x_0 - r e_n, \frac{t(x_0, r)}{2})}{C_0 r} \sim 1.$$

Since $Q_r(x_0) \subset \Sigma$, Proposition 4.3 implies

$$\frac{u(x,t)}{C_0 r} \lesssim KM \quad \text{in} \quad B_r(x_0) \times [t(x_0, r)/2, t(x_0, r)].$$

Motivated from the above estimates, we consider the re-scaled function

$$\tilde{u}(x,t) := \frac{1}{C_0 r} u(rx + x_0, r^2 t + \frac{t(x_0,r)}{2}).$$

The main difficulty in applying the Method of [ACS]-[ACS2] lies in the fact that we cannot guarantee the ϵ-monotonicity of the solution u in time variable (although we can obtain, as above, the $r^{4/3}$-monotonicity of the free boundary $\Gamma(u)$). In [ACS]-[ACS2], it was important that initially the time derivative of the solution was assumed to be controlled by the spatial derivative, i.e.,

$$|u_t| \leq C(|Du^+| + |Du^-|). \tag{5.1}$$

Using (5.1) one can prove that the *direction vectors*

$$\frac{Du^+}{|Du^+|}(-le_n, t), \quad \frac{Du^-}{|Du^-|}(le_n, t)$$

do not change much for $0 \leq t \leq l$. This is pivotal in regularization procedure since then $\Gamma(u)$ regularizes along the direction of the"common gain" obtained by those two direction vectors, the regularity of $\Gamma(u)$ then makes above two vectors line up better in a smaller scale, which contributes to further regularization of $\Gamma(u)$ in a finer scale. In our case we do not have (5.1), which requires an extra care in showing that the vectors do not change their directions too rapidly.

∘ *ϵ-monotonicity of $\Gamma(\tilde{u})$ to full monotonicity of $\tilde{u}$*

First we prove that the ϵ-monotonicity of $\Gamma(\tilde{u})$ improves to Lipschitz continuity. Let $a = C_0 r$. Then in the domain $B_1(0) \times [-\frac{1}{a}, \frac{1}{a}]$, $\tilde{u}(x,t)$ solves

$$\begin{cases} \tilde{u}_t - \Delta \tilde{u} = 0 & \text{in} \quad \{\tilde{u} > 0\} \\ \\ V = a(|D\tilde{u}^+| - |D\tilde{u}^-|) & \text{on} \quad \partial\{\tilde{u} > 0\}. \end{cases}$$

Here note that $r^{7/6} \leq r^\alpha \leq a \leq r^\beta \leq r^{5/6}$. In this scale, since $\tilde{u}$ is Caloric and $\Gamma(\tilde{u})$ is $r^{1/3}$-close to a Lipschitz graph in space and time, it follows that so does $\tilde{u}$ in $B_{1/2}(0) \times [-\frac{1}{a} + 1, \frac{1}{a}]$.

Note that in above step we are losing a lot of information over time: $\Gamma(\tilde{u})$ is in fact $r^{1/3}$-close to a Lipschitz graph moving very slow in time, but this does not guarantee that $\tilde{u}$ also changes slowly in time.

We then follow the iteration process in Lemma 7.2 of [ACS] to show the following:

Lemma 5.1. *If r is sufficiently small, then there exists $0 < c, d < 1/2$ such that the following is true: $\tilde{u}$ is $\lambda r^{1/3}$-monotone in the cone of directions $W_x(\theta_x - r^d, e_n)$ and $W_t(\theta_t - r^d, \nu)$ in the domain $B_{1-r^c}(0) \times [\frac{(-1+r^c)}{a}, \frac{1}{a}]$.*

One can then iterate above lemma to improve the ϵ-monotonicity to full monotonicity, and state the result in terms of $\tilde{u}$:

Lemma 5.2. *$\tilde{u}$ is fully monotone in $B_{1/2}(0)\times[0,\frac{1}{a}]$ for the cone*

$$\mathcal{C}_1 := W_x(\theta_x - r^d, e_n)\cup W_t(\theta_t - r^d,\nu),$$

for some constant $0<d<1/2$.

