A CHARACTERIZATION OF SIMPLICIAL LOCALIZATION FUNCTORS

C. BARWICK AND D.M. KAN

Abstract. We characterize simplicial localization functors among relative functors from relative categories to simplicial categories as any choice of homotopy inverse to the delocalization functor of Dwyer and the second author.

1. An overview

We start with some preliminaries.

1.1. **Relative categories.** As in [BK] we denote by **RelCat** the category of (small) relative categories and relative functors between them, where by a **relative category** we mean a pair $(\boldsymbol{C}, \boldsymbol{W})$ consisting of a category $\boldsymbol{C}$ and a subcategory $\boldsymbol{W} \subset \boldsymbol{C}$ which contains all the objects of $\boldsymbol{C}$ and their identity maps and of which the maps will be referred to as **weak equivalences** and where by a **relative functor** between two such relative categories we mean a weak equivalence preserving functor.

1.2. **Homotopy equivalences between relative categories.** A relative functor $f\colon \boldsymbol{X} \to \boldsymbol{Y}$ between two relative categories (1.1) is called a **homotopy equivalence** if there exists a relative functor $g\colon \boldsymbol{Y} \to \boldsymbol{X}$ (called a **homotopy inverse** of f) such that the compositions gf and fg are naturally weakly equivalent (i.e. can be connected by a finite zigzag of natural weak equivalences) to the identity functors of $\boldsymbol{X}$ and $\boldsymbol{Y}$ respectively.

1.3. **DK-equivalences.** A map in the category **SCat** of *simplicial categories* (i.e. categories enriched over simplicial sets) is [Be] called a DK-**equivalence** if it induces *weak equivalences* between the simplicial sets involved and an *equivalence of categories* between their *homotopy categories*, i.e. the categories obtained from them by replacing each simplicial set by the set of its components.

Furthermore a map in **RelCat** will similarly be called a DK**equivalence** if its image in **SCat** is so under the *hammock localization functor* [DK2]

$$L^H\colon \mathbf{RelCat} \longrightarrow \mathbf{SCat}$$

(or of course the naturally DK-equivalent functors $\mathbf{RelCat} \to \mathbf{SCat}$ considered in [DK] and [DHKS, 35.6]).

We will denote by both

$$\mathbf{DK} \subset \mathbf{SCat} \qquad \text{and} \qquad \mathbf{DK} \subset \mathbf{RelCat}$$

the subcategories consisting of these DK-equivalences.

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Next we define what we mean by

1.4. **Simplicial localization functors.** In defining DK-equivalences in **RelCat** (1.3) we used the hammock localization functor and not one of the other DK-equivalent functors mentioned because, for our purposes here it seemed to be the more convenient one. However in other situations the others are more convenient and it therefore makes sense to define in general a **simplicial localization functor** as any functor **RelCat** → **SCat** which is naturally DK-equivalent to the functors mentioned above (1.3).

We also need

1.5. **The relativization functor.** In contrast with the situation mentioned in 1.4 there *is* a preferred choice for a **relativization functor**

$$R\colon \mathbf{SCat} \longrightarrow \mathbf{RelCat}$$

which is a kind of inverse of the simplicial localization functor, namely the *delocalization* mentioned in [DK3, 2.5] which assigns to an object $\boldsymbol{A} \in \mathbf{SCat}$ its relative *flattening* which is the relative category which consists of

(i) a category which is the Grothendieck construction on $\boldsymbol{A}$, where $\boldsymbol{A}$ is considered as a simplicial diagram of categories, and
(ii) its subcategory obtained by applying the same construction to the subobject of $\boldsymbol{A}$ which consists of its objects only.

Our main result then is

1.6. **Theorem.** *A relative functor*

$$(\mathbf{RelCat}, \mathbf{DK}) \longrightarrow (\mathbf{SCat}, \mathbf{DK})$$

is a simplicial localization functor (1.5) *iff it is a homotopy inverse* (1.2) *of the realization functor* (1.5)

$$\mathrm{Rel}\colon (\mathbf{SCat}, \mathbf{DK}) \longrightarrow (\mathbf{RelCat}, \mathbf{DK}) \ .$$

We end with some

1.7. **Comments on the proof of 1.6.** The proof of theorem 1.6 heavily involves some of the results of [DK] and [DK3, 2.5] and we therefore first (in §2) review some of the results of these papers.

