

BROWNIAN MOTIONS ON METRIC GRAPHS I DEFINITION, FELLER PROPERTY, AND GENERATORS

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ABSTRACT. Brownian motions on a metric graph are defined, their Feller property is proved, and their generators are characterized. This yields a version of Feller's theorem for metric graphs.

1. INTRODUCTION

In his pioneering articles [9–11], Feller raised the problem of characterizing and constructing all Brownian motions on a finite or on a semi-infinite interval. In the sequel this problem stimulated very important research in the field of stochastic processes, and the problem of constructing all such Brownian motions found a complete solution [17, 18] via the combination of the theory of the local time of Brownian motion [31], and the theory of (strong) Markov processes [2, 4–6, 16].

On the other hand, there is a growing interest in metric graphs, that is, piecewise linear varieties where the vertices may be viewed as singularities. Metric graphs arise naturally as models in many domains, such as physics, chemistry, computer science and engineering to mention just a few — we refer the interested reader to [30] for a review of such models and for further references.

Therefore it is natural to extend Feller's problem to metric graphs. The present paper is the first in a series of three articles [24, 25] (together with a more pedagogical one [23], in which the well-known classical cases of finite and semi-infinite intervals are revisited) on the characterization and the construction of all Brownian motions on metric graphs. Stochastic processes, in particular Brownian motions and diffusions, on locally one-dimensional structures, notably on graphs and networks, have already been studied in a number of articles, out of which we want to mention [1, 3, 8, 12–15, 29] in this context.

Heuristically, a metric graph (\mathcal{G}, d) can be thought of as the union of a collection of finite or semi-infinite closed intervals which are glued together at some of their endpoints which form the vertices of the graph, while the intervals are its edges. The metric d is then defined in the canonical way as the length of a shortest path between two points along the edges, and the length along the edges is measured as for usual intervals. For a more formal definition of metric graphs see section 2.1. We will only consider *finite* graphs, that is, those for which the sets of vertices and edges are

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finite. For the definition of a Brownian motion on the metric graph (\mathcal{G}, d) we take a standpoint similar to the one of Knight [22] for the semi-line or a finite interval: It is a strong Markov process with càdlàg paths which are continuous up to the lifetime, and which up to the first passage time at a vertex is a standard Brownian motion on the edge where it started. For the formal definition, cf. section 3.

The crucial problem is then to characterize the behaviour of the stochastic process when it reaches one of the vertices of the graph \mathcal{G} , or in other words, the characterization of the boundary conditions at the vertices of the Laplace operator which generates the stochastic process. We want to mention in passing that in an L^2 -setting all boundary conditions for Laplace operators on \mathcal{G} which make them self-adjoint operators have been characterized in [26, 27]. In this series of papers we shall work with the Banach space of continuous functions on \mathcal{G} which vanish at infinity. The main result of the present paper is Feller's theorem for metric graphs (cf. theorem 5.3). Roughly speaking it states that all possible boundary conditions are *local* boundary conditions of *Wentzell* type, i.e., linear combinations of the value of the function with its first and second (directional) derivatives at each vertex, subject to certain conditions on the coefficients. We want to emphasize here, that the fact that we only obtain local boundary conditions is due to the assumption that the paths of the Brownian motion have no jumps during their lifetime. On a more technical level this assumption entails that we deal with Feller processes — as is proved in section 4 — which is of considerable advantage when we prove the strong Markov property of the processes which we will construct in the follow-up papers [24, 25]. On the other hand, in a forthcoming work we shall embed the situation into the larger framework of Ray processes, and there we shall deal also with non-local boundary conditions, allowing the processes to have jumps from the vertices into \mathcal{G} in addition to the jumps to the cemetery point.

The paper is organized in the following way. In section 2 we recall the pertinent notions of (finite) metric graphs, and of strong Markov processes on metric graphs, at the same time setting up our notation. In section 3 Brownian motions on metric graphs are defined, and some consequences of this definition are discussed. The proof of the statement that Brownian motions on metric graphs are Feller processes is given in section 4. Finally, we state and prove Feller's theorem for metric graphs in section 5. In appendix A we give a short account of Feller semigroups in a form which we find especially convenient for the purposes of the present paper, but which we could not find in this form elsewhere.

The contents of the other two papers in this series are as follows. In the article [24] all Brownian motions on single vertex graphs (roughly speaking, n semi-lines $[0, +\infty)$ glued together at the origin) are constructed, and in the article [25] these Brownian motions are pieced together pathwise to yield all possible Brownian motions on a general metric graph.

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2. PRELIMINARY DEFINITIONS AND NOTATIONS

2.1. Metric Graphs. Throughout this paper we consider a fixed finite *metric graph* (\mathcal{G}, d) . That is, \mathcal{G} is a quadruple $(V, \mathcal{I}, \mathcal{E}, \partial)$, where V is a finite set of *vertices*, \mathcal{I} is a finite set of *internal edges*, \mathcal{E} is a finite set of *external edges*, and ∂ is a map from the set $\mathcal{L} = \mathcal{I} \cup \mathcal{E}$ of *edges* into $(V \times V) \cup V$, which maps an internal edge $i \in \mathcal{I}$ to an ordered pair $(\partial^-(i), \partial^+(i)) \in V \times V$ of vertices, called the *initial* and *final vertex of i* , while $e \in \mathcal{E}$ is mapped to $\partial(e) \in V$, called the *initial vertex of e* . Every edge $l \in \mathcal{L}$ is assumed to be isometrically isomorphic to an interval I_l , namely every $e \in \mathcal{E}$ is in one-to-one correspondence with the half line $[0, +\infty)$, while for every $i \in \mathcal{I}$ there exists $\rho_i > 0$ so that i is isomorphic to $[0, \rho_i]$. Under these isomorphisms, for $i \in \mathcal{I}$ we have that $\partial_-(i)$ corresponds to 0 while $\partial^+(i)$ corresponds to ρ_i , and for $e \in \mathcal{E}$, the vertex $\partial(e)$ corresponds to 0. ρ_i is called the *length* of the internal edge $i \in \mathcal{I}$. Moreover, we suppose that the edges are sets with an ordering as induced by the isomorphisms mentioned above. $\mathcal{L}(v) = \{l \in \mathcal{L}, v \in \partial(l)\}$ is the set of edges incident with v . For notational simplicity we will also use $\partial(l)$, $l \in \mathcal{L}$, to denote the set consisting of $\partial^-(l)$ and $\partial^+(l)$ if $l \in \mathcal{I}$, and of $\partial(l)$ if $l \in \mathcal{E}$.

In [28] the standard notion of a *walk* on a graph (e.g., [19]) has been generalized to graphs of the above type, and therefore we obtain in a natural way a metric d on \mathcal{G} as the minimal length of all walks leading from one point to another, where the length is measured along the edges as induced by the isometry with the corresponding intervals.

In the sequel it will be convenient — and without danger of confusion — to identify the abstract graph \mathcal{G} with its isomorphic *geometric graph* (e.g., [19]). In other words, we also consider \mathcal{G} as a union of the intervals corresponding to the edges, subject to the equivalence relation defined by the combinatorial structure of the graph which identifies those endpoints of the intervals which correspond to vertices to which the respective edges are incident. Similarly, we shall often identify the edges with the intervals they are isomorphic to.

Clearly, (\mathcal{G}, d) is a complete, separable metric space, and hence it is a Polish space. The Borel σ -algebra of \mathcal{G} is denoted by $\mathcal{B}(\mathcal{G})$. We write $B_r(\xi)$ for the open ball with radius $r > 0$ and center $\xi \in \mathcal{G}$.

For $l \in \mathcal{L}$, l° denotes the open interior of l , i.e., l° is the subset of l being isomorphic to $(0, \rho_i)$ if $l = i \in \mathcal{I}$, and to $(0, +\infty)$ if $l \in \mathcal{E}$. We set $\mathcal{G}^\circ = \mathcal{G} \setminus V$ to be the interior of \mathcal{G} , and hence \mathcal{G}° is the pairwise disjoint union of the open edges l° , $l \in \mathcal{L}$. In particular, every $\xi \in \mathcal{G}^\circ$ is in one-to-one correspondence with its *local coordinate* (l, x) , $l \in \mathcal{L}$, $x \in I_l^\circ$, and we may and will write $\xi = (l, x)$.

Assume that f is a real valued function on \mathcal{G} . Then f is in one-to-one correspondence with the family of functions $(f_l, l \in \mathcal{L})$ where f_l is the restriction of f to the edge $l \in \mathcal{L}$. (Of course, if $v \in V$ is a vertex with which the edges $l, l' \in \mathcal{L}$ are

incident, then we have to have $f_l(v) = f_{l'}(v)$.) Sometimes it will also be convenient to write $f_l(x)$ instead of $f(\xi)$, for $\xi \in \mathcal{G}^\circ$ having local coordinates (l, x) .

