The truncated matrix trigonometric moment problem: the operator approach.

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1 Introduction.

The aim of this investigation is to obtain a bijective parameterization of all solutions of the truncated matrix trigonometric moment problem. Recall that the truncated matrix trigonometric moment problem consists of finding a non-decreasing $\mathbb{C}_{N\times N}$-valued function $M(t) = (m_{k,l})_{k,l=0}^{N-1}$, $t \in [0,2\pi]$, $M(0) = 0$, which is left-continuous in $(0,2\pi]$, and such that

$$\int_0^{2\pi} e^{int} dM(t) = S_n, \qquad n = 0,1,...,d, \tag{1}$$

where $\{S_n\}_{n=0}^d$ is a prescribed sequence of $(N\times N)$ complex matrices (moments). Here $N \in \mathbb{N}$ and $d \in \mathbb{Z}_+$ are fixed numbers.
Set

$$T_d = (S_{i-j})_{i,j=0}^d = \begin{pmatrix} S_0 & S_{-1} & S_{-2} & \dots & S_{-d} \\ S_1 & S_0 & S_{-1} & \dots & S_{-d+1} \\ S_2 & S_1 & S_0 & \dots & S_{-d+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ S_d & S_{d-1} & S_{d-2} & \dots & S_0 \end{pmatrix}, \tag{2}$$

where

$$S_n := S_{-n}^*, \qquad n = -d, -d+1, ..., -1. \tag{3}$$

The scalar $(N = 1)$ truncated trigonometric moment problem is well investigated. In 1911, Riesz and Herglotz obtained the necessary and sufficient conditions of the solvability for this moment problem (e.g. [1]). Canonical solutions of the moment problem were described in the positive definite case: $T_d > 0$, by Krein and Nudelman [2]. In 1966, Chumakin described all solutions of the scalar truncated trigonometric moment problem using his results on the generalized resolvents of isometric operators, see [3],[4],[5].

In the general case of an arbitrary N, the following condition:

$$T_d \geq 0, \tag{4}$$

is necessary and sufficient for the solvability of the moment problem (1) (e.g. [6]). In 1969, Inin obtained a description of all solutions of the truncated matrix trigonometric moment problem in the completely indeterminate case: $T_d > 0$ [7]. He used the tools of the theory of pseudo-Hilbert spaces developed by Krein and Berezanskii [8].

The truncated matrix trigonometric moment problem is closely related to (and it is essentially the same as) the Carathéodory matrix coefficient problem (this relation is based on the matrix extension of the Riesz-Herglotz integral representation), see [9],[10] and References therein. In 1998, a parameterization of all solutions of the last problem both in nondegenerate and degenerate cases was for the first time obtained by Chen and Hu [9]. In 2006, another parameterization of all solutions of this problem both in nondegenerate and degenerate cases was obtained by Fritzsche and Kirstein [10]. However, it is not clear whether the above parameterizations are bijective. We shall describe all solutions of the truncated matrix trigonometric moment problem in the general case: $T_d \geq 0$, as well. We shall use an abstract operator approach and the mentioned above results of Chumakin on generalized resolvents of isometric operators. The abstract operator approach allows investigate simultaneously both the nondegenerate and degenerate cases of different moment problems (see [11] and [12],[13]). The obtained parameterization of all solutions is bijective. Also, the operator point of view allows to see the transparent whole picture of the problem at once rather then step-by-step algorithm.

Notations. As usual, we denote by $\mathbb{R}, \mathbb{C}, \mathbb{N}, \mathbb{Z}, \mathbb{Z}_+$, the sets of real numbers, complex numbers, positive integers, integers and non-negative integers, respectively; $\mathbb{D} = \{z \in \mathbb{C} : \ |z| < 1\}$. The set of all complex vectors of size N: $a = (a_0, a_1, \dots, a_{N-1})$, we denote by $\mathbb{C}_N$, $N \in \mathbb{N}$. If $a \in \mathbb{C}^N$, then a^* means its complex conjugate vector. The set of all complex matrices of size $(N \times N)$ we denote by $\mathbb{C}_{N\times N}$.

