On subsemigroups of $\mathbb{N}^e$

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Abstract[1]

Let $\underline{v} = (v_1, \cdots, v_{e+s})$ be a set of vectors of $\mathbb{N}^e$, and assume that v_{e+k} is not in the group generated by $v_1, \ldots, v_{e+k-1}$ for all $k = 1, \cdots, s$. The aim of this paper is to give a formula for the Frobenius number and the conductor of the subsemigroup generated par $\underline{v}$ in $\mathbb{N}^e$.

1 Introduction and Basic Notations

Let $\underline{v} = (v_1, \ldots, v_e, v_{e+1}, \ldots, v_{e+s})$ be a set of nonzero elements of $\mathbb{N}^e$ and let

$$\Gamma(\underline{v}) = \{\sum_{k=1}^{e+s} a_i v_i | a_i \in \mathbb{N}\}$$

be the subsemigroup of $\mathbb{N}^e$ generated by $\underline{v}$. Let $G(\underline{v}) = \{\sum_{k=1}^{e+s} a_i v_i | a_i \in \mathbb{Z}\}$ be the subgroup of $\mathbb{Z}^e$ generated by $\underline{v}$ and let cone$(v_1, \ldots, v_e)$ be the convex cone generated by $v_1, \ldots, v_e$,

$$\text{cone}(v_1, \cdots, v_e) = \{\sum_{k=1}^{e} a_i v_i | a_i \in \mathbb{R}_+\}.$$

Assume that the dimension of cone$(v_1, \ldots, v_e)$ is e -i.e. $(v_1, \ldots, v_e)$ generates $\mathbb{R}^e$- and that $v_{e+1}, \ldots, v_{e+s} \in \text{cone}(v_1, \ldots, v_e)$. The paper deals with the following question:

What is the "smallest" element $w \in \text{cone}(v_1, \ldots, v_e)$ such that for all $v \in w + \text{cone}(v_1, \ldots, v_e)$, if $v \in G(\underline{v})$, then $v \in \Gamma(\underline{v})$?

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Let D_1 be the determinant of the matrix $[v_1^T, \ldots, v_e^T]$ -where T denotes the transpose of a matrix-, and for all $k = 2, \ldots, s+1$, let D_k be the gcd of the (e, e) minors of the matrix $[v_1^T, \ldots, v_e^T, v_{e+1}^T, \ldots, v_{e+k-1}^T]$. Set $e_k = \dfrac{D_k}{D_{k+1}}$ for all $k = 1, \ldots, s$. We shall assume that the two following conditions are satisfied:

(*) $D_1 > D_2 > \cdots > D_{s+1}$ (in particular for all $k = 2, \cdots, s+1, v_{e+k-1}$ is not in the group generated by $(v_1, \cdots, v_e, v_{e+1}, \cdots, v_{e+k-2})$).

(**) $e_k v_{e+k} \in \Gamma(v_1, \ldots, v_e, v_{e+1}, \ldots, v_{e+k-1})$ for all $k = 1, \ldots, s$.

Our main result is the following:

Theorem 1. (See Fig. 1) Let the notations be as above, and let C_e be the unique cell of dimension e of cone$(v_1, \ldots, v_e)$ (more precisely C_e is the interior of cone$(v_1, \ldots, v_e)$). If

$$g(\underline{v}) = \sum_{k=1}^{s} (e_k - 1) v_{e+k} - \sum_{i=1}^{e} v_i$$

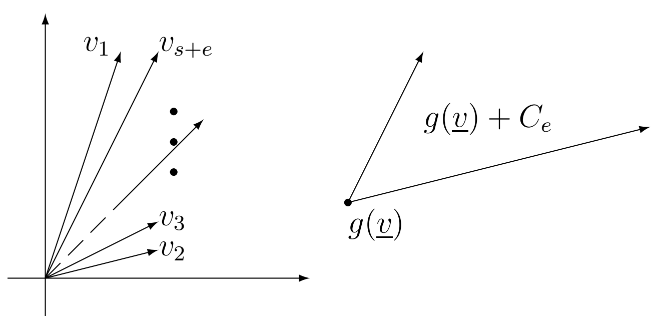
then the following hold:

i) $g(\underline{v}) \notin \Gamma(\underline{v})$.

ii) For all $v \in g + (C_e - \{(0, \cdots, 0)\})$, if $v \in G(\underline{v})$, then $v \in \Gamma(\underline{v})$.

