CYCLIC CONTRACTIONS AND BEST PROXIMITY PAIR THEOREMS

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Abstract. In this paper we introduce a notion called cyclically complete pair for a pair (A, B) of subsets of a metric space. A necessary condition is given for a cyclic contraction T on $A \cup B$ to have a unique point x in A satisfying $d(x, Tx) = \mathrm{dist}(A, B)$, known as best proximity point. We also prove that for any $x_0 \in A$, the Picard's iterates $\{T^{2n}x_0\}$ converges to the unique best proximity point x in A and the Picard's iterates $\{T^{2n+1}x_0\}$ converges to Tx.

1. Introduction and Preliminaries

Let (A, B) be a pair of subsets of a metric space X. We consider a mapping $T : A \cup B \to X$ satisfying $TA \subset B$ and $TB \subset A$ (or $TA \subset A$ and $TB \subset B$). If T is a contraction, that is there is an $\alpha \in (0, 1)$ such that

$$d(Tx, Ty) \leq \alpha d(x, y), \quad \text{for } x \in A \text{ and } y \in B$$

then $A \cap B \neq \emptyset$ and for any $x_0 \in A \cap B$ the iterates $\{T^n x_0\}$ converges to the unique fixed point of T ([5]).

We extend the Banach contraction theorem to a class of mappings, called *cyclic contraction mappings* (see Definition 1.1). Let T be a self map on $A \cup B$ with $TA \subset B$ and $TB \subset A$. In [2], Eldred and Veeramani gave a sufficient condition (Theorem 3.10, [2]) for the existence and uniqueness of a best proximity point for a cyclic contraction map T on a uniformly convex Banach space. In [1], Sadiq Basha introduced a class of mappings called *proximal contraction mappings* (see Definition 1.2) $T : A \to B$, and there by obtained a sequence (Theorem 3.1, [1]) in A, which converges to the unique best proximity point

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under suitable assumptions. It is easy to observe that, results in [1] are not applicable, if $\text{dist}(A,B) \geq \frac{1}{2}\delta(A_0)$. We prove an extension of the Banach contraction theorem for cyclic contraction mappings in a metric space setting. We give a necessary condition for the existence of a unique best proximity point x in A for such a cyclic contraction mapping T. Also we prove that the Picard's iterates $\{T^{2n}x_0\}$, for any $x_0 \in A$, converges to the unique best proximity point x in A for such a mapping T. This recovers the main result of [2]. Further the main theorem of this work (Theorem 3.8) proves that, for any $x_0 \in A$ the sequences $\{T^{2n}x_0\},\ \{T^{2n+1}x_0\}$ converge to $x,\ Tx$ respectively and $x,\ Tx$ are the unique fixed points of T^2 in $A,\ B$ respectively. We also prove that the sequences $\{T^n x_0\}$ and $\{T^n y_0\}$, for any $(x_0, y_0) \in A \times B$, converge to the unique fixed points x and y of a cyclic contraction $T : A \cup B \to A \cup B$ satisfying $TA \subset A,\ TB \subset B$ in A and B respectively with $d(x,y) = \text{dist}(A,B)$.

In this direction we introduce a notion called cyclically complete pair for a pair (A,B) of subsets of a metric space (which coincides with the classical notion of completeness, if $A = B$). We also investigate some of the basic properties of (A,B) in this situation.

Let (X,d) be a metric space and A,B be nonempty subsets of X. We shall say that (A,B) satisfies a property p if each of the sets A and B has the same property p. Also (A,B) is said to be a **semi sharp proximinal pair** if for each $x \in A$ there exists at most one $x' \in B$ such that $d(x,x') = \text{dist}(A,B) := \inf\{d(u,v) : u \in A,\ v \in B\}$. Using a result (Lemma 2.5, [6]) proved in [6] we infer that any closed convex pair (A,B) in a strictly convex Banach space is a semi sharp proximinal pair. Also such examples are given, in section 2, in nonstrictly convex Banach spaces. Let T be a self map on $A \cup B$ with $TA \subset B$ and $TB \subset A$. We say that a point $x \in A \cup B$ is a best proximity point for T, if $d(x,Tx) = \text{dist}(A,B)$. In this case we say that the pair (x,Tx) is best proximity pair for T. If $\text{dist}(A,B) = 0$ then a best proximity point of T turns out to be a fixed point of T. In this work we adopt the following notations and definitions:

$$
\begin{aligned}
A_0 &= \{x \in A : d(x,y) = \text{dist}(A,B),\ \text{for some y in } B\};\\
B_0 &= \{y \in A : d(x,y) = \text{dist}(A,B),\ \text{for some x in } A\};\\
\delta(A,B) &= \sup\{d(x,y) : x \in A,\ y \in B\} \text{ and } \delta(A) = \delta(A,A).
\end{aligned}
$$

Definition 1.1. [2] *A mapping $T : A \cup B \to A \cup B$ is said to be a* **cyclic contraction***, if it satisfies:*

(1) *$TA \subset B$ and $TB \subset A$.*
(2) *For some $\alpha \in (0,1)$ we have $d(Tx,Ty) \leq \alpha d(x,y) + (1-\alpha)\operatorname{dist}(A,B)$, for all $x \in A,\ y \in B$.*

It is easy to see that, if T is a cyclic contraction on $A \cup B$, then $d(Tx,Ty) \leq d(x,y)$, for any $x \in A$ and $y \in B$. Further if $\operatorname{dist}(A,B) < d(x,y)$ then $d(Tx,Ty) < d(x,y)$.

Definition 1.2. [1] *A mapping $T : A \to B$ is said to be a* **proximal contraction***, if there exists a nonnegative real number $\alpha < 1$ such that*

$$d(u,Tx) = d(Tx,Ty) + d(v,Ty) \leq \alpha d(x,y)$$

whenever x and y are distinct elements in A satisfying the conditions

$$d(u,Tx) = \operatorname{dist}(A,B) \text{ and } d(v,Ty) = \operatorname{dist}(A,B)$$

for some $u, v \in A$.

If $A_0 = \{x\}$ then x is the best proximity point of T. Further if $x \neq y \in A_0$, then there exists $u, v \in A$ such that $d(Tx,u) = \operatorname{dist}(A,B)$ and $d(Ty,v) = \operatorname{dist}(A,B)$. In this case, $2\operatorname{dist}(A,B) \leq d(Tx,u) + d(Tx,Ty) + d(Ty,v) \leq \alpha d(x,y) \leq \alpha\delta(A_0)$. Hence under the conditions stated in the above definition (Definition 1.2) and with the assumption $TA_0 \subset B_0$, we have $\operatorname{dist}(A,B) < \frac{1}{2}\delta(A_0)$. In this sense the results obtained in [1] are very restrictive.

