

IN THE CATEGORY OF RELATIVE CATEGORIES THE REZK EQUIVALENCES ARE EXACTLY THE DK-EQUIVALENCES

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ABSTRACT. In a previous paper we lifted Charles Rezk’s complete Segal model structure on the category of simplicial spaces to a Quillen equivalent one on the category of “relative categories” and our aim in this successor paper is to obtain a more explicit description of the weak equivalences in this model structure by showing that *these weak equivalences are exactly the DK-equivalences*, i.e. those maps between relative categories which induce a weak equivalence between their simplicial localizations.

1. A FORMULATION OF THE RESULTS

We start with some preliminaries.

1.1. Relative categories. As in [BK1] we denote by \mathbf{RelCat} the category of (small) relative categories and relative functors between them, where by a **relative category** we mean a pair $(\mathcal{C}, \mathcal{W})$ consisting of a category \mathcal{C} and a subcategory $\mathcal{W} \subset \mathcal{C}$ which contains all the objects of \mathcal{C} and their identity maps and of which the maps will be referred to as **weak equivalences** and where by a **relative functor** between two such relative categories we mean a weak equivalence preserving functor.

1.2. Rezk equivalences. In [BK1] we lifted Charles Rezk’s complete Segal model structure on the category $s\mathcal{S}$ of (small) *simplicial spaces* (i.e. bisimplicial sets) to a Quillen equivalent model structure on the category \mathbf{RelCat} (1.1). We will refer to the weak equivalences in both these model structures as **Rezk equivalences** and denote by both

$$\mathbf{Rk} \subset s\mathcal{S} \quad \text{and} \quad \mathbf{Rk} \subset \mathbf{RelCat}$$

the subcategories consisting of these Rezk equivalences.

1.3. DK-equivalences. A map in the category \mathbf{SCat} of (small) *simplicial categories* (i.e. categories enriched over simplicial sets) is [Be1] called a **DK-equivalence** if it induces *weak equivalences* between the simplicial sets involved and an *equivalence of categories* between their *homotopy categories*, i.e. the categories obtained from them by replacing each simplicial set by the set of its components.

Furthermore a map in \mathbf{RelCat} will similarly be called a **DK-equivalence** if its image in \mathbf{SCat} is so under the *hammock localization functor* [DK2]

$$L^H : \mathbf{RelCat} \longrightarrow \mathbf{SCat}$$

(or of course the naturally DK-equivalent functors $\mathbf{RelCat} \rightarrow \mathbf{SCat}$ considered in [DK] and [DHKS, 35.6]).

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We will denote by both

$$\mathbf{DK} \subset \mathbf{SCat} \quad \text{and} \quad \mathbf{DK} \subset \mathbf{RelCat}$$

the subcategories consisting of these DK-equivalences.

Our main result then is

1.4. Theorem. *A map in \mathbf{RelCat} (1.1) is a Rezk equivalence (1.2) iff it is a DK-equivalence (1.3).*

2. AN OUTLINE OF THE PROOF

The proof of theorem 1.4 heavily involves the notion of

2.1. Homotopy equivalences between relative categories. A relative functor $f: \mathbf{X} \rightarrow \mathbf{Y}$ between two relative categories (1.1) is called a **homotopy equivalence** if there exists a relative functor $g: \mathbf{Y} \rightarrow \mathbf{X}$ (called a **homotopy inverse** of f) such that the compositions gf and fg are naturally weakly equivalent (i.e. can be connected by a finite zigzag of natural weak equivalences) to the identity functors of \mathbf{X} and \mathbf{Y} respectively. This definition readily implies:

- (*) *If $f: \mathbf{X} \rightarrow \mathbf{Y}$ is a homotopy equivalence between relative categories which have the two out of three property, then a map $x: X_1 \rightarrow X_2 \in \mathbf{X}$ is a weak equivalence (1.1) iff the induced map $fx: fX_1 \rightarrow fX_2 \in \mathbf{Y}$ is so.*

We also need the following three results.

2.2. The simplicial nerve functor N . In view of [BK1, 6.1]

- (*) *the simplicial nerve functor $N: \mathbf{RelCat} \rightarrow \mathbf{sS}$ (3.1) is a homotopy equivalence (1.2)*

$$N: (\mathbf{RelCat}, \mathbf{Rk}) \longrightarrow (\mathbf{sS}, \mathbf{Rk}) .$$

2.3. The relativization functor Rel . It was shown in [BK2, 1.6] that the relativization functor $\text{Rel}: \mathbf{SCat} \rightarrow \mathbf{RelCat}$ of the delocalization theorem of [DK3, 2.5] is a homotopy inverse of the simplicial localization functors $(\mathbf{RelCat}, \mathbf{DK}) \rightarrow (\mathbf{SCat}, \mathbf{DK})$ mentioned in 1.3. This clearly implies that

- (*) *the relativization functor $\text{Rel}: \mathbf{SCat} \rightarrow \mathbf{RelCat}$ (3.2) is a homotopy equivalence*

$$\text{Rel}: (\mathbf{SCat}, \mathbf{DK}) \longrightarrow (\mathbf{RelCat}, \mathbf{DK}) .$$

2.4. The flipped nerve functor Z . In view of [Be2, 6.3 and 8.6]

- (*) *the flipped nerve functor $Z: \mathbf{SCat} \rightarrow \mathbf{sS}$ (3.3) is a homotopy equivalence*

$$Z: (\mathbf{SCat}, \mathbf{DK}) \longrightarrow (\mathbf{sS}, \mathbf{Rk}) .$$

These four results marked (*) strongly suggest that theorem 1.4 should be true. To obtain a proof one just has to show that the functors $N \text{ Rel}$ and $Z: \mathbf{SCat} \rightarrow \mathbf{sS}$ are naturally Rezk equivalent.