The space of real valued, bounded measurable functions will be denoted by $B(\mathcal{G})$, while $C_0(\mathcal{G})$ denotes the space of continuous real valued functions on \mathcal{G} which vanish at infinity. Both spaces are equipped with the sup-norm, denoted by $\|\cdot\|$. $(B(\mathcal{G}), \|\cdot\|)$ and $(C_0(\mathcal{G}), \|\cdot\|)$ are Banach spaces, the latter being separable.

2.2. Markov Processes on Metric Graphs. Let (\mathcal{G}, d) be a metric graph as in the previous subsection. Furthermore let (Ω, \mathcal{A}) be a measurable space, equipped with a family $(P_\xi, \xi \in \mathcal{G})$ of probability measures. The expectation with respect to P_ξ , $\xi \in \mathcal{G}$, will be denoted by $E_\xi(\cdot)$. We will say that a statement holds *almost surely* (a.s.), if for all $\xi \in \mathcal{G}$ the statement holds almost surely with respect to P_ξ .

Let Δ be a point not in \mathcal{G} which we will view as a cemetery point. By \mathcal{G}^Δ we denote the union $\mathcal{G} \cup \{\Delta\}$, where Δ is adjoined to \mathcal{G} as an isolated point. We define the σ -algebra $\mathcal{B}(\mathcal{G}^\Delta)$ on \mathcal{G}^Δ as the obvious minimal extension of $\mathcal{B}(\mathcal{G})$. All real valued functions f on \mathcal{G} are understood as being extended to \mathcal{G}^Δ with $f(\Delta) = 0$.

We consider a \mathcal{G}^Δ -valued normal homogeneous Markov process $X = (X_t, t \geq 0)$ on (Ω, \mathcal{A}) relative to a filtration $\mathcal{F} = (\mathcal{F}_t, t \geq 0)$ in \mathcal{A} with càdlàg paths. Throughout, we will suppose that the filtration \mathcal{F} is right continuous and complete for the family $(P_\xi, \xi \in \mathcal{G})$, that is, for all $t \geq 0$, $\mathcal{F}_t = \bigcap_{\epsilon > 0} \mathcal{F}_{t+\epsilon}$, and \mathcal{F}_0 contains all subsets of Ω which are negligible for all P_ξ , $\xi \in \mathcal{G}$. Also Δ is a cemetery state for X , i.e., almost surely $X_s = \Delta$, $s \geq 0$, entails $X_t = \Delta$ for all $t \geq s$. The lifetime ζ of X is defined by $\zeta = \inf\{t \geq 0, X_t = \Delta\}$.

As in [33] we assume that the transition probabilities of X are given in terms of a transition function $P = (P_t, t \geq 0)$, i.e.,

$$P_\xi(X_t \in C) = P_t(\xi, C), \quad \xi \in \mathcal{G}, t \geq 0, C \in \mathcal{B}(\mathcal{G}^\Delta).$$

In particular, for all $t \geq 0$, $C \in \mathcal{B}(\mathcal{G}^\Delta)$, the mapping $\xi \mapsto P_\xi(X_t \in C)$ is measurable. In terms of the transition function P the Markov property of X can be written as follows:

$$P_\xi(X_t \in C \mid \mathcal{F}_s) = P_{t-s}(\xi, C), \quad 0 \leq s \leq t, C \in \mathcal{B}(\mathcal{G}^\Delta), \xi \in \mathcal{G}.$$

It will be convenient and there is no loss of generality to assume the existence of a shift operator $\theta : \mathbb{R}_+ \times \Omega \rightarrow \Omega$, $(s, \omega) \mapsto \theta_s(\omega)$, so that a.s. for all $t, s \geq 0$,

$$(2.1) \quad X_t \circ \theta_s = X_{t+s}.$$

Thus if X is a strong Markov process with respect to \mathcal{F} , its strong Markov property can be expressed in the following way. Let S be an \mathcal{F} -stopping time, and as is usual denote σ -algebra of the past of S by \mathcal{F}_S . If $\xi \in \mathcal{G}$, and Z is a positive or bounded random variable, then

$$(2.2) \quad E_\xi(Z \circ \theta_S \mid \mathcal{F}_S) = E_{X_S}(Z),$$

holds P_ξ -a.s. on the set $\{X_S \neq \Delta\} = \{S < \zeta\}$.

For a subset A of \mathcal{G} we shall denote its hitting time by X by H_A ,

$$H_A = \inf\{t > 0, X_t \in A\}$$

and if $A = \{\xi\}$, $\xi \in \mathcal{G}$, we simply write H_ξ .

We shall occasionally take the liberty to write $X(t)$ for X_t , $t \in \mathbb{R}_+$, or $H(A)$ for H_A , $A \subset \mathcal{G}$, whenever it is typographically more convenient.

The semigroup $U = (U_t, t \geq 0)$ associated with X and acting linearly on $B(\mathcal{G})$ is defined by

$$(2.3) \quad U_t f(\xi) = E_\xi(f(X_t)) = \int_{\mathcal{G}} f(\eta) P_t(\xi, d\eta),$$

for $f \in B(\mathcal{G})$, $\xi \in \mathcal{G}$. The bound $\|U_t f\| \leq \|f\|$ holds for all $f \in B(\mathcal{G})$, $t \geq 0$. The resolvent $R = (R_\lambda, \lambda > 0)$ associated with X , and acting on $B(\mathcal{G})$, is defined by

$$(2.4) \quad R_\lambda f(\xi) = \int_0^\infty e^{-\lambda t} U_t f(\xi) dt, \quad \lambda > 0,$$

and satisfies

$$(2.5) \quad \|R_\lambda f\| \leq \frac{1}{\lambda} \|f\|.$$

We shall denote the restrictions of the semigroup U and the resolvent R to $C_0(\mathcal{G})$ by the same symbols.

Assume that X is a strong Markov process with respect to the filtration \mathcal{F} . A direct consequence of the strong Markov property is the *first passage time formula* for the resolvent (e.g., [32] or [18]): Let $\xi \in \mathcal{G}$, $f \in B(\mathcal{G})$, $\lambda > 0$, and assume that S is a \mathcal{F} -stopping time which is P_ξ -a.s. finite. Then

$$(2.6) \quad R_\lambda f(\xi) = E_\xi \left(\int_0^S e^{-\lambda t} f(X_t) dt \right) + E_\xi (e^{-\lambda S} R_\lambda f(X_S))$$

holds true.

We shall often have occasion to use a standard Brownian family on the real line \mathbb{R} as a family of reference processes: Let (Ω', \mathcal{A}') denote another measurable space with a family $(Q_x, x \in \mathbb{R})$ of probability measures, and for every $x \in \mathbb{R}$, $(B_t, t \in \mathbb{R}_+)$ a standard Brownian motion on \mathbb{R} starting Q_x -a.s. in x . Expectations with respect to Q_x will be denoted by $E_x^Q(\cdot)$. The Brownian family is equipped with a filtration denoted by $\mathcal{J} = (\mathcal{J}_t, t \geq 0)$, and throughout we assume — as we may — that \mathcal{J} is right continuous and complete for the family $(Q_x, x \in \mathbb{R})$. (For example, we can always consider the natural filtration generated by $(B_t, t \in \mathbb{R}_+)$, and then choose its universal augmentation, e.g., [33, Chapter III.2] or [21, Chapter 2.7].) H_A^B denotes the hitting time of the set $A \subset \mathbb{R}$ by B , and as above we simply write H_x^B for $H_{\{x\}}^B$, and occasionally $B(t)$ for B_t , $H^B(A)$ for H_A^B , $A \subset \mathbb{R}$.

Consider an edge $l \in \mathcal{L}$, and the interval $I_l \subset \mathbb{R}_+$ that l is isomorphic to. We set $\partial(I_l) = \{0, \rho_l\}$ if $l \in \mathcal{I}$, and $\partial(I_l) = \{0\}$ if $l \in \mathcal{E}$. For $t \in \mathbb{R}_+$ set

$$B_t^l = B(t \wedge H_{\partial(I_l)}^B)$$

where $s \wedge t = \min\{s, t\}$, $s, t \in \mathbb{R}_+$, i.e., B^l is a Brownian motion on \mathbb{R} with absorption in the endpoint(s) of I_l . Under the family $(Q_x, x \in I_l)$ we call this process the *absorbed Brownian motion on I_l* .

3. DEFINITION OF BROWNIAN MOTIONS ON A METRIC GRAPH

In analogy with [22, Chapter 6] we define a Brownian motion on a metric graph \mathcal{G} as follows.