∘ *Further regularity in space*

Now we suppose $\tilde{u}$ is Lipschitz in space and time. Then in particular, we have the Lipschitz regularity of u in space (and very weak Lipschitz regularity of u in time.) We are interested in proving the following type of statement:

Lemma 5.3 (enlargement for the cone of monotonicity)**.** *There exists $\lambda>0$ such that the following holds: Suppose $\tilde{u}$ is Lipschitz with respect to the cone of monotonicity $\Lambda_x(e_n,\theta_0)$ in $B_1(0)\times[-\frac{1}{a},\frac{1}{a}]$. Then in the half domain $B_{1/2}(0)\times[-\frac{1}{2a},\frac{1}{2a}]$, $\tilde{u}$ is Lipschitz with respect to the cone of monotonicity $\Lambda_x(\nu,(1+\lambda)\theta_0)$ with some unit vector ν.*

To prove the enlargement of the cone, we take a closer look at the change of $\tilde{u}$ over time, in the interior region. More precisely, we need the following lemma which follows the approach taken in [CJK1] and [CJK2].

Lemma 5.4.

$$|\tilde{u}_t|\le a|D\tilde{u}|^2\le Ca \text{ in } [B_{1/2}(e_n)\cup B_{1/2}(-e_n)]\times[-1/2a,1/2a],$$

where C is a dimensional constant.

Proof. 1. The proof is similar to that of Lemma 8.3 of [CJK2]. Note that $\tilde{u}_t$ is a caloric function in $\Omega^+(\tilde{u})$ and $\Omega^-(\tilde{u})$. Let us prove the lemma for $\tilde{u}^+$, since parallel arguments apply to $\tilde{u}^-$.

2. We divide $\tilde{u}_t$ into two parts. More precisely, let

$$\tilde{u}_t = v_1 + v_2,$$

where both v_1 and v_2 are caloric in $\Omega^+(\tilde{u})$, v_1 has initial data zero and the boundary data $a|D\tilde{u}^+|(|D\tilde{u}^+|-|D\tilde{u}^-|)$ on $\Gamma(\tilde{u})$, and v_2 has the initial data $\tilde{u}_t(\cdot,-1/a)$ and the boundary data zero on $\Gamma(\tilde{u})$.

3. As for v_1, we need to use the absolute continuity of the caloric measure with respect to the harmonic measure, as well as the Lipschitz continuity of the free boundary. we proceed as in Lemma 8.3 of [CJK1]. Note that we have

$$|D\tilde{u}^+|\sim|D\tilde{u}^-|\sim 1$$

in $[B_{1/2}(e_n)\cup B_{1/2}(-e_n)]\times[-1/a,1/a]$: this follows from the assumption (4.1), and Lemmas 3.3 and 3.4. Therefore we can proceed as in Lemma 8.3 of [CJK1] to obtain

$$v_1(x,t)\le a\int_{\Gamma(\tilde{u})\cap\{-1/a\le s\le t\}}|D\tilde{u}^+|^2 d\omega^{(x,t)}\le a|D\tilde{u}|^2(x,t)$$

where $\omega^{(x,t)}$ is the caloric measure for $\Omega(\tilde{u})$.

$$v_1(x,t) \geq a \int_{\Gamma(\tilde{u})\cap\{-1/a\leq s\leq t\}} -|D\tilde{u}^-|^2 d\omega^{(x,t)} \geq -a|D\tilde{u}|^2(x,t).$$

4. As for v_2, we conclude that it must be smaller than that of caloric function solved in the whole domain with the absolute value of its initial data. The advantage is that then we can use the heat kernel. Note that the initial data is given at $t = -1/a$ and has a compact support. The initial data is given by $v_t \leq \frac{C}{a} v_{e_n}$, where $v_{e_n}(x,t)$ is comparable to the derivative of harmonic function in Lipschitz domain.

Therefore the heat kernel representation is given as

$$\frac{1}{(t+1/a)^{\frac{n}{2}+1}} \int |x_n - y_n| \exp^{-|x-y|^2/(t+1/a)} v(y,-1/a) dy.$$

Since $t \in [0, 1/a]$, and $k \exp^{-ak^2} \leq C \exp^{-\frac{a}{2}k^2}$, we get the effect of $O(a)$.