We call $g(\underline{v})$ the Frobenius vector of $\Gamma(\underline{v})$.

Fig. 1



When $e = s = 1$ and v_1, v_2 are relatively prime elements of $\mathbb{N}$, Sylvester proved in [9] that the Frobenius number of $\Gamma(v_1, v_2)$ is $(v_1 - 1).(v_2 - 1) - 1$ (note that in this case, $e_1 = v_1$). In [6], M.J. Knight generalized the formula for Sylvester to a system of elements $(v_1, \cdots, v_e, v_{e+1}) \in \mathbb{N}^e$, assuming that $v_{e+1} \in$ cone$(v_1, \cdots, v_e)$, and that $v_1, \cdots, v_e, v_{e+1}$ generate $\mathbb{Z}^e$. Hence, Theorem 1. can be viewed as a generalisation of Knight's result.

Let $e = 1$ and assume that $v_1 < \cdots < v_{s+1}$. Assume, without loos of generality, that $D_{s+1} = 1$, i.e. $v_1, \cdots, v_{s+1}$ are relatively prime. The above theorem says that for all $v \geq g + 1 = \sum_{k=1}^{s} (e_k - 1) v_{1+k} - v_1 + 1, v \in \Gamma(\underline{v})$. The positive integer $g + 1$ is called the conductor of $\Gamma(\underline{v})$ in $\mathbb{N}$. In fact, the ideal (t^c) is the conductor ideal of the algebra $\mathbf{K}[t^{v_1}, \cdots, t^{v_{s+1}}]$ over a field $\mathbf{K}$ into its integral closure $\mathbf{K}[t]$.

2 Lattices in $\mathbb{Z}^e$

Let the notations be as in Section 1. The group $G_k = G(v_1, \cdots, v_e, v_{e+1}, \cdots, v_{e+k}))$ being a subgroup of the free group $\mathbb{Z}^e$ for all $k = 0, \cdots, s$, it follows that G_k is a free group of rank $\leq e$, and the hypothesis on $v_1, \ldots, v_e$ implies that the rank of G_k is e. Let $w_1, \ldots, w_e$ be a basis of G_k, in particular D_{k+1} is the determinant of the (e, e) matrix $[w_1^T, \ldots, w_e^T]$. Furthermore we have the following:

Proposition 2.1 Let v be a nonzero element of $\mathbb{Z}^e$ and denote by D the gcd of the (e, e) minors of the matrix $[v_1^T, \ldots, v_{e+k}^T, v^T]$. Then D is also the gcd of the (e, e) minors of the matrix $[w_1^T, \ldots, w_e^T, v^T]$. We also have the following:

i) D divides D_{k+1} and $v \in G_k$ if and only if $D_{k+1} = D$.

ii) $\dfrac{D_{k+1}}{D}.v \in G_k$ and if $D_{k+1} > D$ then for all $1 \leq i < \dfrac{D_{k+1}}{D}, i.v \notin G_k$.

Proof. i) For all $i = 1, \ldots, e$, let d_i be the determinant of the matrix $[w_1^T, \ldots, w_{i-1}^T, v^T, w_{i+1}^T, \ldots, w_e^T]$ and note that D divides d_i. If $D = D_{k+1}$ then D_{k+1} divides d_i for all $1 \leq i \leq e$. In particular the system $\lambda_1 w_1 + \ldots + \lambda_e w_e = v$ has the unique solution $\lambda_i = \dfrac{d_i}{D_{k+1}} \in \mathbb{Z}$. Conversely, if $v \in G_k$, then there exist unique integers $\lambda_1, \ldots, \lambda_e$ such that $v = \lambda_1 w_1 + \ldots + \lambda_e w_e$, but $(\lambda_1, \ldots, \lambda_e)$ is the unique solution to the $e \times e$ system $a_1 w_1 + \ldots + a_e w_e = v$, in particular $\lambda_i = \dfrac{d_i}{D_{k+1}}$, and D_{k+1} divides d_i for all $i = 1, \ldots, e$. Since $D = \gcd(d_1, \cdots, d_e, D_{k+1})$, then $D = D_{k+1}$.