In §3 we then actually prove theorem 1.6. It turns out however that in addition to the results mentioned in §2 we need a property of the hammock localization of which we will give two proofs. The first is a very short one based on a remark of Toen and Vezzosi [TV, 2.2.1] involving the *homotopy category* of **SCat**. The other, which is due to Bill Dwyer, relies heavily on [DK] and [DK2] and is longer, but has the "advantage" of taking place in the *model category* itself.

2. Preliminaries

In preparation for the proof (in §3) of theorem 1.6 we review here some of the results of [DK], [DK2] and [DK3, 2.3] which will be needed.

2.1. **The hammock localization.** In the proof of 1.6 we will make extensive use of the hammock localization L^{H} of [DK2] because, unlike the other simplicial localization functors, it has the property that every relative category $(\boldsymbol{C}, \boldsymbol{W})$ comes with a *natural embedding* $\boldsymbol{C} \to \mathrm{L}^{\mathrm{H}}(\boldsymbol{C}, \boldsymbol{W})$.

2.2. **The category RelSCat.** This will be the category which has as its objects the pairs $(\boldsymbol{A}, \boldsymbol{U})$ where $\boldsymbol{A} \in \mathbf{SCat}$ and $\boldsymbol{U} \subset \boldsymbol{A}$ is a subobject which contains all the objects of $\boldsymbol{A}$.

One then can consider **RelCat** as a full subcategory of **RelSCat** and [DK2, 2.5] extend the functor $\mathrm{L}^{\mathrm{H}}\colon \mathbf{RelCat} \to \mathbf{SCat}$ to a functor $\mathrm{L}^{\mathrm{H}}\colon \mathbf{RelSCat} \to \mathbf{SCat}$ by sending an object of **RelSCat** to the diagonal of the bisimplicial set obtained from it by dimensionwise application of the hammock localization.

To deal with [DK] and [DK3, 2.5] it will be convenient to introduce a notion of

2.3. **Neglectable categories.** Given an object $(\boldsymbol{A}, \boldsymbol{U}) \in \mathbf{RelSCat}$ (2.2) we will say that $\boldsymbol{U}$ is *neglectable* in $\boldsymbol{A}$ if every map of $\boldsymbol{U}$ goes to an *isomorphism* in $\pi_0\boldsymbol{A}$.

2.4. **Some results from** [DK]**.** [DK, 3.4 and 5.1] imply

(i) *Let $\boldsymbol{A}$ be a category, let $\boldsymbol{U}$ and $\boldsymbol{V} \subset \boldsymbol{A}$ be subcategories which contain all the objects of $\boldsymbol{A}$ and let $\boldsymbol{U} \cup \boldsymbol{V} \subset \boldsymbol{A}$ denote the subcategory spanned by $\boldsymbol{U}$ and $\boldsymbol{V}$ and assume that $\boldsymbol{V}$ is neglectable in $\mathrm{L}^{\mathrm{H}}(\boldsymbol{A}, \boldsymbol{U})$. Then the induced map*

$$\mathrm{L}^{\mathrm{H}}(\boldsymbol{A}, \boldsymbol{U}) \longrightarrow \mathrm{L}^{\mathrm{H}}(\boldsymbol{A}, \boldsymbol{U} \cup \boldsymbol{V}) \in \mathbf{SCat}$$

is a DK-equivalence.

Similarly [DK, 6.4] implies

(ii) *Let $(\boldsymbol{B}, \boldsymbol{V}) \in \mathbf{RelSCat}$ be such that $\boldsymbol{V}$ is neglectable in $\boldsymbol{B}$. Then the induced map (*2.1 *and* 2.2*)*

$$\boldsymbol{B} \longrightarrow \mathrm{L}^{\mathrm{H}}(\boldsymbol{B}, \boldsymbol{V}) \in \mathbf{SCat}$$

is a DK-equivalence.

We end with a brief review of

2.5. **The relativization functor** [DK3, 2.5]**.** The **relativization functor** is the functor

$$\mathrm{Rel}\colon \mathbf{SCat} \longrightarrow \mathbf{RelCat}$$

which sends an object $\boldsymbol{A} \in \mathbf{SCat}$ to the object $(b\boldsymbol{A}, b\mathbf{id}) \in \mathbf{RelCat}$, where $b\boldsymbol{A}$ is the *flattening* of $\boldsymbol{A}$, i.e. the category which has as objects the pairs (A, n), where A is an object of $\boldsymbol{A}$ and n is an integer ≥ 0 and which has as maps $(A_1, n_1) \to (A_2, n_2)$ the pairs (a, q) where a is a map $A_1 \to A_2 \in \boldsymbol{A}_{n_2}$ and q is a simplicial operator from dimension n_1 to dimension n_2 and $b\mathbf{id} \in b\boldsymbol{A}$ is the subcategory consisting of the maps (a, q) for which a is an identity map.