Definition 3.1. A Brownian motion on \mathcal{G} is a normal strong Markov process $X = (X_t, t \geq 0)$ with state space \mathcal{G}^Δ and lifetime ζ . The sample paths of X are right continuous with left limits in \mathcal{G} , and they are continuous on $[0, \zeta)$. Furthermore, for every $\xi \in \mathcal{G}^\circ$ with local coordinates (l, x) , $l \in \mathcal{L}$, $x \in I_l^\circ$, the process $X^{abs} = (X(t \wedge H_V), t \geq 0)$, with start in ξ and absorption in the set of vertices V , is equivalent to an absorbed Brownian motion on I_l with start in x .

Remark 3.2. According to our convention of subsection 2.2, we consider a Brownian motion X on \mathcal{G} as a strong Markov process with respect to a filtration $\mathcal{F} = (\mathcal{F}_t, t \geq 0)$ which is right continuous and complete.

Remark 3.3. Suppose that $\xi \in l$, $l \in \mathcal{L}$, then $H_V = H_{\partial(l)}$, because paths starting at ξ hit the set $\partial(l)$ before any other vertex due to the continuity assumption.

Consider a Brownian motion X on \mathcal{G} . Let us state the last condition in definition 3.1 more explicitly. Fix $l \in \mathcal{L}$, and $\xi \in l^\circ$ with local coordinates (l, x) . Then for all $n \in \mathbb{N}$, $t_1, \dots, t_n \in \mathbb{R}_+$, with $t_1 \leq t_2 \leq \dots \leq t_n$, and all A_1, \dots, A_n in the Borel σ -algebra $\mathcal{B}(l)$ of l ,

$$(3.1) \quad \begin{aligned} P_\xi(X_{t_1} \in A_1, \dots, X_{t_n} \in A_n, t_n \leq H_V) \\ = Q_x(B_{t_1} \in A_1, \dots, B_{t_n} \in A_n, t_n \leq H_{\partial(I_l)}^B). \end{aligned}$$

For simplicity we have identified the set $A_i \subset l$ with its isomorphic image in I_l . Observe that in particular under P_ξ , the stopping time $H_V = H_{\partial(l)}$ has the same law as $H_{\partial(I_l)}^B$ under Q_x , and especially we get that for all $\xi \in \mathcal{G}$, $P_\xi(H_V < +\infty) = 1$.

It follows from definition 3.1 that any discontinuity of the paths of X can only occur at the vertices of \mathcal{G} , and it consists in a jump to the cemetery state Δ . Hence if the process starts in $\xi \in \mathcal{G}^\circ$, it cannot reach the cemetery state Δ before hitting V . On the other hand, if the process starts in $v \in V$, P_v -a.s. it cannot jump right away to Δ , because this would contradict the right continuity of the paths and the requirement $P_\xi(X_0 = \xi) = 1$ for all $\xi \in \mathcal{G}$. In particular, we have for all $\xi \in \mathcal{G}$, $P_\xi(\zeta \geq H_V) = 1$.

For the following discussion we assume that the process X starts at a vertex $v \in V$, and consider the exit time from v , i.e., the stopping time $S_v = H(\mathcal{G}^\circ)$. It is well known (e.g., [7, 22, 33]) that because of the strong Markov property of X , S_v is under P_v exponentially distributed with a rate $\beta_v \in [0, +\infty]$. Thus there are three possibilities:

Case $\beta_v = 0$: In this case the process stays at v forever, i.e., v is a *trap*, and the process is given by $(X(t \wedge H_v), t \in \mathbb{R}_+)$.

Case $0 < \beta_v < +\infty$: In this case the process stays at v P_v -a.s. for a strictly positive, finite moment of time, i.e., v is *exponentially holding*. It is well known (cf., e.g., [22, p. 154], [33, p. 104, Prop. 3.13]) that then the process has to leave v by a jump, and by

our assumption of path continuity on $[0, \eta)$, the process has to jump to the cemetery Δ .

Case $\beta_v = +\infty$: In this case the X leaves the vertex v immediately, and it begins a Brownian excursion into one of the edges incident with the vertex v .

4. FELLER PROPERTY

In this section we prove that the semigroup U associated with every Brownian motion on \mathcal{G} has the Feller property (see, e.g., [22, 33], or definition A.1 in appendix A).

We begin with the following simple lemma.

Lemma 4.1. *Assume that $\xi \in \mathcal{G}^\circ$. Then P_ξ -a.s. H_η converges to zero, whenever η increases or decreases to ξ along the edge to which ξ belongs.*

Proof. First we notice that because up to time ζ the paths of X are continuous, $\eta \mapsto H_\eta$ is P_ξ -a.s. monotone decreasing as η increases or decreases to ξ . Therefore it is enough to show that H_η converges to zero in P_ξ -probability.

Let $\xi \in l^\circ$, $l \in \mathcal{L}$, with local coordinates (l, x) , $x \in I_l^\circ$. Fix $\epsilon > 0$ small enough so that $(l, x \pm \epsilon) \in l^\circ$. Without loss of generality we may assume that $d(\xi, \eta) < \epsilon$. We consider first the case where $\eta \downarrow \xi$, i.e., for the local coordinates (l, y) , $y \in I_l$, of η we have $y \downarrow x$. Let $\delta > 0$, and write

$$(4.1) \quad P_\xi(H_\eta > \delta) = P_\xi(H_\eta > \delta, H_{(l, x-\epsilon)} \geq H_\eta) + P_\xi(H_\eta > \delta, H_{(l, x-\epsilon)} < H_\eta).$$

We estimate the second probability on the right hand side from above by

$$P_\xi(H_{(l, x-\epsilon)} < H_\eta).$$

But this is the probability of the event that the process leaves the set on l which in local coordinates is the interval $(l, [x - \epsilon, y])$ at the end point with local coordinates $(l, x - \epsilon)$. Therefore this is an event which happens before the process hits a vertex, and hence this probability is equal to the corresponding one for a standard Brownian motion (e.g., [18, Problem 6, p. 29]):

$$P_\xi(H_{(l, x-\epsilon)} < H_\eta) = \frac{y - x}{y - x + \epsilon},$$

which converges to zero as $\eta \downarrow \xi$. Similarly, the first probability on the right hand side of equation (4.1) is equal to

$$\begin{aligned} Q_x(H_y^B > \delta, H_{x-\epsilon}^B \geq H_y^B) &\leq Q_x(H_y^B \geq \delta) \\ &= \int_\delta^\infty \frac{y-x}{\sqrt{2\pi t^3}} e^{-(y-x)^2/2t} dt \\ &= \operatorname{erf}\left(\frac{y-x}{\sqrt{2\delta}}\right), \end{aligned}$$

where we used the well-known density of H_y^B under Q_x , e.g., [34, p. 292], [18, section 1.7], or [21, Proposition 2.6.19]. Clearly, the last expression converges to zero as $y \downarrow x$, i.e., as $\eta \downarrow \xi$.

The case $\eta \uparrow \xi$ is treated in an analogous way. \square

Lemma 4.2. *Let $\lambda > 0$, $v \in V$, and suppose that $l \in \mathcal{L}(v)$. Then*

$$(4.2) \quad \lim_{\eta \rightarrow v, \eta \in l} E_\eta(e^{-\lambda H_v}) = 1.$$

Proof. Fix $\epsilon > 0$ in such a way that we have for every $v' \in V$, $v' \neq v$, $d(v, v') > \epsilon$. We may assume that $d(v, \eta) < \epsilon$. Set

$$(4.3) \quad H_{v,\epsilon} = H(B_\epsilon(v)^c),$$

where the superscript c denotes the complement of a set. Write

$$(4.4) \quad 1 - E_\eta(e^{-\lambda H_v}) = E_\eta(1 - e^{-\lambda H_v}; H_v \leq H_{v,\epsilon}) + E_\eta(1 - e^{-\lambda H_v}; H_v > H_{v,\epsilon}),$$

with the notation

$$(4.5) \quad E_\eta(Z; C) = E_\eta(Z 1_C)$$

for positive or P_η -integrable random variables Z , and $C \in \mathcal{A}$. Consider the case where the vertex v corresponds to the point with local coordinates $(l, 0)$, the case where v corresponds to (l, ρ_l) can be dealt with by an analogous argument. Let η have local coordinates (l, y) , $0 \leq y < \epsilon$. The second term on the right hand side of equation (4.4) is less or equal to

$$\begin{aligned} P_\eta(H_v > H_{v,\epsilon}) &= Q_y(H_0^B > H_\epsilon^B) \\ &= \frac{y}{\epsilon}, \end{aligned}$$

which converges to zero with $y \downarrow 0$, i.e., with $\eta \rightarrow v$. On the other hand

$$\begin{aligned} E_\eta(1 - e^{-\lambda H_v}; H_v \leq H_{v,\epsilon}) &= E_y^Q(1 - e^{-\lambda H_0^B}; H_0^B \leq H_\epsilon^B) \\ &\leq E_y^Q(1 - e^{-\lambda H_0^B}) \\ &= 1 - e^{-\sqrt{2\lambda}y}, \end{aligned}$$

where we used the well-known formula for the Laplace transform of the density of H_0^B under Q_y (e.g., [18, p. 26, eq. 5]). Obviously this converges to zero as $y \downarrow 0$, i.e., as $\eta \rightarrow v$. \square

Theorem 4.3. *Every Brownian motion on \mathcal{G} is a Feller process.*

Proof. The proof is based on the first passage time formula (2.6). By theorem A.3 in appendix A it suffices to show that for all $\lambda > 0$, R_λ maps $C_0(\mathcal{G})$ into itself, and that for all $f \in C_0(\mathcal{G})$, $\xi \in \mathcal{G}$, $U_t f(\xi)$ converges to $f(\xi)$ as $t \downarrow 0$.