This will be done in §3 below. In fact we will prove the following somewhat stronger result:

2.5. Proposition. *The functors*

$$N \text{ Rel} \text{ and } Z: \mathbf{SCat} \longrightarrow \mathbf{sS}$$

are naturally Reedy equivalent.

3. COMPLETION OF THE PROOF

Before completing the proof of theorem 1.4, i.e. proving proposition 2.5, we recall first some of the notions involved.

3.1. The simplicial nerve functor N . This is the functor $N: \mathbf{RelCat} \rightarrow \mathbf{sS}$ which sends an object $X \in \mathbf{RelCat}$ to the bisimplicial set which has as its (p, q) -bisimplices ($p, q \geq 0$) the maps

$$\hat{p} \times \tilde{q} \longrightarrow X \in \mathbf{RelCat}$$

where \hat{p} denotes the category $0 \rightarrow \cdots \rightarrow p$ in which only the identity maps are weak equivalences and \tilde{q} denotes the category $0 \rightarrow \cdots \rightarrow q$ in which all maps are weak equivalences.

3.2. The relativization functor Rel . This is the functor $\text{Rel}: \mathbf{SCat} \rightarrow \mathbf{sS}$ which sends an object $\mathbf{A} \in \mathbf{SCat}$ to the pair $(b\mathbf{A}, \text{bid}) \in \mathbf{RelCat}$ in which

- (i) $b\mathbf{A}$ is the Grothendieck construction on \mathbf{A} obtained by considering \mathbf{A} as a simplicial diagram of categories, i.e. the category
 - (i)' which has as objects the pairs (p, A) consisting of an integer $p \geq 0$ and an object $A \in \mathbf{A}$,
 - (i)'' which has as maps $(p_1, A_1) \rightarrow (p_2, A_2)$ the pairs (t, a) consisting of a simplicial operator t from dimension p_1 to dimension p_2 and a map $a: A_1 \rightarrow A_2 \in \mathbf{A}_{p_2}$, and
 - (i)''' in which the composition is given by the formula

$$(t', a')(t, a) = (t't, a'(ta))$$

and in which

- (ii) id denotes the subobject of \mathbf{A} consisting of the maps of the form (t, id) .

3.3. The flipped nerve functor Z . This is the functor $Z: \mathbf{SCat} \rightarrow \mathbf{sS}$ which sends an object $\mathbf{A} \in \mathbf{SCat}$ to the simplicial space $Z\mathbf{A}$ of which the space in dimension $k \geq 0$ is the simplicial set $(Z\mathbf{A})_k$ which is the disjoint union, taken over all ordered sequences A_0, \dots, A_k of objects of \mathbf{A} , of the products

$$\text{hom}(A_0, A_1) \times \cdots \times \text{hom}(A_{k-1}, A_k) .$$

3.4. The opposite Γ^{op} of the category of simplices functor Γ . This is the functor $\Gamma^{\text{op}}: \mathcal{S} \rightarrow \mathbf{Cat}$ which sends a simplicial set $X \in \mathcal{S}$ to its category of simplices, i.e. the category which has

- (i) as objects the pair (p, x) consisting of an integer $p \geq 0$ and a p -simplex of X , and
- (ii) as maps $(p_1, x_1) \rightarrow (p_2, x_2)$ the simplicial operators t from dimension p_1 to dimension p_2 such that $tx_1 = x_2$.

We also need

3.5. Some auxiliary notions. For every object $\mathbf{A} \in \mathbf{SCat}$, denote

- by $Y\mathbf{A}$ the simplicial diagram of categories of which the category $(Y\mathbf{A})_k$ in dimension $k \geq 0$ has as objects the sequences of maps in $b\mathbf{A}$ (3.2) of the form

$$(p_0, A_0) \xrightarrow{(t_1, a_1)} \cdots \xrightarrow{(t_k, a_k)} (p_k, A_k)$$

and as maps the commutative diagrams in $b\mathbf{A}$ of the form

$$\begin{array}{ccc} (p_0, A_0) & \xrightarrow{(t_1, a_1)} \cdots \xrightarrow{(t_k, a_k)} & (p_k, A_k) \\ (u_0, \text{id}) \downarrow & & \downarrow (u_k, \text{id}) \\ (p'_0, A_0) & \xrightarrow{(t'_1, a'_1)} \cdots \xrightarrow{(t'_k, a'_k)} & (p'_k, A_k) \end{array}$$

and

- by $\overline{Y\mathbf{A}} \subset Y\mathbf{A}$ the subobject of which the category $(\overline{Y\mathbf{A}})_k$ in dimension k is the subcategory of $(Y\mathbf{A})_k$ consisting of the above maps for which *the t_i 's and the t'_i 's are identities and hence all p_i 's are the same, all p'_i 's are the same and all u_i 's are the same.*