It then was noted in [DK3, 2.5] that, for every object $\boldsymbol{A} \in \mathbf{SCat}$, there exists an object $\overline{\boldsymbol{A}} \in \mathbf{SCat}$ with the same object set as $b\boldsymbol{A}$ with the following properties:

(i) *There is a natural monomorphism $\boldsymbol{A} \to \overline{\boldsymbol{A}}$ which is a DK-equivalence.*

(ii) *There is a natural embedding* $b\boldsymbol{A} \to \overline{\boldsymbol{A}}$ *with the property that (if the image of* $b\mathbf{id} \in b\boldsymbol{A}$ *in* $\overline{\boldsymbol{A}}$ *is also denoted by* $b\mathbf{id}$*) the induced map*

$$\mathrm{L^H}(b\boldsymbol{A}, b\mathbf{id}) \longrightarrow \mathrm{L^H}(\overline{\boldsymbol{A}}, b\mathbf{id}) \in \mathbf{SCat}$$

is a DK-equivalence.

(iii) $b\mathbf{id}$ *is neglectable in* $\overline{\boldsymbol{A}}$ (2.3) *and hence the embedding (*2.1 *and* 2.2*)*

$$\overline{\boldsymbol{A}} \longrightarrow \mathrm{L^H}(\overline{\boldsymbol{A}}, b\mathbf{id}) \in \mathbf{SCat}$$

is a DK-equivalence.

It follows that

(iv) $\boldsymbol{A}$ *and* $\mathrm{L^H}\,\mathrm{Rel}\,\boldsymbol{A}$ *can be connected by the natural zigzag of DK-equivalences*

$$\boldsymbol{A} \longrightarrow \overline{\boldsymbol{A}} \longrightarrow \mathrm{L^H}(\overline{\boldsymbol{A}}, b\mathbf{id}) \longleftarrow \mathrm{L^H}(b\boldsymbol{A}, b\mathbf{id}) = \mathrm{L^H}\,\mathrm{Rel}\,\boldsymbol{A}$$

which in turn implies that

(v) Rel *is a relative functor* (1.3)

$$\mathrm{Rel}\colon (\mathbf{SCat}, \mathbf{DK}) \longrightarrow (\mathbf{RelCat}, \mathbf{DK}) \ .$$

3. A proof of theorem 1.6

To prove theorem 1.6 is suffices, in view of 2.5(iv) and (v), to prove

3.1. **Proposition.** *Every object* $(\boldsymbol{C}, \boldsymbol{W}) \in \mathbf{RelCat}$ *is naturally DK-equivalent to* $\mathrm{Rel}\,\mathrm{L^H}(\boldsymbol{C}, \boldsymbol{W})$.

Proof. Consider the commutative diagram in **RelSCat** (2.2)

$$\begin{array}{ccccc}
(\boldsymbol{C}, \boldsymbol{W}) & & & & \big(bL^H(\boldsymbol{C}, \boldsymbol{W}), b\mathbf{id}\big) = RL^H(\boldsymbol{C}, \boldsymbol{W}) \\
 & \overset{a}{\searrow} & & \overset{b}{\swarrow} & \\
\downarrow c & & \big(bL^H(\boldsymbol{C}, \boldsymbol{W}), b\mathbf{id} \cup \boldsymbol{W}\big) & & \downarrow e \\
 & & \downarrow d & & \\
\big(L^H(\boldsymbol{C}, \boldsymbol{W}), \boldsymbol{W}\big) & & & & \big(\overline{L^H(\boldsymbol{C}, \boldsymbol{W})}, b\mathbf{id}\big) \\
 & \overset{f}{\searrow} & & \overset{g}{\swarrow} & \\
 & & \big(\overline{L^H(\boldsymbol{C}, \boldsymbol{W})}, b\mathbf{id} \cup \boldsymbol{W}\big) & &
\end{array}$$

in which

- c is as in 2.1, f is as in 2.5(i), d and e are as in 2.5(ii) and a is the unique map such that $da = fc$, and
- the symbol $\cup$ is as in 2.4(i) and, in the formulas which involve two $\boldsymbol{W}$'s, the second $\boldsymbol{W}$ is the image of the $\boldsymbol{W}$ in the upper left $(\boldsymbol{C}, \boldsymbol{W})$.