First we show that for every $\lambda > 0$, R_λ maps $C_0(\mathcal{G})$ into itself. Assume that $f \in C_0(\mathcal{G})$. Consider the case $\xi \in \mathcal{G}^\circ$. Then it follows from lemma 4.1 as in [18, Section 3.6] that $R_\lambda f$ is continuous at ξ . Consider now the case $\xi = v \in V$, let l belong to the set $\mathcal{L}(v)$ of edges incident with v , and let $\eta \in l$. Note that P_η -a.s. H_v is finite (cf. section 3). Therefore we can employ equation (2.6) with $S = H_v$:

$$R_\lambda f(\eta) = E_\eta\left(\int_0^{H_v} e^{-\lambda t} f(X_t) dt\right) + E_\eta(e^{-\lambda H_v}) R_\lambda f(v).$$

Using (2.5) we estimate in the following way

$$\begin{aligned} |R_\lambda f(\eta) - R_\lambda f(v)| &\leq \left| E_\eta \left(\int_0^{H_v} e^{-\lambda t} f(X_t) dt \right) \right| \\ &\quad + \left(1 - E_\eta(e^{-\lambda H_v}) \right) |R_\lambda f(v)| \\ &\leq \frac{2}{\lambda} \|f\| \left(1 - E_\eta(e^{-\lambda H_v}) \right). \end{aligned}$$

By lemma 4.2 this term converges to zero as η converges to v along l . Therefore $R_\lambda f$ is also continuous at v .

Next we prove that for all $\lambda > 0$, $f \in C_0(\mathcal{G})$, $R_\lambda f$ vanishes at infinity. If \mathcal{G} has no external edges there is nothing to prove, and so we assume that $e \in \mathcal{E}$ is an external edge of \mathcal{G} which is incident with the vertex $v \in V$, $\partial(e) = \{v\}$. Let $\lambda, \epsilon > 0$ be given. We choose $r_1 \geq 0$ large enough so that for all $\xi \in e$ with $d(v, \xi) > r_1$ we have $|f(\xi)| < \epsilon\lambda/2$. Choose $r_2 > r_1$, and consider $\xi \in e$ with $d(v, \xi) \geq r_2$. Denote by ξ_1 the point on e which has distance r_1 to v . Then we have that P_ξ -a.s., $H_{\xi_1} \leq H_v$, and consequently $P_\xi(H_{\xi_1} < +\infty) = 1$. Hence we can use the first passage time formula (2.6) in the form

$$R_\lambda f(\xi) = E_\xi \left(\int_0^{H_{\xi_1}} e^{-\lambda t} f(X_t) dt \right) + E_\xi(e^{-\lambda H_{\xi_1}}) R_\lambda f(\xi_1).$$

For $t \in [0, H_{\xi_1}]$ we have $d(v, X_t) \geq r_1$, and therefore the absolute value of the first term on the right hand side is bounded from above by $\epsilon/2$. For the second term we can compute the expectation as for the corresponding expression of the standard Brownian motion B on \mathbb{R} , and we obtain (again with (2.5))

$$\begin{aligned} \left| E_\xi(e^{-\lambda H_{\xi_1}}) R_\lambda f(\xi_1) \right| &= e^{-\sqrt{2\lambda} d(\xi, \xi_1)} |R_\lambda f(\xi_1)| \\ &\leq e^{-\sqrt{2\lambda}(r_2 - r_1)} \frac{\|f\|}{\lambda}. \end{aligned}$$

Now choose r_2 large enough so as to make the last term less than $\epsilon/2$, and we are done. Thus we have shown that for every $\lambda > 0$, R_λ maps $C_0(\mathcal{G})$ into itself.

Finally, consider for $f \in C_0(\mathcal{G})$, $\xi \in \mathcal{G}$, $t > 0$,

$$U_t f(\xi) = E_\xi(f(X_t)).$$

By definition, X has right continuous sample paths, and $P_\xi(X_0 = \xi) = 1$. Since f is continuous and bounded, an application of the dominated convergence theorem shows that $U_t f(\xi)$ converges to $f(\xi)$ as t decreases to 0. \square

5. GENERATORS AND FELLER'S THEOREM

Let $V_{\mathcal{L}}$ denote the subset of $V \times \mathcal{L}$ given by

$$V_{\mathcal{L}} = \{(v, l), v \in V \text{ and } l \in \mathcal{L}(v)\}.$$

We shall also write v_l for $(v, l) \in V_{\mathcal{L}}$. We remark in passing that

$$|V_{\mathcal{L}}| = |\mathcal{E}| + 2|\mathcal{I}|.$$

Consider a real valued function f on \mathcal{G} , let $v \in V$ and let $l \in \mathcal{L}(v)$ be an edge incident with v . We define the *directional derivative of f at v in direction $l \in \mathcal{L}(v)$* as follows:

$$(5.1) \quad f'(v_l) = \begin{cases} \lim_{\xi \rightarrow v, \xi \in l^\circ} f'(\xi), & \text{if } v \text{ is an initial vertex of } l, \\ - \lim_{\xi \rightarrow v, \xi \in l^\circ} f'(\xi), & \text{if } v \text{ is a final vertex of } l, \end{cases}$$

whenever the corresponding limit on the right hand side exists. Geometrically this directional derivative is just the inward normal derivative which makes it an intrinsic definition, independent of the orientation chosen on the edge.

Definition 5.1. $C_0^{0,2}(\mathcal{G})$ denotes the subspace of functions f in $C_0(\mathcal{G})$ which are twice continuously differentiable on \mathcal{G}° , and such that for every $v \in V$ and all $l \in \mathcal{L}(v)$ the limit

$$(5.2) \quad f''(v_l) = \lim_{\xi \rightarrow v, \xi \in l^\circ} f''(\xi)$$

exists. $C_0^2(\mathcal{G})$ denotes the subspace of those functions f in $C_0^{0,2}(\mathcal{G})$ so that f'' extends from \mathcal{G}° to a continuous function on \mathcal{G} .

Thus $C_0^2(\mathcal{G})$ consists of all $f \in C_0^{0,2}(\mathcal{G})$ so that for every $v \in V$, the $f''(v_l)$, $l \in \mathcal{L}(v)$, are all equal. Assume that $f \in C_0^2(\mathcal{G})$, and let $v \in V$. The continuous extension of f'' to v will simply be denoted by $f''(v)$. Consider an edge $l \in \mathcal{L}(v)$ incident with v . Then it is easy to see that $f'(v_l)$ exists (and is finite). On the other hand, in general for $l, l' \in \mathcal{L}(v)$, $l \neq l'$, we have $f'(v_l) \neq f'(v_{l'})$. In other words, in general f' does *not* have a continuous extension from \mathcal{G}° to \mathcal{G} . Also, it is not hard to check that f' vanishes at infinity.

The proof of the following lemma can be taken over with minor modifications from the standard literature, e.g., from [22, Chapter 6.1].

Lemma 5.2. *For every Brownian motion X on the metric graph \mathcal{G} , the generator A of its semigroup U acting on $C_0(\mathcal{G})$ has a domain $\mathcal{D}(A)$ contained in $C_0^2(\mathcal{G})$. Moreover, for every $f \in \mathcal{D}(A)$, $Af = 1/2 f''$.*

Consider data of the following form

$$(5.3) \quad \begin{aligned} a &= (a_v, v \in V) \in [0, 1]^V \\ b &= (b_{v_l}, v_l \in V_{\mathcal{L}}) \in [0, 1]^{V_{\mathcal{L}}} \\ c &= (c_v, v \in V) \in [0, 1]^V \end{aligned}$$

subject to the condition

$$(5.4) \quad a_v + \sum_{l \in \mathcal{L}(v)} b_{v_l} + c_v = 1, \quad \text{for every } v \in V.$$

We define a subspace $\mathcal{H}_{a,b,c}$ of $C_0^2(\mathcal{G})$ as the space of those functions f in $C_0^2(\mathcal{G})$ which at every vertex $v \in V$ satisfy the Wentzell boundary condition

$$(5.5) \quad a_v f(v) - \sum_{l \in \mathcal{L}(v)} b_{v_l} f'(v_l) + \frac{1}{2} c_v f''(v) = 0.$$

Now we can state and prove the analogue of *Feller's theorem* for metric graphs.