ii) Let the notations be as in i) and $1 \leq i < \dfrac{D_{k+1}}{D}$. Let $\tilde{D}$ be the gcd of the (e, e) minors of the matrix $[w_1^T, \cdots, w_e^T, (i.v)^T]$. If $i.v \in G_k$, then $\tilde{D} = D_{k+1}$. But $\tilde{D} = \gcd(id_1, \cdots, id_e, D_{k+1})$, in particular D_{k+1} divides $\gcd(id_1, \cdots, id_e, iD_{k+1}) = i.D$ which is a contradiction because $i.D < D_{k+1}$.■

Since $D_1 > \cdots > D_{s+1}$, it follows that $G_0 \subset G_1 \subset \cdots \subset G_s$. We also have the following:

Proposition 2.2 i) For all $1 \leq k \leq s, e_k$ is the index of G_{k-1} in G_k.

ii) For all $k = 1, \cdots, s, e_k v_{e+k} \in G_{k-1}$ and $(e_k - i)v_{e+k} \notin G_{k-1}$ for all $1 \leq i < e_k$.

iii) Given $0 \leq k \leq s$ and $v \in G_k$, there exist unique integers $\lambda_1, \ldots, \lambda_e, \lambda_{e+1}, \ldots, \lambda_{e+k}$ such that $v = \sum_{i=1}^{e+k} \lambda_i v_i$ and $0 \leq \lambda_i < e_i$ for all $i = e+1, \ldots, e+k$ (we call this representation the standard representation with respect to $v_1, \cdots, v_{e+k}$).

Proof. i) is obvious and ii) results from Proposition 2.1. ii). To prove iii), we first prove the existence: let $v = \sum_{i=1}^{e+k} c_i v_i$ where $c_i \in \mathbb{Z}$ for all $1 \leq i \leq e+k$. If $k = 0$, then the assertion is clear. Assume that $k \geq 1$, and that $c_{e+k} < 0$. Let $p \in \mathbb{N}^*$ be such that $0 \leq pe_k + c_{e+k} < e_k$. We have:

$$v = \sum_{i=1}^{e+k-1} c_i v_i + (c_{e+k} + pe_k - pe_k) v_{e+k}$$

since $e_k v_{e+k} \in G_{k-1}$, then so is for $-pe_k v_{e+k}$. In particular we can rewrite v as:

$$v = \sum_{i=1}^{e+k-1} \tilde{c}_i v_i + (\tilde{c}_{e+k}) v_{e+k}$$

and $0 \leq \tilde{c}_{e+k} = pe_k + c_{e+k} < e_k$. Since $\sum_{i=1}^{e+k-1} \tilde{c}_i v_i \in G_{k-1}$, then we get the result by induction on k.

To prove the uniqueness, let $v = \sum_{i=1}^{e+k} a_i v_i = \sum_{i=1}^{e+k} b_i v_i$ where for all $i = e+1, \ldots, e+k, 0 \leq a_i, b_i < e_i$ and let j be the greatest integer such that $a_j - b_j \neq 0$. Suppose that $j \geq e+1$ and also that $a_j - b_j > 0$, then

$$(a_j - b_j) v_j = \sum_{i=1}^{e} (b_i - a_i) + (b_{e+1} - a_{e+1}) v_1 + \ldots + (b_{j-1} - a_{j-1}) v_{j-1} \in G_{j-1}$$

and $0 < a_j - b_j < e_j$. This contradicts ii). ■

Note that the results of Propositions 2.1. and 2.2. hold assuming only that the condition (*) of page 2 is satisfied. This will not be the case in the following Corollary.

Corollary 2.3 Let $0 \leq k \leq s$ and let $v \in G_k$. Let

$$v = \sum_{i=1}^{e+k} \lambda_i v_i$$

be the standard representation with respect to $v_1, \cdots, v_{e+k}$. The vector $v \in \Gamma(v_1, \cdots, v_{e+k})$ if and only if $\lambda_i \geq 0$ for all $i = 1, \cdots, e$.