Then it suffices to show that a and b are DK-equivalences in **RelCat** or equivalently that $\mathrm{L^H}\,a$ and $\mathrm{L^H}\,b$ are DK-equivalences in **SCat**.

This is done as follows:

The map f admits a factorization

$$\big(\mathrm{L^H}(\boldsymbol{C}, \boldsymbol{W}), \boldsymbol{W}\big) \xrightarrow{\ x\ } \big(\overline{\mathrm{L^H}(\boldsymbol{C}, \boldsymbol{W})}, \boldsymbol{W}\big) \xrightarrow{\ y\ } \big(\overline{\mathrm{L^H}(\boldsymbol{C}, \boldsymbol{W})}, b\mathbf{id} \cup \boldsymbol{W}\big)$$

in which clearly $\boldsymbol{W}$ is neglectable (2.3) in $\mathrm{L}^{\mathrm{H}}(\boldsymbol{C}, \boldsymbol{W})$ and hence, in view of 2.4(ii) and 2.5(i), $\mathrm{L}^{\mathrm{H}}\, x$ is a DK-equivalence and $\boldsymbol{W}$ is neglectable in $\overline{\mathrm{L}^{\mathrm{H}}(\boldsymbol{C}, \boldsymbol{W})}$. It follows that (2.5(iii)) $\mathrm{L}^{\mathrm{H}}\, y$ is a DK-equivalence and hence (2.4(ii)) so is $\mathrm{L}^{\mathrm{H}}\, g$.

Furthermore, in view of 2.5(ii), $\mathrm{L}^{\mathrm{H}}\, e$ is a DK-equivalence and consequently $\mathrm{L}^{\mathrm{H}}\, d$ is a DK-equivalence and $\boldsymbol{W}$ is neglectable in $\mathrm{L}^{\mathrm{H}}\big(b\,\mathrm{L}^{\mathrm{H}}(\boldsymbol{C}, \boldsymbol{W}), b\mathbf{id}\big)$ which implies that $\mathrm{L}^{\mathrm{H}}\, b$ is a DK-equivalence.

It thus remains to prove that $\mathrm{L}^{\mathrm{H}}\, a$ is a DK-equivalence, but this now follows from

3.2. **Proposition.** $\mathrm{L}^{\mathrm{H}}\, c\colon\ \mathrm{L}^{\mathrm{H}}(\boldsymbol{C}, \boldsymbol{W}) \to \mathrm{L}^{\mathrm{H}}\big(\mathrm{L}^{\mathrm{H}}(\boldsymbol{C}, \boldsymbol{W}), \boldsymbol{W}\big)$ *is a DK-equivalence.*

We will give two proofs of this proposition.

The first is short and is based on a remark by Toen and Vezzosi involving the *homotopy category* $\mathrm{Ho}\,\mathbf{SCat}$ of $\mathbf{SCat}$.

The other, due to Bill Dwyer, is longer but takes place inside the *model category* $\mathbf{S}O$-$\mathbf{Cat}$ of the simplicial categories with a fixed object set O (in this case the object set of $\boldsymbol{C}$).

They both involve the commutative diagram

(i)
$$\begin{array}{ccc} \boldsymbol{C} & \longrightarrow & \mathrm{L}^{\mathrm{H}}(\boldsymbol{C}, \boldsymbol{W}) \\ \downarrow & & \downarrow \\ \mathrm{L}^{\mathrm{H}}(\boldsymbol{C}, \boldsymbol{W}) & \xrightarrow{\mathrm{L}\, c} & \mathrm{L}^{\mathrm{H}}\big(\mathrm{L}^{\mathrm{H}}(\boldsymbol{C}, \boldsymbol{W}), \boldsymbol{W}\big) \end{array}$$

in which the unmarked maps are as in 2.1, and

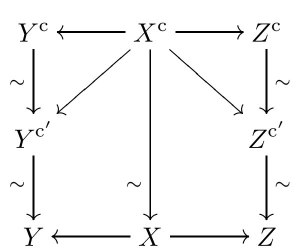
(ii) in which, in view of 2.4(ii) the right hand map is a DK-equivalence.

