CHARACTERISTIC NUMBERS OF RATIONAL NODAL CURVES IN $\mathbb{P}^3$

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Abstract. In this paper we compute the number of rational curves with one node passing through a given number of points, lines and tangent to a given number of planes in $\mathbb{P}^3$.

1. introduction

Computing charateristic numbers of curves in projective space is a classical problem in algebraic geometry: how many curves in projective space that pass through a general set of linear subspaces, and are tangent to a general set of hyperplanes? In this paper we will compute the characteristic numbers of rational curves with one node in $\mathbb{P}^3$.

This project is the first step by the author in an attempt to complete the program of computing all characteristic numbers of elliptic curves and of elliptic curves with fixed j-invanriant in projective spaces. An appropriate generalization of the result in this paper will give the characteristic numbers of rational nodal curves with condition on the node. Using a degeneration argument as in [P2], we can show that any characteristic number of elliptic curves with fixed j-invariant is a linear combinations of charateristic numbers of rational nodal curves, with or without condition on the node, thus completing the program for elliptic curves with fixed j-invariant. Then we can use equation (6), section 5.7 in [V1] to complete the program for elliptic curves.

In section 2, we describe various definitions and conventions that are used throughout the paper. In section 3 and section 5, we study the enumerative geometry of boundary divisors of the space of rational nodal curves. In section 4, we study the enumerative geometry of rational smooth curves with a special tangent condition. In section 6, we derive a recursive formula for the characteristic number of rational nodal curves. Examples will be given throughout the paper, and at the end of section 6, numerical results of degree up to 5 will be given. We will also discuss possible generalizations at the end of this paper.

The author would like to thank R.Vakil for numerous helpful conversations and insights, and also for introducing him to the beatiful subject of enumerative geometry. A C++ program implementing the algorithms described here are available on the author's webpage.

2. Definitions and Notations

2.1. **The moduli space of rational curves in $\mathbb{P}^3$.** As usual, $\overline{M}_{0,n}(3, d)$ will denote the Kontsevich compactification of the moduli space of genus zero curves with n marked points of degree d in $\mathbb{P}^3$. Let S be the set of markings, so $|S| = n$. We will also use the notation $\overline{M}_{0,S}(r, d)$ when we want to specify the markings. The followings are Weil divisors on $\overline{M}_{0,S}(3, d)$:

- The divisor $(U \mid\mid V)$ of $\overline{M}_{0,S}(3, d)$ is the closure in $\overline{M}_{0,S}(r, d)$ of the locus of curves with two components such that $U \cup V = S$ is a partition of the marked points over the two components.
- The divisor (d_1, d_2) is the closure in $\overline{M}_{0,S}(3, d)$ of the locus of curves with two components, sucht that $d_1 + d_2 = d$ is the degree partition over the two components.
- The divisor $(U, d_1 \mid\mid V, d_2)$ is the closure in $\overline{M}_{0,S}(3, d)$ of the locus of curves with two components, where $U \cup V = S$ and $d_1 + d_2 = d$ are the partition of markings and of degree over the two components respectively.

2.2. **The constraints and the ordering of constraints.** We will be interested in the number of curves satisfying a constraint, and each constraint is denoted by a $4-$tuple Δ as follows :

- $\Delta(0)$ is the number of planes that the curves need to be tangent to.
- For $0 < i \leq 4$, $\Delta(i)$ is the number of subspaces of codimension i that the curves need to pass through.

Note that because in general a curve of degree d will always intersect a plane at d points, introducing an incident condition with a plane has the same effect as that of multiplying the enumerative number by d. For example, if we ask how many genus zero curves of degree 4 pass through the constraint $\Delta = (1, 2, 3, 4)$, that means we ask how many genus zero curves of degree d pass through three lines, four points, are tangent to one plane, and then multiply that answer by 4^2. Let $|\Delta| = \Delta[1] + \Delta[2] + \Delta[3]$. We will also refer to Δ as a set of linear spaces, hence we can say for example, pick a space A in Δ.

We consider the following ordering on the set of constraints, in order to prove that our algorithm will terminate later on. Let $r(\Delta) = \sum_{i>1} \Delta[i] \cdot i^2$, and this will be our rank function. We compare two constraints Δ, Δ' use the following criteria, whose priority are in the following order :

- If Δ has fewer non-plane elements than Δ' does, then $\Delta < \Delta'$,
- $s(\Delta) > s(\Delta')$ then $\Delta < \Delta'$.

We write $\Delta = \Delta_1\Delta_2$ if $\Delta = \Delta_1 \cup \Delta_2$ as a parition of a set of linear spaces.

2.3. **The stacks $\mathcal{R}$,$\mathcal{N}$,$\mathcal{NR}$ and $\mathcal{RR}$.** We define $\mathcal{N}(d)$ to be the closure in $\overline{M}_{0,\{A,B\}}(3,d)$ of the locus of maps of smooth rational curves γ such that $\gamma(A) = \gamma(B)$. Informally, $\mathcal{N}$ parametrize degree d rational nodal curves in $\mathbb{P}^3$.

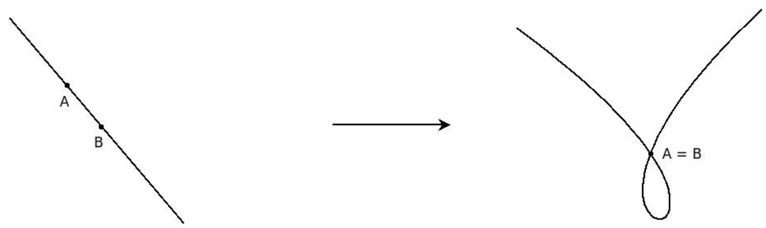

Fig 1. A general map in $\mathcal{N}(d)$

For $d_1, d_2 > 0$ we define $\mathcal{NR}(d_1, d_2)$ to be the closure in $\overline{M}_{0,\{A,B,C\}}(3, d_1) \times_{\mathbb{P}^3} \overline{M}_{0,\{C\}}(3, d)$ (the projections are evaluation maps ev_C) of the locus of maps γ such that $\gamma(A) = \gamma(B)$.

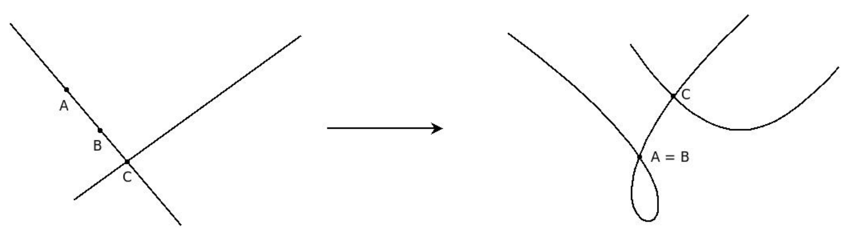

Fig 2. A general map in $\mathcal{NR}(d_1, d_2)$

For $d_1, d_2 > 0$ we define $\mathcal{RR}(d_1, d_2)$ to be the closure in $\overline{M}_{0,\{A,C\}}(3, d_1) \times_{\mathbb{P}^3} \overline{M}_{0,\{B,C\}}(3, d_2)$ (the projections are evaluation maps e_C) of the locus of maps γ such that $\gamma(A) = \gamma(B)$.

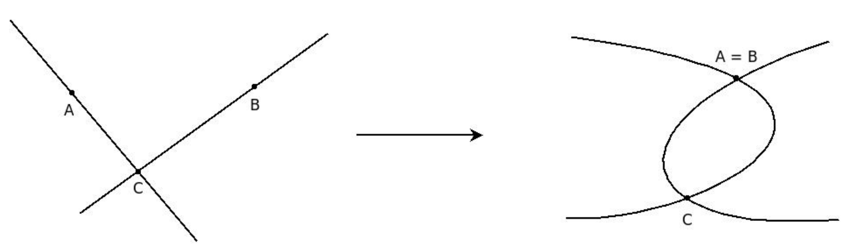

Fig 3. A general map in $\mathcal{RR}(d_1, d_2)$

2.4. **Special Tangent Condition.** It turns out that it is necessary to understand the enumerative geometry of rational curves, now with extra conditions of the form: there is a fixed marked point A on the curve, and the projective tangent line at A passes through a given line L. We would also need to consider the case where there is a condition on A, which means it could be specified to lie on a certain linear subspace.

