

Fractional Derivatives and Integrals on Time Scales via the Inverse Generalized Laplace Transform*

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Abstract

We introduce a fractional calculus on time scales using the theory of delta (or nabla) dynamic equations. The basic notions of fractional order integral and fractional order derivative on an arbitrary time scale are proposed, using the inverse Laplace transform on time scales. Useful properties of the new fractional operators are proved.

Keywords: fractional derivatives and integrals, time scales, Laplace transform.

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1 Introduction

The fractional calculus deals with extensions of derivatives and integrals to noninteger orders. It represents a powerful tool in applied mathematics to study a myriad of problems from different fields of science and engineering, with many break-through results found in mathematical physics, finance, hydrology, biophysics, thermodynamics, control theory, statistical mechanics, astrophysics, cosmology and bioengineering [10, 13, 14, 16].

The fractional calculus may be approached via the theory of linear differential equations. Analogously, starting with a linear difference equation, we are led to a definition of fractional difference of an arbitrary order [3]. Our main objective is to introduce here a fractional calculus on an arbitrary time scale, using the theory of delta (or nabla) differential equations.

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The analysis on time scales is a fairly new area of research. It was introduced in 1988 by Stefan Hilger and his Ph.D. supervisor Bernd Aulbach [2,9]. It combines the traditional areas of continuous and discrete analysis into one theory. After the publication of three textbooks in this area [5,7,11], more and more researchers are getting involved in this fast-growing field.

Recently, two attempts have been made to provide a general definition of fractional derivative on an arbitrary time scale [1,15]. These two works address a very interesting question, but unfortunately there is a small inconsistency in the very beginning of both studies. Indeed, investigations of [1,15] are based on the following definition of generalized polynomials on time scales $h_\alpha : \mathbb{T} \times \mathbb{T} \rightarrow \mathbb{R}$:

$$\begin{aligned} h_0(t, s) &= 1, \\ h_{\alpha+1}(t, s) &= \int_s^t h_\alpha(\tau, s) \Delta\tau. \end{aligned} \tag{1}$$

Recursion (1) provides a definition only in the case $\alpha \in \mathbb{N}_0$, and there is no hope to define polynomials h_α for real or complex indices α with (1). Here we propose a different approach to the subject based on the Laplace transform [8].

The paper is organized as follows. In Sect. 2 we briefly review the basic notions of Riemann–Liouville and Caputo fractional integration and differentiation (Sect. 2.1) as well as necessary tools from time scales (Sect. 2.2). Our results are then given in Sect. 3: we introduce the concept of fractional integral and fractional derivative on an arbitrary time scale \mathbb{T} (Sect. 3.1); we then prove some important properties of the fractional integrals and derivatives (Sect. 3.2).

2 Preliminaries

The following definitions and basic results serve essentially to fix notations. The reader is assumed to be familiar with both fractional calculus and calculus on time scales. For an introduction to fractional and time scale theories we refer to the books [10,13,14,16] and [5,7,11].

2.1 Caputo and Riemann–Liouville fractional derivatives

Let f be an arbitrary integrable function. By $\mathcal{D}_a^{-\alpha} f$ we denote the fractional integral of f of order $\alpha \in (0,1)$ on $[0,t]$, defined as

$$(\mathcal{D}_a^{-\alpha} f)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}} d\tau$$

with Γ the well-known gamma function. For an arbitrary real number α , the Riemann–Liouville and Caputo fractional derivatives are defined, respectively, as in [12]:

$$(D_a^\alpha f)(t) = \left(\frac{d^{[\alpha]+1}}{dt^{[\alpha]+1}} \left(\mathcal{D}_a^{-([\alpha]-\alpha+1)} f \right) \right) (t)$$

and

$$({}^C D_a^\alpha f)(t) = \left(\mathcal{D}_a^{-([\alpha]-\alpha+1)} \left(\frac{d^{[\alpha]+1}}{dt^{[\alpha]+1}} f \right) \right) (t),$$

where $[\alpha]$ is the integer part of α .

The notion of Riemann–Liouville derivative is the preferred fractional derivative among mathematicians, while Caputo fractional derivative is the preferred one among engineers. If $f(a) = f'(a) = \dots = f^{(n-1)}(a) = 0$, then both Riemann–Liouville and Caputo derivatives coincide. In particular, for $\alpha \in (0, 1)$ and $f(a) = 0$ one has ${}^C D_a^\alpha f(t) = D_a^\alpha f(t)$.