Let $M(x)$ be a left-continuous non-decreasing matrix function $M(x) = (m_{k,l}(x))_{k,l=0}^{N-1}$ on $[0, 2\pi]$, $M(0) = 0$, and $\tau_M(x) := \sum_{k=0}^{N-1} m_{k,k}(x)$; $\Psi(x) = (dm_{k,l}/d\tau_M)_{k,l=0}^{N-1}$. By $L^2(M)$ we denote a set (of classes of equivalence) of $\mathbb{C}_N$-valued functions f on $[0, 2\pi]$, $f = (f_0, f_1, \dots, f_{N-1})$, such that (see, e.g., [14])

$$\|f\|^2_{L^2(M)} := \int_0^{2\pi} f(x)\Psi(x)f^*(x)d\tau_M(x) < \infty.$$

The space $L^2(M)$ is a Hilbert space with a scalar product

$$(f, g)_{L^2(M)} := \int_0^{2\pi} f(x)\Psi(x)g^*(x)d\tau_M(x), \qquad f, g \in L^2(M).$$

If H is a Hilbert space then $(\cdot, \cdot)_H$ and $\| \cdot \|_H$ mean the scalar product and the norm in H, respectively. Indices may be omitted in obvious cases. For a linear operator A in H, we denote by $D(A)$ its domain, by $R(A)$ its range, by $\mathrm{Ker}\, A$ its null subspace (kernel), and A^* means the adjoint operator if it exists. If A is invertible then A^{-1} means its inverse. $\overline{A}$ means the closure of the operator, if the operator is closable. If A is bounded then $\|A\|$ denotes its norm. For a set $M \subseteq H$ we denote by $\overline{M}$ the closure of M in the norm of H. For an arbitrary set of elements $\{x_n\}_{n\in I}$ in H, we denote by $\mathrm{Lin}\{x_n\}_{n\in I}$ the set of all linear combinations of elements x_n, and $\mathrm{span}\{x_n\}_{n\in I} := \overline{\mathrm{Lin}\{x_n\}_{n\in I}}$. Here I is an arbitrary set of indices. By E_H we denote the identity operator in H, i.e. $E_H x = x$, $x \in H$. If H_1 is a subspace of H, then $P_{H_1} = P_{H_1}^H$ is an operator of the orthogonal projection on H_1 in H.

2 Solvability and a description of solutions.

Let the moment problem (1) be given. Suppose that the moment problem has a solution $M(t)$. Define S_n, $n = -d, -d+1, ..., -1$, by relation (3). Observe that

$$S_{-n} = S_n^* = \left(\int_0^{2\pi} e^{int} dM(t) \right)^* = \int_0^{2\pi} e^{-int} dM(t), \qquad n = 0, 1, ..., d.$$

Therefore

$$\int_0^{2\pi} e^{int} dM(t) = S_n, \qquad -d \leq n \leq d. \tag{5}$$

Consider an arbitrary vector-valued polynomial $P(t)$ of the following form:

$$P(t) = \sum_{k=0}^{d} (\alpha_{k,0}, \alpha_{k,1}, ..., \alpha_{k,N-1}) e^{ikt} = \sum_{k=0}^{d} \sum_{s=0}^{N-1} \alpha_{k,s} e^{ikt} \vec{e}_s, \ \ \alpha_{k,s} \in \mathbb{C}, \tag{6}$$

where

$$\vec{e}_s = (\delta_{0,s}, \delta_{1,s}, ..., \delta_{N-1,s}). \tag{7}$$

Then

$$0 \leq \int_0^{2\pi} P(t) dM(t) P^*(t) = \sum_{k,r=0}^{d} \sum_{s,l=0}^{N-1} \alpha_{k,s} \overline{\alpha_{r,l}} \int_0^{2\pi} e^{i(k-r)t} \vec{e}_s dM(t) \vec{e}_l^{\,*}$$

$$= \sum_{k,r=0}^{d} \sum_{s,l=0}^{N-1} \alpha_{k,s} \overline{\alpha_{r,l}} \vec{e}_s S_{k-r} \vec{e}_l^{\,*}.$$

Thus, for arbitrary complex numbers $\alpha_{k,s}$, $0 \leq k \leq d$, $0 \leq s \leq N-1$, it holds

$$\sum_{k,r=0}^{d} \sum_{s,l=0}^{N-1} \alpha_{k,s} \overline{\alpha_{r,l}} \vec{e}_s S_{k-r} \vec{e}_l^{\,*} \geq 0. \tag{8}$$

Define the matrix T_d by (2) and set

$$\vec{\alpha} = (\alpha_{0,1}, \alpha_{0,2}, ..., \alpha_{0,N}, \alpha_{1,1}, \alpha_{1,2}, ..., \alpha_{1,N}, ..., \alpha_{d,1}, \alpha_{d,2}, ..., \alpha_{d,N}). \quad (9)$$