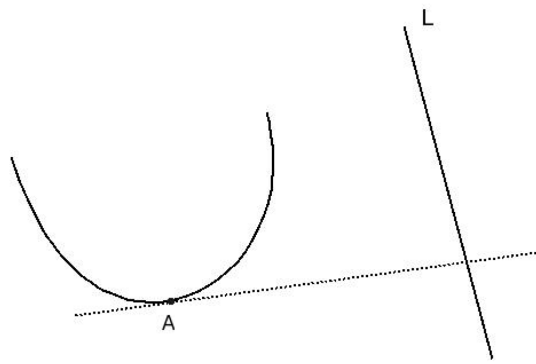

Fig 4. A curve with a special tangent condition

2.5. **Stacks of stable maps with constraints.** Let $\mathcal{F}$ be a stack of stable maps of curves in $\mathbb{P}^3$. For a constraint Δ, we define $(\mathcal{F}, \Delta)$ be the closure in $\mathcal{F}$ of the locus of maps that satisfy the constraint Δ. If the stack of maps $\mathcal{F}$ has two marked points A and B, we define $(\mathcal{F}, \mathcal{T}^m \mathcal{H}^n \mathcal{P}^k \mathcal{L}_A^u \mathcal{L}_B^v)$ to be the closure in $\mathcal{F}$ of the locus of maps γ such that:

- γ is tangent to m general planes.
- γ passes through n general lines and k general points.
- $\gamma(A)$ lies on u general planes, and $\gamma(B)$ lies on v general planes.

If $\mathcal{F}$ has one marked point A then we define $(\mathcal{F}, \mathcal{T}^m \mathcal{H}^n \mathcal{P}^k \mathcal{L}_A^u \mathcal{W}_A)$ to be the closure of maps γ such that:

- γ is tangent to m general planes.
- γ passes through n general lines and k general points.
- $\gamma(A)$ lies on u general planes,
- The tangent line to the image of γ at $\gamma(A)$ passes through a general line.

If a stack of $\mathcal{F}$ consists of a finite number of reduced points then we denote $\#\mathcal{F}$ to be the cardinality of $\mathcal{F}$. Our objective in this paper is to compute

$$\frac{1}{2}\#(\mathcal{N}(d), \Delta)$$

which is the number of rational nodal curves of degree d satisfying the constraint Δ. The factor $1/2$ is due to the fact that every rational nodal curve in $\mathbb{P}^3$ corresponds to 2 maps in $\mathcal{N}(d)$ since we can interchange the two marked points A and B.

Note that if $u, v > 0$, then for the constraint Δ that consists of m tangency planes, n incident lines, k incident points, and another one subspace of codimension u, and another one subspace of codimension v, we have the following equation:

$$\#(\mathcal{F}, \mathcal{T}^m \mathcal{H}^n \mathcal{P}^k \mathcal{L}_A^u \mathcal{L}_B^v) = \#(\mathcal{F}, \Delta)$$

if one of them is finite.

If $\mathcal{F}$ is a closed substack of $\mathcal{NR}(d_1, d_2)$ then we denote $(\mathcal{F}, \Gamma_1, \Gamma_2, k)$ to be the closure in $\mathcal{F}$ of the locus of maps γ such that the restriction of γ on the $i-$th component satisfies constraint Γ_i and that $\gamma(C)$ lies on k general planes. Here Γ_i can be a constraint either in the 4-tuple form or in the form of products of enumerative classes. Similary, we use the notation $(\mathcal{F}, \Delta, k)$ if we don't want to distinguish the conditions on each component.

If $\mathcal{F}$ is a closed substack of $\mathcal{RR}(d_1, d_2)$ then we denote $(\mathcal{F}, \Gamma_1, \Gamma_2, k, l)$ to be the closure in $\mathcal{F}$ of the locus of maps γ such that the restriction of γ on the $i-$th component satisfies constraint Γ_i and that $\gamma(C)$ lies on k general planes, and that $\gamma(A) = \gamma(B)$ lies on l general planes. Similary, we use the notation $(\mathcal{F}, \Delta, k, l)$ if we don't want to distinguish the conditions on each component.

2.6. **Some Examples.**

Example 2.1. *Let* $\Delta = (1, 0, 3, 2)$. *Then*

$$\#(\overline{M}_{0,\{A,B\}}(3, 2), \mathcal{T}\mathcal{H}\mathcal{P}^2\mathcal{L}_A^2\mathcal{L}_B^2) = \#(\overline{M}_{0,0}(3, 2), \Delta)$$

as both are the number of conics in $\mathbb{P}^3$ *that are tangent to one plane, that pass through* 3 *lines and* 2 *points.*

□

Example 2.2. *Let* $\Gamma_1 = (0, 0, 4, 3)$ *and* $\Gamma_2 = (0, 0, 3, 0)$. *Then*

$$\frac{1}{2}\#(\mathcal{NR}(3, 1), \Gamma_1, \Gamma_2, 1)$$

is the number of pairs of nodal cubic-line (γ, l) *in* $\mathbb{P}^3$ *intersecting at one point such that:*

- *The point in common lies on a fixed plane.*
- *The cubic* γ *passes through* 4 *lines and* 3 *points.*
- *The line* l *passes through* 3 *other lines.*

□

Example 2.3.
$$\#(\overline{M}_{0,\{A\}}(3,2), \mathcal{T}^3\mathcal{P}^2\mathcal{L}_A\mathcal{W}_A)$$
is the number of conics in $\mathbb{P}^3$ *that have a marked point* A*, that are tangent to* 3 *planes, that pass thourgh* 2 *points, and that the tangent line to the conic at* A *passes through a general line, and that* A *lies on a fixed plane.*

□

Example 2.4.
$$\#(\mathcal{R}\mathcal{R}(1,2), \mathcal{H}\mathcal{P}, \mathcal{T}^3\mathcal{P}^2)$$
is the number of pairs of line-conic (L, C) *in* $\mathbb{P}^3$ *such that they intersect at two distinguished points, and that* L *passes through* 1 *line and* 1 *point, and that* C *is tangent to* 3 *lines and passes through* 2 *points.*

□

3. Counting maps in $\mathcal{NR}(d_1, d_2)$.

In this section we discuss how to count maps in $\mathcal{NR}(d_1, d_2)$. This is simple if we know the enumerative geometry of $\mathcal{N}(d_1)$. Recall that $\mathcal{NR}(d_1, d_2)$ is a substack of $\overline{M}_{0,\{A,B,C\}}(3, d_1) \times_{\mathbb{P}^3} \overline{M}_{0,\{C\}}(3, d_2)$ of maps that map A and B to a same point. Let $\mathcal{N}(d_1)^{\{C\}}$ be the locus of maps in $\overline{M}_{0,\{A,B,C\}}(3, d_1)$ that map A and B to a same point.

Proposition 3.1. *We have*

$$\#(\mathcal{NR}(d_1, d_2), \Gamma_1, \Gamma_2, k) = \#(\mathcal{N}(d_1), \Delta_1') \cdot \#(\overline{M}_{0,0}(3, d), \Delta_2')$$

where Δ_i' are determined as follows. Let e_1 be the dimension of the pushforward under ev_C of $(\mathcal{N}^{\{C\}}, \Gamma_1)$ into $\mathbb{P}^3$. Let e_2 be the dimension of the pushforward under ev_C of $(\overline{M}_{0,\{C\}}(3, d_2), \Gamma_2)$ into $\mathbb{P}^3$. Then Δ_i' is obtained from Γ_i by adding a subspace of codimension e_i.

Proof. Let α_i be the class of $ev_{C*}(\mathcal{N}^{\{C\}}, \Gamma_1)$ in $\mathbb{P}^3$. Let α_2 be the class of $ev_{C*}(\overline{M}_{0,\{C\}}(3, d_2), \Gamma_2)$. Let β be the class of a linear subspace of codimension k. Then the answer to the enumerative problem in the proposition is the intersection product :

$$\alpha_1 \cdot \alpha_2 \cdot \beta = \deg(\alpha_1)\deg(\alpha_2).$$

To compute $\deg(\alpha_1)$, we intersect it with a linear subspace of codimension e_1. Thus $\deg(\alpha_1) = \#(\mathcal{N}(d_1), \Delta_1')$. Similarly, $\deg(\alpha_2) = \#(\overline{M}_{0,0}(3, d), \Delta_2')$. We have proved the proposition. ☐

Example 3.2. *Let $\Gamma_1 = (0, 0, 4, 3)$ and $\Gamma_2 = (0, 0, 3, 0)$. Then we have*

$$\frac{1}{2}\#(\mathcal{NR}(3, 1), \Gamma_1, \Gamma_2, 1) = 24.$$

The dimension of the family of cubics in $\mathbb{P}^3$ with a marked point C is 12. Thus the condition Γ_1 cuts out a 2 dimensional family in $\mathcal{N}^{\{C\}}(3)$. Hence $e_1 = 2$, and $\Delta_1' = (0, 0, 5, 3)$. The dimension of the family of lines in $\mathbb{P}^3$ with a marked point C is 5. Thus the condition Γ_2 cuts out a 2 dimensional family in $\overline{M}_{0,\{C\}}(3, 1)$. Hence $e_2 = 2$, and $\Delta_2' = (0, 0, 4, 0)$.

$\#(\mathcal{N}(3), \Delta_1')$ is the number of nodal cubics in $\mathbb{P}^3$ passing through 5 lines and 3 points . This is 12 because the 3 points determine the plane of the nodal cubics, so this number is the same as the number of nodal cubics in $\mathbb{P}^2$ passing through 8 points. $\#(\overline{M}_{0,0}(3, 1), \Delta_2')$ is the number of lines passing through 4 general lines in $\mathbb{P}^3$, which is 2. Thus the total number is $12 \cdot 2 = 24$. ☐

The following lemma is useful because it allow us to express the tangency condition on maps of reducible curves in terms of tangency conditions on maps on each component plus an incident condition on the node.