Proof. If $\lambda_i \geq 0$ for all $i = 1, \cdots, e$, then clearly $v \in \Gamma(v_1, \cdots, v_{e+k})$. Conversely, suppose that $v \in \Gamma(v_1, \cdots, v_{e+k})$, then $v = \sum_{i=1}^{e+k} \mu_i v_i$ where $\mu_i \geq 0$ for all $i = 1, \cdots, e+k$. We shall construct the standard representation of v as in the Proposition above: if $0 \leq \mu_i < e_i$ for all $i = e+1, \ldots, e+k$, then it is over. Assume that $\mu_i \geq e_i$ for some $i \geq e+1$ and let $e+j$ be the greatest element with this property. Write $\mu_j = pe_j + \tilde{\mu_j}$, where $0 \leq \tilde{\mu_j} < e_j$. But $e_j v_j \in \Gamma(v_1, \ldots, v_e, v_{e+1}, \ldots, v_{j-1})$, in particular $e_j v_j = \sum_{i=1}^{j-1} \tilde{\lambda}_i v_i$. We finally rewrite v in the following form:

$$v=\sum_{i=1}^{e+k}\tilde{\lambda}_i v_i$$

where $\tilde{\lambda_i}\geq 0$ and $0\leq \tilde{\lambda_i}<e_i$ for all $i=j,\dots e+k$. Finally, we get the result by an easy induction.■

3 Proof of Theorem 1. and applications

Proof of Theorem 1. Let the notations be as in Section 1. and let $g(\underline{v})=\sum_{k=1}^{s}(e_k-1)v_{e+k}-\sum_{i=1}^{e}v_i$. Clearly $g(\underline{v})\in G(\underline{v})$, and by corollary 2.3., $g(\underline{v})\notin\Gamma(\underline{v})$. Let $u\in C_e-\{(0,\cdots,0)\}$ and let $v=g(\underline{v})+u$. Assume that $v\in G(\underline{v})$ and let

$$v=\sum_{k=1}^{e+s}\theta_k v_k$$

be the standard representation of v and recall that $0\leq\theta_{e+k}<e_k$ for all $k=1,\cdots,s$. We have:

$$\sum_{k=1}^{s}(e_k-1-\theta_{e+k})v_{e+k}+u=(\theta_1+1)v_1+\cdots+(\theta_e+1)v_e$$

But $\sum_{k=1}^{s}(e_k-1-\theta_{e+k})v_{e+k}+u\in C_e$, which implies that $\theta_k+1>0$ for all $k=1,\cdots,e$, in particular $\theta_k\geq 0$ for all $k=1,\cdots,e$, consequently $g(\underline{v})+u\in\Gamma(\underline{v})$.■

Definition 3.1 Suppose that $D_{s+1}=1$, i.e. $G(\underline{v})=\mathbb{Z}^e$, and let $\mathrm{N}(C_e)$ be the set of the compact faces of the convex hull of $\bigcup_{w\in C_e}w+C_e$. Let $w_1,\cdots,w_r\in\mathbb{N}^e$ be the set of integral vectors of $\mathrm{N}(C_e)$. For all $v\in C_e$, there is $1\leq k\leq r$ such that $v\in w_k+C_e$. In particular, for all $v\in g(\underline{v})+C_e$, if $v\in C_e$, then there exists $1\leq k\leq r$ such that $v\in(g+w_k)+C_e$. The set $\{g(\underline{v})+w_1,\cdots,g(\underline{v})+w_r\}$ is called the conductor of $\Gamma(\underline{v})$.