By the rules of the multiplication of block matrices relation (8) means that

$$\vec{\alpha} T_d \vec{\alpha}^* \geq 0, \quad (10)$$

and therefore

$$T_d \geq 0. \quad (11)$$

Conversely, let the moment problem (1) with $d \in \mathbb{N}$ be given and relation (11) holds with T_d defined by (2). Let

$$T_d = (\gamma_{n,m})_{n,m=0}^{(d+1)N-1}, \ S_k = (S_{k;s,l})_{s,l=0}^{N-1}, \quad -d \leq k \leq d, \quad (12)$$

where $\gamma_{n,m}, S_{k;s,l} \in \mathbb{C}$.
Observe that

$$\gamma_{kN+s,rN+l} = S_{k-r;s,l}, \qquad 0 \leq k, r \leq d, \quad 0 \leq s, l \leq N-1. \quad (13)$$

By the well-known construction (see, e.g., [13, Theorem 1]), from (11) it follows that there exist a Hilbert space H and elements $\{x_n\}_{n=0}^{(d+1)N-1}$ in H such that

$$(x_n, x_m) = \gamma_{n,m}, \qquad 0 \leq n, m \leq (d+1)N-1, \quad (14)$$

and $\mathrm{span}\{x_n\}_{n=0}^{(d+1)N-1} = H$. Set $H_0 := \mathrm{Lin}\{x_n\}_{n=0}^{dN-1}$.
Consider the following operator:

$$Ax = \sum_{k=0}^{dN-1} \alpha_k x_{k+N}, \quad x = \sum_{k=0}^{dN-1} \alpha_k x_k, \ \alpha_k \in \mathbb{C}. \quad (15)$$

Let us check that this definition is correct. Suppose that $x \in H_0$ has two representations:

$$x = \sum_{k=0}^{dN-1} \alpha_k x_k, \quad x = \sum_{k=0}^{dN-1} \beta_k x_k, \quad \alpha_k, \beta_k \in \mathbb{C}.$$

Using relations (13),(14) we may write

$$\left\| \sum_{k=0}^{dN-1} \alpha_k x_{k+N} - \sum_{k=0}^{dN-1} \beta_k x_{k+N} \right\|^2 = \left(\sum_{k=0}^{dN-1} (\alpha_k - \beta_k) x_{k+N}, \sum_{r=0}^{dN-1} (\alpha_r - \beta_r) x_{r+N} \right)$$

$$= \sum_{k,r=0}^{dN-1} (\alpha_k - \beta_k)\overline{(\alpha_r - \beta_r)}\gamma_{k+N,r+N} = \sum_{k,r=0}^{dN-1} (\alpha_k - \beta_k)\overline{(\alpha_r - \beta_r)}\gamma_{k,r}$$

$$= \left(\sum_{k=0}^{dN-1} (\alpha_k - \beta_k) x_k, \sum_{r=0}^{dN-1} (\alpha_r - \beta_r) x_r \right) = (x - x, x - x) = 0.$$

Thus, the definition of A is correct and $D(A) = H_0$.
Let $x, y \in H_0$, $x = \sum_{k=0}^{dN-1} \alpha_k x_k$, $y = \sum_{k=0}^{dN-1} \gamma_k x_k$, $\alpha_k, \gamma_k \in \mathbb{C}$. Then

$$(Ax, Ay) = \sum_{k,r=0}^{dN-1} \alpha_k \overline{\gamma_r} (x_{k+N}, x_{r+N}) = \sum_{k,r=0}^{dN-1} \alpha_k \overline{\gamma_r} \gamma_{k+N,r+N}$$

$$= \sum_{k,r=0}^{dN-1} \alpha_k \overline{\gamma_r} \gamma_{k,r} = \sum_{k,r=0}^{dN-1} \alpha_k \overline{\gamma_r} (x_k, x_r) = (x, y).$$