Next proposition gives the Laplace transform of the Caputo fractional derivative.

Proposition 2.1 ([10]). *Let $\alpha > 0$, n be the integer such that $n-1 < \alpha \leq n$, and f a function satisfying $f \in C^n(\mathbb{R}^+)$, $f^{(n)} \in L_1(0, t_1)$, $t_1 > 0$, and $|f^{(n)}(t)| \leq B e^{q_0 t}$, $t > t_1 > 0$. If the Laplace transforms $\mathcal{L}[f](z)$ and $\mathcal{L}[D^n f](z)$ exist, and $\lim_{t \rightarrow +\infty} D^k f(t) = 0$ for $k = 0, \dots, n-1$, then*

$$\mathcal{L} [{}^C D_0^\alpha f] (z) = z^\alpha \mathcal{L} [f] (z) - \sum_{k=0}^{n-1} f^{(k)}(0) z^{\alpha-k-1}.$$

Remark 2.1. If $\alpha \in (0, 1]$, then $\mathcal{L} [{}^C D_0^\alpha f] (z) = z^\alpha \mathcal{L} [f] (z) - f(0) z^{\alpha-1}$.

2.2 The Laplace transform on time scales

Throughout the paper, \mathbb{T} is an arbitrary time scale with bounded graininess, i.e., $0 < \mu_{\min} \leq \mu(t) \leq \mu_{\max}$ for all $t \in \mathbb{T}$. Let $t_0 \in \mathbb{T}$ be fixed. We define functions $h_k(\cdot, t_0) : \mathbb{T} \rightarrow \mathbb{R}$, $k \in \mathbb{N}_0$, recursively as in (1): $h_0(t, t_0) \equiv 1$, $h_{k+1}(t, t_0) = \int_{t_0}^t h_k(\tau, t_0) \Delta \tau$. Functions $h_k(\cdot, t_0)$ are called *generalized polynomials* on the time scale \mathbb{T} . By $h_k^\Delta(t, t_0)$ we denote the delta derivative of $h_k(t, t_0)$ with respect to t . For $t \in \mathbb{T}^\kappa$ one has $h_0^\Delta(t, t_0) = 0$ and $h_k^\Delta(t, t_0) = h_{k-1}(t, t_0)$, $k \in \mathbb{N}$.

Example 2.2. In [5] one can find exact formulas of generalized polynomials for some particular time scales \mathbb{T} : if $\mathbb{T} = \mathbb{R}$, then $h_k(t, t_0) = \frac{(t-t_0)^k}{k!}$; if $\mathbb{T} = \mathbb{Z}$, then $h_k(t, t_0) = \frac{(t-t_0)^{(k)}}{k!} = \binom{t-t_0}{k}$, where $(t-t_0)^{(0)} = 1$ and $(t-t_0)^{(k)} = (t-t_0)(t-t_0-1) \cdots (t-t_0-k+1)$, $k \in \mathbb{N}$.

The Laplace transform on time scales is studied in [8].

Definition 2.1 ([8]). For $f : \mathbb{T} \rightarrow \mathbb{R}$, the generalized Laplace transform of f , denoted by $\mathcal{L}_{\mathbb{T}}[f]$, is defined by

$$\mathcal{L}_{\mathbb{T}}[f](z) = F(z) := \int_0^\infty f(t) e_{\ominus z}(\sigma(t), 0) \Delta t.$$

Remark 2.2. In view of Definition 2.1, the Laplace transform \mathcal{L} of Proposition 2.1 can be written as $\mathcal{L}_{\mathbb{R}}$.

Definition 2.2 ([8]). The function $f : \mathbb{T} \rightarrow \mathbb{R}$ is said to be of *exponential type I* if there exist constants $M, c > 0$ such that $|f(t)| \leq M e^{ct}$. Furthermore, f is said to be of *exponential type II* if there exist constants $M, c > 0$ such that $|f(t)| \leq M e_c(t, 0)$.

The time scale exponential function $e_c(t, 0)$ is of type II while generalized polynomials $h_k(t, 0)$ are of type I.