Lemma 3.3. *Let $\mathcal{M}_1, \mathcal{M}_2$ be stacks of stable maps of degree d_1, d_2 into $\mathbb{P}^3$. Assume each map in each family carries at least one marked point C. Let $\mathcal{M} = \mathcal{M}_1 \times_{\mathbb{P}^3} \mathcal{M}_2$ where the fibre product is taken over the evaluation maps ev_C. Let $\mathcal{T}$ be the tangency divisor on $\mathcal{M}$, and $\mathcal{T}_i$ be the pull-back of the tangency divisor on the $i-th$ component. Then on $\mathcal{M}$ we have this divisorial equation: $\mathcal{T} = \mathcal{T}_1 + \mathcal{T}_2 + 2\mathcal{L}_C$.*

Proof. Let $\mathcal{C}$ be a general curve in $\mathcal{M}$. $\mathcal{C}$ has the following description. There is a family of nodal curves over $\pi : S \to \mathcal{C}$ such that S is the union of two family of nodal curves M_1, M_2 along a section $s : \mathcal{C} \to S$. The section s represents the marked point C of the family. There is also a map $\mu : S \to \mathbb{P}^3$ such that the restriction of μ on each fiber is an element (a map) of $\mathcal{M}_1 \times_{\mathbb{P}^3} \mathcal{M}_2$. Now choose a general plane H in $\mathbb{P}^3$. Then the restriction of the tangency divisor $\mathcal{T}$ on $\mathcal{C}$ is the branched divisor of the map $\pi : \mu^{-1}(H) = \mathcal{D} \to \mathcal{C}$. This map is a $d_1 + d_2$ sheet covering of $\mathcal{C}$. The ramification points of this map come from three sources :

- The ramification points on $\mu^{-1}(H)_{|M_1}$.
- The ramification points on $\mu^{-1}(H)_{|M_2}$.
- The intersections $\mu^{-1}(H) \cap s$.

The first two sources contribute to the pull backs $\mathcal{T}_1 \cdot \mathcal{C}$ and $\mathcal{T}_2 \cdot C$ respectively. The intersections points $\mu^{-1}(H) \cap s$ correspond precisely to the maps γ with $\gamma(C) \in H$. These points are the nodes of the curve, because through each of them, there are two branches : one from $\mu^{-1}(H)_{|M_1}$, one from $\mu^{-1}(H)_{|M_2}$. If $P \in \mathcal{D}$ is one of such points, then the branched divisor of π contains $\pi(P)$ with multiplicity 2. Thus we have $\mathcal{T} \cdot \mathcal{C} = \mathcal{T}_1 \cdot \mathcal{C} + \mathcal{T}_2 \cdot \mathcal{C} + 2\mathcal{L}_C \cdot \mathcal{C}$.

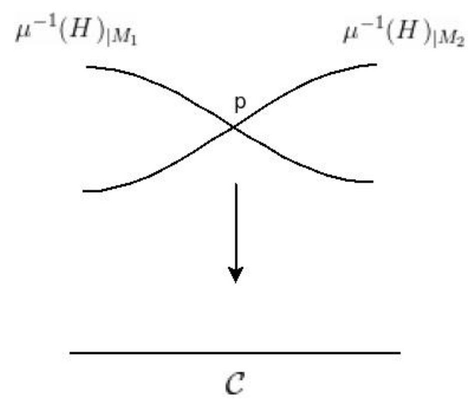

Fig 5. The picture in a neighborhood of a point $p \in \mu^{-1}(H) \cap s$

□

Using Lemma 3.3, we can "expand" the tangency conditions on $\mathcal{NR}(d_1, d_2)$ until we have tangency conditions only on each individual component.

Proposition 3.4. *Let Δ be a constraint and let Δ_i be the constraint obtained from Δ by removing i tangency conditions. Then we have the following equality :*

$$\begin{aligned}
\#(\mathcal{NR}(d_1,d_2),\Delta) &= \sum_{\Gamma_1\Gamma_2=\Delta} \#(\mathcal{NR}(d_1,d_2),\Gamma_1,\Gamma_2) \\
&+ 2n \sum_{\Gamma_1\Gamma_2=\Delta_1} \#(\mathcal{NR}(d_1,d_2),\Gamma_1,\Gamma_2,1) \\
&+ 4\frac{n(n-1)}{2} \sum_{\Gamma_1\Gamma_2=\Delta_2} \#(\mathcal{NR}(d_1,d_2),\Gamma_1,\Gamma_2,2) \\
&+ 8\frac{n(n-1)(n-2)}{6} \sum_{\Gamma_1\Gamma_2=\Delta_3} \#(\mathcal{NR}(d_1,d_2),\Gamma_1,\Gamma_2,3).
\end{aligned}$$

□

4. Counting Curves With A Special Tangent Condition

In this section, we will attempt to count curves with one special tangent classes in $\mathbb{P}^3$ Formally, the problem is as follows : how many rational curves in the family $\mathcal{X}$ of degree d with a marked point A in $\mathbb{P}^3$ passing through the constraint Δ and also satisfy the condition that A lies on a given subspace of codimension k and the projective tangent line at A passes through a general line. Here are some examples whose answer we know immediately :

Example 4.1. *How many lines in $\mathbb{P}^3$ that pass through 2 general lines, having one marked point A, the tangent line at which passes through a general line, and A also has to lie on a given line?*

This number is easily seen to be the same as the number of lines passing through 4 general lines, which is 2. □

Example 4.2. *How many conics in $\mathbb{P}^3$ that pass through 3 points, that have a marked point A which must lie on a fixed line M, and that the tangent line at A to the curve passes through a fixed line L?*

Because the three points that the conic passes through determine its plane H, this problem reduces to an enumerative problem in $\mathbb{P}^2$: how many conics in $\mathbb{P}^2$ that pass through 3 points and is tangent to a line at a fixed point? The answer is therefore 1. □

The following example is helpful in showing how we could get a hold of the special tangent class.

Example 4.3. *Consider the problem of counting curves in the family $\mathcal{X}$ in $\mathbb{P}^3$, that have a marked point A, that satisfy the constraint Δ, such that the tangent line at A to the curves passes through a line L. Let $\mathcal{X}'$ be the image of $\mathcal{X}$ via the forgetful morphism that forgets the marked point A. Then we have*

$$\#(\mathcal{X}, \Delta\mathcal{W}_A) = 2(d-1)\#(\mathcal{X}', \Delta)$$

Proof. The constraint Δ cuts out an one-dimensional family $\mathcal{F}$ of curves on $\mathcal{X}$. There is no condition on the marked point A, which means that the one dimensional family consists of a finite number of curves in $\mathbb{P}^3$ with a choice of a marked point A on each of them. The number $\#(\mathcal{F}, \mathcal{W}_A)$ can then be computed as follows. For each curve $\gamma \in \mathcal{F}$, we find the number p of points $A \in \gamma$ such that the projective tangent line to γ at A passes through a fixed line L. Then we multiply p with the number of curves $\gamma \in \mathcal{F}$, which is $\#(\mathcal{X}', \Delta)$.

The number p is the ramification number of the projection map $\pi_L : \gamma \to K$, where K is a line in $\mathbb{P}^3$ and π_L is the projection from L to K. By Riemann-Hurwitz formula

$$p = \deg R = (2g_\gamma - 2) - d(2g_K - 2) = 2d - 2.$$

Thus we have shown

$$\#(\mathcal{X}, \Delta\mathcal{W}_A) = (2d-2)\#(\mathcal{X}', \Delta).$$

$\square$

Following the notation in [P1], let $\mathcal{K}^{j,A}$ be the boundary divisor of $\overline{\mathcal{M}_{0,1}(3,d)}$ whose points represent reducible curves in which the component containing A is mapped with degree j. Using a similar idea as in the above example, we can prove:

Lemma 4.4. *The following equation holds in the group* $Pic(\overline{\mathcal{M}}_{0,\{A\}}(3,d)) \otimes \mathbb{Q}$*, for* $r > 2$ *:*

$$\mathcal{W}_A = 2\mathcal{L}_A + \psi_A$$

where ψ_A *is the psi-class, and* $\mathcal{L}_A$ *is the pull back of* $\mathcal{O}(1)$ *on* $\mathbb{P}^3$ *via the evaluation map* ev_A*. In particular, we have*

$$\mathcal{W}_A = \left(2 - \frac{2}{d}\right)\mathcal{L}_A + \frac{1}{d^2}\mathcal{H} + \sum_{j=1}^{j<d} \frac{(d-j)^2}{d^2}\mathcal{K}^{j,A}$$