Corollary 3.2 Let $\underline{v}=(v_1,\cdots,v_{e+s})$ be as in Section 1. and let $A=[v_1^T,\cdots,v_{e+s}^T]$. Consider the Diophantine equation $A.X=B$ where $B\in\mathbb{N}^e$. By Theorem 1., if $B\in g(\underline{v})+C_e$, then $B\in\Gamma(\underline{v})$, in particular the Diophantine equation $A.X=B$ has a solution in $\mathbb{N}^{e+s}$

3.1 The semigroup of a curve singularity

Let $\mathbf{K}$ be an algebraically closed field of characteristic zero and let $f=y^n+a_1(x)y^{n-1}+\cdots+a_n(x)$ be a nonzero element of $\mathbf{K}[[x]][y]$. Suppose that f is irreducible. By Newton theorem, there exists $y(t)=\sum_p c_pt^p\in\mathbf{K}[[t]]$ such that $f(t^n,y(t))=0$. Furthermore, $f(t^n,y)=\prod_{w\in U_n}(y-y(wt))$, where U_n denotes the group of roots of unity in $\mathbf{K}$. Given a nonzero polynomial $g \in \mathbf{K}[[x]][y]$, we set $\mathrm{int}(f,g) = O_t g(t^n, y(t))$, where O_t denotes the t-order. The set of $\mathrm{int}(f,g), 0 \neq g \in \mathbf{K}[[x]][y]$ is a numerical semigroup, denoted $\Gamma(f)$. Let $m_0 = n = d_1$ and for all $k \geq 1$, let $m_k = \inf\{p | c_p \neq 0 \text{ and } m_{k-1} \text{ does not divide } p\}$. There exists $h \geq 1$ such that $d_{h+1} = 1$. The set $\{m_1, \ldots, m_h\}$ is called the set of Newton-Puiseux exponents of f. With these notations, $\Gamma(f)$ is generated by $r_0, r_1, \cdots, r_h$, where $r_0 = n, r_1 = m_1$ and for all $2 \leq k \leq h$:

$$r_k = r_{k-1}\frac{d_{k-1}}{d_k} + m_k - m_{k-1}$$

and it is well known that $r_k e_k \in \Gamma(r_0, \ldots, r_{k-1})$ for all $1 \leq k \leq h$. Conversely, let $r_0 < r_1, \cdots < r_h$ be a given sequence of relatively prime nonnegative integers. Let $d_1 = r_0$ and for all $1 \leq k \leq h$ let $d_{k+1} = \gcd(r_k, d_k)$. If $r_k > r_{k-1}\dfrac{d_{k-1}}{d_k}$ for all $1 \leq k \leq h$, then $r_0, \cdots, r_h$ generate the semigroup of an irreducible element of $\mathbf{K}[[x]][y]$ (see [11]).

Let f be as above, and let $r_0, \cdots, r_h$ be the set of generators of $\Gamma(f)$. Let $e_k = \dfrac{d_k}{d_{k+1}}, k = 1, \cdots, h$, and let:

$$c = \sum_{k=1}^{h}(e_k - 1)r_k - r_0 + 1$$

then c is the conductor of $\Gamma(f)$, i.e. $g = c - 1 \notin \Gamma(f)$ and $c + \mathbb{N} \subseteq \Gamma(f)$. The ideal (t^c) is the conductor ideal of $\mathbf{K}[[x]][y]/(f)$ into its intergal closure $\mathbf{K}[[t]]$, and c is also the Milnor number of f, i.e. $c = \mathrm{rank}_{\mathbf{K}}\mathbf{K}[[x,y]]/(f_x, f_y)$, where f_x (resp. f_y) denotes the x-derivative (resp. the y-derivative) of f. Furthermore, the cardinality of $\mathbb{N} - \Gamma(f)$ (the set of gaps of $\Gamma(f)$) is $\dfrac{c}{2}$.

3.2 The semigroup of a quasi-ordinary polynomial

Let $\mathbf{K}$ be an algebraically closed field of characteristic zero and let $f = y^n + a_1(\underline{x})y^{n-1} + \cdots + a_n(\underline{x})$ be an irreducible element of $\mathbf{K}[[\underline{x}]][y] = \mathbf{K}[[x_1, \cdots, x_e]][y]$ and assume that the discriminant $D_y(f)$ of f, defined to be the y-resultant of f and its y-derivative f_y, is of the form $x_1^{N_1} \cdots x_e^{N_e}(a + u(\underline{x}))$, where $N_1, \cdots, N_e \in \mathbb{N}, a \in \mathbf{K}^*$, and $u(\underline{0}) = 0$ (such a polynomial is called a quasi-ordinary polynomial). By [1], there exists $y(\underline{t}) = y(t_1, \cdots, t_e) = \sum_{p \in \mathbb{N}^e} c_p \underline{t}^p \in \mathbf{K}[[t_1, \cdots, t_e]]$ such that $f(t_1^n, \cdots, t_e^n, y(\underline{t})) = 0$. Furthermore, there exist n distinct elements $(\underline{w}^1, \cdots, \underline{w}^n) \in U_n^e$, where U_n denotes the group of roots of unity in $\mathbf{K}$, such that $f(t_1^n, \cdots, t_e^n, y) = \prod_{k=1}^n (y - y(w_1^k t_1, \cdots, w_e^k t_e))$.