Theorem 2.3 ([8]). *If f is of exponential type II with exponential constant c , then the delta integral $\int_0^\infty f(t) e_{\ominus}(\sigma(t), 0) \Delta t$ converges absolutely for $z \in D$.*

From [8, Theorem 1.3] and mathematical induction one obtains the following result:

Proposition 2.4. *Let F be the generalized Laplace transform of $f : \mathbb{T} \rightarrow \mathbb{R}$. Then,*

$$\mathcal{L}_{\mathbb{T}}[f^{\Delta^n}](z) = z^n F(z) - \sum_{k=0}^{n-1} z^{n-k-1} f^{\Delta^k}(0). \quad (2)$$

Remark 2.3. If F is the Laplace transform of $f : \mathbb{T} \rightarrow \mathbb{R}$, then the Laplace transform of the delta derivative of f is given by

$$\mathcal{L}_{\mathbb{T}}[f^{\Delta}](z) = zF(z) - f(0).$$

Our work is motivated by the following result:

Theorem 2.5 (Inversion formula of the Laplace transform [8]). *Suppose that F is analytic in the region $\operatorname{Re}_{\mu}(z) > \operatorname{Re}_{\mu}(c)$ and $F(z) \rightarrow 0$ uniformly as $|z| \rightarrow \infty$ in this region. Assume F has finitely many regressive poles of finite order $\{z_1, z_2, \dots, z_n\}$ and $\tilde{F}_{\mathbb{R}}(z)$ is the transform of the function $\tilde{f}(t)$ on \mathbb{R} that corresponds to the transform $F(z) = F_{\mathbb{T}}(z)$ of $f(t)$ on \mathbb{T} . If*

$$\int_{c-i\infty}^{c+i\infty} \left| \tilde{F}_{\mathbb{R}}(z) \right| |dz| < \infty,$$

then

$$f(t) = \sum_{i=1}^n \operatorname{Res}_{z=z_i} e_z(t, 0) F(z)$$

has transform $F(z)$ for all z with $\operatorname{Re}(z) > c$.

Remark 2.4. The inverse transform of $F(z) = \frac{1}{z^{k+1}}$, $k \in \mathbb{N}$, is $h_k(\cdot, 0)$. Moreover, the inversion formula gives the claimed inverses for any of the elementary functions that were presented in the table of Laplace transform in [6].

Using the series $e_z(t, 0) = \sum_{k=0}^{+\infty} z^k h_k(t, 0)$, we see that generalized polynomials $h_k(t, 0)$ give us the difference between time scales for the inverse Laplace images.

3 Main Results

We begin by introducing the definition of fractional integral and fractional derivative on an arbitrary time scale \mathbb{T} .

3.1 Fractional derivative and integral on time scales

Similarly to the classical case, the Laplace transform of a Caputo fractional derivative of order $\alpha \in (0, 1]$ is given by $\mathcal{L} [{}^C D_{0^+}^{\alpha} f](z) = z^{\alpha} \mathcal{L}[f](z) - f(0^+) z^{\alpha-1}$. Simultaneously, the generalized Laplace transform on time scales gives unification and extension of the classical results. Important to us, the Laplace transform of the Δ -derivative is given by the formula $\mathcal{L}_{\mathbb{T}}[f^{\Delta}](z) = zF(z) - f(0)$. Our idea is to define the fractional derivative on time scales via the inverse Laplace transform formula for the complex function $G(z) = z^{\alpha} \mathcal{L}_{\mathbb{T}}[f](z) - f(0^+) z^{\alpha-1}$. Furthermore, for $\alpha \in (n-1, n]$, $n \in \mathbb{N}$, we use a generalization of (2) to define fractional derivatives on time scales for higher orders α .

Definition 3.1 (Fractional integral on time scales). Let $\alpha > 0$, \mathbb{T} be a time scale, and $f : \mathbb{T} \rightarrow \mathbb{R}$. The fractional integral of f of order α on the time scale \mathbb{T} , denoted by $I_{\mathbb{T}}^{\alpha} f$, is defined by

$$I_{\mathbb{T}}^{\alpha} f(t) = \mathcal{L}_{\mathbb{T}}^{-1} \left[\frac{F(z)}{z^{\alpha}} \right] (t).$$