Thus, A is an isometric operator. Every isometric operator admits a unitary extension ([15]). Let $U \supseteq A$ be a unitary extension of A in a Hilbert space $\widetilde{H} \supseteq H$. Choose an arbitrary non-negative integer n:

$$n = rN + l, \qquad 0 \le r \le d,\ 0 \le l \le N - 1.$$

By induction one easily derives the following relation:

$$x_{rN+l} = A^r x_l. \tag{16}$$

Choose an arbitrary m:

$$m = kN + s, \qquad 0 \le k \le d,\ 0 \le s \le N - 1.$$

Using (13) we may write

$$S_{k-r;s,l} = \gamma_{kN+s,rN+l} = (x_m, x_n)_H = (A^k x_s, A^r x_l)_H = (U^k x_s, U^r x_l)_{\widetilde{H}}$$

$$= (U^{k-r} x_s, x_l)_{\widetilde{H}} = \int_0^{2\pi} e^{i(k-r)t} d(E_t x_s, x_l)_{\widetilde{H}},$$

where $\{E_t\}_{t \in [0,2\pi]}$ is the left-continuous orthogonal resolution of unity of the operator U. Thus, we have

$$S_{j;s,l} = \int_0^{2\pi} e^{ijt} d(P_H^{\widetilde{H}} E_t x_s, x_l)_H, \quad -d \le j \le d,\ 0 \le s, l \le N - 1. \tag{17}$$

Set

$$M_U(t) = \left((P_H^{\widetilde{H}} E_t x_s, x_l)_H \right)_{s,l=0}^{N-1}, \qquad t \in [0, 2\pi]. \tag{18}$$

Then $M_U(t)$ is a solution of the moment problem (1) (the fact that it is non-decreasing follows easily from the properties of the orthogonal resolution of unity).

We can state the following well-known fact:

Theorem 1 *Let the truncated matrix trigonometric moment problem (1) be given. The moment problem has a solution if and only if relation (4) with T_d from (2) holds.*

Proof. The required result for the case $d \in \mathbb{N}$ was proved above. For the case $d = 0$ the following function is a solution:

$$M(t) = \begin{cases} 0, & t = 0 \\ S_0, & t \in (0, 2\pi] \end{cases}.$$

□

Remark. For the case $d = 0$, an arbitrary left-continuous non-decreasing $\mathbb{C}_{N \times N}$-valued function M, $M(0) = 0$, $M(2\pi) = S_0$ is a solution of the moment problem (1). Therefore we shall investigate the case $d \in \mathbb{N}$.

Let the moment problem (1) be given with $d \in \mathbb{N}$ and condition (4) holds. As it was done above, we construct a Hilbert space H, a sequence $\{x_n\}_{n=0}^{(d+1)N-1}$ in H and the isometric operator A. Let $\widehat{U} \supseteq A$ be an arbitrary unitary extension of A in a Hilbert space $\widehat{H} \supseteq H$. Let $\{\widehat{E}_t\}_{t \in [0,2\pi]}$ be the left-continuous orthogonal resolution of unity of $\widehat{U}$. Recall [4],[5] that the following function:

$$\mathbf{E}_t = P_H^{\widehat{H}} \widehat{E}_t, \qquad t \in [0, 2\pi], \tag{19}$$

is said to be a *spectral function* of A. The operator-valued function

$$\mathbf{R}_\zeta = P_H^{\widehat{H}} (E_{\widehat{H}} - \zeta \widehat{U})^{-1}, \qquad \zeta \in \mathbb{C} : |\zeta| \neq 1, \tag{20}$$

is said to be a *generalized resolvent* of A. If $\mathbf{E}_t$ and $\mathbf{R}_\zeta$ correspond to the same unitary extension of A, they are said to be related. The related left-continuous spectral function and generalized resolvent of A are in a bijective correspondence:

$$(\mathbf{R}_\zeta h, g)_H = \int_0^{2\pi} \frac{1}{1 - \zeta e^{it}} d(\mathbf{E}_t h, g), \qquad \forall h, g \in H. \tag{21}$$

The function $(\mathbf{E}_t h, g)$ can be found by the inversion formula ([5]).

As we have seen above, an arbitrary left-continuous spectral function of the isometric operator A generates a solution of the moment problem (1) by relation (18).

On the other hand, let $\widehat{M}$ be an arbitrary solution of the moment problem (1). A set of all (classes of equivalence which include) polynomials of the form (6) in $L^2(\widehat{M})$ we shall denote by $L^2_{0,d}(\widehat{M})$. Choose an arbitrary

$$Q(t) = \sum_{r=0}^{d} \sum_{l=0}^{N-1} \beta_{r,l} e^{irt} \vec{e}_l,\ \ \beta_{r,l} \in \mathbb{C}. \tag{22}$$