2. Cyclically completeness

Let (A,B) be a pair of nonempty subsets of a metric space X. In this section we give some properties of cyclically Cauchy sequences. The notion of *cyclically Cauchy sequences* (see Definition 2.1) was introduced in [4]. Also the author proposed a version of completeness on (A,B). In this paper an extension of the Banach contraction principle for cyclic contraction mappings is given. To achieve this we introduce a notion of cyclically complete pair and investigate some basic properties for such pairs.

Definition 2.1. [4] *Let X be a metric space, A and B nonempty of subsets of X. A sequence $\{x_n\}_{n=0}^{\infty}$ in $A \cup B$ with $x_{2n} \in A$ and $x_{2n+1} \in B$ for all $n \in \mathbb{N}$ is said to be* **cyclically Cauchy sequence** *if, for every $\epsilon > 0$ there exist an $N \in \mathbb{N}$ such that*

$$d(x_n, x_m) < \operatorname{dist}(A, B) + \epsilon, \text{ when } n \text{ is even, } m \text{ is odd and } n, m \geq N.$$

Remark 2.2. *If $\operatorname{dist}(A, B) = 0$, then a sequence $\{x_n\}$ in $A \cup B$ is cyclically Cauchy if and only if the sequence $\{x_n\}$ is a Cauchy sequence.*

Before stating some properties of cyclically Cauchy sequences we look at some example.

Example 2.3. *Let $X = (l_p, \|\cdot\|_p),\ 1 \leq p \leq \infty$ and $A = \{0\},\ B = \{x \in X : \|x\| \geq 1\}$. Then the sequence $\{x_n\}$ defined as*

$$x_n := \begin{cases} (1 + \frac{1}{n})e_n & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

is a cyclically Cauchy sequence.

Example 2.4. *Let $A = \{(x, y) : x \leq 0,\ y \in \mathbb{R}\}$ and $B = \{(x, y) : x \geq 1,\ y \in \mathbb{R}\}$ in $(\mathbb{R}^2, \|\cdot\|_2)$. Then the sequence $\{x_n\}$ is not cyclically Cauchy even though $d(x_n, x_{n+1}) \to \operatorname{dist}(A, B)$, as $n \to \infty$, if $x_{2n} = (-\frac{1}{n}, n)$ and $x_{2n+1} = (1, n + \frac{1}{n})$ for all $n \in \mathbb{N}$.*

The following Lemma ensures the boundedness of a cyclically Cauchy sequence.

Lemma 2.5. *Any cyclically Cauchy sequence in a pair (A, B) of subsets of metric space is bounded.*

Proof. Let $\{x_n\}$ be a cyclically Cauchy sequence in $A \cup B$. There exists $N \in \mathbb{N}$, such that $d(x_{2n}, x_{2N+1}) < \operatorname{dist}(A, B) + 1$ for all $n \geq N$. Therefore for all $n \in \mathbb{N},\ x_{2n} \in B(x_{2N+1}, r)$, where $r = \max\{d(x_2, x_{2N+1}), d(x_4, x_{2N+1}), \ldots, d(x_{2N}, x_{2N+1}), \operatorname{dist}(A, B) + 1\}$. So that $\{x_{2n}\}$ is bounded. similarly one can prove that the sequences $\{x_{2n+1}\}$ is a bounded sequence and hence $\{x_n\}$ is bounded.

□

In general the converse of the above statement need not be true.

Example 2.6. *Let* $A = \{\lambda(0,0)+(1-\lambda)(0,1) : \lambda \in [0,1]\}$ *and* $A = \{\lambda(1,0)+(1-\lambda)(1,1) : \lambda \in [0,1]\}$ *in* $(\mathbb{R}^2, \|\cdot\|_2)$*. The sequences* $\{x_n\}$ *is a bounded sequence but not a cyclically Cauchy sequence, where*

$$x_n := \begin{cases} (0, 1-\frac{1}{n}) & \text{if } n \text{ is even,} \\ (1, \frac{1}{n}) & \text{if } n \text{ is even.} \end{cases}$$

It is to be noted that a cyclically Cauchy sequence need not have convergent subsequence even if A and B are closed subsets of a complete metric space.

Example 2.7. *Let* $X = (l_p, \|\cdot\|_p)$*, for* $1 \leq p \leq \infty$ *and* $A = \{e_{2n} : n \in \mathbb{N}\}$*,* $B = \{e_{2n+1} : n \in \mathbb{N}\}$*. Then The cyclic Cauchy sequence* $\{x_n\}$ *does not have any convergent subsequence, where* $x_n = e_n$*, for all* $n \in \mathbb{N}$*.*

Now we define the notion of cyclically complete pair for a pair of sets in a metric space..

Definition 2.8. *A pair* (A, B) *of subsets of a metric space is said to be* **cyclically complete** *if every cyclically Cauchy sequence* $\{x_n\}$ *in* $A \cup B$ *has one of the following:*

(1) *Both sequences* $\{x_{2n}\}$ *and* $\{x_{2n+1}\}$ *have convergent subsequences in* A *and* B *respectively.*
(2) *There exists* $N \in \mathbb{N}$ *such that* $d(x_{2n}, x_{2m+1}) = \text{dist}(A, B),\ \forall\ n, m \geq N$*.*

Before proving main properties of cyclically complete pair, we look at some examples.

Examples 2.9.

(1) *If* A *and* B *are closed subset of a complete metric space* X *with* $\text{dist}(A, B) = 0$*, then* (A, B) *is a cyclically complete pair (in particular* (A, A) *is a cyclically complete pair, if* A *is a closed subset of* X*).*
(2) *Any boundedly compact pair in a metric space is cyclically complete.*
(3) *The pair in Example 2.7 is a cyclically complete because* $d(x, y) = \text{dist}(A, B)$ *for all* $x \in A$ *and* $y \in B$*.*
(4) *Let* $(\mathbb{R}^2, \|\cdot\|_2)$ *and* $A := \{(x, 0) : x \in \mathbb{R}\}$*,* $B := \{(x, y) : y \geq \frac{1}{x} \text{ and } x \geq 0\}$*. One can notice that even though* $\text{dist}(A, B) = 0$*, there is no cyclically Cauchy sequence in* $A \cup B$ *and hence* (A, B) *is cyclically complete.*

(5) *Let* $(\mathbb{R}^2, \|\cdot\|_2)$ *and* $A := \{(x,y) : y \geq \frac{1}{-x}$ *and* $x < 0\}$, $B := \{(x, 1+y) : y \geq \frac{1}{x}$ *and* $x \geq 0\}$. *One can notice that even though* $\text{dist}(A,B) = 1$, *there is no cyclically Cauchy sequence in* $A \cup B$ *and hence* (A,B) *is cyclically complete.*

(6) *The pair* (A,B) *in Example 2.3 is not a cyclically complete pair, because neither the sequence* $\{x_{2n+1}\}$ *has a convergent subsequence nor* $d(x_{2n}, x_{2m+1}) = 0$ *for any* $n, m \in \mathbb{N}$.