Definition 3.2 (Fractional derivative on time scales). Let \mathbb{T} be a time scale, $F(z) = \mathcal{L}_{\mathbb{T}}[f](z)$, and $\alpha \in (n-1, n]$, $n \in \mathbb{N}$. The fractional derivative of function f of order α on the time scale \mathbb{T} , denoted by $f^{(\alpha)}$, is defined by

$$f^{(\alpha)}(t) = \mathcal{L}_{\mathbb{T}}^{-1} \left[z^{\alpha} F(z) - \sum_{k=0}^{n-1} f^{\Delta^k}(0^+) z^{\alpha-k-1} \right] (t). \quad (3)$$

Remark 3.1. For $\alpha \in (0, 1]$ we have

$$f^{(\alpha)}(t) = \mathcal{L}_{\mathbb{T}}^{-1} [z^{\alpha} F(z) - f(0^+) z^{\alpha-1}] (t).$$

3.2 Properties

We begin with two trivial but important remarks about the fractional integral and the fractional derivative operators just introduced.

Remark 3.2. As the inverse Laplace transform is linear, we also have linearity for the new fractional integral and derivative:

$$\begin{aligned} I_{\mathbb{T}}^{\alpha}(af + bg)(t) &= aI_{\mathbb{T}}^{\alpha} f(t) + bI_{\mathbb{T}}^{\alpha} g(t), \\ (af + bg)^{(\alpha)}(t) &= af^{(\alpha)}(t) + bg^{(\alpha)}(t). \end{aligned}$$

Remark 3.3. Since $h_0(t) \equiv 1$ for any time scale \mathbb{T} , from the definition of Laplace transform and the fractional derivative we conclude that $h_0^{(\alpha)}(t) = 0$. For $\alpha > 1$ one has $h_1^{(\alpha)}(t) = 0$.

We now prove several important properties of the fractional integrals and fractional derivatives on arbitrary time scales.

Proposition 3.1. *Let $\alpha \in (n-1, n]$, $n \in \mathbb{N}$. If $k \leq n-1$, then*

$$h_k^{(\alpha)}(t, 0) = 0.$$

Proof. From (3) it follows that

$$\begin{aligned} h_k^{(\alpha)}(t, 0) &= \mathcal{L}_{\mathbb{T}}^{-1} \left[\frac{z^{\alpha}}{z^{k+1}} - \sum_{i=0}^{n-1} h_k^{\Delta^i}(0) z^{\alpha-i-1} \right] (t) = \mathcal{L}_{\mathbb{T}}^{-1} \left[\frac{z^{\alpha}}{z^{k+1}} - \sum_{i=0}^k h_{k-i}(0) z^{\alpha-i-1} \right] (t) \\ &= \mathcal{L}_{\mathbb{T}}^{-1} \left[\frac{z^{\alpha}}{z^{k+1}} - z^{\alpha-k-1} \right] (t) = 0. \end{aligned}$$

□

Proposition 3.2. *Let $\alpha \in (n-1, n]$, $n \in \mathbb{N}$. If $k \geq n$, then*

$$h_k^{(\alpha)}(t, 0) = \mathcal{L}_{\mathbb{T}}^{-1} \left[\frac{1}{z^{k+1-\alpha}} \right] (t).$$

Proof. From (3) we have

$$\begin{aligned} h_k^{(\alpha)}(t) &= \mathcal{L}_{\mathbb{T}}^{-1} \left[\frac{z^\alpha}{z^{k+1}} - \sum_{i=0}^{n-1} h_k^{\Delta^i}(0) z^{\alpha-i-1} \right] (t) = \mathcal{L}_{\mathbb{T}}^{-1} \left[\frac{z^\alpha}{z^{k+1}} - \sum_{i=0}^{n-1} h_{k-i}(0) z^{\alpha-i-1} \right] (t) \\ &= \mathcal{L}_{\mathbb{T}}^{-1} \left[\frac{z^\alpha}{z^{k+1}} \right] (t) = \mathcal{L}_{\mathbb{T}}^{-1} \left[\frac{1}{z^{k+1-\alpha}} \right] (t). \end{aligned}$$

□

Proposition 3.3. *Let $\alpha \in (n-1, n]$, $n \in \mathbb{N}$. If $c(t) \equiv m$, $m \in \mathbb{R}$, then*

$$c^{(\alpha)}(t) = 0.$$

Proof. From the linearity of the inverse Laplace transform and the fact that $h_0^{(\alpha)}(t, 0) = 0$, it follows that $c^{(\alpha)}(t) = (m \times 1)^{(\alpha)} = (m \times h_0(t, 0))^{(\alpha)} = m \times h_0^{(\alpha)}(t, 0) = m \times 0 = 0$. □