Then

$$(P(t), Q(t))_{L^2(\widehat{M})} = \sum_{k,r=0}^{d} \sum_{s,l=0}^{N-1} \alpha_{k,s} \overline{\beta_{r,l}} \int_0^{2\pi} e^{i(k-r)t} \vec{e}_s d\widehat{M}(t) \vec{e}_l^{\,*}$$

$$= \sum_{k,r=0}^{d} \sum_{s,l=0}^{N-1} \alpha_{k,s} \overline{\beta_{r,l}} \vec{e}_s S_{k-r} \vec{e}_l^{\,*} = \sum_{k,r=0}^{d} \sum_{s,l=0}^{N-1} \alpha_{k,s} \overline{\beta_{r,l}} S_{k-r;s,l}$$

$$= \sum_{k,r=0}^{d} \sum_{s,l=0}^{N-1} \alpha_{k,s} \overline{\beta_{r,l}} \gamma_{kN+s,rN+l} = \sum_{k,r=0}^{d} \sum_{s,l=0}^{N-1} \alpha_{k,s} \overline{\beta_{r,l}} (x_{kN+s}, x_{rN+l})_H$$

$$= \left(\sum_{k=0}^{d} \sum_{s=0}^{N-1} \alpha_{k,s} x_{kN+s}, \sum_{r=0}^{d} \sum_{l=0}^{N-1} \beta_{r,l} x_{rN+l} \right)_H . \tag{23}$$

Consider the following operator:

$$WP(t) = \sum_{k=0}^{d} \sum_{s=0}^{N-1} \alpha_{k,s} x_{kN+s}. \tag{24}$$

Let us check that this operator is correctly defined as an operator from $L^2_{0,d}(\widehat{M})$ to H. Let $P(t)$ and $Q(t)$ are two polynomials of the forms (6) and (22), respectively. Suppose that they belong to the same class of equivalence in $L^2(\widehat{M})$:

$$(P(t) - Q(t), P(t) - Q(t))_{L^2(\widehat{M})} = 0. \tag{25}$$

Then

$$0 = \left(\sum_{k=0}^{d} \sum_{s=0}^{N-1} (\alpha_{k,s} - \beta_{k,s}) e^{ikt} \vec{e}_s, \sum_{r=0}^{d} \sum_{l=0}^{N-1} (\alpha_{r,l} - \beta_{r,l}) e^{irt} \vec{e}_l \right)_{L^2(\widehat{M})}$$

$$= \sum_{k,r=0}^{d} \sum_{s,l=0}^{N-1} (\alpha_{k,s} - \beta_{k,s}) \overline{(\alpha_{r,l} - \beta_{r,l})} \int_0^{2\pi} e^{i(k-r)t} \vec{e}_s d\widehat{M}(t) \vec{e}_l^{\,*}$$

$$= \left(\sum_{k=0}^{d} \sum_{s=0}^{N-1} (\alpha_{k,s} - \beta_{k,s}) x_{kN+s}, \sum_{r=0}^{d} \sum_{l=0}^{N-1} (\alpha_{r,l} - \beta_{r,l}) x_{rN+l} \right)_H = \| WP - WQ \|_H.$$

Thus, the operator W is defined correctly. Relation (23) shows that W is an isometric operator. It maps $L^2_{0,d}(\widehat{M})$ onto H. Denote

$$L^2_1(\widehat{M}) := L^2(\widehat{M}) \ominus L^2_{0,d}(\widehat{M}).$$

The operator

$$U := W \oplus E_{L^2_1(\widehat{M})},$$

is a unitary operator which maps $L^2(\widehat{M}) = L^2_{0,d}(\widehat{M}) \oplus L^2_1(\widehat{M})$ onto $H_1 := H \oplus L^2_1(\widehat{M})$.

Consider the following unitary operator:

$$U_0 f(t) = e^{it} f(t), \qquad f(t) \in L^2(\widehat{M}).$$

Then

$$\widetilde{U}_0 := U U_0 U^{-1},$$

is a unitary operator in H_1. Observe that

$$\widetilde{U}_0 x_{kN+s} = U U_0 e^{ikt} \vec{e}_s = U e^{i(k+1)t} \vec{e}_s = x_{(k+1)N+s} = A x_{kN+s},$$

where $0 \leq k \leq d-1,\ 0 \leq s \leq N-1$. Therefore $\widetilde{U}_0 \supseteq A$. Let $\{\widetilde{E}_t\}_{t\in[0,2\pi]}$ be the left-continuous orthogonal resolution of unity of $\widetilde{U}_0$, and $\mathbf{E}_t$, $\mathbf{R}_\zeta$, be a spectral function and a generalized resolvent of A which correspond to the unitary extension $\widetilde{U}_0$, respectively. Let us check that