Theorem 5.3. *Suppose that X is a Brownian motion on a metric graph \mathcal{G} , and that $\mathcal{D}(A)$ is the domain of the generator A of its semigroup. Then there are a, b, c as in (5.3), (5.4), so that $\mathcal{D}(A) = \mathcal{H}_{a,b,c}$.*

Remark 5.4. The case $a_v = 1, v \in V$, would correspond to (zero) Dirichlet boundary conditions at the vertex v . The paths of the process associated with this boundary condition have to jump instantaneously to Δ when reaching the vertex, and by our requirement that the paths are right continuous this means that the process will never be at the vertex. But this is in contradiction to our assumption (cf. definition 3.1) that the process with absorption at the vertex is equivalent to a Brownian motion with absorption in the endpoint (endpoints, resp.) of the corresponding interval. Therefore this stochastic process is *not* a Brownian motion on \mathcal{G} in the sense of definition 3.1, and this case has to be excluded from our discussion. Also note that in this case the semigroup does not act strongly continuously on $C_0(\mathcal{G})$, and therefore is in particular not Feller.

The proof of theorem 5.3 has two rather distinct parts, and therefore we split it by proving the following two lemmas:

Lemma 5.5. *Suppose that X is a Brownian motion on a metric graph \mathcal{G} , and that $\mathcal{D}(A)$ is the domain of the generator A of its semigroup. Then there are a, b, c as in (5.3), (5.4), so that $\mathcal{D}(A) \subset \mathcal{H}_{a,b,c}$.*

Lemma 5.6. *Suppose that A is the generator of a Brownian motion X on \mathcal{G} with domain $\mathcal{D}(A) \subset \mathcal{H}_{a,b,c}$ for some a, b, c as in (5.3), (5.4). Then $\mathcal{D}(A) = \mathcal{H}_{a,b,c}$.*

Proof of lemma 5.5. Our proof follows the one in [22, Chapter 6.1] quite closely — actually, it is sufficient to consider a special case of the proof given there.

We show that for every vertex $v \in V$ there are constants $a_v \in [0, 1], b_{v_l} \in [0, 1], l \in \mathcal{L}(v), c_v \in [0, 1]$ satisfying (5.4), and such that all f in the domain $\mathcal{D}(A)$ of the generator satisfy the boundary condition (5.5). To this end, we let $f \in \mathcal{D}(A)$, fix a vertex $v \in V$, and compute $Af(v)$. Let us consider the three cases for β mentioned in section 3.

If $\beta = 0$, v is a trap, and $U_t f(v) = f(v)$ for all $t \geq 0$. Consequently, $Af(v) = 0$, and therefore $1/2 f''(v) = 0$. Thus f satisfies the boundary condition (5.5) at v with $a_v = 0, c_v = 1$, and $b_{v_l} = 0$ for all $l \in \mathcal{L}(v)$.

Next we consider the case where $\beta \in (0, +\infty)$, i.e., v is exponentially holding. We know from the discussion in section 3 that then after expiration of the holding time the process jumps directly to the cemetery state. Therefore we get for $t > 0, U_t f(v) =$

$\exp(-\beta t)f(v)$, and thus $Af(v) + \beta f(v) = 0$, and the boundary condition (5.5) holds for the choice

$$(5.6) \quad a_v = \frac{\beta}{1 + \beta}, \quad c_v = \frac{1}{1 + \beta}, \quad b_{v_l} = 0, \quad l \in \mathcal{L}(v).$$

Finally we consider the case that $\beta = +\infty$, i.e., the process leaves v immediately, and in particular, v is not a trap. Therefore we may compute $Af(v)$ in Dynkin's form, e.g., [5, p. 140, ff.], [18, p. 99]. As in (4.3) we let $H_{v,\epsilon}$ denote the hitting time of the complement of $B_\epsilon(v)$. Then

$$(5.7) \quad Af(v) = \lim_{\epsilon \downarrow 0} \frac{E_v\left(f(X(H_{v,\epsilon}))\right) - f(v)}{E_v(H_{v,\epsilon})}.$$

Recall the notation $f_l(\epsilon)$ for $f(\xi)$ with $\xi \in \mathcal{G}$ having local coordinates (l, ϵ) , $l \in \mathcal{L}$, $\epsilon \in I_l$. Then we get

$$\begin{aligned} E_v\left(f(X(H_{v,\epsilon}))\right) &= \sum_{l \in \mathcal{L}(v)} f_l(\epsilon) P_v(X(H_{v,\epsilon}) \in l) + f(\Delta) P_v(X(H_{v,\epsilon}) = \Delta) \\ &= \sum_{l \in \mathcal{L}(v)} f_l(\epsilon) P_v(X(H_{v,\epsilon}) \in l), \end{aligned}$$

where the last equality follows from $f(\Delta) = 0$. Let us denote

$$\begin{aligned} r_l(\epsilon) &= \frac{P_v(X(H_{v,\epsilon}) \in l)}{E_v(H_{v,\epsilon})}, \quad l \in \mathcal{L}(v), \\ r_\Delta(\epsilon) &= \frac{P_v(X(H_{v,\epsilon}) = \Delta)}{E_v(H_{v,\epsilon})}, \\ K(\epsilon) &= 1 + r_\Delta(\epsilon) + \epsilon \sum_{l \in \mathcal{L}(v)} r_l(\epsilon). \end{aligned}$$

The continuity of the paths of X up to the lifetime ζ yields

$$\sum_{l \in \mathcal{L}(v)} P_v(X(H_{v,\epsilon}) \in l) + P_v(X(H_{v,\epsilon}) = \Delta) = 1,$$

and therefore equation (5.7) can be rewritten as

$$\lim_{\epsilon \downarrow 0} \left(Af(v) + r_\Delta(\epsilon)f(v) - \sum_{l \in \mathcal{L}(v)} r_l(\epsilon)(f_l(\epsilon) - f(v)) \right) = 0.$$

Since for all $\epsilon > 0$, $K(\epsilon)^{-1} \leq 1$, it follows that

$$\lim_{\epsilon \downarrow 0} \left(\frac{1}{K(\epsilon)} Af(v) + \frac{r_\Delta(\epsilon)}{K(\epsilon)} f(v) - \sum_{l \in \mathcal{L}(v)} \frac{\epsilon r_l(\epsilon)}{K(\epsilon)} \frac{f_l(\epsilon) - f(v)}{\epsilon} \right) = 0,$$

which by lemma 5.2 we may rewrite as

$$\lim_{\epsilon \downarrow 0} \left(a_v(\epsilon) f(v) + \frac{1}{2} c_v(\epsilon) f''(v) - \sum_{l \in \mathcal{L}(v)} b_{v_l}(\epsilon) \frac{f_l(\epsilon) - f(v)}{\epsilon} \right) = 0,$$

where we have introduced the non-negative quantities

$$\begin{aligned} a_v(\epsilon) &= \frac{r_\Delta(\epsilon)}{K(\epsilon)}, \\ c_v(\epsilon) &= \frac{1}{K(\epsilon)}, \\ b_{v_l}(\epsilon) &= \frac{\epsilon r_l(\epsilon)}{K(\epsilon)}, \quad l \in \mathcal{L}(v). \end{aligned}$$

Observe that for every $\epsilon > 0$,

$$a_v(\epsilon) + c_v(\epsilon) + \sum_{l \in \mathcal{L}(v)} b_{v_l}(\epsilon) = 1.$$

Therefore every sequence $(\epsilon_n, n \in \mathbb{N})$ with $\epsilon_n > 0$ and $\epsilon_n \downarrow 0$ has a subsequence so that $a_v(\epsilon)$, $c_v(\epsilon)$ and $b_{v_l}(\epsilon)$, $l \in \mathcal{L}(v)$, converge along this subsequence to numbers a_v , c_v , and b_{v_l} respectively in $[0, 1]$, and the relation (5.4) holds true. From the remark after definition 5.1 it follows that

$$\frac{f_l(\epsilon) - f(v)}{\epsilon}$$

converges with $\epsilon \downarrow 0$ to $f'(v_l)$, and therefore we obtain that for every vertex $v \in V$, $f \in \mathcal{D}(A)$ satisfies the boundary condition (5.5) with data a, b, c as in (5.3), (5.4). \square

Before we can prove lemma 5.6 we have to introduce some additional formalism.