Proposition 3.4. *Let $\alpha, \beta > 0$. Then,*

$$I_{\mathbb{T}}^\beta (I_{\mathbb{T}}^\alpha f)(t) = I_{\mathbb{T}}^{\alpha+\beta} f(t).$$

Proof. By definition we have

$$I_{\mathbb{T}}^\beta (I_{\mathbb{T}}^\alpha f)(t) = \mathcal{L}_{\mathbb{T}}^{-1} \left[z^{-\beta} \mathcal{L}_{\mathbb{T}} [I_{\mathbb{T}}^\alpha f] \right] (t) = \mathcal{L}_{\mathbb{T}}^{-1} \left[s^{-\alpha-\beta} F(z) \right] (t) = I_{\mathbb{T}}^{\alpha+\beta} f(t).$$

□

Proposition 3.5. *Let $\alpha, \beta \in (0, 1]$.*

(a) *If $\alpha + \beta \leq 1$, then*

$$\left(f^{(\alpha)} \right)^{(\beta)}(t) = f^{(\alpha+\beta)}(t) - \mathcal{L}_{\mathbb{T}}^{-1} \left[z^{\beta-1} f^{(\alpha)}(0) \right] (t).$$

(b) *If $1 < \alpha + \beta \leq 2$, then*

$$\left(f^{(\alpha)} \right)^{(\beta)}(t) = f^{(\alpha+\beta)}(t) - \mathcal{L}_{\mathbb{T}}^{-1} \left[z^{\beta-1} f^{(\alpha)}(0) \right] (t) + \mathcal{L}_{\mathbb{T}}^{-1} \left[z^{\alpha+\beta-2} f^{\Delta}(0) \right] (t).$$

(c) *If $\beta \in (0, 1]$, then*

$$\left(f^{\Delta} \right)^{(\beta)}(t) = f^{(\beta+1)}(t).$$

Proof. (a) Let $\alpha + \beta \leq 1$. Then

$$\begin{aligned} \left(f^{(\alpha)} \right)^{(\beta)}(t) &= \mathcal{L}_{\mathbb{T}}^{-1} \left[z^\beta \mathcal{L} \left[f^{(\alpha)} \right] (z) - z^{\beta-1} f^{(\alpha)}(0) \right] (t) \\ &= \mathcal{L}_{\mathbb{T}}^{-1} \left[z^{\alpha+\beta} F(z) - z^{\alpha+\beta-1} f(0) \right] (t) - \mathcal{L}_{\mathbb{T}}^{-1} \left[z^{\beta-1} f^{(\alpha)}(0) \right] (t) \\ &= f^{(\alpha+\beta)}(t) - \mathcal{L}_{\mathbb{T}}^{-1} \left[z^{\beta-1} f^{(\alpha)}(0) \right] (t). \end{aligned}$$

(b) For $1 < \alpha + \beta \leq 2$ we have

$$f^{(\alpha+\beta)}(t) = \mathcal{L}_{\mathbb{T}}^{-1} \left[z^{\alpha+\beta} \mathcal{L}[f](z) - z^{\alpha+\beta-1} f(0) - z^{\alpha+\beta-2} f^{\Delta}(0) \right] (t).$$

Hence,

$$\left(f^{(\alpha)} \right)^{(\beta)}(t) = f^{(\alpha+\beta)}(t) + \mathcal{L}_{\mathbb{T}}^{-1} \left[z^{\alpha+\beta-2} f^{\Delta}(0) \right] (t) - \mathcal{L}_{\mathbb{T}}^{-1} \left[z^{\beta-1} f^{(\alpha)}(0) \right] (t).$$

(c) The intended relation follows from (b) by choosing $\alpha = 1$. \square

In general $\left(f^{(\alpha)} \right)^{(\beta)}(t) \neq \left(f^{(\beta)} \right)^{(\alpha)}(t)$. However, the equality holds in particular cases:

Proposition 3.6. *If $\alpha + \beta \leq 1$ and $f(0) = 0$, then $\left(f^{(\alpha)} \right)^{(\beta)}(t) = \left(f^{(\beta)} \right)^{(\alpha)}(t)$.*