3.3. **The short proof.** In view of [TV, 2.2.1] the maps at the right and the bottom in 3.2(i) have the same image in $\mathrm{Ho}\,\mathbf{SCat}$ and as (3.2(ii)) the one on the right is a DK-equivalence, so is the one at the bottom.

3.4. **The longer proof.** We start with a brief discussion of

(i) **Homotopy pushouts**

Given a model category together with a choice of *cofibrant replacement functor* and a choice of *functorial factorization* of maps into a cofibration followed by a trivial fibration, associate with every zigzag $Y \leftarrow X \to Z$ a commutative diagram



as follows. The pentagon is obtained by applying the cofibrant approximation functor and the two triangles by means of the functorial factorization. Consequently the maps indicated $\sim$ are weak equivalences.

Then the pushout $Y^{\mathrm{c}} \amalg_{X^{\mathrm{c}}} Z^{\mathrm{c}}$ is a homotopy pushout of the zigzag $Y \leftarrow X \to Z$.

Clearly this construction is functorial in the sense that every diagram of the form

$$\begin{array}{ccccc} Y_0 & \longleftarrow & X_0 & \longrightarrow & Z_0 \\ {\scriptstyle y}\big\downarrow & & {\scriptstyle x}\big\downarrow & & {\scriptstyle z}\big\downarrow \\ Y_1 & \longleftarrow & X_1 & \longrightarrow & Z_1 \end{array}$$

induces a map

$$Y_0^{\mathrm{c}} \amalg_{X_0^{\mathrm{c}}} Z_0^{\mathrm{c}} \longrightarrow Y_1^{\mathrm{c}} \amalg_{X_1^{\mathrm{c}}} Z_1^{\mathrm{c}}$$

which is a weak equivalence whenever y, x and z are.

Next we discuss

(ii) **The original simplicial localization functor** L [DK]

We will work in the model category **S**O**-Cat** [DK] of the simplicial categories with a fixed object set O (which will be the object set of $\boldsymbol{C}$). The weak equivalences in the model structure are the DK-equivalences.

As [DK, 4.1] $\mathrm{L}(\boldsymbol{C},\boldsymbol{W})$ is the pushout of the zigzag

$$\mathrm{F}_*\boldsymbol{C} \longleftarrow \mathrm{F}_*\boldsymbol{W} \longrightarrow \mathrm{F}_*\boldsymbol{W}[\mathrm{F}_*\boldsymbol{W}^{-1}]$$

and the map $\mathrm{F}_*\boldsymbol{W} \to \mathrm{F}_*\boldsymbol{C}$ is a cofibration, it follows from [DK, 8.1] that this pushout is also a homotopy pushout. Consequently

($*$) $\mathrm{L}(\boldsymbol{C},\boldsymbol{W})$ is naturally DK-equivalent to

$$(\mathrm{F}_*\boldsymbol{C})^{\mathrm{c}} \amalg_{(\mathrm{F}_*\boldsymbol{W})^{\mathrm{c}}} \big(\mathrm{F}_*\boldsymbol{W}[\mathrm{F}_*\boldsymbol{W}^{-1}]\big)^{\mathrm{c}} \ .$$

Now we turn to

(iii) **The hammock localization** L^{H} [DK2]

In view of [DK2, 2.5] the functors L and L^{H} are naturally DK-equivalent and there exists a diagram of the form

$$\begin{array}{ccccccc} \mathrm{F}_*\boldsymbol{C} & \longleftarrow & \mathrm{F}_*\boldsymbol{W} & \longrightarrow & \mathrm{F}_*\boldsymbol{W}[\mathrm{F}_*\boldsymbol{W}^{-1}] & = & \mathrm{L}(\boldsymbol{W},\boldsymbol{W}) \\ \approx\big\uparrow & & \approx\big\uparrow & & \big\uparrow & & \\ \mathrm{F}_*\boldsymbol{C} & \longleftarrow & \mathrm{F}_*\boldsymbol{W} & \longrightarrow & \mathrm{L}^{\mathrm{H}}(\mathrm{F}_*\boldsymbol{W},\mathrm{F}_*\boldsymbol{W}) & & \\ \big\downarrow & & \big\downarrow & & \big\downarrow & & \\ \boldsymbol{C} & \longleftarrow & \boldsymbol{W} & \longrightarrow & \mathrm{L}^{\mathrm{H}}(\boldsymbol{W},\boldsymbol{W}) & & \end{array}$$