The following Theorem gives a necessary and sufficient condition for A_0 to be a nonempty.

Theorem 2.10. *Let* (A,B) *be a cyclically complete pair of subsets of a metric space* X. *Then there exists a cyclically Cauchy sequence if and only if* A_0 *is a non empty subset of* X.

Proof. Let $\{x_n\}$ be a cyclically Cauchy sequence in $A \cup B$. Suppose there exists convergent subsequences $\{x_{2n_k}\}$ and $\{x_{2m_k+1}\}$ of $\{x_{2n}\}$ and $\{x_{2n+1}\}$, that converges to $x \in A$ and $y \in B$ respectively. Then $\text{dist}(A,B) \leq d(x,y) \leq \lim_{k\to\infty} d(x_{2n_k}, x_{2m_k+1}) = \text{dist}(A,B)$. That is $d(x,y) = \text{dist}(A,B)$. Also if there exists $N \in \mathbb{N}$ such that $d(x_{2n}, x_{2m+1}) = \text{dist}(A,B)$. Hence $A_0 \neq \emptyset$ and so is B_0. For sufficiency, let $x \in A$ and $y \in B$ be such that $d(x,y) = \text{dist}(A,B)$. If for $n \in \mathbb{N}$, define $x_{2n} = x$ and $x_{2n+1} = y$, then $\{x_n\}$ is a cyclically Cauchy sequence. □

If A and B are closed subsets in a complete metric space, the pair (A,B) need not be a cyclically complete pair. The following example illustrates the same.

Example 2.11. *Let* $(l_p, \|\cdot\|_p)$ *for* $1 \leq p < \infty$ *and* $A := \{0\}$, $B := \{(1 + \frac{1}{n})e_n : n \in \mathbb{N}\}$. *It is easy to see that* (A,B) *is closed pair of* l_p. *It is easy see that* (A,B) *is not cyclically complete, because for the cyclically Cauchy sequence* $\{x_n\} \in A \cup B$, *neither the sequence* $\{x_{2n+1}\}$ *has any convergent subsequence nor* $d(x_{2n}, x_{2m+1}) = \text{dist}(A,B)$, *where*

$$x_n := \begin{cases} (1 + \frac{1}{n})e_n & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even .} \end{cases}$$

The following Theorem ensure the closedness of A_0 and B_0, for a cyclically complete pair.

Theorem 2.12. *Let A and B be a subsets of a metric space. If (A, B) is cyclically complete, then A_0 and B_0 are closed subsets of X.*

Proof. Let $\{x_n\}$ be a sequence in A_0 such that $x_n \to x$ in X. For $n \in \mathbb{N}$ get $x'_n \in B_0$ such that $d(x_n, x'_n) = \operatorname{dist}(A, B)$. For $n \in \mathbb{N}$ define

$$y_n := \begin{cases} x_m & \text{if } n = 2m \text{ for some } m \in \mathbb{N}, \\ x'_m & \text{if } n = 2m+1 \text{ for some } m \in \mathbb{N}. \end{cases}$$

Now $d(y_{2n}, y_{2m+1}) = d(x_n, x'_m) \leq d(x_n, x) + d(x, x_m) + d(x_m, x'_m)$ and hence $\{y_n\}$ is a cyclically Cauchy sequence. Suppose $\{x_n\}$ and $\{x'_n\}$ has convergent subsequences, converges to x and y respectively, then $d(x, y) = \operatorname{dist}(A, B)$. If not, there exists $N \in \mathbb{N}$ such that $d(x_n, x'_N) = \operatorname{dist}(A, B)$ for all $n \geq N$, then $d(x, x'_N) = \lim_{n\to\infty} d(x_n, x'_N) = \operatorname{dist}(A, B)$. That is $x \in A_0$. In a similar fashion one can prove B_0 is also a closed set. □

Example 2.3 show that the converse of Theorem 2.12 need not be true. For a cyclically complete pair (A, B) in a metric space X, there may exist a cyclically Cauchy sequence $\{x_n\}$ in $A \cup B$ such that either $\{x_{2n}\}$ have two different convergent subsequences which converges to different points. Following Examples illustrates the same.

Examples 2.13.

(1) *Let $X = (\mathbb{R}^3, \|\cdot\|_2)$. $A := \{(x, y, z) \in X : x \leq 0,\ y^2+z^2 = 1\}$, $B := \{(0, 0, 0)\}$. It is easy to see that $\operatorname{dist}(A, B) = 1$. Then $\{x_{2n}\}$ has two different different convergent subsequences $\{(\frac{-1}{n}, 1, 0)\}$ and $\{(\frac{1}{n}, -1, 0)\}$ which converge to $(0, 1, 0)$ and $(0, -1, 0)$ respectively, for the cyclically Cauchy sequence $\{x_n\}$, where*

$$x_{2n} := \begin{cases} (\frac{-1}{n}, 1, 0) & \textit{if } n \textit{ is odd}, \\ (\frac{1}{n}, -1, 0) & \textit{if } n \textit{ is even}, \end{cases}$$

$$x_{2n+1} = (0, 0, 0) \textit{ for all } n \in \mathbb{N}$$

.

(2) *Let $A := \{e_{2n} : n \in \mathbb{N}\}$, $B := \{e_{2n+1} : n \in \mathbb{N}\}$ be subsets in $(l_\infty, \|\cdot\|_\infty)$. The sequence $\{x_n\}$ is a cyclically Cauchy, where $x_n = e_n,\ \forall\ n$. Also one can observe that neither $\{x_{2n}\}$ nor $\{x_{2n+1}\}$ have a convergent subsequence. Also the sequence*

$\{y_n\}$ is cyclically Cauchy, where

$$y_n := \begin{cases} e_n & \text{if } n \text{ is even} \\ e_2 & \text{if } n \text{ is odd} \end{cases}$$

Then one can observe that $\{y_{2n}\}$ does not have convergent subsequence even though $\{y_{2n+1}\}$ is a convergent sequence in B.