Proof. We use the method as described in [P1], intersecting the two sides of the equations with a general curve $\mathcal{C}$ in $\overline{\mathcal{M}_{0,1}(3,d)}$. Let γ denote the image of $\mathcal{C}$ under the evaluation map ev_A. Let L be the line in $\mathbb{P}^3$ corresponding to the special tangent condition $\mathcal{W}_A$. Beccause $\mathcal{C}$ is a general curve, we can assume γ is smooth. Let M be a general line in $\mathbb{P}^3$, and let $\pi_L : \mathbb{P}^3 - L \to M$ be the projection onto M from L. Let ϕ_A be the line bundle on γ described as follows. For each point $p \in \gamma$, $ev_A^{-1}(p)$ is a map $\gamma \in \mathcal{C}$. The fibre of ϕ_A over p is then the tangent vector to the image of α at $= \alpha(A)$. Let R be the zero scheme of the bundle map $\phi_A \to {\pi_L}^*(T_M)$. Geometrically, R represents the locus pf points $p \in \gamma$, such that the map $ev_A^{-1}(p)$ satisfies special tangent condition with respect to the line L. Thus

$$\deg R = R \cap [\gamma] = \mathcal{C} \cap \mathcal{W}_A.$$

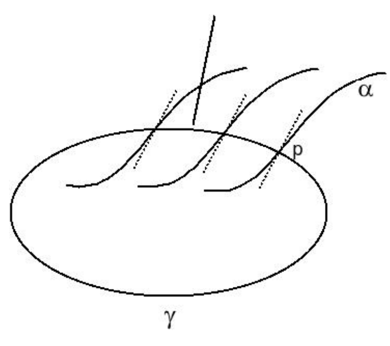

Fig 6. The curve γ with line bundle ϕ_A

We have

$$\deg R = -c_1(\phi_A) + \deg(\pi_{L|\gamma})c_1(T_M).$$

Now $c_1(T_M) = 2$[class of a point], and $\deg(\pi_{L|\gamma}) = \deg\gamma = \mathcal{L}_A \cap \mathcal{C}$. The pullback of ϕ_A by ev_A is isomorphic to the line bundle on C obtained by attaching to each map the tangent vector at A to the source curve. Hence $-c_1(\phi_A)\cap\gamma = -c_1(ev_A^*(\phi_A))\cap\mathcal{C} = \psi_A\cap\mathcal{C}$ is the usual psi class. In short, we have

$$\mathcal{W}_A = 2\mathcal{L}_A + \psi_A.$$

The second equality follows from the fact that $\psi_A = -\pi_*(s_A^2)$ on $\overline{M}_{0,\{A\}}(3,d)$ and Lemma 2.2.2 in [P1]. $\square$

The rest of this section will be examples.

Example 4.5. *Verifying Example 3.1.*

Proof. We need to compute $\#(\overline{M}_{0,\{A\}}(3,1), \mathcal{H}^2\mathcal{L}_A^2\mathcal{W}_A)$. There is no boundary divisor on $\overline{M}_{0,\{A\}}(3,1)$, hence

$$\mathcal{W}_A = \mathcal{H}$$

Thus

$$\#(\overline{M}_{0,\{A\}}(3,1), \mathcal{H}^2\mathcal{L}_A^2\mathcal{W}_A) = \#(\overline{M}_{0,\{A\}}(3,1), \mathcal{H}^3\mathcal{L}_A^2) = \#(\overline{M}_{0,0}(3,1), \mathcal{H}^4) = 2$$

$\square$

Example 4.6. *Verifying Example 3.2.*

Proof. We need to compute $\#(\overline{M}_{0,\{A\}}(3,2), \mathcal{P}^3\mathcal{L}_A^2 W_A)$. On $\overline{M}_{0,\{A\}}(3,2)$, there is one boundary divisor, $\mathcal{K} = (\emptyset, 1 \mid\mid \{A\}, 1)$, which parametrize pair of lines intersecting at one point, and the marked point A is on one of them. Using lemma 3.4 we have

$$\mathcal{W}_A = \mathcal{L}_A + \frac{\mathcal{H}}{4} + \frac{\mathcal{K}}{4}$$

Thus

$$\begin{aligned}\#(\overline{M}_{0,\{A\}}(3,2), \mathcal{P}^3\mathcal{L}_A^2 W_A) &= \#(\overline{M}_{0,\{A\}}(3,2), \mathcal{P}^3\mathcal{L}_A^3) + \frac{1}{4}\#(\overline{M}_{0,\{A\}}(3,2), \mathcal{H}\mathcal{P}^3\mathcal{L}_A^2) \\ &\quad + \frac{1}{4}\#(\mathcal{K}, \mathcal{P}^3\mathcal{L}_A^2) \\ &= 0 + \frac{1}{4} + \frac{1}{4}3 = 1\end{aligned}$$

The first "#" term of the right hand side is the number of conics in $\mathbb{P}^3$ passing through 4 points. The second "#" term is the number of conics in $\mathbb{P}^3$ passing through 3 points and 2 lines. The last "#" term is the number of pair of lines in $\mathbb{P}^3$ with one common point, that pass through 3 points, and that the component with the marked point A intersect a line at A. $\square$

Example 4.7. *There are* 16 *conics in* $\mathbb{P}^3$ *that have a marked point* A*, that are tangent to* 3 *planes, that pass thourgh* 2 *points, and that the tangent line to the conic at* A *passes through a general line, and that* A *lies on a fixed plane.*

Proof. We need to show that $\#(\overline{M}_{0,\{A\}}(3,2), \mathcal{T}^3\mathcal{P}^2\mathcal{L}_A\mathcal{W}_A) = 16$.

$$\begin{aligned}
\#(\overline{M}_{0,\{A\}}(3,2), \mathcal{T}^3\mathcal{P}_A^2 L_A W_A) &= \#(\overline{M}_{0,\{A\}}(3,2), \mathcal{T}^3\mathcal{P}^2\mathcal{L}_A^2) + \frac{1}{4}\#(\overline{M}_{0,\{A\}}(3,2), \mathcal{H}\mathcal{T}^3\mathcal{P}^2\mathcal{L}_A) \\
&\quad + \frac{1}{4}\#(\mathcal{K}, \mathcal{T}^3\mathcal{P}^2\mathcal{L}_A) \\
&= 8 + \frac{1}{4}16 + \frac{1}{4}16 = 16.
\end{aligned}$$

$\#(\overline{M}_{0,\{A\}}(3,2), \mathcal{T}^3\mathcal{P}^2\mathcal{L}_A^2)$ is 8 because there are 8 conics that pass through 2 points, 1 line, that are tangent to 3 planes. $\#(\overline{M}_{0,\{A\}}(3,2), \mathcal{H}\mathcal{T}^3\mathcal{P}^2\mathcal{L}_A)$ is the same enumerative problem, but now there is two choices of the marked point A, thus is 16. To compute the term $\#(\mathcal{K}, \mathcal{T}^3\mathcal{P}^2\mathcal{L}_A) = 16$, we note that in order to satisfy the constraint $\mathcal{T}^3\mathcal{P}^2\mathcal{L}_A$, a pair of lines in $\mathcal{K}$ must have there intersection lying on all three tangency planes (multiplicity 8), and each of the line must pass through one of the point conditions (2 ways to assign the point conditions to each line). Thus the total number is 16.

Example 4.8. *How many degree twisted cubics in* $\mathbb{P}^3$ *that pass through* 3 *points,* 5 *lines, that have a marked point* A *that lies on a fixed plane, and that the tangent line at* A *passing through a general line* L*? In formula, the problem is the intersection number* $\#(\overline{M}_{0,\{A\}}(3,3), H^5\mathcal{P}^3\mathcal{L}_A\mathcal{W}_A)$.

Put $\mathcal{X} = \overline{M}_{0,\{A\}}(3,3)$ and $\mathcal{X}' = \overline{M}_{0,0}(3,3)$. Let $\mathcal{K}^{1,A}$ and $K^{2,A}$ be boundary divisors of $\mathcal{X}$. Let $\mathcal{K}$ be the unique boundary divisor of $\mathcal{X}'$. Applying the second formula in Lemma 3.4 :

$$\mathcal{W}_A = \frac{4}{3}\mathcal{L}_A + \frac{1}{9}\mathcal{H} + \frac{4}{9}\mathcal{K}^{1,A} + \frac{1}{9}\mathcal{K}^{2,A}$$

Thus we have

$$\begin{aligned}
\#(\mathcal{X}, \mathcal{H}^5\mathcal{P}^3\mathcal{W}_A\mathcal{L}_A) &= \frac{4}{3}\#(\mathcal{X}, \mathcal{H}^5\mathcal{P}^3\mathcal{L}_A^2) + \frac{1}{9}\#(\mathcal{X}, \mathcal{H}^6\mathcal{P}^3\mathcal{L}_A) \\
&\quad + \frac{4}{9}\#(\mathcal{K}^{1,A}, \mathcal{H}^5\mathcal{P}^3\mathcal{L}_A) + \frac{1}{9}\#(\mathcal{K}^{2,A}, \mathcal{H}^5\mathcal{P}^3\mathcal{L}_A) \\
&= \frac{4}{3}\#(\mathcal{X}', H^6\mathcal{P}^3) + 3\frac{1}{9}(\mathcal{X}', H^6\mathcal{P}^3) \\
&\quad + \frac{4}{9}\#(\mathcal{K}, H^5\mathcal{P}^3) + 2\frac{1}{9}\#(\mathcal{K}, H^5\mathcal{P}^3) \\
&= \frac{4}{3}190 + \frac{1}{3}190 + \frac{4}{9}344 + \frac{2}{9}344 \\
&= 546.
\end{aligned}$$