$\square$

Now we change the scale, and consider the function

$$v(x,t) := \frac{1}{C_0 r} u(rx + x_0, \frac{r}{C_0} t + 1) \tag{5.2}$$

Then this function is Lipschitz continuous, in space and time, *away* from the free boundary. The following lemma suggests that the cone of monotonicity improves away from the free boundary, as we look at smaller scales. The proof is parallel to that of Lemma 8.4 in [ACS2].

Lemma 5.5. *Let v given by* (5.2)*. Suppose that there exists constants $\delta > 0$ and $0 \leq A \leq B, \mu := B - A$ such that*

$$\alpha(Dv, -e_n) \leq \delta \text{ and } A \leq \frac{v_t}{-e_n \cdot Dv} \leq B$$

in $B_{1/6}(-\frac{3}{4}e_n) \times (-\delta/\mu, \delta/\mu)$ with $\frac{\delta}{\mu} < r$. Then there exist a unit vector $\nu \in \mathbb{R}^n$ and positive constants $r_0, b_0 < 1$ depending only on A, B and n such that

$$\alpha(Dv(x,t), \nu) \leq b_0 \delta \text{ in } B_{1/8}(-\frac{3}{4}e_n) \times (-r_0 \frac{\delta}{\mu}, r_0 \frac{\delta}{\mu}).$$

Now we can proceed as in section 6 of [CJK2] to obtain further regularity, using Lemma 5.4 instead of the uniform upper bound on $|Du|$ up to the free boundary.

Theorem 5.6. *$\Gamma(v)$ is C^1 in space in $Q_{1/2}$. In particular, three exist constants $l_0, C_0 > 0$ depending only on L, n and M such that for a free boundary point $(x_0, t_0) \in \Gamma(v)$, $\Gamma(v) \cap B_{2^{-l}}(x_0, t_0)$ is a Lipschitz graph with Lipschitz constant less than $\dfrac{C_0}{l}$ if $l \geq l_0$.*

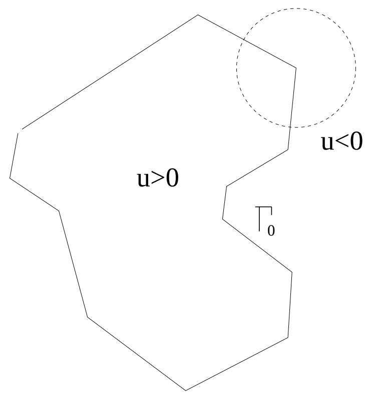

Figure 4: Locally Lipschitz initial domain

∘ *Regularity in time*

Lastly, proceeding as in section 7-8 of [CJK2] yields the differentiability of $\Gamma(v)$ in time. The main step in the argument is the following proposition: the statement and its proof is parallel to those of Theorem 7.2 in [CJK2].

Proposition 5.7. *There exist constants $l_0 > 0$ and $1 < \gamma < 2$ depending only on L, n, M such that for $(x_0, t_0) \in \Gamma(v) \cap Q_1$, if $l > l_0$ then $\Gamma(v) \cap B_{2^{-l}}(x_0, t_0)$ is a Lipschitz graph with Lipschitz constant less than $l^{-\gamma}$.*

Above proposition and the blow-up argument in section 8 of [CJK2] yields the desired result:

Theorem 5.8. $\Gamma(v)$ *is differentiable in time. Moreover*

$$C^{-1} \leq |Dv|(x,t) \leq C \text{ in } \Omega(\tilde{u}) \cap Q_{1/2},$$

where $C = C(M, n)$.

6 General case: solutions with Locally Lipschitz Initial data

In this section, we present how to extend the result of the main theorem to solutions with locally Lipschitz initial data. Our setting is as follows. Suppose Ω_0 is a bounded region in $B_R(0)$. Suppose u is a solution of (ST2) with $u_0 \geq -1$, $u_0 = -1$ on $B_R(0)$ and $u_0 \leq M_0$. Further suppose that Ω_0 is locally Lipschitz: that is, for any $x_0 \in \Gamma_0$, $\Gamma_0 \cap B_1(x_0)$ is Lipschitz with a Lipschitz constant $L \leq L_n$.