Now we prove that every closed convex pair is cyclically complete in the setting of a uniformly convex Banach space.

Proposition 2.14. *Any nonempty closed convex pair (A, B) in a uniformly convex Banach spaces is cyclically complete. Further, for any of cyclic Cauchy sequence $\{x_n\}$, the sequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ converge to $x \in A$ and $y \in B$ respectively.*

Proof. Let $\{x_n\}$ be a cyclically Cauchy sequence in $A \cup B$. Suppose $\{x_{2n}\}$ is not a Cauchy sequence. Then there exists $\epsilon_0 > 0$ and subsequences $\{x_{2n_k}\}$ and $\{x_{2m_k}\}$ of $\{x_{2n}\}$ such that

$$d(x_{2n_k}, x_{2m_k}) \geq \epsilon_0, \quad \text{for all } k \in \mathbb{N}.$$

Also one can observe that $d(x_{2n_k}, x_{2k+1}) \to \text{dist}(A, B)$ and $d(x_{2m_k}, x_{2k+1}) \to \text{dist}(A, B)$, as $k \to \infty$. By Lemma 3.7, in [2], there exists $N_1 \in \mathbb{N}$ such that $d(x_{2n_k}, x_{2m_k}) < \epsilon_0$ for all $k \geq N_1$, a contradiction. That is $\{x_{2n}\}$ is a Cauchy sequence, and hence $\{x_{2n}\}$ converges in A. Therefore $x_{2n} \to x$ for some $x \in A$. In a similar fashion one can prove that $x_{2n+1} \to y$ in A. □

Theorem 2.15. *Let (A, B) be a cyclically complete semi sharp proximinal pair in a metric space X. If $\{x_n\}$ is a cyclically Cauchy sequence then $x_{2n} \to x$, for some $x \in A$ and $x_{2n+1} \to y$, for some $y \in B$. Further $d(x, y) = \text{dist}(A, B)$.*

Proof. Let $\{x_n\}$ be a cyclically Cauchy sequence in $A \cup B$. If there exist an $N \in \mathbb{N}$ such that $d(x_{2n}, x_{2m+1}) = \text{dist}(A, B)$, for all $n, m \geq N$. Then by semi sharp proximinality of (A, B), $x_{2n} = x_{2N}$ and $x_{2n+1} = x_{2N+1}$ for all $n \geq N$. Therefore $x_{2n} \to x_{2N}$ and $x_{2n+1} \to x_{2N+1}$, as $n \to \infty$. Hence it is enough to prove in the case when $\{x_{2n}\}$ and $\{x_{2n+1}\}$ have convergent subsequences. Fix a convergent subsequence $\{x_{2n_k+1}\}$ of $\{x_{2n+1}\}$, that converge to $y \in B$. Let $\{x_{2m_k}\}$ and $\{x_{2l_k}\}$ be convergent subsequence of $\{x_{2n}\}$, that converges to x_1 and $x_2 \in B$ respectively. Now $d(x_1, y) = \lim_{k\to\infty} d(x_{2m_k}, y) = \text{dist}(A, B) = \lim_{k\to\infty} d(x_{2l_k}, y) = d(x_2, y)$. By the semi sharp proximinality of (A, B), $x_1 = x_2$. That is any two convergent subsequences of $\{x_{2n}\}$ converges to a point say to x, with $d(x, y) = \text{dist}(A, B)$. Suppose $\{x_{2n}\}$ is not Cauchy, then there exists $\epsilon_0 > 0$ and two subsequences $\{x_{2n_p}\}, \{x_{2m_p}\}$ of $\{x_{2n}\}$ such that

$$d(x_{2n_p}, x_{2m_p}) \geq \epsilon_0, \text{ for all } p \in \mathbb{N}.$$

Now consider the sequence $\{y_p\}$, where

$$y_p := \begin{cases} x_{2n_p} & \text{if } p \text{ is even} \\ x_p & \text{if } p \text{ is odd} \end{cases}$$

Then it is easy to see that the sequence $\{y_p\}$ is a cyclically Cauchy sequence and hence $\{x_{2n_p}\}$ has a convergent subsequence. Similarly $\{x_{2m_p}\}$ has a convergent subsequence. Since (A, B) is a cyclically complete pair, $\{x_{2n_p}\}$ and $\{x_{2m_p}\}$ have convergent subsequences, that converges to x. Hence there exists $P \in \mathbb{N}$ such that $d(x_{2n_P}, x) < \frac{\epsilon_0}{2}$ and $d(x_{2m_P}, x) < \frac{\epsilon_0}{2}$. Now $d(x_{2n_P}, x_{2m_P}) \leq d(x_{2n_P}, x) + d(x_{2m_P}, x) < \frac{\epsilon_0}{2} + \frac{\epsilon_0}{2} = \epsilon_0$, a contraction. That is $\{x_{2n}\}$ Cauchy. Also $\{x_{2n}\}$ has a convergent subsequence and hence $x_{2n} \to x$ in A. In a similar fashion one can show $x_{2n+1} \to y$ in B. $\square$

As a particular case we get the following Corollary.

Corollary 2.16. *Let (A, B) be a nonempty convex cyclically complete pair in a strictly convex Banach space X. If $\{x_n\}$ is cyclically Cauchy then $x_n \to x$, for some $x \in A$ and $x_{2n+1} \to y$, for some $y \in B$, with $d(x, y) = \text{dist}(A, B)$.*

We conclude this section by giving an example to illustrate Theorem 2.15

Examples 2.17.

(1) *Let X be $\mathbb{R}^3$ with the l_1 norm. If A is the line segment joining points $(0, 0, 0)$ and $(0, 1, 0)$ and B is the line segment joining points $(0, 0, 1)$ and $(1, 1, 0)$, then it is shown in [3] that for each $x \in A($ or $\in B)$ there exists a unique $x' \in B($ respectively $\in B)$ such that $d(x, x') = \text{dist}(A, B)$. Hence (A, B) is a semi*

sharp proximinal pair. The sequence $\{x_n\}$ is a cyclically Cauchy sequence and $x_{2n} \to (0,1,0)$ and $x_{2n+1} \to (1,1,0)$, where

$$x_n := \begin{cases} (0, 1-\frac{1}{n}, 0) & \textit{if } n \textit{ is even} \\ (1-\frac{1}{n}, 1-\frac{1}{n}, 0) & \textit{if } n \textit{ is odd} \end{cases}$$

Hence (A, B) is a cyclically complete pair in $(\mathbb{R}^3, l_1)$ and for any cyclically Cauchy sequence $\{x_n\}$, the subsequence $\{x_{2n}\}$ and $\{x_{2n+1}\}$ converges in A and B respectively.