$$\widehat{M}(t) = ((\mathbf{E}_t x_s, x_l)_H)_{s,l=0}^{N-1}. \tag{26}$$

In fact, we may write

$$\int_0^{2\pi} \frac{1}{1-\zeta e^{it}} d(\mathbf{E}_t x_s, x_l)_H = (\mathbf{R}_\zeta x_s, x_l)_H = \left((E_{H_1} - \zeta \widetilde{U}_0)^{-1} x_s, x_l \right)_{H_1}$$

$$= \left(U (E_{L^2(\widehat{M})} - \zeta U_0)^{-1} U^{-1} x_s, x_l \right)_{H_1} = \left((E_{L^2(\widehat{M})} - \zeta U_0)^{-1} \vec{e}_s, \vec{e}_l \right)_{L^2(\widehat{M})}$$

$$= \int_0^{2\pi} \frac{1}{1-\zeta e^{it}} \vec{e}_s d\widehat{M}(t) \vec{e}_l^{\,*}.$$

By the inversion formula we conclude that relation (26) is true.

Theorem 2 *Let the truncated matrix trigonometric moment problem (1) with $d \in \mathbb{N}$ be given and condition (4) is true. Let an operator A be constructed for the moment problem as in (15). All solutions of the moment problem have the following form*

$$M(t) = (m_{k,j}(t))_{k,j=0}^{N-1}, \quad m_{k,j}(t) = (\mathbf{E}_t x_k, x_j)_H, \tag{27}$$

where $\mathbf{E}_t$ is a left-continuous spectral function of the isometric operator A. Conversely, an arbitrary left-continuous spectral function of A generates by formula (27) a solution of the moment problem (1).

Moreover, the correspondence between all left-continuous spectral functions of A and all solutions of the moment problem is bijective.

Proof. It remains to prove that different left-continuous spectral functions of the operator A produce different solutions of the moment problem (1). Set

$$H_\zeta := (E_H - \zeta A)D(A) = (E_H - \zeta A)H_0, \quad \zeta \in \mathbb{C} : |\zeta| \neq 1; \tag{28}$$

$$L_N := \mathrm{Lin}\{x_k\}_{k=0}^{N-1}. \tag{29}$$

Choose an arbitrary element $x \in H$, $x = \sum_{k=0}^{dN+N-1} \alpha_k x_k$, $\alpha_k \in \mathbb{C}$. Let us check that for an arbitrary $\zeta \in \mathbb{C}\backslash\{0\} : |\zeta| \neq 1$, there exists a representation

$$x = v + y, \qquad v \in H_\zeta,\ y \in L_N, \tag{30}$$

where elements v, y may depend on the choice of ζ.
In fact, choose an arbitrary $\zeta \in \mathbb{C}\backslash\{0\} : |z| \neq 1$. Set

$$c_r := -\frac{1}{\zeta}\alpha_{r+N}, \qquad r = dN-N, dN-N+1, ..., dN-1. \tag{31}$$

Then we set

$$c_r := \frac{1}{\zeta}(c_{r+N} - \alpha_{r+N}), \qquad r = dN-N-1, dN-N-2, ..., 0. \tag{32}$$

Let

$$u := \sum_{k=0}^{dN-1} c_k x_k \in D(A); \tag{33}$$

$$v := (E_H - \zeta A)u \in H_\zeta. \tag{34}$$

Then

$$v = \sum_{k=0}^{dN-1} c_k x_k - \zeta \sum_{k=0}^{dN-1} c_k x_{k+N} = \sum_{k=0}^{dN-1} c_k x_k - \zeta \sum_{k=N}^{dN+N-1} c_{k-N} x_k$$

$$= \sum_{k=0}^{N-1} c_k x_k + \sum_{k=N}^{dN-1} (c_k - \zeta c_{k-N}) x_k - \zeta \sum_{k=dN}^{dN+N-1} c_{k-N} x_k$$
$$= \sum_{k=0}^{N-1} c_k x_k + \sum_{k=N}^{dN+N-1} \alpha_k x_k = \sum_{k=0}^{N-1} (c_k - \alpha_k) x_k + x.$$

Finally, we set $y := -\sum_{k=0}^{N-1}(c_k - \alpha_k)x_k \in L_N$, and obtain $x = v + y$. Thus, relation (30) holds.