∎

5. Counting maps in $\mathcal{RR}(d_1, d_2)$

Recall that $\mathcal{RR}(d_1, d_2)$ is a substack of $\overline{M}_{0,\{A,C\}}(3, d_1) \times_{\mathbb{P}^3} \overline{M}_{0,\{B,C\}}(3, d_2)$ of maps γ such that $\gamma(A) = \gamma(B)$. We rephrase the problem of counting maps in $\mathcal{RR}(d_1, d_2)$ as follows :

Given two families $\mathcal{F}_1$ and $\mathcal{F}_2$ of maps of rational curves with two marked points A, C. How many times a map γ_1 from $\mathcal{F}_1$ and a map γ_2 from $\mathcal{F}_2$ intersect in such a way that :

- *$\gamma_1(A) = \gamma_2(A)$ and $\gamma_1(C) = \gamma_2(C)$.*
- *$\gamma_i(A)$ lies on a fixed linear space of codimension p.*
- *$\gamma_i(C)$ lies on a fixed linear space of codimension q.*

We consider the evaluation map

$$ev_{AC} : \mathcal{F}_i \longrightarrow Bl_{\mathcal{D}}(\mathbb{P}^3 \times \mathbb{P}^3)$$

Let T_i be the pushforward of $\mathcal{F}_i$ under the evaluation map. Let h, k be the hyperplane classes of the first and second factor in $Bl_{\mathcal{D}}(\mathbb{P}^3 \times \mathbb{P}^3)$. Then the answer to our enumerative problem above is the intersection number

$$T_1 T_2 h^p k^q$$

where the product is evaluated in the Chow ring of $Bl_{\mathcal{D}}(\mathbb{P}^3 \times \mathbb{P}^3)$.

The next natural thing to do is then to investigate the Chow ring of $Bl_{\mathcal{D}}(\mathbb{P}^3 \times \mathbb{P}^3)$. Compared to $\mathbb{P}^3 \times \mathbb{P}^3$, this space has an additional piece which is the projective normal cone of the diagonal $\mathcal{D}$ in $\mathbb{P}^3 \times \mathbb{P}^3$, which is isomorphic to the projective tangent bundle $P(T_{\mathbb{P}^3})$.

Proposition 5.1. *The Chow ring of $P(T_{\mathbb{P}^3})$ is generated by H, the pull back of the hyperplane class of $\mathbb{P}^3$, and $E = c_1(\mathcal{O}_{T_{\mathbb{P}^3}}(1))$ with the following relations :*

$$H^4 = 0, \;\; E^3 = 4HE^2 - 6H^2E + 4H^3$$

Proof. The Chow ring of $P(T_{\mathbb{P}^3})$ is generated by two classes : the pull back of hyperplane class in $\mathbb{P}^3$ which we denote H and the class $E = c_1(\mathcal{O}_{T_{\mathbb{P}^3}}(1))$. By theorem 3.3 in [Fu] there exist $a_i : 0 < i \leq 3$ such that

$$E^3 = a_1 HE^2 + a_2 H^2 E + a_3 H^3$$

Let $p : T_{\mathbb{P}^3} \to \mathbb{P}^3$ be the projection. The coefficients can be found inductively as follows. We have

$$p_*(E^{3+j}) = \sum_i a_i H^i p_*(E^{3-i+j})$$

for $0 \leq j \leq 2$. But if $3-i+j < 2$ or equivalently $j < i-1$ then $p_*(E^{3-i+j}) = 0$ for dimension reason (if the image is of lower dimension then the pushforward is zero). Let s_i and c_i be the i-th Segre classes and Chern classes of $T_{\mathbb{P}^3}$. Then we have the following equalities :

$$\begin{aligned}
s_1 &= p_*(E^3) = a_1 H p_*(E^2) = a_1 H s_0 \\
s_2 &= p_*(E^4) = a_1 H p_*(E^3) + a_2 H^2 p_*(E^2) = a_1 H s_1 + a_2 H^2 s_0 \\
s_3 &= p_*(E^5) = a_1 H p_*(E^4) + a_2 H^2 p_*(E^3) + a_3 H^3 p_*(E^2) \\
&= a_1 H s_2 + a_2 H^2 s_1 + a_3 H^3 s_0
\end{aligned}$$

¿From those and the equality $(s_0 + s_1 + s_2 + s_3)(c_0 + c_1 + c_2 + c_3) = 1$ we can easily find that $a_i H^i = -c_i$. From the Euler exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3} \longrightarrow \mathcal{O}_{\mathbb{P}^3}(1)^{\oplus 4} \longrightarrow T_{\mathbb{P}^3} \longrightarrow 0$$

we deduce that $c(T_{\mathbb{P}^3}) = (1 + H)^4$ hence $c_i = \binom{4}{i} H^i$. In particular, $a_1 = 4$, $a_2 = -6$, $a_3 = 4$. □

Now we can calculate the Chow ring of $Bl_{\mathcal{D}}(\mathbb{P}^3 \times \mathbb{P}^3)$.

Proposition 5.2. *The Chow ring of $Bl_{\mathcal{D}}(\mathbb{P}^3 \times \mathbb{P}^3)$ is generated by h, k, the hyperplane class of the first and the second factor, and the exceptional divisor e with the following relations :*

$$\begin{aligned}
h^4 &= k^4 = 0, \\
he &= ke, \\
e^3 &= 4he^2 - 6h^2 e + h^3 + h^2 k + hk^2 + k^3
\end{aligned}$$

The ring $A^*(Bl_{\mathcal{D}}(\mathbb{P}^3 \times \mathbb{P}^3))$ is generated by 3 elements : h, k, the hyperplane classes of the first and second factor respectively, and e, the exceptional divisor. As a scheme, $e \cong P(T_{\mathbb{P}^3})$. The followings are obvious relations :

$$he = ke, h^4 = k^4 = 0$$

Restricting to $e = P(T_{\mathbb{P}^3})$, we have :

$$h = k = H, -e = E$$

Thus we have this equality from Proposition 4.1:

$$\begin{aligned}
e \cdot \left((-e)^3\right) &= -e \cdot \left(\sum_{i>0}^{i \leq 3} \tbinom{4}{i} h^i (-e)^{3-i} \right) \qquad \Leftrightarrow \\
\Leftrightarrow \qquad e^4 &= e \cdot \left(\sum_{i>0}^{i \leq 3} (-1)^{i-1} \tbinom{4}{i} h^i e^{3-i} \right) \qquad (*)
\end{aligned}$$

Since the Betti numbers of $Bl_{\mathcal{D}}(\mathbb{P}^3 \times \mathbb{P}^3)$ have to be symmetrical, it follows that 3 must be the least number r so that e^r is expressible as linear combinations of monomials in which the power of e is less than r. So let

$$e^3 = \sum_{i>0}^{i \leq 3} \tau_i h^i e^{3-i} + \sum_{i \geq 0}^{i \leq 3} A_i h^i k^{3-i} \qquad (**)$$

Multiplying the equation $(**)$ by e and compare to the equation $(*)$ above we get $\tau_i = (-1)^{i-1}\binom{4}{i}$ and $\sum_i A_i = (-1)^3 4$. If we pushforward the equation $(**)$ onto the $\mathbb{P}^3 \times \mathbb{P}^3$:

$$\begin{aligned} p_*(e^3) &= p_*\left(e\cdot(-1)^2(-e)^2\right) = p_*(E^2) = \mathcal{D} \\ \sum_{i>0}^{i\leq 3} p_*(h^i e^{3-i}) &= 0 \qquad \text{(dimension reason)} \\ p_*\left(\sum_{i\geq 0}^{i\leq 3} A_i h^i k^{3-i}\right) &= \left(\sum_{i\geq 0}^{i\leq 3} A_i h^i k^{3-i}\right) \end{aligned}$$

hence we deduce that

$$\mathcal{D} = \left(\sum_{i\geq 0}^{i\leq 3} A_i h^i k^{3-i}\right)$$

as classes in the Chow ring of $\mathbb{P}^3 \times \mathbb{P}^3$. But in $A^*(\mathbb{P}^3 \times \mathbb{P}^3)$

$$\mathcal{D} = \sum_{i\geq 0}^{i\leq 3} h^i k^{3-i}$$

hence all the A_i's must be 1. □

To count maps in $\mathcal{RR}(d_1, d_2)$ satisfying the constraint (Δ, p, q), we first consider all the partitions $\Delta = \Gamma_1 \Delta \Gamma_2$, and for each such partition , and assigning constraint Γ_i to the i-th component. Then the constraint Γ_1 cuts out a family $\mathcal{F}_1$ on $\overline{M}_{0,\{A,C\}}(3, d_1)$. Similarly, γ_2 cuts out a family $\mathcal{F}_2$ on $\overline{M}_{0,\{A,C\}}(3, d_2)$. Let T_i be the pushforward of $\mathcal{F}_i$ under the evaluation map ev_{AC}. We then calculate the product

$$T_1 T_2 h^p k^q$$

in the Chow ring $A^*(Bl_{\mathcal{D}}(\mathbb{P}^3 \times \mathbb{P}^3))$. Then take the sum over all partitions $\Delta = \Gamma_1\Gamma_2$ to get the number of maps $\#(\mathcal{RR}(d_1, d_2), \Delta, p, q)$. We need a result to calculate the classes of $(ev_{AC})_*(\mathcal{F}_i)$ in $A^*(Bl_{\mathcal{D}}(\mathbb{P}^3 \times \mathbb{P}^3))$. The following lemma is useful:

Lemma 5.3. *Let $\mathcal{F}$ be a family of stable maps in $\overline{M}_{0,\{A,C\}}(3, d)$. Let T be the pushforward of $\mathcal{F}$ under the evaluation map $ev_{AC} : \mathcal{F} \to Bl_D(\mathbb{P}^3 \times \mathbb{P}^3)$. Let $\mathcal{G}$ be the family of stable maps in $\overline{M}_{0,\{A\}}(3, d)$ that is the image of $\mathcal{F}$ under the forgetful morphism $\overline{M}_{0,\{A,C\}}(3, d) \to \overline{M}_{0,\{A\}}(3, d)$ that forgets the marked point C. Assume $\dim T \leq 2r$. Then we have*

- *For m, n such that $m + n = \dim T$:*
$$Th^m k^n = \#(\mathcal{F}, \mathcal{L}_A^m \mathcal{L}_C^n).$$
- *For m such that $m + 1 = \dim T$:*
$$Th^m e = \#(\mathcal{G}, \mathcal{L}_A^m).$$
- *For m, n such that $m + 2 = \dim T$, we have*
$$Th^m e(h + k - e) = \#(\mathcal{G}, \mathcal{L}_A^m \mathcal{W}_A)$$

Proof. The first equality is trivial. The number $Th^m k^n$ is the number of maps $\gamma \in \mathcal{F}$ such that $\gamma(A)$ belongs to h planes, and that $\gamma(C)$ belongs to k planes. That is precisely the number $\#(\mathcal{F}, \mathcal{L}_A^m \mathcal{L}_C^n)$. The second equality follows from the fact that multiplying with e is the same as replacing the family $\mathcal{F}$ by the family G.

Now we prove the third equality. Let

$$[x_0 : x_1 : \cdots : x_n] \times [y_0 : y_1 : \cdots : y_n]$$

be a homogeneous coordinate system of $\mathbb{P}^3 \times \mathbb{P}^3$. Let H be the hypersurface

$$x_0 y_n = x_n y_0$$

in $\mathbb{P}^3 \times \mathbb{P}^3$. H contains $\mathcal{D}$ with multiplicity one and $T = h + k$ in $A^*(\mathbb{P}^3 \times \mathbb{P}^3)$, hence the proper transformation $\widetilde{T}$ of T in $Bl_{\mathcal{D}}(\mathbb{P}^3 \times \mathbb{P}^3)$ satisfies

$$\widetilde{T} = h + k - e.$$

Let us examine what it means to intersect $\tilde{T}$ with e and $\widetilde{H}$. Let $\pi : Bl_{\mathcal{D}}(\mathbb{P}^3 \times \mathbb{P}^3) \to \mathbb{P}^3 \times \mathbb{P}^3$ be the blow up. We have a map $f : T \to T \cap \mathcal{D}$ defined as folows. For each point $x \in T$, let P_x be the subspace $\{p\} \times \mathbb{P}^3 \subset \mathbb{P}^3 \times \mathbb{P}^3$, where $\{p\} \in \mathbb{P}^3$ is chosen so that $x \in P_x$. The intersection $T \cap P_x$ is a genus zero curve f_x in P_x, and f maps the entire curve f_x onto x. The intersection $H \cap P_x$ is a plane in P_x which is the span of x and the codimension 2 subspace $x_o = y_0 = 0$. Then for a point $y \in \widetilde{T}$ with $\pi(y) = x$, we have $y \in \widetilde{T} \cap e \cap \widetilde{H}$ iff f_x, as a curve in the projective space P_x is tangent to H_x at x. Thus intersecting with $\widetilde{T}$ (after intersecting with e) has the effect of imposing one special tangent condition on the family $\mathcal{G}$.

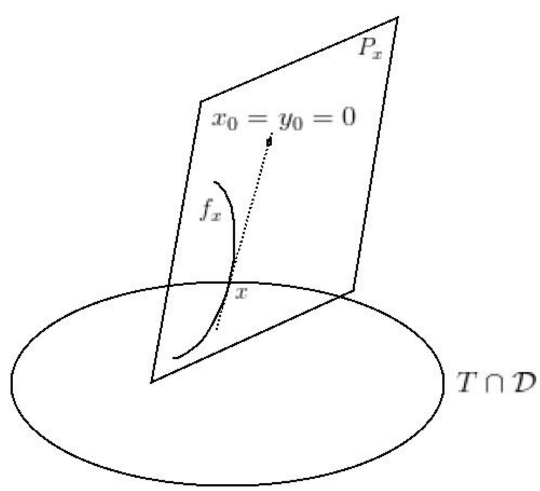

Fig 7.

$\square$

Now we have enough to be able compute the class of $T = ev_{AC*}(\mathcal{F})$ in $A^*(Bl_{\mathcal{D}}(\mathbb{P}^3 \times \mathbb{P}^3))$. The formal statement of that fact is the following proposition, whose proof is trivial.

Proposition 5.4. *Let $T \in A^*(Bl_{\mathcal{D}}(\mathbb{P}^3 \times \mathbb{P}^3))$ be a class of codimension $d, 0 \le d \le 6$. Then the following intersection products determine T :*

- $Th^m k^{d-m}$ *with* $0 \le m \le d$.
- $Th^{d-2}e(h+k-e)$ *with* $m+2=d$.
- $Th^{d-1}e$.

□

Example 5.5. *How many pair of lines (L_1, L_2) in $\mathbb{P}^3$ such that they intersect twice, and that each of them passes through 3 lines? The answer is* 0.

Proof. The answer is obvious because two distinct lines can never intersect twice. But our algorithm does not know that. We need to compute

$$\frac{1}{2}\#(\mathcal{RR}(1,1), \mathcal{H}^3, \mathcal{H}^3).$$

The factor 1/2 accounts for the fact the statement of the problem does not distinguish the two intersection points. Let $\mathcal{F}_i$ be the family of lines L_i with a choice of two marked points A, Con them. Let T_i be the pushforward of $\mathcal{F}_i$ under the evaluation maps $ev_{AC} : \mathcal{F}_i \to Bl_D(\mathbb{P}^3 \times \mathbb{P}^3)$. T_1 is three dimensional, so we can assume

$$T_1 = \alpha(h^3 + k^3) + \beta(h^2k + hk^2) + \gamma eh^2 + \mu e^2 h.$$

The coefficients of h^3 and k^3 must be the same due to symmetry. Similarly the coefficients of h^2k and hk^2 must be the same.

$$\begin{aligned}
\alpha &= \alpha h^3k^3 = T_1k^3 = \#(\overline{M}_{0,\{A,C\}}(3,1), \mathcal{H}^3\mathcal{L}_A^3) = 0 \\
\beta &= \beta h^3k^3 = T_1kh^2 = \#(\overline{M}_{0,\{A,C\}}(3,1), \mathcal{H}^3\mathcal{L}_A^2\mathcal{L}_C) = 2 \\
\mu &= \mu h^3e^3 = T_1h^2e = \#(\overline{M}_{0,\{A\}}(3,1), \mathcal{H}^3\mathcal{L}_A^2) = 2
\end{aligned}$$

Computation of γ is a little bit lengthier. First we have

$$\begin{aligned}
\gamma &= \gamma h^3k^3 = T_1he^2 - \mu e^4h^2 = \big(2T_1h^2e - T_1he(k+k-e)\big) - 4\mu \\
&= -2\mu - T_1he(h+k-e).
\end{aligned}$$

Now $T_1he(h+k-e) = \#(\overline{M}_{0,\{A\}}(3,1), \mathcal{H}^3\mathcal{L}_A\mathcal{W}_A)$ is the number of lines with a marked point A in $\mathbb{P}^3$ that pass through 3 lines, such that A lies on a fixed plane, and such that the tangent line at A passes through a general line. This number is the same as the number of lines passing through 4 general lines in $\mathbb{P}^3$, which is 2. Thus $\gamma = -2\mu - T_1hk(h+k-e) = -4-2 = -6$. Therefore

$$T_1 = 2(h^2k + hk^2) - 6h^2e + 2he^2$$

Obviously $T_1 = T_2$, so after a bit of algebra we have

$$T_1T_2 = \big(2(h^2k + hk^2) - 6h^2e + 2he^2\big)^2 = 0.$$

□

Example 5.6. *How many pair of line-conic (L, C) in $\mathbb{P}^3$ such that they intersect twice, and that L passes through 1 line and 1 point, and that C is tangent to 3 lines and passes through 2 points. The answer is 4.*