For given data a, b, c as in (5.3), (5.4), it will be convenient to consider $\mathcal{H}_{a,b,c}$ equivalently as being the subspace of $C_0^{0,2}(\mathcal{G})$ so that for its elements f at every $v \in V$ the boundary conditions (5.5) as well as the boundary condition

$$(5.8) \quad f''(v_l) = f''(v_k), \quad \text{for all } l, k \in \mathcal{L}(v)$$

hold true. Relation (5.8) is just another way to express that f'' is continuous on \mathcal{G} .

We consider the sets V, \mathcal{E} , and \mathcal{I} as being ordered in some arbitrary way. With the convention that in \mathcal{L} the elements of \mathcal{E} come first this induces also an order relation on \mathcal{L} .

Suppose that $f \in C_0^{0,2}(\mathcal{G})$. With the given ordering of \mathcal{E} and \mathcal{I} we define the following column vectors of length $|\mathcal{E}| + 2|\mathcal{I}|$:

$$\begin{aligned} f(V) &= \left((f_e(0), e \in \mathcal{E}), (f_i(0), i \in \mathcal{I}), (f_i(\rho_i), i \in \mathcal{I}) \right)^t, \\ f'(V) &= \left((f'_e(0), e \in \mathcal{E}), (f'_i(0), i \in \mathcal{I}), (-f'_i(\rho_i), i \in \mathcal{I}) \right)^t, \\ f''(V) &= \left((f''_e(0), e \in \mathcal{E}), (f''_i(0), i \in \mathcal{I}), (f''_i(\rho_i), i \in \mathcal{I}) \right)^t, \end{aligned}$$

where the superscript “ t ” indicates transposition.

We want to write the boundary conditions (5.5), (5.8) in a compact way, and to this end we introduce the following order relation on $V_{\mathcal{L}}$: For $v_l, v_{l'} \in V_{\mathcal{L}}$ we set $v_l \preceq v_{l'}$ if and only if $v \prec v'$ or $v = v'$ and $l \preceq l'$ (where for V and \mathcal{L} we use the order relations introduced above). For f as above set

$$\begin{aligned}\tilde{f}(V) &= (f(v_l), v_l \in V_{\mathcal{L}})^t, \\ \tilde{f}'(V) &= (f'(v_l), v_l \in V_{\mathcal{L}})^t, \\ \tilde{f}''(V) &= (f''(v_l), v_l \in V_{\mathcal{L}})^t.\end{aligned}$$

Then there exists a permutation matrix P so that

$$\tilde{f}(V) = Pf(V), \quad \tilde{f}'(V) = Pf'(V), \quad \tilde{f}''(V) = Pf''(V).$$

In particular, P is an orthogonal matrix which has in every row and in every column exactly one entry equal to one while all other entries are zero.

For every $v \in V$ we define the following $|\mathcal{L}(v)| \times |\mathcal{L}(v)|$ matrices:

$$\begin{aligned}\tilde{A}(v) &= \begin{pmatrix} a_v & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \\ \tilde{B}(v) &= \begin{pmatrix} -b_{v_{l_1}} & -b_{v_{l_2}} & -b_{v_{l_3}} & \cdots & -b_{v_{l_{|\mathcal{L}(v)|}}} \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \\ \tilde{C}(v) &= \begin{pmatrix} 1/2 c_v & 0 & 0 & 0 & \cdots & 0 \\ 1 & -1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -1 \end{pmatrix},\end{aligned}$$

where we have labeled the elements in $\mathcal{L}(v)$ in such a way that in the above defined ordering we have $l_1 \prec l_2 \prec \cdots \prec l_{|\mathcal{L}(v)|}$. Observe that $\tilde{C}(v)$ is invertible if and only if $c_v \neq 0$. Define block matrices \tilde{A} , \tilde{B} , and \tilde{C} by

$$\tilde{A} = \bigoplus_{v \in V} A(v), \quad \tilde{B} = \bigoplus_{v \in V} B(v), \quad \tilde{C} = \bigoplus_{v \in V} C(v).$$

Then we can write the boundary conditions (5.5), (5.8) simultaneously for all vertices as

$$(5.9) \quad \tilde{A}\tilde{f}(V) + \tilde{B}\tilde{f}'(V) + \tilde{C}\tilde{f}''(V) = 0.$$

Consequently the boundary conditions can equivalently be written in the form

$$(5.10) \quad Af(V) + Bf'(V) + Cf''(V) = 0,$$

with

$$(5.11) \quad A = P^{-1}\tilde{A}P, \quad B = P^{-1}\tilde{B}P, \quad C = P^{-1}\tilde{C}P.$$

We bring in the following two matrix-valued functions on the complex plane

$$(5.12) \quad \hat{Z}_{\pm}(\kappa) = A \pm \kappa B + \kappa^2 C, \quad \kappa \in \mathbb{C}.$$

Lemma 5.7. *There exists $R > 0$ so that for all $\kappa \in \mathbb{C}$ with $|\kappa| \geq R$ the matrices $\hat{Z}_{\pm}(\kappa)$ are invertible, and there are constants $C, p > 0$ so that*

$$(5.13) \quad \|\hat{Z}_{\pm}(\kappa)^{-1}\| \leq C |\kappa|^p, \quad |\kappa| \geq R.$$

Remark 5.8. The bound (5.13) is actually rather crude, but sufficient for our purposes.

Proof of lemma 5.7. Since we have

$$(5.14) \quad \hat{Z}_{\pm}(\kappa) = P^{-1}(\tilde{A} \pm \kappa \tilde{B} + \kappa^2 \tilde{C})P$$

for an orthogonal matrix P , for the proof of the first statement it suffices to show that there exists $R > 0$ such that

$$\tilde{A} \pm \kappa \tilde{B} + \kappa^2 \tilde{C}$$

are invertible for complex κ outside of the open ball of radius R . For this in turn it suffices to show that for every vertex $v \in V$ the matrices

$$\begin{aligned} & \tilde{A}(v) \pm \kappa \tilde{B}(v) + \kappa^2 \tilde{C}(v) \\ &= \begin{pmatrix} a_v \pm \kappa b_{v_{l_1}} + \kappa^2/2 c_v & \pm \kappa b_{v_{l_2}} & \pm \kappa b_{v_{l_3}} & \pm \kappa b_{v_{l_4}} & \cdots & \pm \kappa b_{v_{|\mathcal{L}(v)|}} \\ \kappa^2 & -\kappa^2 & 0 & 0 & \cdots & 0 \\ 0 & \kappa^2 & -\kappa^2 & 0 & \cdots & 0 \\ 0 & 0 & \kappa^2 & -\kappa^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -\kappa^2 \end{pmatrix} \end{aligned}$$

are invertible for all $\kappa \in \mathbb{C}$ with $|\kappa| \geq R$. An elementary calculation gives

$$\det(\tilde{A}(v) \pm \kappa \tilde{B}(v) + \kappa^2 \tilde{C}(v)) = \left(a_v \pm \kappa \sum_{l \in \mathcal{L}(v)} b_{v_l} + \frac{\kappa^2}{2} c_v \right) (-\kappa^2)^{|\mathcal{L}(v)|-1}.$$

The choices $\kappa = \pm 1$ together with condition (5.4) show that the polynomial of second order in κ in the first factor on the right hand side does not vanish identically. Therefore, it is non-zero in the exterior of an open ball with some radius $R_v > 0$. Hence, we obtain the first statement for the choice $R = \max_{v \in V} R_v$. Moreover, from the calculation of the determinants above we also get for every $v \in V$ and all $\kappa \in \mathbb{C}$ with $|\kappa| \geq R$ an estimate of the form

$$(5.15) \quad |\det(\tilde{A}(v) \pm \kappa \tilde{B}(v) + \kappa^2 \tilde{C}(v))|^{-1} \leq \text{const.}$$

Thus, using the co-factor formula for

$$(\tilde{A}(v) \pm \kappa \tilde{B}(v) + \kappa^2 \tilde{C}(v))^{-1}$$

we find with (5.15) the estimate

$$\|(\tilde{A}(v) \pm \kappa \tilde{B}(v) + \kappa^2 \tilde{C}(v))^{-1}\| \leq C_v |\kappa|^{p_v}, \quad |\kappa| \geq R,$$

for some constants $C_v, p_v > 0$. Consequently we get

$$\|(\tilde{A} \pm \kappa \tilde{B} + \kappa^2 \tilde{C})^{-1}\| \leq C |\kappa|^p, \quad |\kappa| \geq R,$$

for some constants $C, p > 0$, and by (5.14) we have proved inequality (5.13). \square

With these preparations we can enter the

Proof of lemma 5.6. Let the data a, b, c be given as in(5.3), (5.4). We have to show that the inclusion $\mathcal{D}(A) \subset \mathcal{H}_{a,b,c}$ is not strict. Let $R = (R_\lambda, \lambda > 0)$ be the resolvent of A . Then for every $\lambda > 0$, R_λ is a bijection from $C_0(\mathcal{G})$ onto $\mathcal{D}(A)$, that is, R_λ^{-1} is a bijection from $\mathcal{D}(A)$ onto $C_0(\mathcal{G})$.