(2) *Consider the space X of all complex valued continuous functions on $[0,1]$ with sup norm, i.e., $X = (\mathcal{C}[0,1], \|.\|_\infty)$.*
$A := \{f_\alpha : \alpha \in [0,1]\}$ and $B := \{g_\alpha : \alpha \in [0,1]\}$, where

$$f_\alpha(t) := \begin{cases} 2i\alpha t, & \textit{if } t \in [0, \frac{1}{2}] \\ 2i\alpha(1-t), & \textit{if } t \in [\frac{1}{2}, 1] \end{cases}$$

$$g_\alpha(t) := \begin{cases} 1 + \alpha(t - \frac{1}{2}) + 2i\alpha t, & \textit{if } t \in [0, \frac{1}{2}] \\ 1 - \alpha(t - \frac{1}{2}) + 2i\alpha(1-t), & \textit{if } t \in [\frac{1}{2}, 1] \end{cases}$$

It is shown in [6] that for each $x \in A($ or $\in B)$ there exists a unique $x' \in B($ respectively $\in B)$ such that $d(x, x') = \operatorname{dist}(A, B)$ and the pair (A, B) is a compact convex. Hence (A, B) is a cyclically complete pair in $\mathcal{C}[0,1]$ and for any cyclically Cauchy sequence $\{x_n\}$, the subsequence $\{x_{2n}\}$ and $\{x_{2n+1}\}$ converges in A and B respectively.

3. Existence of Best proximity points

Let (A, B) be a pair of subsets of a metric space X. Suppose $T : A \cup B \to A \cup B$ is a map satisfying $TA \subset B$ and $TB \subset A$. We prove the existence of a best proximity point for such a cyclic contraction T.

Theorem 3.1. *Let (A, B) be a pair of subsets of a metric space X and let $T : A \cup B \to A \cup B$ be a cyclic contraction satisfying $TA \subset B$ and $TB \subset A$. If (A, B) is cyclically complete then there exists $(x, y) \in A \times B$ such that $d(x, Tx) = \operatorname{dist}(A, B)$ and $d(y, Ty) = \operatorname{dist}(A.B)$ with $d(x, y) = \operatorname{dist}(A, B)$.*

Proof. Let $x_0 \in A$, define $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$. It is clear that $\{x_{2n}\} \subset A$ and $\{x_{2n+1}\} \subset B$.

claim: Any convergent subsequences of $\{x_{2n}\}$ and $\{x_{2n+1}\}$ converges to best proximity points say $x \in A$ and $y \in B$ respectively with $d(x, y) = \operatorname{dist}(A, B)$.

Let $\{x_{2n_k}\}$ be a convergent subsequence of $\{x_{2n}\}$, which converges to $x \in A$. Now $\operatorname{dist}(A, B) \leq d(x_{2n_k-1}, x) \leq d(x_{2n_k-1}, x_{2n_k}) + d(x_{2n_k}, x)$, that is $d(x_{2n_k-1}, x) \to \operatorname{dist}(A, B)$ as $k \to \infty$. Now $\operatorname{dist}(A, B) \leq d(x, Tx) = \lim_{k\to\infty} d(x_{2n_k}, Tx) \leq \lim_{k\to\infty} d(x_{2n_k-1}, x) = \operatorname{dist}(A, B)$. In a similar fashion one can prove, if $\{x_{2m_k+1}\}$ is a convergent subsequence of $\{x_{2n+1}\}$, which converges to $y \in A$ then $d(y, Ty) = \operatorname{dist}(A, B)$. Also $d(x, y) = \lim_{k\to\infty} d(x_{2n_k}, x_{2m_k+1}) = \operatorname{dist}(A, B)$. Now we prove that the sequence $\{x_{2n}\}$ is bounded. Suppose not, for $M = \dfrac{2\alpha^2 d(x_1, x_2)}{1-\alpha^2} + \operatorname{dist}(A, B)$, there exists $n \in \mathbb{N}$, such that

$$d(x_3, x_{2n-2}) \leq M \text{ and } d(x_3, x_{2n}) > M.$$

Now

$$\begin{aligned}
M < d(x_3, x_{2n}) &\leq \alpha^2 d(x_1, x_{2n-2}) + (1-\alpha^2)\operatorname{dist}(A, B) \\
&\leq \alpha^2 (d(x_1, x_2) + d(x_2, x_{2n-2})) + (1-\alpha^2)\operatorname{dist}(A, B) \\
&\leq \alpha^2 (d(x_1, x_2) + d(x_2, x_3) + M) + (1-\alpha^2)\operatorname{dist}(A, B) \\
&\leq \alpha^2 (d(x_1, x_2) + d(x_1, x_2) + M) + (1-\alpha^2)\operatorname{dist}(A, B) \\
&\leq \alpha^2 (2d(x_0, x_1) + M) + (1-\alpha^2)\operatorname{dist}(A, B) \\
&\leq \alpha^2 M + (1-\alpha^2)M = M
\end{aligned}$$

a contradiction. A similar way one can prove $\{x_{2n+1}\}$ is bounded and hence the sequence $\{x_n\}$ is bounded. For any $n \geq m$ in $\mathbb{N}$, $d(x_{2n}, x_{2m+1}) \leq \alpha^m d(x_0, x_{2(n-m)+1}) + (1-\alpha^m)\operatorname{dist}(A, B)$. Therefore the sequence $\{x_n\}$ is a cyclically Cauchy sequence in $A \cup B$ as $0 < \alpha < 1$. Since (A, B) is cyclically complete, either both the sequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ have convergent subsequences or there exists $N \in \mathbb{N}$ such that $d(x_{2n}, x_{2m+1}) = \operatorname{dist}(A, B)$, $\forall\ n, m \geq N$. For the second case, the pair (x_{2N}, x_{2N+1}) satisfies the conclusions. Suppose both the sequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ have convergent subsequences, converges to x and y respectively, then by claim, $d(x, Tx) = \operatorname{dist}(A, B) = d(y, Ty)$ and $d(x, y) = \operatorname{dist}(A, B)$. □

The following examples illustrates Theorem 3.1.