Proof. We need to compute

$$\frac{1}{2}\#(\mathcal{RR}(1,2), \mathcal{HP}, \mathcal{T}^3\mathcal{P}^2)$$

The pair (L, C) is planar, thus the three points that they pass through determine their plane. Thus this reduces to an enumerative problem on $\mathbb{P}^2$: L passes through 2 points, and C is tangent to 3 lines and passes through 2 points, and they intersect twice (which is redundant). Thus it is easy to see that the answer is 4. Let T_1 and T_2 be as above, using a similar procedure, we determine that, in $Bl_{\mathcal{D}}(\mathbb{P}^3 \times \mathbb{P}^3)$

$$T_1 = h^2k + hk^2 - 3h^2e + he^2.$$

Now we will compute T_2. T_2 is 3 dimensional, so we can write $T_2 = \alpha(h^3 + k^3) + \beta(h^2k6 + hk^2) + \gamma eh^2 + \mu e^2h$.

$$\begin{aligned}
\alpha &= \alpha h^3k^3 = T_2k^3 = \#(\overline{M}_{0,\{A,C\}}(3,2), \mathcal{T}^3\mathcal{P}^2\mathcal{L}_A^3) = 0.\\
\beta &= \beta h^3k^3 = T_2kh^2 = \#(\overline{M}_{0,\{A,C\}}(3,2), \mathcal{T}^3\mathcal{P}^2\mathcal{L}_A^2\mathcal{L}_C) = 16.\\
\mu &= \mu h^3e^3 = T_2h^2e = \#(\overline{M}_{0,\{A\}}(3,2), \mathcal{T}^3\mathcal{P}^2\mathcal{L}_A^2) = 8.\\
\gamma &= -2\mu - T_2he(h + k - e) = -16 - \#(\overline{M}_{0,\{A\}}(3,2), \mathcal{T}^3\mathcal{P}^2\mathcal{L}_A\mathcal{W}_A) = -32.
\end{aligned}$$

The last equality follows from Example 3.7. Thus

$$T_2 = 16(h^2k + hk^2) - 32h^2e + 16he^2.$$

Using simple algebra we conclude that the answer to our enumerative problem is

$$\frac{1}{2}T_1T_2 = \frac{1}{2}(h^2k + hk^2 - 3h^2e + he^2)(16(h^2k + hk^2) - 32h^2e + 16he^2) = 4$$

□

Example 5.7. *How many pair of conics (C_1, C_2) in $\mathbb{P}^3$ such that they intersect at two distinguished points N_1, N_2 and that :*

- *N_1 lies on a fixed plane.*
- *N_2 lies on a fixed line.*
- *C_1 is tangent to 5 planes.*
- *C_2 passes through 6 lines.*

The answer is 988.

Proof. Using the procedure above we have

$$\begin{aligned}
T_1 &= 8(h + k) - 2e\\
T_2 &= 36(h^2 + k^2) + 92hk - 112he + 18e^2
\end{aligned}$$

Thus the answer is

$$T_1T_2hk^2 = 988.$$

□

We end this section with a proposition similar to Proposition 3.3.

Proposition 5.8. *Let Δ be a constraint and let Δ_i be a constraint obtained from Δ by removing i tangency conditions. Then we have the following equality :*

$$\begin{aligned}
\#(\mathcal{NR}(\Delta_1, d_2), \Delta, 0, k) &= \sum_{\Gamma_1\Gamma_2=\Delta} \#(\mathcal{RR}(d_1, d_2), \Gamma_1, \Gamma_2, 0, k) \\
&+ 2n \sum_{\Gamma_1\Gamma_2=\Delta_1} \#(\mathcal{RR}(d_1, d_2), \Gamma_1, \Gamma_2, 1, k) \\
&+ 4\frac{n(n-1)}{2} \sum_{\Gamma_1\Gamma_2=\Delta_2} \#(\mathcal{RR}(d_1, d_2), \Gamma_1, \Gamma_2, 2, k) \\
&+ 8\frac{n(n-1)(n-2)}{6} \sum_{\Gamma_1\Gamma_2=\Delta_3} \#(\mathcal{RR}(d_1, d_2), \Gamma_1, \Gamma_2, 3, k).
\end{aligned}$$

6. Characteristic numbers of rational nodal curves in $\mathbb{P}^3$

Recall that $\mathcal{N}(d)$ is the closure in $\overline{M}_{0,\{A,B\}}(3,d)$ of the locus of maps γ such that $\gamma(A) = \gamma(B)$.

Theorem 6.1. *Let Δ be a constraint such that $\Delta(0) = 0$. Choose a subspace u in Δ which is not a plane, such that the dimension of u is largest possible. Then choose any two other subspaces s, t in Δ. The following constraints are derived from Δ :*
0) $\widetilde{\Delta}$ by removing u, s, t from Δ.
1) Δ_0 by replacing u with two subspaces : a plane p and a subspace q such that $p \cap q = u$.
2) Δ_1 is derived from d_0, by replacing p and s with $h \cap s$.
3) Δ_2 is derived from d_0, by replacing q and t with $q \cap t$.
4) Δ_3 is derived from d_0, by replacing s and t with $s \cap t$.

If Γ is a set of linear spaces, and a and b are two linear spaces, denote $\Gamma^{(a,b)}$ the set obtained from Γ by adding a and b.

Then the following formula holds :

$$\begin{aligned}
\#(\mathcal{N}(d) \cdot \Delta) &= -\sum_{d_1+d_2=d}^{\Gamma_1\Gamma_2=\widetilde{\Delta}} \#(\mathcal{NR}(d_1,d_2), \Gamma_1^{(s,t)}, \Gamma_2^{(p,q)}) \\
&- \sum_{d_1+d_2=0}^{\Gamma_1\Gamma_2=\widetilde{\Delta}} \#(\mathcal{NR}(d_1,d_2), \Gamma_1^{(p,q)}, \Gamma_2^{(s,t)}) \\
&- 2\sum_{d_1+d_2=d}^{\Gamma_1\Gamma_2=\widetilde{\Delta}} \#(\mathcal{RR}(d_1,d_2), \Gamma_1^{(p,q)}, \Gamma_2^{(s,t)}) \\
&+ \sum_{d_1+d_2=d}^{\Gamma_1\Gamma_2=\widetilde{\Delta}} \#(\mathcal{NR}(d_1,d_2), \Gamma_1^{(q,t)}, \Gamma_2^{(p,s)}) \\
&+ \sum_{d_1+d_2=d}^{\Gamma_1\Gamma_2=\widetilde{\Delta}} \#(\mathcal{NR}(d_1,d_2), \Gamma_1^{(p,s)}, \Gamma_2^{(q,t)}) \\
&+ 2\sum_{d_1+d_2=d}^{\Gamma_1\Gamma_2=\widetilde{\Delta}} \#(\mathcal{RR}(d_1,d_2), \Gamma_1^{(p,s)}, \Gamma_2^{(q,t)}) \\
&- \#(\mathcal{N}(d), \Delta_3) + \#(\mathcal{N}(d), \Delta_1) + \#(\mathcal{N}(d), \Delta_2).
\end{aligned}$$

Furthermore, each $\Delta_i, i \geq 1$ is of rank lower than that of Δ.

Proof. Let S be a set of markings that is in one-to-one correspondence $\mu : \Delta_0 \to S$ with the linear spaces in Δ_0. Let $\mathcal{X}$ be the moduli space $\overline{M}_{0,\{A,B\}\cup S}(3,d)$, and let $\mathcal{N}^{(S)}(d)$ be the closure of the locus in $\mathcal{X}$ of maps γ such that $\gamma(A) = \gamma(B)$. Let $\mathcal{Y}$ be the closure in $\mathcal{N}^{(S)}$ of the locus of maps γ such that $\gamma(\mu(m)) \in m$ for all $m \in \Delta_0$. Because $\#(\mathcal{N}(d), \Delta)$ is finite, $\mathcal{Y}$

is one-dimensional. We consider two equivalent divisors on $\mathcal{X}$:

$$(\mu(p), \mu(q) \mid\mid \mu(s), \mu(t)) = (\mu(p), \mu(s) \mid\mid \mu(q), \mu(t)).$$

Let $\mathcal{K}_1 = (\mu(p), \mu(q) \mid\mid \mu(s), \mu(t))$, and let $\mathcal{K}_2 = (\mu(p), \mu(s) \mid\mid \mu(q), \mu(t))$. Then we have

$$\#\left(\mathcal{Y} \cap \mathcal{K}_1\right) = \#\left(\mathcal{Y} \cap \mathcal{K}_2\right).$$