Let the initial data u_0 solve $\Delta u_0 = 0$ in $B_1(x_0)$. Then we claim that the parallel statements as in Theorem 1.2 hold in $B_{2d_0}(x_0) \times [t(x_0, d_0)/2, t(x_0, d_0)]$, where d_0 is a constant depending on n and M_0. More precisely:

Theorem 6.1. *Suppose u is a solution of (ST2) with initial data u_0 such that $-1 \le u_0 \le M_0$. Further suppose that for $x_0 \in \Gamma_0$, $\Gamma_0 \cap B_1(x_0)$ is Lipschitz with a Lipschitz constant $L \le L_n$ and $\Delta u_0 = 0$ in the positive and negative phases of u_0 in $B_1(x_0)$. Then there exists a constant $d_0 > 0$ depending on n and M_0 such that (a) and (b) of Theorem 1.2 hold for u and $d \le d_0$.*

The proof of the above theorem is parallel to that of Theorem 1.2 in section 5, after proving the following lemma.

Lemma 6.2. *There exists a solution v of (ST2) with a star-shaped initial data such that the level sets of u and v are ϵd_0-close to each other in $B_{2d_0}(x_0)$ up to the time $t(x_0, d_0; u)$, where $d_0 > 0$ is sufficiently small. In particular, u and $\Gamma(u)$ is ϵ-monotone in a cone of W_x and W_t in $B_{2d_0}(x_0) \times [t(x_0, d_0)/2, t(x_0, d_0)]$.*

Even though our equation is nonlocal, the behavior of far-away region would not affect much the behavior of solution in the unit ball, if the solution behaves "reasonably" outside the unit ball. For example, in the star-shaped case, we know at least that the free boundary is almost locally Lipschitz at each time. In the locally Lipschitz case, we control the solution by putting an upper bound M_0 on the initial data u_0. We will argue that in a sufficiently small subregion of $B_1(x_0) \times [0, 1]$, the solution is mostly determined by the local initial data in $B_1(x_0)$. The perturbation method in the proof of Lemma 2.4 in [CJK1] will be adopted here. Denote $B_1(x_0) = B_1$.

1. Construct a star-shaped region $\Omega' \subset B_R(0)$ such that

(a) $\Omega' \cap B_1 = \Omega_0 \cap B_1$.

(b) Ω' is star-shaped with respect to every $x \in K \subset \Omega'$ for a sufficiently large ball K.

Let v_0^+ be the harmonic function in $\Omega' - K$ with boundary data 1 on ∂K, and 0 on $\partial\Omega'$. Next, let v_0^- be the harmonic function in $B_R(0) - \Omega'$ with boundary data 1 on $\partial B_R(0)$, and 0 on $\partial\Omega'$. Let B_2 be a concentric ball in B_1 with the radius of ϵ^{k_0}, i.e.,

$$B_2 = B_{\epsilon^{k_0}}(x_0) \subset B_1(x_0) = B_1.$$

Let k_0 be sufficiently large. Then by Lemma 2.6, a normalization of $v_0^\pm$ by a suitable constant multiple yields that for any $x \in B_2$

$$1 - \epsilon \le \frac{u_0(x)}{v_0(x)} \le 1 + \epsilon. \tag{6.1}$$

Let v solve (ST2) with initial data $v_0 = v_0^+ - v_0^-$. Then Theorem 1.2 applies for v since v_0 is star-shaped with respect to K.

For the proof of the claim, we will find a sufficiently small d_0 such that v is ϵd_0-close to u in $B_{2d_0}(x_0)$ up to the time $t(x_0, d_0)$. More precisely, we will construct a supersolution w_1 and a subsolution w_2 of (ST2) such that in some small ball $B_h(x_0)$, we have

$$w_2 \le u \le w_1$$

and the level sets of w_1 and w_2 are $h\epsilon$ close to the level sets of v.