Given a nonzero element $g \in \mathbf{K}[[\underline{x}]][y]$, we define $O(f,g)$ to be the maximal element with respect to the lexicographical order of the initial form of $g(t_1^n, \cdots, t_e^n, y(\underline{t}))$. The set of $O(f,g), 0 \neq g \in \mathbf{K}[[\underline{x}]][y]$, is a semigroup of $\mathbb{N}^e$, denotes $\Gamma(f)$. Let $\mathrm{Supp}(y(\underline{t})) = \{p | c_p \neq 0\}$. In [7], J. Lipman proved the existence of $m_1, \cdots, m_h \in \mathrm{Supp}(y(\underline{t}))$ such that the following hold:

i) $m_1 < m_2 < \cdots < m_h$, where $<$ means $<$ coordinate-wise.

ii) Let $M_0 = (n\mathbb{Z})^e$ and for all $k = 1, \cdots, h$, let $M_k = M_0 + \sum_{i=1}^{k} m_i\mathbb{Z}$. We have $M_0 \subset M_1 \subset \cdots \subset M_h$. Furthermore, for all $p \in \mathrm{Supp}(y(\underline{t})), p \in \sum_{p\in m_k+\mathbb{N}^e} m_k\mathbb{Z}$.

Let $D_1 = n^e$ and for all $k = 1, \cdots, h$, let D_{k+1} be the gcd of $e \times e$ minors of the matrix $[nI_e, m_1^T, \cdots, m_k^T]$, where I_e denotes the (e, e) unit matrix. By conditions i), ii), we have $D_1 > \cdots > D_{h+1}$, furthermore $D_{h+1} = n^{e-1}$ (see [3]). Let $r_0^1, \cdots, r_0^e$ be the canonical basis of $(n\mathbb{Z})^e$, and define the sequence $r_1, \ldots, r_h$ by $r_1 = m_1$ and for all $k = 2, \cdots, h$:

$$r_k = r_{k-1}\frac{D_{k-1}}{D_k} + (m_k - m_{k-1})$$

then $r_0^1, \cdots, r_0^e, r_1, \cdots, r_h$ generate $\Gamma(f)$ and $r_k\dfrac{D_k}{D_{k+1}} \in \Gamma(r_0^1, \ldots, r_0^e, r_1, \ldots, r_{k-1})$ for all $1 \le k \le h$. Furthermore, for all $k = 1, \cdots, h$, if $\tilde{D}_{k+1}$ denotes the gcd of the (e, e) minors of the matrix $[nI_e, r_1^T, \cdots, r_k^T]$, then $\tilde{D}_{k+1} = D_{k+1}$.

Note that in this situation, the convex cone generated by $r_0^1, \cdots, r_0^e$ is nothing but $\mathbb{R}_+^e$, and $C_e = (\mathbb{R}_+^*)^e$.

Set $e_k = \dfrac{D_k}{D_{k+1}}$ for all $k = 1, \cdots, h$. The Frobenius vector of $\Gamma(f)$ is:

$$g = \sum_{k=1}^{h}(e_k - 1)r_k - \sum_{k=1}^{e} r_0^k$$

in particular, for all $u \in (\mathbb{R}_+^*)^e$, if $g + u \in G(r_0^1, \cdots, r_0^e, r_1, \cdots, r_h)$, then $g + u \in \Gamma(f)$.