Example 3.2. *Let* $X = (l_\infty, \|\cdot\|_\infty)$ *and let* $A = \{e_{2n} : n \in \mathbb{N}\}$ *and* $B = \{e_{2n+1} :\in \mathbb{N}\}$*. Since* $d(x,y) = \operatorname{dist}(A, B,$ *for all* $x \in A$ *and* $y \in B$*any map* $T : A \cup B \to A \cup B$ *satisfying* $TA \subset B$*,* $TB \subset A$ *is a cyclic contraction. One can notice that each point of* A *is a best proximity point for* T*..*

Now we look at a generalization of the Banach contraction theorem in this situation. For $x \in A$, define $[x] = \{y \in B : d(x,y) = \operatorname{dist}(A,B)\}$ and a similar way for we have $[y] = \{u \in A : d(u,y) = \operatorname{dist}(A,B)\}$, for $y \in B$. It is easy to see that, if $x_i \in [x]$ for $i = 1,2$ for some $x \in A \cup B$, Then $x \in \bigcap_{i=1,2} [x_i]$. Suppose $T : A \cup B \to A \cup B$ is a cyclic contraction satisfying $TA \subset B$ and $TB \subset A$. The following Proposition gives a necessary condition for the existence of a unique best proximity point of T.

Proposition 3.3. *Let* (A, B) *be a pair of subsets of a metric space* X*. If there exists* $x \in A$ *such that* $[x]$ *contains two different points say* x_1 *and* x_2 *with* $\bigcap_{i=1,2} [x_i]$ *contains a point other then* x*. Then there exists a map* $T : A \cup B \to A \cup B$ *such that* $TA \subset B$ *and* $TB \subset A$ *satisfying:*

(1) T *is a cyclic contraction mapping.*
(2) T *has two distinct best proximity points in* A*.*
(3) *For any* $x_0 \in A$*, neither of the sequences* $\{T^{2n}x_0\}$ *and* $\{T^{2n+1}x_0\}$ *converges.*

Proof. Let $x \in A$ such that $x_1 \neq x_2$ in $[x]$ and $\cap_{i=1,2}[x_i]$ contains an element say $y \neq x$. Define $T : A \cup B \to A \cup B$ as

$$T(u) := \begin{cases} x_1 & \text{if } u \in A \text{ and } u = x \\ x_2 & \text{if } u \in A \text{ and } u \neq x \\ y & \text{if } u \in B \text{ and } u = x_1 \\ x & \text{if } u \in B \text{ and } u \neq x_1. \end{cases}$$

Notice that $TA \subset B$ and $TB \subset A$. Also for any $u \in A$ and $v \in B$, $d(Tu, Tv) = \operatorname{dist}(A,B) = \alpha \operatorname{dist}(A,B) + (1-\alpha)\operatorname{dist}(A,B) \leq \alpha d(u,v) + (1-\alpha)\operatorname{dist}(A,B)$, for all $\alpha \in [0,1]$. Hence T is a cyclic contraction. It is easy see that x and y are different best proximity points of T in A. For any fixed $u \in A$, either $Tu = x_1$ or $Tu = x_2$. If $Tu = x_1$, then $T^2u = Tx_1 = y,\ T^3u = Ty = x_2,\ T^4u = Tx_2 = x, \cdots$. Therefore the sequence $\{T^{2n}u\}_{n=1}^{\infty}$ is $(u, y, x, y, x, \cdots)$ and hence $\{T^{2n}u\}$ diverges. In a similar fashion, one can prove $\{T^{2n}u\}_{n=1}^{\infty}$ is $(u, x, y, x, y, \cdots)$ and $\{T^{2n}u\}$ diverges, if $Tu = x_2$. Hence $\{T^{2n}u\}$ is a divergent sequence for all $u \in A$. In a similar fashion one can show that $\{T^{2n+1}u\}$ is a divergent sequence for all $u \in A$. □

In the same way one can see that if there exists $y \in B$ such that $y_1 \neq y_2 \in [y]$ and $y \neq z$ in $\bigcap\limits_{i-1,2} [y_i]$ then for any $v \in B$ the sequence $\{T^{2n}v\}$ does not converge.

The following Examples illustrates Proposition 3.3.

Examples 3.4.

(1) *Let A be the line segment joining the points $(0, 0),\ (0, 1)$ and B be the line segment joining the points $(1, 0),\ (1, 1)$ in $(\mathbb{R}^2, \|\cdot\|_\infty)$. One can observe that $\operatorname{dist}(A, B) = 1$ and for any $x \in A,\ [x] = B$ and $\bigcap\limits_{x' \in [x]} [x'] = A$. It is easy to see that every map T on $A \cup B$ satisfying $TA \subset B$ and $TB \subset A$ is a cyclic contraction. Also $d(x, y) = \operatorname{dist}(A, B)$ for all $x \in A,\ y \in B$ and hence each point in A is a best proximity point for T.*

(2) *Let A be the line segment joining the points $(0, 1, 0),\ (\frac{1}{2}, \frac{1}{2}, 0)$ and B be the line segment joining the points $(0, 0, 0),\ (\frac{1}{2}, 1, \frac{1}{2})$ in $(\mathbb{R}^3, \|\cdot\|_1)$. Then it is easy to see that $\operatorname{dist}(A, B) = 1$. For $(0, 0, 0) \in B,\ d\big((0, 0, 0), (0, 1, 0)\big) = 1 = \operatorname{dist}(A, B) = d\big((0, 0, 0), (\frac{1}{2}, \frac{1}{2}, 0)\big)$. Also $d\big((\frac{1}{2}, 1, \frac{1}{2}), (0, 1, 0)\big) = 1 = \operatorname{dist}(A, B) = d\big((\frac{1}{2}, 1, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2}, 0)\big)$. That is $(0, 1, 0), (\frac{1}{2}, \frac{1}{2}, 0) \in [(0, 0, 0)]$ and $(\frac{1}{2}, 1, \frac{1}{2}) \in [(0, 1, 0)] \bigcap [(\frac{1}{2}, \frac{1}{2}, 0)]$. Hence one one constrict a cyclic contraction T on $A \cup B$ with $TA \subset B$ and $TB \subset A$, which satisfies the conclusion of Theorem 3.3.*

Theorem 3.5. *Let (A, B) be a cyclically complete semi sharp proximinal pair in a metric space X. Suppose $T : A \cup B \to A \cup B$ is a cyclic contraction such that $TA \subset B$ and $TB \subset A$, then following holds:*

(1) *There exists a unique $x \in A$ such that $d(x, Tx) = \operatorname{dist}(A, B)$.*
(2) *For any $x_0 \in A$, the sequence $T^{2n}x_0$ and $T^{2n+1}x_0$ converge to x and Tx respectively.*
(3) *x and Tx are the unique fixed points of T^2 in A and B respectively.*