Then there is a *strong deformation retraction* of $Y\mathbf{A}$ onto $\overline{Y\mathbf{A}}$ which to each object of $Y\mathbf{A}$ as above assigns the map

$$\begin{array}{ccc} (p_0, A_0) & \xrightarrow{(t_1, a_1)} \cdots \xrightarrow{(t_k, a_k)} & (p_k, A_k) \\ (t_k \cdots t_1, \text{id}) \downarrow & & \downarrow (\text{id}, \text{id}) \\ (p_k, A_0) & \xrightarrow{(\text{id}, t_k \cdots t_1 a_1)} \cdots \xrightarrow{(\text{id}, a_k)} & (p_k, A_k) \end{array}$$

the existence of which implies that

- (i) *the inclusion $\overline{Y\mathbf{A}} \subset Y\mathbf{A}$ is a dimensionwise weak equivalence of categories.*

One also readily verifies that there is a canonical 1-1 correspondence between the objects of $(\overline{Y\mathbf{A}})_k$ and the simplices of $(Z\mathbf{A})_k$ (3.3) and that in effect

- (ii) *there is a canonical isomorphism $\overline{Y\mathbf{A}} \approx \Gamma^{\text{op}} Z\mathbf{A}$ (3.4).*

Now we are ready for the

3.6. Completion of the proof. Let $n: \mathbf{Cat} \rightarrow \mathbf{S}$ denote the *classical nerve functor*.

Then clearly $N \operatorname{Rel} \mathbf{A} = nY\mathbf{A}$ (3.5) and if one defines $\overline{N \operatorname{Rel} \mathbf{A}} \subset N \operatorname{Rel} \mathbf{A}$ by $\overline{N \operatorname{Rel} \mathbf{A}} \subset n\overline{Y\mathbf{A}}$, then it follows from 3.5(i) that

(i) *the inclusion $\overline{N \operatorname{Rel} \mathbf{A}} \rightarrow N \operatorname{Rel} \mathbf{A} \in \mathbf{sS}$ is a Reedy equivalence.*

Moreover it follows from 3.5(ii) that

(ii) *there is a canonical isomorphism $\overline{N \operatorname{Rel} \mathbf{A}} \approx n\Gamma^{\operatorname{op}}Z\mathbf{A}$*

and to complete the proof of proposition 2.5 and hence of theorem 1.4 it thus suffices, in view of the fact that clearly

(iii) *the functors $n\Gamma^{\operatorname{op}}$ and $n\Gamma: \mathbf{S} \rightarrow \mathbf{S}$ (3.4) are naturally weakly equivalent,*

to show that

(iv) *there exists a natural Reedy equivalence $n\Gamma Z\mathbf{A} \rightarrow Z\mathbf{A} \in \mathbf{sS}$.*

But this follows immediately from the observation of Dana Latch [L] (see also [H, 18.9.3]) that there exists a natural weak equivalence

$$n\Gamma \longrightarrow 1 .$$

REFERENCES

- [BK1] C. Barwick and D. M. Kan, *Relative categories; another model for the homotopy theory of homotopy theories*, To appear.
- [BK2] ———, *A characterization of simplicial localization functors*, To appear.
- [Be1] J. E. Bergner, *A model category structure on the category of simplicial categories*, Trans. Amer. Math. Soc. **359** (2007), no. 5, 2043–2058.
- [Be2] ———, *Three models for the homotopy theory of homotopy theories*, Topology **46** (2007), no. 4, 397–436.
- [H] P. S. Hirschhorn, *Model categories and their localizations*, Mathematical Surveys and Monographs, vol. 99, American Mathematical Society, Providence, RI, 2003.
- [DHKS] W. G. Dwyer, P. S. Hirschhorn, D. M. Kan, and J. H. Smith, *Homotopy limit functors on model categories and homotopical categories*, Mathematical Surveys and Monographs, vol. 113, American Mathematical Society, Providence, RI, 2004.
- [DK1] W. G. Dwyer and D. M. Kan, *Simplicial localizations of categories*, J. Pure Appl. Algebra **17** (1980), no. 3, 267–284.
- [DK2] ———, *Calculating simplicial localizations*, J. Pure Appl. Algebra **18** (1980), no. 1, 17–35.
- [DK3] ———, *Equivalences between homotopy theories of diagrams*, Algebraic topology and algebraic K-theory (Princeton, N.J., 1983), Ann. of Math. Stud., vol. 113, Princeton Univ. Press, Princeton, NJ, 1987, pp. 180–205.
- [L] D. M. Latch, *The uniqueness of homology for the category of small categories*, J. Pure Appl. Algebra **9** (1976/77), no. 2, 221–237.

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