A FIXED POINT THEOREM FOR L^1 SPACES

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1. Introduction

Andrés Navas asked us if there is a fixed point theorem for all isometries of L^1 that preserve a given bounded set. Unlike many known cases where a geometric argument applies, there is a fundamental obstruction in L^1: *any infinite group G admits a fixed-point-free isometric action on a bounded convex subset of L^1.* (This can be seen by examining the G-action on the affine subspace of summable functions of sum one on G.)

Thus, we have to search for fixed points possibly outside the convex set, indeed outside the affine subspace it spans. We shall do this more generally for any *L-embedded* Banach space V, that is, a space whose bidual can be decomposed as $V^{**} = V \oplus_1 V_0$ for some $V_0 \subseteq V$ (and $\oplus_1$ indicates that the norm is the sum of the norms on V and V_0). Recall that L^1 is L-embedded by the Yosida–Hewitt decomposition and that this holds more generally for the predual of any von Neumann algebra [7, III.2.14] (in particular, for the dual of any C*-algebra).

Theorem A. *Let A be a non-empty bounded subset of an L-embedded Banach space V.*

Then there is a point in V fixed by every isometry of V preserving A. Moreover, one can choose a fixed point which minimises $\sup_{a\in A} \|v - a\|$ over all $v \in V$.

We recall that an isometric action of a group G on a Banach space V is given by a linear part and a cocycle $b : G \to V$. The cocycle is the orbital map of $0 \in V$ and a fixed point v corresponds to a trivialisation $b(g) = v - g.v$, where $g.v$ is the linear action. The above norm statement implies that one can arrange $\|v\| \leq \sup_g \|b(g)\|$ by considering $A = b(G) \ni 0$.

As a special case, we recover the main theorem of [4], but with an improved (indeed optimal) norm estimate:

Corollary B. *Let G be a group acting by homeomorphisms on a locally compact space X. Then any bounded cocycle $b : G \to M(X)$ to the space of measures on X is trivial. More precisely, there is a measure μ with $\|\mu\| \leq \sup_{g\in G} \|b(g)\|$ such that $b(g) = \mu - g.\mu$ for all $g \in G$.* □

Indeed, $M(X)$ is the dual of the C*-algebra $C_0(X)$ and hence the predual of a von Neumann algebra.

Numerous consequences of Corollary B are listed in [4]; let us only recall that it settles the so-called *derivation problem* whose history began in the 1960's: If G is a locally compact group, then any derivation from the convolution algebra $L^1(G)$ to $M(G)$ is inner. This is often phrased in terms of derivations "of $L^1(G)$" since any derivation $L^1(G) \to M(G)$ must range in $L^1(G)$ by Paul Cohen's factorisation theorem. It also follows that any derivation of $M(G)$ is inner. Our estimate is optimal by Remark 7.2(a) in [4].

As observed by Uffe Haagerup, Theorem A also yields a new proof that all C*-algebras are weakly amenable, which was proved in [2] using the Grothendieck–Haagerup–Pisier inequality. In fact, our theorem immediately implies that a *bounded* derivation from any abstract algebra A to a predual M_* of a von Neumann algebra is inner as soon as A is spanned by the elements represented as invertible isometries of M_* (see the proof of the corollary below). In the particular case of C*-algebras, we obtain the following general statement.

Corollary C. *Let A be a unital C*-algebra. Let M_* be the predual of a von Neumann algebra. Assume M_* is a Banach bimodule over A. Then any arbitrary derivation $D : A \to M_*$ is inner.*

Moreover, we can choose $v \in M_*$ with $D(a) = v.a - a.v$ such that $\|v\| \leq \|D\|$.

The weak amenability of A is given by the special case $M_* = A^*$. Our definition of Banach bimodule demands $\|a.v.b\| \leq \|a\| \cdot \|v\| \cdot \|b\|$ $(a, b \in A,\ v \in M_*)$.