Proof. By Theorem 3.1, there exists $x \in A$ such that $d(x, Tx) = \text{dist}(A, B)$. Notice that $\text{dist}(A, B) \leq d(Tx, T^2x) \leq d(x, Tx) = \text{dist}(A, B)$. Since (A, B) is a semi sharp proximinal, $T^2x = x$. For uniqueness, if there exists $x \neq x' \in A$ satisfying $d(x', Tx') = \text{dist}(A, B)$ then $T^2x' = x'$. Since $x \neq x'$, and by semi sharp proximinality of (A, B), $d(x', Tx) > \text{dist}(A, B)$. Hence $d(T^2x, Tx') < d(Tx, x')$. Now $d(x, Tx') = d(T^2x, Tx') < d(Tx, x') = d(Tx, T^2x) \leq d(x, Tx')$, a contradiction. Hence there a unique $x \in A$ such that $d(x, Tx) = \text{dist}(A, B)$. Fix $x_0 \in A$. Define $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$. It has been proved in Theorem 3.1 that $\{x_n\}$ is a cyclically Cauchy sequence and hence by Theorem 2.15 $\{x_{2n}\}$ and $\{x_{2n+1}\}$ are convergent sequences. Also by the claim in the proof of Theorem 3.1 $\{x_{2n}\}$ converge to the best proximity point x in A and $\{x_{2n+1}\}$ converges to the best proximity point y in B, with $d(x, y) = \text{dist}(A, B)$. By semi sharp proximinality of (A, B), $y = Tx$. Now we prove that x is a unique fixed point of T^2 in A. We have $T^2x = x$. If $y \in A$ satisfying $T^2y = y$ then $T^{2n}y = y$ for all $n \in \mathbb{N}$. We have $T^{2n}y \to x$ and hence $y = x$. In a similar fashion one can show that Tx is a unique fixed point of T^2 in B. □

It is to be noticed that, if $\text{dist}(A, B) = 0$ then the pair (A, B) is a semi sharp proximinal pair. The above theorem 3.5 generalizes the Banach contraction theorem for cyclic contraction mappings satisfying $TA \subset B$ and $TB \subset A$. As a particular case we get the following:

Corollary 3.6. [2] *Let A and B be nonempty closed and convex subsets of a uniformly convex Banach space. Suppose $T : A \cup B \to A \cup B$ is a cyclic contraction map, then there exists a unique best proximity point x in A (that is with $\|x - Tx\| = \text{dist}(A, B)$). Further, if $x_0 \in A$ and $x_{n+1} = Tx_n$, then $\{x_{2n}\}$ converges to the best proximity point.*

Let $T : A \cup B \to A \cup B$ satisfying $TA \subset A$ and $TB \subset B$. The following Proposition 3.7 gives a necessary condition for the existence of a fixed point such a mapping T.

Proposition 3.7. *Let (A, B) be a pair of subsets of a metric space X. If there exists $x \in A \cup B$ such that $[x]$ contains two different points say x_1 and x_2 with $\bigcap_{i=1,2} [x_i]$ contains* *a point other then x then there exists a map $T : A \cup B \to A \cup B$ such that $TA \subset A$ and $TB \subset B$ satisfying:*

(1) *T is a cyclic contraction mapping.*
(2) *There is no fixed point for T.*
(3) *For any $x_0 \in A$, the sequences $\{T^n x_0\}$ diverges.*

Proof. Let $x \in A$ such that $x_1 \neq x_2$ in $[x]$ and $\cap_{i=1,2}[x_i]$ contains more than one element say $x \neq y$ (It is easy to see that $x \in \cap_{i=1,2}[x_i]$). Define $T : A \cup B \to A \cup B$ as

$$T(u) := \begin{cases} y & \text{if } u \in A \text{ and } u = x \\ x & \text{if } u \in A \text{ and } u \neq x \\ x_2 & \text{if } u \in B \text{ and } u = x_1 \\ x_1 & \text{if } u \in B \text{ and } u \neq x_1. \end{cases}$$

Notice that $TA \subset A$ and $TB \subset B$. Also for any $u \in A$ and $v \in B$, $d(Tu, Tv) = \operatorname{dist}(A, B) \leq \alpha d(u, v) + (1 - \alpha) \operatorname{dist}(A, B)$, for all $\alpha \in [0, 1]$. Hence T is cyclic contraction. Also it is easy to see that there is no fixed point for T in $A \cup B$. For any fixed $u \in A$, either $Tu = x$ or $Tu = y$. If $Tu = x$, then $T^2u = Tx = y,\ T^3u = Ty = x,\ T^4u = Tx = y, \cdots$. Therefore the sequence $\{T^n u\}_{n=1}^{\infty}$ is $(x, y, x, y, x, \cdots)$. In a similar fashion one can show that, If $Tu = y,\ \{T^n u\}_{n=1}^{\infty}$ is $(y, x, y, x, y, \cdots)$. Hence $\{T^n u\}$ diverges. □

In the same way one can see that, with the same assumptions of Proposition 3.7, for any $v \in B$ the sequence $\{T^n v\}$ diverges.

Theorem 3.8. *Let (A, B) be a cyclically complete semi sharp proximinal pair in a metric space X. Suppose $T : A \cup B \to A \cup B$ is a cyclic contraction such that $TA \subset A$ and $TB \subset B$, then the following holds:*

(1) *There exists a unique pair $(x, y) \in A \times B$ such that x and y are fixed points of T.*
(2) *$d(x, y) = \operatorname{dist}(A, B)$.*
(3) *For any $(x_0, y_0) \in A \times B$, $T^n x_0 \to x$ and $T^n y_0 \to y$.*

Proof. Let $(x_0, y_0) \in A \times B$. Define $x_{2n} := T^n x_0$ and $x_{2n+1} := T^n y_0$, for all $n \in \mathbb{N}$. First we prove that the sequence $\{T^n x_0\}$ is bounded. Suppose not, for $M = \dfrac{\alpha d(y_0, Ty_0)}{1 - \alpha} +$