2. Let k_1 and k_2 be large constants which will be determined later. Define

$$H^\pm := (\Gamma_0(v) \pm \epsilon^{k_0+k_1} e_n) \cap B_2.$$

Let

$$d_0 := \epsilon^{k_0+k_1+k_2}.$$

and let $t(d_0) := t(x_0, d_0; v) = t(x_0, d_0; u)$. First note that

$$t(d_0) \geq d_0^{2-\beta} \geq \epsilon^{7(k_0+k_1+k_2)/6}.$$

Hence for v to be almost harmonic in a scale much larger than $\epsilon^{k_0+k_1}$, we need $\sqrt{t(d_0)} > \epsilon^{k_0}$, i.e.,

$$7(k_0 + k_1 + k_2)/12 < k_0.$$

Observe that by the construction of $H^\pm$ and d_0,

$$\sqrt{t(d_0)} \gg \text{radius}(B_2) \gg \text{dist}(H^\pm, \Gamma_0) \gg \max_{x \in \Gamma_t \cap B_2, 0 \leq t \leq t(d_0)} \text{dist}(x, \Gamma_0) \tag{6.2}$$

where the last inequality follows from Lemma 2.11 if we choose $k_2 \geq 2k_1$. If k_2 is sufficiently large, then one can prove from the last inequality of (6.2) and the bound on v_t that

$$1 - \epsilon \leq \frac{|v(x,t)|}{|v_0(x)|} = \frac{|v(x,t)|}{|u_0(x)|} \leq 1 + \epsilon \text{ on } H^\pm \times [0, t(d_0)]. \tag{6.3}$$

3. We do have an estimate, Lemma 2.11, on how far the boundaries move away for the local one-phase case. If we take the one-phase versions with initial data u_0^+ and u_0^-, and compare with u, then we obtain that $\Gamma(u) \cap B_2$ stays in the $d_0^{\frac{2-\alpha}{2-\beta}}$-neighborhood of $\Gamma_0(u) \cap B_2$ up to the time $t(d_0) = t(x_0, d_0)$. In other words, the free boundary of u moves less than $d_0^{5/7}$ in B_2 up to the time $t(d_0)$.

Now we let S be the region between H^+ and H^-. To construct a sub (or super) solution in S, we take the fixed boundary data $(1-\epsilon)v_0(x)$ on H^- (or H^+), and $(1+\epsilon)v_0(x)$ on H^+ (or H^-). To control the effect from the side $\partial B_2 \cap S$, we bend the free boundary $\Gamma_t(v)$ by $d_0^{5/7}$ on each side of $\partial B_2 \cap S$, using the conformal mapping $\hat{\Phi}$ (or $\breve{\Phi}$). (See section 4 of for the definition of $\hat{\Phi}$ and $\breve{\Phi}$.) More precisely, we bend the free boundary of v downward (or upward) using the conformal map $\hat{\Phi}$ (or $\breve{\Phi}$), and solve the heat equation in there. Then similar arguments as in Lemmas 4.1 and 4.3 of [CK] yield that the solution is still (almost) a supersolution, and it stays close to the original solution.