Assume to the contrary that the inclusion $\mathcal{D}(A) \subset \mathcal{H}_{a,b,c}$ is strict. We will derive a contradiction. For $\lambda > 0$ consider the linear mapping $H_\lambda : f \mapsto \lambda f - 1/2 f''$ from $\mathcal{H}_{a,b,c}$ to $C_0(\mathcal{G})$. On $\mathcal{D}(A)$ this mapping coincides with R_λ^{-1} , and R_λ^{-1} is a bijection from $\mathcal{D}(A)$ onto $C_0(\mathcal{G})$. Therefore our assumption entails that H_λ cannot be injective. Hence for any $\lambda > 0$ there exists $f(\lambda) \in \mathcal{H}_{a,b,c}$, $f(\lambda) \neq 0$, with

$$(5.16) \quad H_\lambda f(\lambda) = \lambda f(\lambda) - \frac{1}{2} f''(\lambda) = 0.$$

We will show that $f(\lambda) \in \mathcal{H}_{a,b,c}$ satisfying (5.16) can only hold when $f(\lambda) = 0$ on \mathcal{G} . It will be convenient to change the variable λ to $\kappa = \sqrt{2\lambda}$, and there will be no danger of confusion that we shall simply write $f(\kappa)$ for $f(\lambda)$ from now on. Then the solution of (5.16) is necessarily of the form given by

$$(5.17) \quad f_e(\kappa, x) = r_e(\kappa) e^{-\kappa x} \quad e \in \mathcal{E}, x \in \mathbb{R}_+,$$

$$(5.18) \quad f_i(\kappa, x) = r_i^+(\kappa) e^{\kappa x} + r_i^-(\kappa) e^{\kappa(\rho_i - x)} \quad i \in \mathcal{I}, x \in [0, \rho_i],$$

and we want to show that for some $\kappa > 0$, the boundary conditions (5.5) and (5.8) entail that $r_e(\kappa) = r_i^+(\kappa) = r_i^-(\kappa) = 0$ for all $e \in \mathcal{E}, i \in \mathcal{I}$. For $\kappa > 0$, define a column vector $r(\kappa)$ of length $|\mathcal{E}| + 2|\mathcal{I}|$ by

$$r(\kappa) = ((r_e(\kappa), e \in \mathcal{E}), (r_i^+(\kappa), i \in \mathcal{I}), (r_i^-(\kappa), i \in \mathcal{I}))^t,$$

and introduce the $(|\mathcal{E}| + 2|\mathcal{I}|) \times (|\mathcal{E}| + 2|\mathcal{I}|)$ matrices

$$X_\pm(\kappa) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \pm e^{\kappa \rho} \\ 0 & \pm e^{\kappa \rho} & 1 \end{pmatrix}$$

— appropriately modified in case that \mathcal{E} or \mathcal{I} is the empty set — with the $|\mathcal{I}| \times |\mathcal{I}|$ diagonal matrices

$$e^{\kappa \rho} = \text{diag}(e^{\kappa \rho_i}, i \in \mathcal{I}).$$

Then the boundary conditions (5.5), (5.8) for $f(\kappa)$ read

$$(5.19) \quad Z(\kappa)r(\kappa) = 0,$$

with

$$(5.20) \quad Z(\kappa) = (A + \kappa^2 C)X_+(\kappa) + \kappa B X_-(\kappa).$$

Thus, if we can show that for some $\kappa > 0$ the matrix $Z(\kappa)$ is invertible, the proof of the theorem is finished. Note that the matrix-valued function Z is entire in κ , and therefore so is its determinant. Thus, if can show that $\kappa \mapsto \det Z(\kappa)$ does not vanish identically, then it can only vanish on a discrete subset of the complex plane, and for κ in the complement of this set $Z(\kappa)$ is invertible. Write

$$X_{\pm}(\kappa) = 1 \pm \delta X(\kappa),$$

with

$$\delta X(\kappa) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & e^{\kappa\rho} \\ 0 & e^{\kappa\rho} & 0 \end{pmatrix},$$

so that we can write

$$Z(\kappa) = \hat{Z}_+(\kappa)(1 + \delta Z(\kappa)),$$

with

$$\delta Z(\kappa) = \hat{Z}_+(\kappa)^{-1} \hat{Z}_-(\kappa) \delta X(\kappa).$$

Observe that in case that $\mathcal{I} = \emptyset$, we obtain $\delta Z(\kappa) = 0$, and in this case the invertibility of $Z(\kappa)$ for all κ with $\kappa \geq R$ follows from lemma 5.7. Hence we assume from now on that $\mathcal{I} \neq \emptyset$. Lemma 5.7 provides us with the bound

$$\|\hat{Z}_+(\kappa)^{-1} \hat{Z}_-(\kappa)\| \leq \text{const. } |\kappa|^q,$$

for all $\kappa \in \mathbb{C}$, $|\kappa| \geq R$, and for some $q > 0$. On the other hand, we get

$$\|\delta X(\kappa)\| \leq e^{\kappa\rho_0},$$

for all $\kappa \leq 0$ where $\rho_0 = \min_{i \in \mathcal{I}} \rho_i$. Therefore, there exists a constant $R' > 0$ so that for all $\kappa \leq -R'$ we have $\|\delta Z(\kappa)\| < 1$, and therefore for such κ , $Z(\kappa)$ is invertible, i.e., $\det Z(\kappa) \neq 0$. Hence there also exists $\kappa > 0$ so that $Z(\kappa)$ is invertible, and the proof is finished. \square

APPENDIX A. FELLER SEMIGROUPS AND RESOLVENTS

In this appendix we give an account of the Feller property of semigroups and resolvents. The material here seems to be quite well-known, and our presentation of it owes very much to [32], most notably the inversion formula for the Laplace transform, equation (A.3) in connection with lemma A.6. On the other hand, we were not able to locate a reference where the results are collected and stated in the form in which we employ them in the present paper. Therefore we also provide proofs for some of the statements.

Assume that (E, d) is a locally compact separable metric space with Borel σ -algebra denoted by $\mathcal{B}(E)$. $B(E)$ denotes the space of bounded measurable real valued

functions on E , $C_0(E)$ the subspace of continuous functions vanishing at infinity. $B(E)$ and $C_0(E)$ are equipped with the sup-norm $\|\cdot\|$.

The following definition is as in [33]:

Definition A.1. A Feller semigroup is a family $U = (U_t, t \geq 0)$ of positive linear operators on $C_0(E)$ such that

- (i) $U_0 = id$ and $\|U_t\| \leq 1$ for every $t \geq 0$;
- (ii) $U_{t+s} = U_t \circ U_s$ for every pair $s, t \geq 0$;
- (iii) $\lim_{t \downarrow 0} \|U_t f - f\| = 0$ for every $f \in C_0(E)$.

Analogously we define

Definition A.2. A Feller resolvent is a family $R = (R_\lambda, \lambda > 0)$ of positive linear operators on $C_0(E)$ such that

- (i) $\|R_\lambda\| \leq \lambda^{-1}$ for every $\lambda > 0$;
- (ii) $R_\lambda - R_\mu = (\mu - \lambda)R_\lambda \circ R_\mu$ for every pair $\lambda, \mu > 0$;
- (iii) $\lim_{\lambda \rightarrow \infty} \|\lambda R_\lambda f - f\| = 0$ for every $f \in C_0(E)$.

In the sequel we shall focus our attention on semigroups U and resolvents R associated with an E -valued Markov process, and which are *a priori* defined on $B(E)$. (In our notation, we shall not distinguish between U and R as defined on $B(E)$ and their restrictions to $C_0(E)$.)

Let $X = (X_t, t \geq 0)$ be a Markov process with state space E , and let $(P_x, x \in E)$ denote the associated family of probability measures on some measurable space (Ω, \mathcal{A}) so that $P_x(X_0 = x) = 1$. $E_x(\cdot)$ denotes the expectation with respect to P_x . We assume throughout that for every $f \in B(E)$ the mapping

$$(t, x) \mapsto E_x(f(X_t))$$

is measurable from $\mathbb{R}_+ \times E$ into \mathbb{R} . The semigroup U and resolvent R associated with X act on $B(E)$ as follows. For $f \in B(E)$, $x \in E$, $t \geq 0$, and $\lambda > 0$ set

$$(A.1) \quad U_t f(x) = E_x(f(X_t)),$$

$$(A.2) \quad R_\lambda f(x) = \int_0^\infty e^{-\lambda t} U_t f(x) dt.$$

Property (i) of Definitions A.1 and A.2 is obviously satisfied. The semigroup property, (ii) in Definition A.1, follows from the Markov property of X , and this in turn implies the resolvent equation, (ii) of Definition A.2. Moreover, it follows also from the Markov property of X that the semigroup and the resolvent commute. On the other hand, in general neither the property that U or R map $C_0(E)$ into itself, nor the strong continuity property (iii) in Definitions A.1, A.2 hold true on $B(E)$ or on $C_0(E)$.