Suppose to the contrary that two different left-continuous spectral functions of A produce the same solution of the moment problem (1). That means that there exist two unitary extensions $U_j \supseteq A$, in Hilbert spaces $\widetilde{H}_j \supseteq H$, such that

$$\mathbf{E}_{1,t} = P_H^{\widetilde{H}_1} E_{1,t} \neq P_H^{\widetilde{H}_2} E_{2,t} = \mathbf{E}_{2,t}; \tag{35}$$

and

$$(P_H^{\widetilde{H}_1} E_{1,t} x_k, x_j)_H = (P_H^{\widetilde{H}_2} E_{2,t} x_k, x_j)_H, \qquad 0 \leq k, j \leq N-1, \quad t \in [0, 2\pi], \tag{36}$$

where $\{E_{j,t}\}_{t\in[0,2\pi]}$ are orthogonal resolutions of unity of operators U_j, $j = 1, 2$. By linearity we get

$$(P_H^{\widetilde{H}_1} E_{1,t} x, y)_H = (P_H^{\widetilde{H}_2} E_{2,t} x, y)_H, \qquad x, y \in L_N, \quad t \in [0, 2\pi]. \tag{37}$$

Set

$$R_{j,\zeta} := (E_{\widetilde{H}_j} - \zeta U_j)^{-1},\ \mathbf{R}_{j,\zeta} := P_H^{\widetilde{H}_j} R_{j,\zeta}, \qquad j = 1, 2,\ \zeta \in \mathbb{C} : |\zeta| \neq 1.$$

From (37),(21) it follows that

$$(\mathbf{R}_{1,\zeta} x, y)_H = (\mathbf{R}_{2,\zeta} x, y)_H, \qquad x, y \in L_N, \quad \zeta \in \mathbb{C} : |\zeta| \neq 1. \tag{38}$$

Choose an arbitrary $\zeta \in \mathbb{C} : |\zeta| \neq 1$. Since for $j = 1, 2$, we may write

$$R_{j,\zeta}(E_H - \zeta A)x = (E_{\widetilde{H}_j} - \zeta U_j)^{-1}(E_{\widetilde{H}_j} - \zeta U_j)x = x, \qquad x \in H_0 = D(A),$$

we get

$$R_{1,\zeta} u = R_{2,\zeta} u \in H, \qquad u \in H_\zeta,\ \zeta \in \mathbb{C} : |\zeta| \neq 1; \tag{39}$$
$$\mathbf{R}_{1,\zeta} u = \mathbf{R}_{2,\zeta} u, \qquad u \in H_\zeta,\ \zeta \in \mathbb{C} : |\zeta| \neq 1. \tag{40}$$

Suppose additionally that $\zeta \neq 0$. We may write

$$(\mathbf{R}_{j,\zeta} x, u)_H = (R_{j,\zeta} x, u)_{\widetilde{H}_j} = (x, R_{j,\zeta}^* u)_{\widetilde{H}_j}$$

$$= (x,(E_{\widetilde{H}_j} - R_{j,\frac{1}{\overline{\zeta}}})u)_{\widetilde{H}_j} = (x,u)_H - (x,\mathbf{R}_{j,\frac{1}{\overline{\zeta}}}u)_H, \quad x\in L_N,\ u\in H_{\frac{1}{\overline{\zeta}}},\ j=1,2. \tag{41}$$

Therefore we get

$$(\mathbf{R}_{1,\zeta}x,u)_H = (\mathbf{R}_{2,\zeta}x,u)_H, \qquad x\in L_N,\ u\in H_{\frac{1}{\overline{\zeta}}},\ \zeta\in\mathbb{C}\backslash\{0\}: |\zeta|\neq 1. \tag{42}$$

Choose an arbitrary $\zeta\in\mathbb{C}:\ 0<|\zeta|<1$. By (30) an arbitrary element $y\in H$ can be represented as $y = y_{\frac{1}{\overline{\zeta}}} + y'$, $y_{\frac{1}{\overline{\zeta}}}\in H_{\frac{1}{\overline{\zeta}}}$, $y'\in L_N$. Using (38) and (42) we get

$$(\mathbf{R}_{1,\zeta}x,y)_H = (\mathbf{R}_{1,\zeta}x, y_{\frac{1}{\overline{\zeta}}} + y')_H = (\mathbf{R}_{2,\zeta}x, y_{\frac{1}{\overline{\zeta}}} + y')_H = (\mathbf{R}_{2,z}x,y)_H,$$

where $x\in L_N,\ y\in H$. Thus, we obtain

$$\mathbf{R}_{1,\zeta}x = \mathbf{R}_{2,\zeta}x, \qquad x\in L_N,\ \zeta\in\mathbb{C}:\ 0<|\zeta|<1. \tag{43}$$