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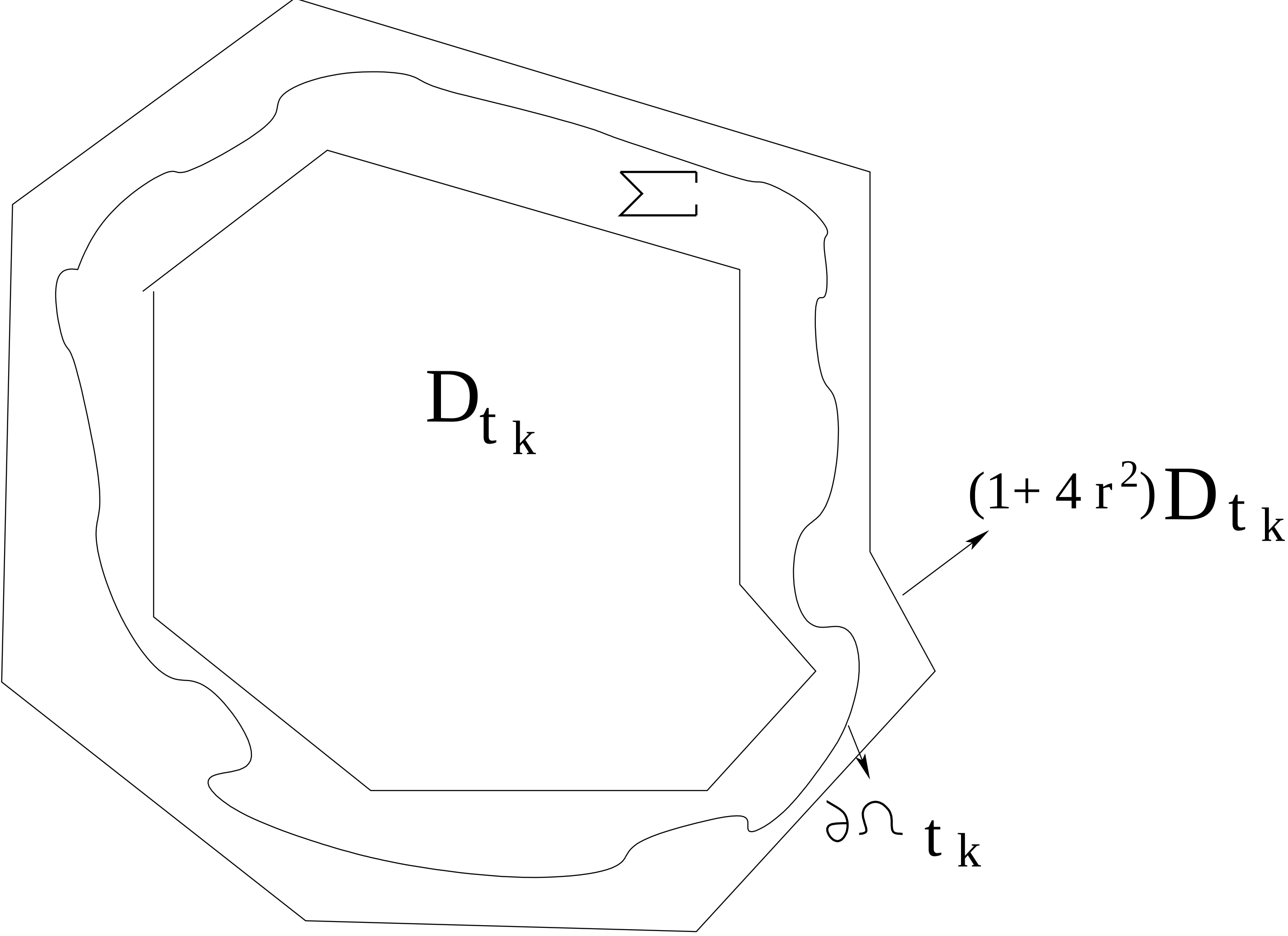
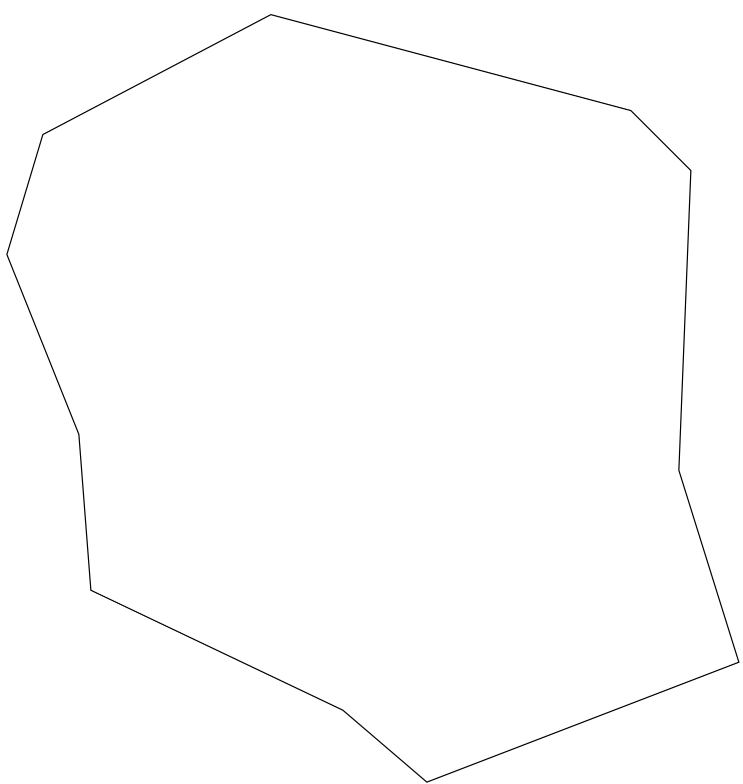