If W is a subspace of $B(E)$ the resolvent equation shows that the image of W under R_λ is independent of the choice of $\lambda > 0$, and in the sequel we shall denote the image by RW . Furthermore, for simplicity we shall write $UC_0(E) \subset C_0(E)$, if $U_t f \in C_0(E)$ for all $t \geq 0$, $f \in C_0(E)$.

Theorem A.3. *The following statements are equivalent:*

- (i) U is Feller.
- (ii) R is Feller.
- (iii) $UC_0(E) \subset C_0(E)$, and for all $f \in C_0(E)$, $x \in E$, $\lim_{t \downarrow 0} U_t f(x) = f(x)$.
- (iv) $UC_0(E) \subset C_0(E)$, and for all $f \in C_0(E)$, $x \in E$, $\lim_{\lambda \rightarrow \infty} \lambda R_\lambda f(x) = f(x)$.
- (v) $RC_0(E) \subset C_0(E)$, and for all $f \in C_0(E)$, $x \in E$, $\lim_{t \downarrow 0} U_t f(x) = f(x)$.
- (vi) $RC_0(E) \subset C_0(E)$, and for all $f \in C_0(E)$, $x \in E$, $\lim_{\lambda \rightarrow \infty} \lambda R_\lambda f(x) = f(x)$.

We prepare a sequence of lemmas. The first one follows directly from the dominated convergence theorem:

Lemma A.4. *Assume that for $f \in B(E)$, $U_t f \rightarrow f$ as $t \downarrow 0$. Then $\lambda R_\lambda f \rightarrow f$ as $\lambda \rightarrow +\infty$.*

Lemma A.5. *The semigroup U is strongly continuous on $RB(E)$.*

Proof. If strong continuity at $t = 0$ has been shown, strong continuity at $t > 0$ follows from the semigroup property of U , and the fact that U and R commute. Therefore it is enough to show strong continuity at $t = 0$.

Let $f \in B(E)$, $\lambda > 0$, $t > 0$, and consider for $x \in E$ the following computation

$$\begin{aligned}
U_t R_\lambda f(x) - R_\lambda f(x) &= \int_0^\infty e^{-\lambda s} E_x(f(X_{t+s})) ds - \int_0^\infty e^{-\lambda s} E_x(f(X_s)) ds \\
&= e^{\lambda t} \int_t^\infty e^{-\lambda s} E_x(f(X_s)) ds - \int_0^\infty e^{-\lambda s} E_x(f(X_s)) ds \\
&= (e^{\lambda t} - 1) \int_t^\infty e^{-\lambda s} E_x(f(X_s)) ds - \int_0^t e^{-\lambda s} E_x(f(X_s)) ds
\end{aligned}$$

where we used Fubini's theorem and the Markov property of X . Thus we get the following estimation

$$\begin{aligned}
\|U_t R_\lambda f - R_\lambda f\| &\leq \left((e^{\lambda t} - 1) \int_t^\infty e^{-\lambda s} ds + \int_0^t e^{-\lambda s} ds \right) \|f\| \\
&= \frac{2}{\lambda} (1 - e^{-\lambda t}) \|f\|,
\end{aligned}$$

which converges to zero as t decreases to zero. □

For $\lambda > 0$, $t \geq 0$, $f \in B(E)$, $x \in E$ set

$$(A.3) \quad U_t^\lambda f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!} n \lambda e^{n \lambda t} R_{n \lambda} f(x).$$

Observe that, because of $n \lambda \|R_{n \lambda} f\| \leq \|f\|$, the last sum converges in $B(E)$.

For the proof of the next lemma we refer the reader to [32, p. 477 f]:

Lemma A.6. *For all $t \geq 0$, $f \in RB(E)$, $U_t^\lambda f$ converges in $B(E)$ to $U_t f$ as λ tends to infinity.*

Lemma A.7. *If $U_t C_0(E) \subset C_0(E)$ for all $t \geq 0$, then $R_\lambda C_0(E) \subset C_0(E)$, for all $\lambda > 0$. If $R_\lambda C_0(E) \subset C_0(E)$, for some $\lambda > 0$, and $R_\lambda C_0(E)$ is dense in $C_0(E)$, then $U_t C_0(E) \subset C_0(E)$ for all $t \geq 0$.*

Proof. Assume that $U_t C_0(E) \subset C_0(E)$ for all $t \geq 0$, let $f \in C_0(E)$, $x \in E$, and suppose that $(x_n, n \in \mathbb{N})$ is a sequence converging in (E, d) to x . Then a straightforward application of the dominated convergence theorem shows that for every $\lambda > 0$, $R_\lambda f(x_n)$ converges to $R_\lambda f(x)$. Hence $R_\lambda f \in C_0(E)$.

Now assume that $R_\lambda C_0(E) \subset C_0(E)$, for some and therefore for all $\lambda > 0$, and that $R_\lambda C_0(E)$ is dense in $C_0(E)$. Consider $f \in RC_0(E)$, $t > 0$, and for $\lambda > 0$ define $U_t^\lambda f$ as in equation (A.3). Because $R_{n\lambda} f \in C_0(E)$ and the series in formula (A.3) converges uniformly in $x \in E$, we get $U_t^\lambda f \in C_0(E)$. By lemma A.6, we find that $U_t^\lambda f$ converges uniformly to $U_t f$ as $\lambda \rightarrow +\infty$. Hence $U_t f \in C_0(E)$. Since $RC_0(E)$ is dense in $C_0(E)$, U_t is a contraction and $C_0(E)$ is closed, we get that $U_t C_0(E) \subset C_0(E)$ for every $t \geq 0$. \square

The following lemma is proved as a part of Theorem 17.4 in [20] (cf. also the proof of Proposition 2.4 in [33]).

Lemma A.8. *Assume that $RC_0(E) \subset C_0(E)$, and that for all $x \in E$, $f \in C_0(E)$, $\lim_{\lambda \rightarrow \infty} \lambda R_\lambda f(x) = f(x)$. Then $RC_0(E)$ is dense in $C_0(E)$.*

If for all $f \in C_0(E)$, $x \in E$, $U_t f(x)$ converges to $f(x)$ as t decreases to zero, then similarly as in the proof of lemma A.4 we get that $\lambda R_\lambda f(x)$ converges to $f(x)$ as $\lambda \rightarrow +\infty$. Thus we obtain the following

Corollary A.9. *Assume that $RC_0(E) \subset C_0(E)$, and that for all $x \in E$, $f \in C_0(E)$, $\lim_{t \downarrow 0} U_t f(x) = f(x)$. Then $RC_0(E)$ is dense in $C_0(E)$.*

Now we can come to the

Proof of theorem A.3. We show first the equivalence of statements (i), (ii), (iv), and (vi):

“(i) \Rightarrow (ii)” Assume that U is Feller. From lemma A.7 it follows that $R_\lambda C_0(E) \subset C_0(E)$, $\lambda > 0$. Let $f \in C_0(E)$. Since U is strongly continuous on $C_0(E)$, lemma A.4 implies that $\lambda R_\lambda f$ converges to f as λ tends to $+\infty$. Hence R is Feller.

“(ii) \Rightarrow (vi)” This is trivial.

“(vi) \Rightarrow (iv)” By lemma A.8, $RC_0(E)$ is dense in $C_0(E)$, and therefore lemma A.7 entails that $UC_0(E) \subset C_0(E)$.

“(iv) \Rightarrow (i)” By lemmas A.7 and A.8, $RC_0(E)$ is dense in $C_0(E)$, and therefore by lemma A.5 U is strongly continuous on $C_0(E)$. Thus U is Feller.

Now we prove the equivalence of (i), (iii), and (v):

“(i) \Rightarrow (iii)” This is trivial.

“(iii) \Rightarrow (v)” This follows directly from Lemma A.7.

“(v) \Rightarrow (i)” By corollary A.9, $RC_0(E)$ is dense in $C_0(E)$, hence it follows from lemma A.7 that $UC_0(E) \subset C_0(E)$. Furthermore, lemma A.5 implies the strong continuity of U on $RC_0(E)$, and by density therefore on $C_0(E)$. (i) follows. \square

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