$\operatorname{dist}(A,B)$, there exists $n \in \mathbb{N}$, such that

$$d(Ty_0, T^n x_0) \leq M \text{ and } d(Ty_0, T^{n+1}x_0) > M.$$

Now

$$\begin{aligned} M < d(Ty_0, T^{n+1}x_0) &\leq \alpha d(y_0, T^n x_0) + (1-\alpha)\operatorname{dist}(A,B) \\ &\leq \alpha(d(y_0, Ty_0) + d(Ty_0, T^n x_0)) + (1-\alpha)\operatorname{dist}(A,B) \\ &\leq \alpha(d(y_0, Ty_0) + M) + (1-\alpha)\operatorname{dist}(A,B) \\ &= \alpha M + \alpha(d(y_0, Ty_0)) + (1-\alpha)\operatorname{dist}(A,B)) \\ &= \alpha M + (1-\alpha)M = M \end{aligned}$$

a contradiction. A similar way one can prove $\{T^n y_0\}$ is bounded and hence the sequence $\{x_n\}$ is bounded. For any $n \geq m$ in $\mathbb{N}$, $d(x_{2n}, x_{2m+1}) \leq \alpha^m d(x_0, x_{2(n-m)+1}) + (1 - \alpha^m)\operatorname{dist}(A,B)$. Hence $\{x_n\}$ is a cyclically Cauchy sequence in $A \cup B$, as $\alpha \in (0,1)$. Since (A,B) is cyclically complete, either both the sequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ have convergent subsequences or there exists $N \in \mathbb{N}$ such that $d(x_{2n}, x_{2m+1}) = \operatorname{dist}(A,B)$, $\forall\, n, m \geq N$. For the second case $x_{2n} = x_{2N}$ and $x_{2n+1} = x_{2N+1}$ for all $n \geq N$ and so $\{x_{2N}\}, \{x_{2N+1}\}$ satisfies the conclusions. Therefore it is enough to prove in the case that the both sequences $\{T^n x_0\}$ and $\{T^n y_0\}$ have convergent subsequences. By Theorem 3.1, one can show that the both sequences $\{T^n x_0\}$ and $\{T^n y_0\}$ convergent sequences, say converges to $x \in A$ and $y \in B$ respectively. Also $d(x,y) = \lim_{n\to\infty} d(T^n x_0, T^n y_0) = \operatorname{dist}(A,B)$. Now

$$\begin{aligned} \operatorname{dist}(A,B) \leq d(x, Ty) &\leq \lim_{n\to\infty} d(T^n x_0, Ty) \\ &\leq \lim_{n\to\infty} d(T^{n-1}x_0, y) = d(x,y) = \operatorname{dist}(A,B). \end{aligned}$$

That is $d(x,Ty) = \operatorname{dist}(A,B)$, and by sharp proximinality of (A,B) $y = Ty$. In a similar fashion one can prove $x = Tx$. □

The above Theorem 3.8 generalizes the Banach contraction theorem for cyclic contraction mappings satisfying $TA \subset A$ and $TB \subset B$. As a particular case we get the following:

Corollary 3.9. *Let A and B be nonempty closed and convex subsets of a uniformly convex Banach space. Suppose $T : A \cup B \to A \cup B$ is a cyclic contraction map satisfying $TA \subset A$ and $TB \subset B$, then the following holds:*

1. *There exists a unique fixed point x in A and a unique fixed point $y \in B$ for T;*
2. $\|x - y\| = \operatorname{dist}(A, B)$*;*
3. *For any $(x_0, y_0) \in A \times B$, $T^n x_0 \to x$ and $T^n y_0 \to y$.*

Now we prove the existence and uniqueness of a best proximity point for a map T, if T^n is a cyclic contraction.

Theorem 3.10. *Let (A, B) be a cyclically complete semi sharp proximinal pair in a metric space X. Suppose $T : A \cup B \to A \cup B$ is a map satisfying $TA \subset B$ and $TB \subset A$. If there exists $n \in \mathbb{N}$ such that T^n is cyclic contraction then there exists a unique $x \in A$ such that $d(x, Tx) = \operatorname{dist}(A, B)$.*

Proof. If $n = 2m + 1$ for some $m \in \mathbb{N}$, then by Theorem 3.5 there exists a unique $x \in A$ such that $d(x, T^{2m+1}x) = \operatorname{dist}(A, B)$. By semi sharp proximinality of (A, B) one can get $T^{2(2m+1)}x = x$. Suppose if $d(x, Tx) > \operatorname{dist}(A, B)$ then $d(T^{2m+1}x, T^{2m+2}x) < d(x, Tx)$. Now

$$\begin{aligned} d(x, Tx) &= d(T^{2(2m+1)}x, T^{2(2m+1)+1}x) \\ &\leq d(T^{2m+1}x, T^{2m+2}x) \\ &< d(x, Tx) \end{aligned}$$

a contradiction. That is $d(x, Tx) = \operatorname{dist}(A, B)$. If $n = 2m$ for some $m \in \mathbb{N}$, then by Theorem 3.8 there exists a unique $x \in A$ such that $x = T^{2m}x$. Suppose if $d(x, Tx) > \operatorname{dist}(A, B)$ then $d(T^{2m}x, T^{2m+1}x) < d(x, Tx)$. Now

$$\begin{aligned} d(x, Tx) &= d(T^{2m}x, T^{2m+1}x) \\ &< d(x, Tx) \end{aligned}$$

a contradiction. That is $d(x, Tx) = \operatorname{dist}(A, B)$. For uniqueness, if there exists $x \neq x' \in A$ such that $d(x', Tx') = \operatorname{dist}(A, B)$, then $T^{2n}x' = x'$. Also $T^{2n}x = x$. By semi sharp proximinality of (A, B), $d(x', Tx) > \operatorname{dist}(A, B)$ so that $d(T^n x', T^{n+1}x) < d(x', Tx)$. Now $d(x', Tx) = d(T^{2n}x', T^{2n+1}x) \leq d(T^n x', T^{n+1}x) < d(x', Tx)$, a contradiction. □

Now we prove the existence and uniqueness of a fixed point in A for a map T on $A \cup B$ satisfying $TA \subset A$ and $TB \subset B$ if T^n is a cyclic contraction. .

Theorem 3.11. *Let (A, B) be a cyclically complete semi sharp proximinal pair in a metric space X. Suppose $T : A \cup B \to A \cup B$ is a map satisfying $TA \subset A$ and $TB \subset B$. If there exists $n \in \mathbb{N}$ such that T^n is cyclic contraction then there exists a unique pair $(x, y) \in A \times B$ such that x and y are fixed points for T and $d(x, y) = \operatorname{dist}(A, B)$.*

Proof. By Theorem 3.8 there exists a unique pair $(x, y) \in A \times B$ such that $T^n x = x$, $T^n y = y$ and $d(x, y) = \operatorname{dist}(A, B)$. If $x \neq Tx$, by semi sharp proximinality $d(Tx, y) > \operatorname{dist}(A, B)$ and hence $d(T^{n+1}x, Ty) > d(Tx, y)$. Now $d(Tx, y) = d(T^{n+1}x, T^n yx) < d(Tx, y)$, a contradiction. In a similar fashion one can prove that $y = Ty$. □

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