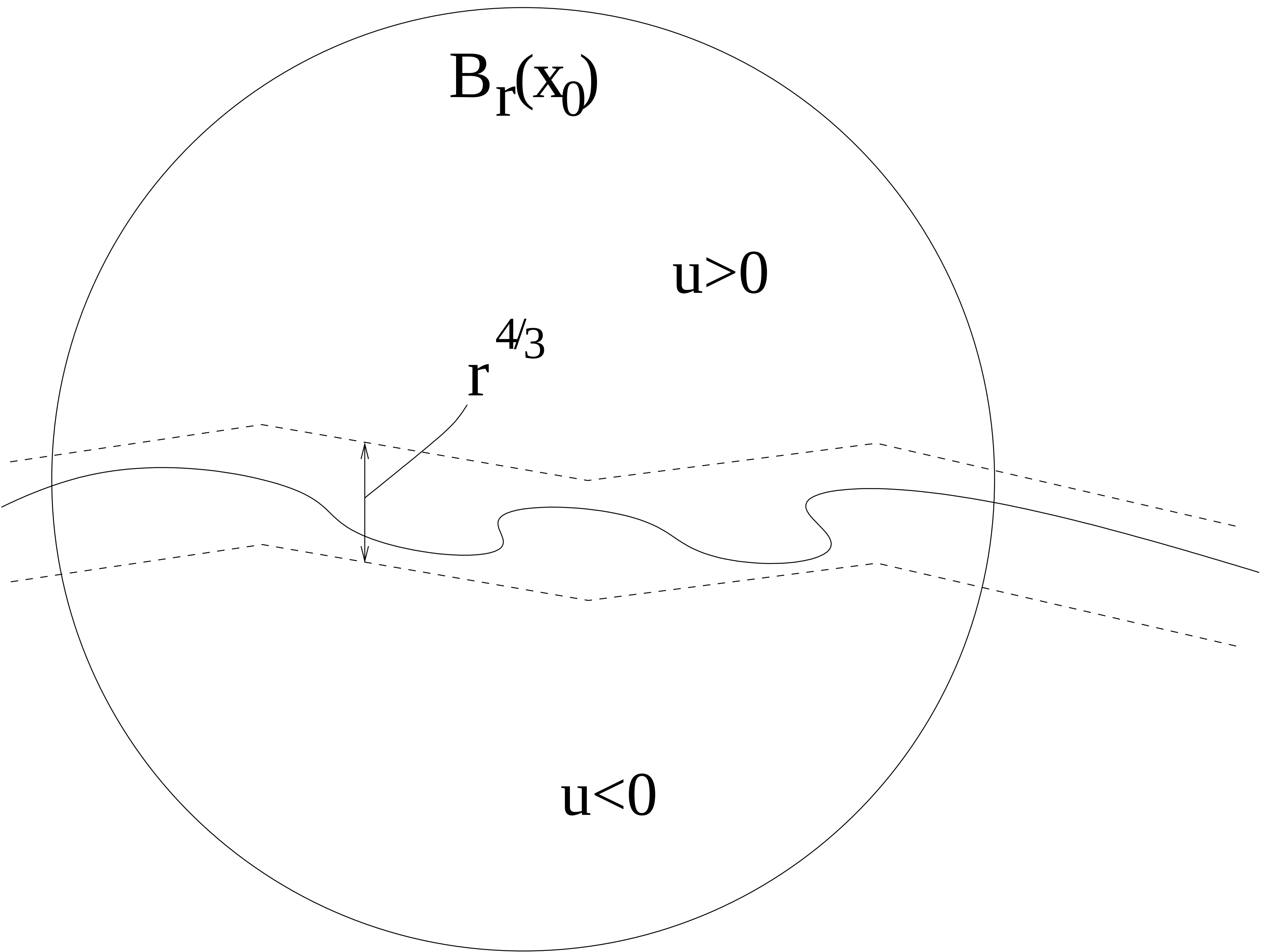
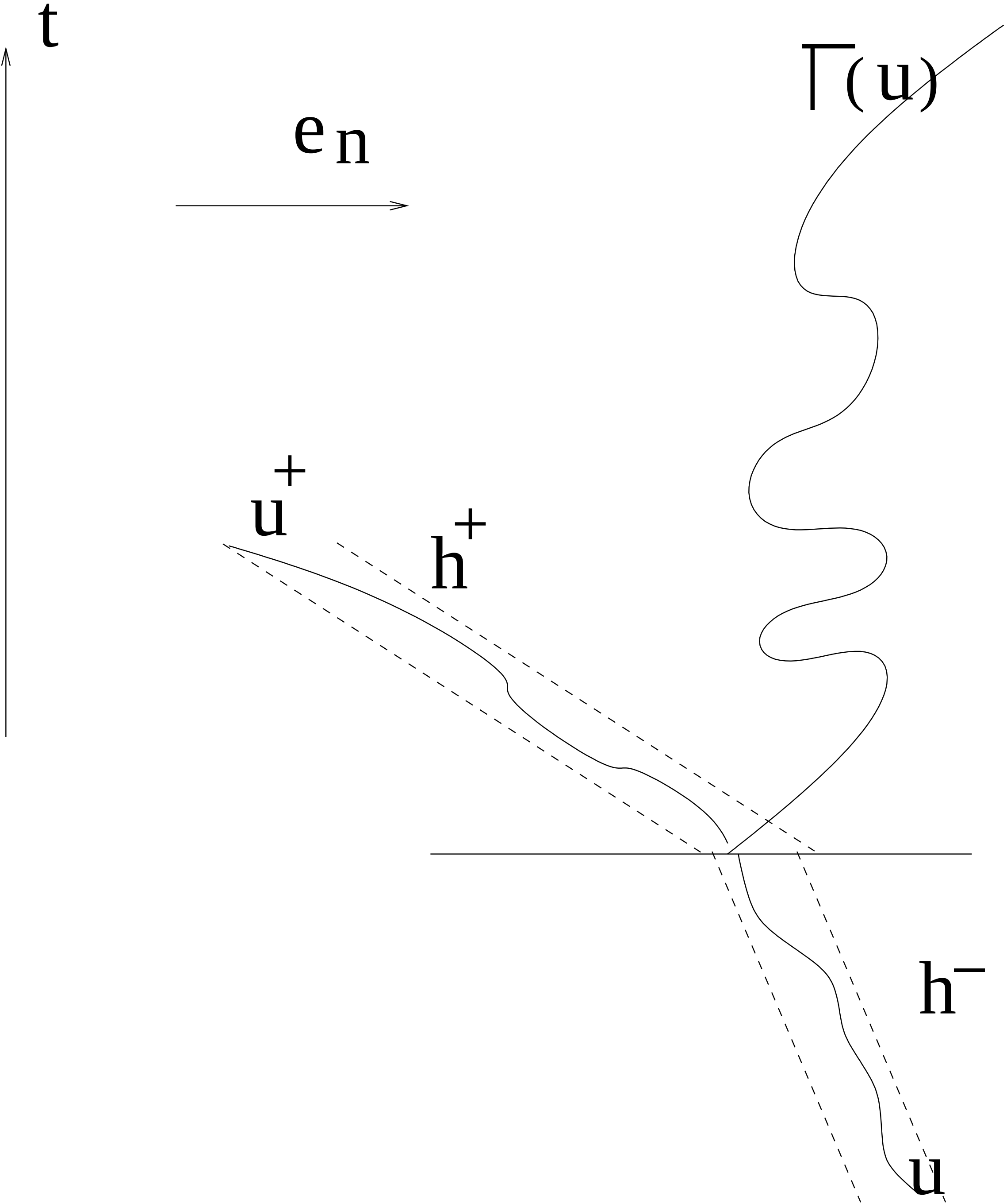


Σ
D_{t_k}
$(1+4r^2)D_{t_k}$
$\partial\Omega_{t_k}$





$B_r(x_0)$
u>0
$r^{4/3}$
u<0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