Choose an arbitrary $\zeta\in\mathbb{C}:\ 0<|\zeta|<1$. For an arbitrary $h\in H$, by (30) we may write

$$h = a+b, \qquad a\in L_N, \quad b\in H_\zeta.$$

Using relations (43),(40) we obtain

$$\mathbf{R}_{1,\zeta}h = \mathbf{R}_{1,\zeta}a + \mathbf{R}_{1,\zeta}b = \mathbf{R}_{2,\zeta}a + \mathbf{R}_{2,\zeta}b = \mathbf{R}_{2,\zeta}h.$$

Therefore

$$\mathbf{R}_{1,\zeta} = \mathbf{R}_{2,\zeta}, \qquad \zeta\in\mathbb{C}:\ 0<|\zeta|<1. \tag{44}$$

Observe that $\mathbf{R}_{1,0} = E_H = \mathbf{R}_{2,0}$, and the following relation holds [5]:

$$\mathbf{R}^*_{j,\zeta} = \mathbf{R}_{j,\frac{1}{\overline{z}}}, \qquad \zeta\in\mathbb{C}\backslash\{0\}:\ |\zeta|\neq 1, \quad j=1,2.$$

Therefore

$$\mathbf{R}_{1,\zeta} = \mathbf{R}_{2,\zeta}, \qquad \zeta\in\mathbb{C}:\ |\zeta|\neq 1. \tag{45}$$

By the inversion formula, we obtain $\mathbf{E}_{1,t} = \mathbf{E}_{2,t}$. The obtained contradiction completes the proof. □

We shall use the following result:

Theorem 3 *[5, Theorem 3] An arbitrary generalized resolvent $\mathbf{R}_\zeta$ of a closed isometric operator U in a Hilbert space H has the following representation:*

$$\mathbf{R}_\zeta = \left[E - \zeta(U\oplus\Phi_\zeta)\right]^{-1}, \qquad \zeta\in\mathbb{D}. \tag{46}$$

Here Φ_ζ is an analytic in $\mathbb{D}$ operator-valued function which values are linear contractions (i.e. $\|\Phi_\zeta\|\leq 1$) from $H\ominus D(U)$ into $H\ominus R(U)$.
Conversely, each analytic in $\mathbb{D}$ operator-valued function with above properties generates by relation (46) a generalized resolvent $\mathbf{R}_\zeta$ of U.

Observe that relation (46) also shows that different analytic in $\mathbb{D}$ operator-valued functions with above properties generate different generalized resolvents of U.

Comparing the last two theorems we obtain the following result.

Theorem 4 *Let the truncated matrix trigonometric moment problem (1) be given and condition (4) is true. Let an operator A be constructed for the moment problem as in (15). All solutions of the moment problem have the following form*

$$M(t) = (m_{k,j}(t))_{k,j=0}^{N-1}, \qquad t \in [0, 2\pi], \tag{47}$$

where $m_{k,j}$ are obtained from the following relation:

$$\int_0^{2\pi} \frac{1}{1-\zeta e^{it}} dm_{k,j}(t) = (\mathbf{R}_\zeta x_k, x_j)_H, \qquad z \in \mathbb{C}: \ |z| \neq 1; \tag{48}$$

and

$$\mathbf{R}_\zeta = \left[E - \zeta(U \oplus \Phi_\zeta)\right]^{-1}, \quad \mathbf{R}_{\frac{1}{\overline{\zeta}}} = E_H - \mathbf{R}_\zeta^*, \qquad \zeta \in \mathbb{D}. \tag{49}$$

Here Φ_ζ is an analytic in $\mathbb{D}$ operator-valued function which values are linear contractions from $H \ominus D(A)$ into $H \ominus R(A)$.

Conversely, each analytic in $\mathbb{D}$ operator-valued function with above properties generates by relations (47)-(49) a solution of the moment problem (1).

Moreover, the correspondence between all analytic in $\mathbb{D}$ operator-valued functions with above properties and all solutions of the moment problem (1) is bijective.

Proof. The proof is obvious. □

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The truncated matrix trigonometric moment problem: the operator approach.

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In this paper we study the truncated matrix trigonometric moment problem. We obtained a bijective parameterization of all solutions of this moment problem (both in nondegenerate and degenerate cases) via an operator approach. We use important results of M.E. Chumakin on generalized resolvents of isometric operators.