Proof. It follows from item (a) of Proposition 3.5. \square

Proposition 3.7. *Let $\alpha \in (n-1, n]$, $n \in \mathbb{N}$, and $\lim_{t \rightarrow 0^+} f^{\Delta^k}(t) = f^{\Delta^k}(0^+)$ exist, $k = 0, \dots, n-1$. The following equality holds:*

$$I_{\mathbb{T}}^{\alpha} \left(f^{(\alpha)} \right) (t) = f(t) - \sum_{k=0}^{n-1} f^{\Delta^k}(0^+) h_k(t).$$

Proof. Let $F(z) = \mathcal{L}_{\mathbb{T}}[f](z)$. Then,

$$\begin{aligned} I_{\mathbb{T}}^{\alpha} \left(f^{(\alpha)} \right) (t) &= \mathcal{L}_{\mathbb{T}}^{-1} \left[z^{-\alpha} \mathcal{L}_{\mathbb{T}} \left[f^{(\alpha)} \right] (z) \right] (t) = \mathcal{L}_{\mathbb{T}}^{-1} \left[F(z) - \sum_{k=0}^{n-1} f^{\Delta^k}(0^+) \frac{z^{\alpha-k-1}}{z^{\alpha}} \right] (t) \\ &= f(t) - \sum_{k=0}^{n-1} f^{\Delta^k}(0^+) \mathcal{L}_{\mathbb{T}}^{-1} \left[\frac{1}{z^{k+1}} \right] (t) = f(t) - \sum_{k=0}^{n-1} f^{\Delta^k}(0^+) h_k(t). \end{aligned}$$

\square

Proposition 3.8. *Let $\alpha \in (n-1, n]$, $n \in \mathbb{N}$, and $F(z) = \mathcal{L}_{\mathbb{T}}[f](z)$. If $\lim_{z \rightarrow \infty} F(z) = 0$ and $\lim_{z \rightarrow \infty} \frac{F(z)}{z^{\alpha-k}} = 0$, $k \in \{1, \dots, n\}$, then $(I_{\mathbb{T}}^{\alpha} f)^{(\alpha)}(t) = f(t)$.*

Proof. Firstly let us notice that $\lim_{t \rightarrow 0^+} (I_{\mathbb{T}}^{\alpha} f)^{\Delta^k}(t) = 0$ for $k \in \{0, \dots, n-1\}$. For that we check that $\lim_{z \rightarrow \infty} z \mathcal{L} \left[(I_{\mathbb{T}}^{\alpha} f) \right] (z) = \lim_{z \rightarrow \infty} z \frac{F(z)}{z^{\alpha}} = 0$. Nextly let us assume that it holds for $i = 0, \dots, k-1$ and let us calculate

$$\begin{aligned} \lim_{z \rightarrow \infty} z \mathcal{L} \left[(I_{\mathbb{T}}^{\alpha} f)^{\Delta^k} \right] (z) &= \lim_{z \rightarrow \infty} z \left(z^k \frac{F(z)}{z^{\alpha}} - \sum_{i=0}^{k-1} z^{k-1-i} (I_{\mathbb{T}}^{\alpha})^{\Delta^i}(0) \right) \\ &= \lim_{z \rightarrow \infty} z \left(z^k \frac{F(z)}{z^{\alpha}} \right) = \lim_{z \rightarrow \infty} \frac{F(z)}{z^{\alpha-k-1}} = 0. \end{aligned}$$

Then we easily conclude that

$$(I_{\mathbb{T}}^{\alpha} f)^{(\alpha)}(t) = \mathcal{L}_{\mathbb{T}}^{-1} \left[z^{\alpha} \frac{F(z)}{z^{\alpha}} \right] (t) = f(t).$$

\square

The convolution of two functions $f : \mathbb{T} \rightarrow \mathbb{R}$ and $g : \mathbb{T} \times \mathbb{T} \rightarrow \mathbb{R}$ on time scales, where g is rd-continuous with respect to the first variable, is defined in [4, 8]:

$$(f * g)(t) = \int_0^t f(\tau)g(t, \sigma(\tau))\Delta\tau.$$

As function g we can consider, e.g., $e_c(t, t_0)$ or $h_k(t, t_0)$.

Proposition 3.9. *Let $t_0 \in \mathbb{T}$. If $\alpha \in (0, 1)$, then*

$$(f * g(\cdot, t_0))^{(\alpha)}(t) = \left(f^{(\frac{\alpha}{2})} * g^{(\frac{\alpha}{2})}(\cdot, t_0)\right)(t) = \left(f^{(\alpha)} * g(\cdot, t_0)\right)(t),$$

where we assume the existence of the involved derivatives.