in which the vertical maps are DK-equivalences. Hence

($*$) $\mathrm{L}^{\mathrm{H}}(\boldsymbol{C},\boldsymbol{W})$ is naturally DK-equivalent to

$$\boldsymbol{C}^{\mathrm{c}} \amalg_{\boldsymbol{W}^{\mathrm{c}}} \mathrm{L}^{\mathrm{H}}(\boldsymbol{W},\boldsymbol{W})^{\mathrm{c}}$$

and $\mathrm{L}^{\mathrm{H}}\big(\mathrm{L}^{\mathrm{H}}(\boldsymbol{C},\boldsymbol{W}),\boldsymbol{W}\big)$ is naturally DK-equivalent to

$$\boldsymbol{C}^{\mathrm{c}} \amalg_{\boldsymbol{W}^{\mathrm{c}}} \mathrm{L}^{\mathrm{H}}(\boldsymbol{W},\boldsymbol{W})^{\mathrm{c}} \amalg_{\boldsymbol{W}^{\mathrm{c}}} \mathrm{L}^{\mathrm{H}}(\boldsymbol{W},\boldsymbol{W})^{\mathrm{c}}$$

in which the two middle maps $\boldsymbol{W}^{\mathrm{c}} \to \mathrm{L}^{\mathrm{H}}(\boldsymbol{W}, \boldsymbol{W})^{\mathrm{c}}$ are the same and which therefore is the same as

$$\boldsymbol{Q} = \operatorname{colim} \left(\begin{array}{ccc} & & \mathrm{L}^{\mathrm{H}}(\boldsymbol{W}, \boldsymbol{W})^{\mathrm{c}} \\ \boldsymbol{C}^{\mathrm{c}} \longleftarrow & \boldsymbol{W}^{\mathrm{c}} & \begin{array}{c} \nearrow \\ \searrow \end{array} \\ & & \mathrm{L}^{\mathrm{H}}(\boldsymbol{W}, \boldsymbol{W})^{\mathrm{c}} \end{array} \right) .$$

It follows that diagram 3.2(i) is DK-equivalent to the commutative diagram

$$\begin{array}{ccc} \boldsymbol{C}^{\mathrm{c}} & \longrightarrow & \boldsymbol{C}^{\mathrm{c}} \amalg_{\boldsymbol{W}^{\mathrm{c}}} \mathrm{L}^{\mathrm{H}}(\boldsymbol{W}, \boldsymbol{W})^{\mathrm{c}} \\ \downarrow & & \downarrow u \\ \boldsymbol{C}^{\mathrm{c}} \amalg_{\boldsymbol{W}^{\mathrm{c}}} \mathrm{L}^{\mathrm{H}}(\boldsymbol{W}, \boldsymbol{W})^{\mathrm{c}} & \xrightarrow{\ v\ } & \boldsymbol{Q} \end{array}$$

in which u is obtained by mapping the zigzag $\boldsymbol{C}^{\mathrm{c}} \leftarrow \boldsymbol{W}^{\mathrm{c}} \to \mathrm{L}^{\mathrm{H}}(\boldsymbol{W}, \boldsymbol{W})^{\mathrm{c}}$ to the upper zigzag in the diagram whose colimit is $\boldsymbol{Q}$ and v is obtained by mapping it to the lower zigzag.

In view of 3.2(ii) the map u is a DK-equivalence and as $v = Tu$ where $T \colon \boldsymbol{Q} \to \boldsymbol{Q}$ denotes the automorphism which switches the two copies of $\mathrm{L}^{\mathrm{H}}(\boldsymbol{W}, \boldsymbol{W})^{\mathrm{c}}$, so is the map v and therefore also the desired map

$$\mathrm{L}\, c \colon\ \mathrm{L}^{\mathrm{H}}(\boldsymbol{C}, \boldsymbol{W}) \longrightarrow \mathrm{L}^{\mathrm{H}}\big(\mathrm{L}^{\mathrm{H}}(\boldsymbol{C}, \boldsymbol{W}), \boldsymbol{W}\big) .$$

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Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139

E-mail address: `clarkbar@gmail.com`

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139