Let us analyze the left-hand side of the equation. Let γ be a general point of $\mathcal{Y} \cap \mathcal{K}_1$. Then γ is a stable map whose source curve has two components C_1, C_2 joined at a node, such that $\mu(p), \mu(q) \in C_1$ and $\mu(s), \mu(t) \in C_2$. There are several cases to consider:

- $\deg \gamma_{|C_1} = 0$. If $A, B \in C_1$ then the image of γ has a cusp at A. A quick dimension count shows that this case has no enumerative contribution due to dimension reason. If one of A, B is $\in C_1$, or if any other marked point is $\in C_1$, simple dimension count also shows that none of those cases has enumerative contribution. Thus we only need to consider $A, B \in C_2$, and $\gamma_{|C_2}$ is a rational nodal curve and satisfies the constraint Δ (but these conditions are marked). The contribution to $\#(\mathcal{Y} \cap \mathcal{K}_1)$ in this case is $\#(\mathcal{N}(d), \Delta)$.
- $\deg \gamma_{|C_2} = 0$. Arguing similarly, we have that the contribution to $\#(\mathcal{Y} \cap \mathcal{K}_1)$ is $\#(\mathcal{N}(d), \Delta_3)$.
- γ has positive degree d_i component C_i. There are three subcases :
  - $A, B \in C_1$: In this case, $\gamma_{|C_1}$ is a rational nodal curve and $\gamma_{|C_2}$ is a rational curve. The contribution in this case is

$$\sum_{d_1+d_2=0}^{\Gamma_1\Gamma_2=\widetilde{\Delta}} \#(\mathcal{NR}(d_1, d_2), \Gamma_1^{(p,q)}, \Gamma_2^{(s,t)}).$$

  - $A, B \in C_2$: The contribution is

$$\sum_{d_1+d_2=d}^{\Gamma_1\Gamma_2=\widetilde{\Delta}} \#(\mathcal{NR}(d_1, d_2), \Gamma_1^{(s,t)}, \Gamma_2^{(p,q)}).$$

  - $A \in C_1, B \in C_2$ or vice versa. In this case the image of γ is a curve having two components that intersect twice at distinguished points. The contribution is therefore

$$2 \sum_{d_1+d_2=d}^{\Gamma_1\Gamma_2=\widetilde{\Delta}} \#(\mathcal{RR}(d_1, d_2), \Gamma_1^{(p,q)}, \Gamma_2^{(s,t)}).$$

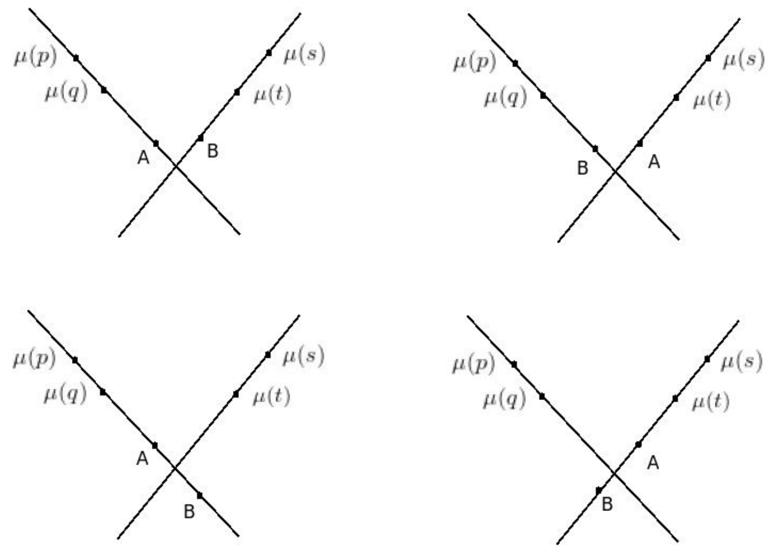

Fig 8. Four types of source curves of a general map in $\mathcal{Y} \cap \mathcal{K}_1$.

We can analyze $\mathcal{Y} \cap \mathcal{K}_2$ in the same way and after rearranging the terms, we derive the equation in the statement of the theorem. The last statement is just simple algebra. $\square$

Using a program implementing the above formula, we derive the following numbers of rational nodal curves in $\mathbb{P}^3$ passing through a lines and b points, with increasing a (which are also the number of elliptic curves with fixed $j-$invariant passing through the same constraints).

- Cubics: $0, 0, 12, 144, 1392, 12960$.
- Quartics: $24, 192, 1800, 17808, 18223, 1935936, 21422448, 247191840$.
- Quintics: $1344, 13440, 144324, 1625184, 19092384, 233875584, 2987074368,$ $39753459648, 550666856448, 7928395296768$.

It is now possible to use the results so far to compute the full characteristic number of rational nodal curves.

Theorem 6.2. *Let Δ be a constraint such that $\Delta(0) > 0$. Let Δ'' be the constraint obtained from Δ by removing a tangency plane. Let Δ' be the constraint obtained from Δ'' by adding an incident line. Then we have the following equality, provided that the left hand side is finite.*

$$
\begin{aligned}
\#(\mathcal{N}(d),\Delta) &= \frac{d-1}{d}\#(\mathcal{N}(d),\Delta') \\
&+ \sum_{d_1+d_2=d}\big(\#(\mathcal{NR}(d_1,d_2),\Delta'')+\#(\mathcal{RR}(d_1,d_2),\Delta'')\big).
\end{aligned}
$$

Proof. We have the following equality of divisors on $\overline{M}_{0,\{A.B\}}(3,d)$

$$
\mathcal{T}=\frac{d-1}{d}\mathcal{H}+\sum_{d>0}^{j\le d/2}\frac{j(d-j)}{d}(j,d-j).
$$

Thus

$$
\begin{aligned}
\#(\mathcal{N}(d),\Delta) &= \#\left((\mathcal{N}(d),\Delta''),\mathcal{T}\right) \\
&= \frac{d-1}{d}\#\left((\mathcal{N}(d),\Delta''),\mathcal{H}\right)+\sum_{j>0}^{j\le d/2}\#\left(\mathcal{N}(d)\cap(j,d-j),\Delta''\right).
\end{aligned}
$$

Now we will analyze $\#(\mathcal{N}(d)\cap(j,d-j),\Delta'')$. A general map $\gamma\in\mathcal{N}(d)\cap(j,d-j)$ has two-component source curve. There are two cases:

- A,B belong to a same component. The contribution is $\#(\mathcal{NR}(j,d-j),\Delta'')+\#(\mathcal{NR}(d-j,j),\Delta'')$ if $j<d-j$ depending on whether A,B are in the component of lower or higher degree. If $j=d-j$, the contribution is just $\#(\mathcal{NR}(j,d-j),\Delta'')$.
- A,B belong to different components. The contribution is $2\#(\mathcal{RR}(j,d-j),\Delta'')$ if $j<d-j$ and is $\#(\mathcal{RR}(j,d-j),\Delta'')$ if $j=d-j$.

Sum up all possibilities, we derive the formula in Theorem 6.4. $\square$

To compute $\#(\mathcal{RR}(d_1,d_2),\Delta)$ we use Proposition 5.8, Proposition 5.4 and Lemma 5.3. To compute $\#(\mathcal{NR}(d_1,d_2),\Delta)$ we use Propostion 3.4 and Proposition 3.1. Using a computer program implementing the above algorithm, we can calculate the following numbers of rational nodal curves of degree $3,4,5$ that pass through a lines and are tangent to b planes, with increasing b :

- Cubics: $12960, 29520, 61120, 109632, 167616, 214400, 230240, 211200, 170192, 124176, 85440, 56960$.
- Quartics: $247191840, 519424512, 1034619648, 1932171072, 3353134848, 5361957120, 7841572992, 10431095808, 2599060192, 13851211968, 13948252800, 12986719872, 11309818368, 9330496512, 7394421888, 5703866880$.
- Quintics : $7928395296768, 16913986557696, 34645178529792, 67838371113984, 126457160317440, 223497303825408, 373019416582656, 585746158152192, 862662145899264, 1188950459771520, 1532051249787648, 1846789046255616, 2086982230651392, 2218873455750144, 2230284906086400, 2131793502847488, 1950604658328576, 1720926352507392, 1475336408154624, 1239040055854080$.

6.3. Possible generalizations. All of the results in sections $3, 6$ generalize without change to curves in $\mathbb{P}^r$. Results in section 5 will also carry over to to $\mathbb{P}^r$ if generalizations of results in section 4 are achieved. The main difficulty in doing so is the fact that when we count curves with multiple special tangent conditions, excess intersection will arise. The intersection of all special tangent conditions will contain all the loci of maps γ such that $\gamma(A)$ is a node of the image.

Another possible generalization is to family of rational nodal curves with possible condition on the node. Theorem 6.1 will generalize without change. However, Theorem 6.2 will need adjustment. That is because when counting rational nodal curves with tangency conditions and condition on the node, such curves of degree 2 must be correctly interpreted and counted with correct multiplicities. These generalizations will be discussed in [N].

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