Proof of Corollary C. By Theorem 2 in [6], D is continuous; thus it is bounded (by $\|D\| < \infty$) on the group G of unitaries of A. The map $G \to M_*$ given by $g \mapsto D(g).g^{-1}$ is a cocycle for the Banach G-module structure defined by the rule $v \mapsto g.v.g^{-1}$. Theorem A thus yields v, with norm bounded by $\|D\|$, such that $D(g) = v.g - g.v$ for all $g \in G$. The statement follows since any element of A is a combination of four unitaries (in fact, three [3]). □

Finally, returning to the case $V = L^1$ of Theorem A, we recall that any isometric action of a Kazhdan group on an L^1 space has bounded orbits because of a Fock space argument (see e.g. [1, 1.3(2)]). Therefore, we deduce:

Corollary D. *Let Ω by any measure space. Then any isometric action of a Kazhdan group on $L^1(\Omega)$ has a fixed point.* □

By the Kakutani representation theorem, this corollary applies unchanged to abstract L^1 spaces, for instance to $M(X)$ for any locally compact space X. Moreover, it follows that the fixed point property on L^1 *characterises* Kazhdan's property (T) for countable groups, see [1, 1.3].

2. Proof

We first recall the concept of Chebyshev centre. Let A by a non-empty bounded subset of a metric space V. The *circumradius* of A in V is

$$\varrho_V(A) = \inf\left\{r \geq 0 : \exists\, x \in V \text{ with } A \subseteq \overline{B}(x,r)\right\},$$

where $\overline{B}(x,r)$ denotes the closed r-ball around x. The *Chebyshev centre* of A in V is the (possibly empty) set

$$C_V(A) = \left\{c \in V : A \subseteq \overline{B}(c, \varrho_V(A))\right\}.$$

Notice that $C_V(A)$ can be written as an intersection of closed balls as follows:

$$C_V(A) = \bigcap_{r > \varrho_V(A)} C_V^r(A) \quad \text{where} \quad C_V^r(A) = \bigcap_{a \in A} \overline{B}(a, r).$$

Thus, when V is a Banach space, $C_V(A)$ is a bounded closed convex set. More importantly, when V is a dual Banach space, we deduce from Alaoğlu's theorem that $C_V(A)$ is weak-* compact and that it is non-empty because the non-empty sets $C_V^r(A)$ are monotone in r.

Proposition. *Let A be a non-empty bounded subset of an L-embedded Banach space V. Then the convex set $C_V(A)$ is weakly compact and non-empty.*

Proof. Consider A as a subset of V^{**} under the canonical embedding $V \subseteq V^{**}$. In view of the above discussion, $C_{V^{**}}(A)$ is a non-empty weak-* compact convex set. We claim that it lies in V and coincides with $C_V(A)$; the proposition then follows. Let thus $c \in C_{V^{**}}(A)$ and write $c = c_V + c_{V_0}$ according to the decomposition $V^{**} = V \oplus_1 V_0$. Then, for any $a \in A$, we have

$$\|a - c\| = \|a - c_V\| + \|c_{V_0}\|$$

since $A \subseteq V$. Therefore,

$$\varrho_{V^{**}}(A) = \sup_{a \in A} \|a - c\| = \sup_{a \in A} \|a - c_V\| + \|c_{V_0}\| \geq \varrho_V(A) + \|c_{V_0}\|.$$

Since $\varrho_{V^{**}}(A) \leq \varrho_V(A)$ anyway, we deduce $c_{V_0} = 0$ and $\varrho_{V^{**}}(A) = \varrho_V(A)$, whence the claim □

We now complete the proof of Theorem A. Since the definition of $C_V(A)$ is metric, it is preserved by any isometry preserving A. By the proposition, we can apply the Ryll-Nardzewski theorem and deduce that there is a point of $C_V(A)$ fixed by all isometries preserving A. The norm condition follows from the definition of centres. □

We remind the reader that in the present context the Ryll-Nardzewski theorem has a particularly short geometric proof relying on the dentability of weakly compact sets [5].

Remark. The above proof works with slightly weaker assumptions on the decomposition of the bidual V^{**}, e.g. a p-summand decomposition, $p < \infty$. However, a canonical norm one projection $V^{**} \to V$ is not enough. Indeed, any dual space is canonically complemented in its own bidual, but the fixed point property in all duals characterises amenability. Specifically, any non-amenable group G has a fixed-point-free action with bounded orbits in $(\ell^\infty(G)/\mathbf{R})^*$.

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