Proof. From the convolution theorem for the generalized Laplace transform [8, Theorem 2.1],

$$\mathcal{L}_{\mathbb{T}} \left[(f * g(\cdot, t_0))^{(\alpha)} \right] (z) = z^{\alpha} F(z)G(z).$$

Hence,

$$\begin{aligned} (f * g(\cdot, t_0))^{(\alpha)}(t) &= \mathcal{L}_{\mathbb{T}}^{-1} [z^{\alpha} F(z)G(z)](t) = \mathcal{L}_{\mathbb{T}}^{-1} \left[z^{\frac{\alpha}{2}} F(z) \right] (t) \mathcal{L}_{\mathbb{T}}^{-1} \left[z^{\frac{\alpha}{2}} G(z) \right] (t) \\ &= \left(f^{(\alpha/2)} * g^{(\alpha/2)}(\cdot, t_0) \right) (t). \end{aligned}$$

Equivalently, we can write

$$\mathcal{L}_{\mathbb{T}}^{-1} [z^{\alpha} F(z)G(z)](t) = \mathcal{L}_{\mathbb{T}}^{-1} [z^{\alpha} F(z)](t) \mathcal{L}_{\mathbb{T}}^{-1} [G(z)](t) = \left(f^{(\alpha)} * g(\cdot, t_0) \right) (t).$$

□

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References

- [1] G. A. Anastassiou, Principles of delta fractional calculus on time scales and inequalities, *Math. Comput. Modelling* **52** (2010), no. 3-4, 556–566.
- [2] B. Aulbach and S. Hilger, A unified approach to continuous and discrete dynamics, in *Qualitative theory of differential equations (Szeged, 1988)*, 37–56, *Colloq. Math. Soc. János Bolyai*, 53, North-Holland, Amsterdam, 1990.
- [3] N. R. O. Bastos, R. A. C. Ferreira and D. F. M. Torres, Discrete-time fractional variational problems, *Signal Process.* **91** (2011), no. 3, 513–524. [arXiv:1005.0252](https://arxiv.org/abs/1005.0252)

- [4] M. Bohner and D. A. Lutz, Asymptotic expansions and analytic dynamic equations, *ZAMM Z. Angew. Math. Mech.* **86** (2006), no. 1, 37–45.
- [5] M. Bohner and A. Peterson, *Dynamic equations on time scales*, Birkhäuser Boston, Boston, MA, 2001.
- [6] M. Bohner and A. Peterson, Laplace transform and Z -transform: unification and extension, *Methods Appl. Anal.* **9** (2002), no. 1, 151–157.
- [7] M. Bohner and A. Peterson, *Advances in dynamic equations on time scales*, Birkhäuser Boston, Boston, MA, 2003.
- [8] J. M. Davis, I. A. Gravagne, B. J. Jackson, R. J. M. Marks and A. A. Ramos, The Laplace transform on time scales revisited, *J. Math. Anal. Appl.* **332** (2007), no. 2, 1291–1307.
- [9] S. Hilger, Analysis on measure chains—a unified approach to continuous and discrete calculus, *Results Math.* **18** (1990), no. 1-2, 18–56.
- [10] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and applications of fractional differential equations*, Elsevier, Amsterdam, 2006.
- [11] V. Lakshmikantham, S. Sivasundaram and B. Kaymakçalan, *Dynamic systems on measure chains*, Kluwer Acad. Publ., Dordrecht, 1996.
- [12] Y. Li, Y. Q. Chen and I. Podlubny, Mittag-Leffler stability of fractional order nonlinear dynamic systems, *Automatica* **45** (2009), no. 8, 1965–1969.
- [13] K. B. Oldham and J. Spanier, *The fractional calculus*, Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York, 1974.
- [14] I. Podlubny, *Fractional differential equations*, Academic Press, San Diego, CA, 1999.
- [15] M. R. S. Rahmat and M. S. Md. Noorani, Fractional integrals and derivatives on time scales with an application, *Comput. Math. Appl.* (2009), DOI: 10.1016/j.camwa.2009.10.013
- [16] S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional integrals and derivatives*, Translated from the 1987 Russian original, Gordon and Breach, Yverdon, 1993.