3.3 Numerical examples

Example 3.3 (See Fig. 2) Let $\underline{v} = (v_1, v_2, v_3) = (4, 6, 7)$. With the notations of Section 1. we have $D_1 = 4, D_2 = 2, D_3 = 1, e_1 = \dfrac{D_1}{D_2} = 2, e_2 = \dfrac{D_2}{D_3} = 2$. The Frobenius vector of $\Gamma(\underline{v})$ is:

$$g(\underline{v}) = (e_1 - 1)v_2 + (e_2 - 1)v_3 - v_1 = 9$$

and the conductor of $\Gamma(\underline{v})$ is $c = g(\underline{v}) + 1 = 10$. Note that $\mathbb{N} - \Gamma(\underline{v}) = \{1, 2, 3, 5, 9\}$, whose cardinality is $\dfrac{c}{2} = 5$.

Example 3.4 (See Fig. 3) Let $\underline{v} = (v_1, v_2, v_3, v_4) = ((8, 0), (0, 8), (2, 2), (12, 8))$. With the notations of Section 1. we have $D_1 = 64$, D_2-the gcd of the $(2, 2)$ minors of the matrix $[8I_2, (2, 2)^T]$- is 16, and D_3-the gcd of the $(2, 2)$ minors of the matrix $[8I_2, (2, 2)^T, (12, 8)^T]$- is 8. Finally, $e_1 = \dfrac{D_1}{D_2} = 4, e_2 = \dfrac{D_2}{D_3} = 2$. The Frobenius vector of $\Gamma(\underline{v})$ is:

$$g(\underline{v}) = (e_1 - 1)v_3 + (e_2 - 1)v_4 - v_1 - v_2 = 3v_3 + v_4 - v_1 - v_2 = (10, 6)$$

Let $u = v_2 = (2, 2)$, then $g + u = g + v_2 = (e_1 - 1)v_3 + (e_2 - 1)v_4 - v_1 = (10, 14) \notin \Gamma(\underline{v})$. In fact, u belongs to a cell of cone(v_1, v_2) of dimension 1.

Example 3.5 (See Fig. 4) Let $\underline{v} = (v_1, v_2, v_3, v_4) = ((4, 6), (6, 3), (8, 10), (3, 4))$. With the notations of Section 1. we have $D_1 = 24$, D_2-the gcd of the $(2, 2)$ minors of the matrix $[(4, 6)^T, (6, 3)^T, (8, 10)^T]$- is 4, and D_3-the gcd of the $(2, 2)$ minors of the matrix $[(4, 6)^T, (6, 3)^T, (8, 10)^T, (3, 4)^T$ is 1. Finally, $e_1 = \dfrac{D_1}{D_2} = 6, e_2 = \dfrac{D_2}{D_3} = 4$. The Frobenius vector of $\Gamma(\underline{v})$ is:

$$g(\underline{v}) = (e_1 - 1)v_3 + (e_2 - 1)v_4 - v_1 - v_2 = (49, 52) - (10, 9) = (39, 53)$$

In this example, since $D_3 = 1$, then $G(\underline{v}) = \mathbb{Z}^2$. In particular, for all $v \in g(\underline{v}) + C_e, v \in \Gamma(\underline{v})$. Furthermore, for all $v \in C_e$, $v \in (1, 1) + C_e$, hence the conductor of $\Gamma(\underline{v})$ is $g(\underline{v}) + (1, 1) = (40, 54)$.

Example 3.6 (See Fig. 5) Let $\underline{v} = (v_1, v_2, v_3) = ((1, 3), (3, 2), (1, 1))$. With the notations of Section 1. we have $D_1 = 7$, D_2-the gcd of the $(2, 2)$ minors of the matrix $[(1, 3)^T, (3, 2)^T, (1, 1)^T]$- is 1. Finally, $e_1 = \dfrac{D_1}{D_2} = 7$. The Frobenius number of $\Gamma(\underline{v})$ is:

$$g(\underline{v}) = (e_1 - 1)v_3 - v_1 - v_2 = (6, 6) - (4, 5) = (2, 1)$$

In this example, since $D_2 = 1$, then $G(\underline{v}) = \mathbb{Z}^2$. In particular, for all $v \in g(\underline{v}) + C_e, v \in \Gamma(\underline{v})$. Furthermore, for all $v \in C_e$, $v \in [(1, 1) + C_e] \cup [(1, 2) + C_e]$, hence the conductor of $\Gamma(\underline{v})$ is $\{g(\underline{v}) + (1, 1), g(\underline{v}) + (1, 2)]\} = \{(3, 2), (3, 3)\}$.

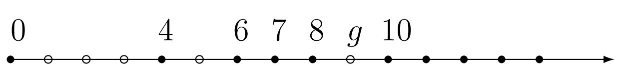


Fig. 2

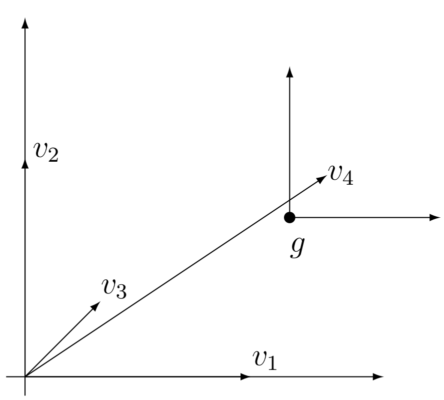


Fig. 3

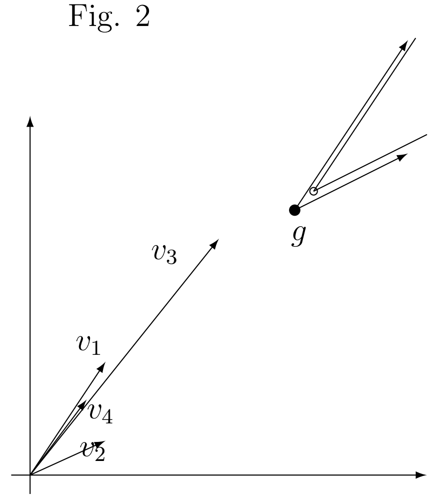


Fig. 4

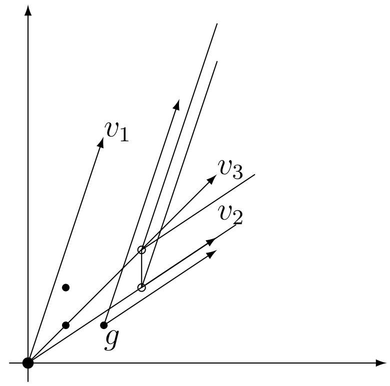


Fig. 5

Remark 3.7 i) Let $\underline{v} = (v_1, \cdots, v_{e+s})$ et let the notations be as in Section 1. and suppose that $D_1 \geq D_2 \geq \cdots \geq D_{s+1}$. Let $(v_{e+i_1}, \cdots, v_{e+i_t})$ be the maximal set of $\{v_{e+1}, \cdots, v_{e+s}\}$ such that for all $1 \leq k \leq t, v_{e+i_k} \notin G(v_1, \cdots, v_e, \cdots, v_{e+i_k-1})$ and let $g = \sum_{k=1}^{t}(e_{i_k} - 1)v_{e+i_k} - \sum_{k=1}^{e} v_k$. For all $v \in g + C_e$, if $v \in G(\underline{v}) = G(v_1, \cdots, v_e, v_{i_1}, \cdots, v_{i_t})$, then $v \in \Gamma(v_1, \cdots, v_e, v_{i_1}, \cdots, v_{i_t}) \subseteq \Gamma(\underline{v})$. In general, the vector g need not to be the "smallest" one with this property. Let for example $\underline{v} = (4, 6, 7, 9)$: for all $v \geq 9, v \in \Gamma(\underline{v})$, but the Frobenius number of $\Gamma(\underline{v})$ is 5. General subsemigroups of $\mathbb{N}$ and their Frobenius numbers have been studied by many authors (see [8] and references).

ii) If $\underline{v} = (8, 10, 11)$, then $g(\underline{v}) = (e_1 - 1)v_2 + (e_{\_}1)v_3 - v_1 = 30 + 11 - 8 = 33 = 3.11 \in \Gamma(8, 10, 11)$. In this example, condition (**) of page 2 is not satisfied, since $e_2 v_3 = 22 \notin \Gamma(8, 10)$.

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