Proper Resolutions and Auslander-Type Conditions of Modules *†

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Abstract

We obtain some methods to construct a (strongly) proper resolution (resp. coproper coresolution) of one end term in a short exact sequence from that of the other two terms. By using this method, we prove that for a left and right Noetherian ring R, ${}_RR$ satisfies the Auslander condition if and only if so does every flat left R-module, if and only if the injective dimension of the ith term in a minimal flat resolution of any injective left R-module is at most $i-1$ for any $i \geq 1$, if and only if the flat (resp. injective) dimension of the ith term in a minimal injective (resp. flat) resolution of any left R-module M is at most the flat (resp. injective) dimension of M plus $i-1$ for any $i \geq 1$, if and only if the flat (resp. injective) dimension of the injective envelope (resp. flat cover) of any left R-module M is at most the flat (resp. injective) dimension of M, and if and only if any of the opposite versions of the above conditions hold true. Furthermore, we prove that for an Artinian algebra R satisfying the Auslander condition, R is Gorenstein if and only if the subcategory consisting of finitely generated modules satisfying the Auslander condition is contravariantly finite. As applications, we get some equivalent characterizations of Auslander-Gorenstein rings and Auslander-regular rings.

1. Introduction

It is well known that commutative Gorenstein rings are fundamental and important research objects in commutative algebra and algebraic geometry. Bass proved in [B2] that a commutative Noetherian ring R is a Gorenstein ring (that is, the self-injective dimension of R is finite) if and only if the flat dimension of the ith term in a minimal injective coresolution of R as an R-module is at most $i-1$ for any $i \geq 1$. In non-commutative case, Auslander proved that this condition is left-right symmetric ([FGR, Theorem 3.7]). In this case, R is said to satisfy the *Auslander condition*. Motivated by this philosophy, Huang and Iyama introduced the notion of certain Auslander-type conditions as follows. For any $m, n \geq 0$, a left and right Noetherian ring is said to be $G_n(m)$ if the flat dimension of the ith term

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in a minimal injective coresolution of R_R is at most $m + i - 1$ for any $1 \leq i \leq n$. The Auslander-type conditions are non-commutative analogs of commutative Gorenstein rings. Such conditions play a crucial role in homological algebra, representation theory of algebras and non-commutative algebraic geometry ([AR3], [AR4], [Bj], [EHIS], [FGR], [H], [HI], [IS], [I1], [I2], [I3], [I4], [M], [Ro], [S], [W], and so on). In particular, by constructing an injective coresolution of the last term in an exact sequence of finite length from that of the other terms, Miyachi obtained in [M] an equivalent characterization of the Auslander condition in terms of the relation between the flat dimensions of any module and its injective envelope. Then he got some properties of Auslander-Gorenstein rings and Auslander-regular rings.

This paper is organized as follows.

In Section 2, we give some terminology and some preliminary results.

In Section 3, we introduce the notion of strongly proper (co)resolutions of modules, and then give a method to construct a (strongly) proper resolution (resp. coproper coresolution) of the first (resp. last) term in a short exact sequence from that of the other two terms. We will prove the following two theorems and their dual results.

Theorem 1.1. *Let $\mathscr{C}$ be a full subcategory of $\operatorname{Mod} R$ closed under finite direct sums and under kernels of epimorphisms, and let*

$$0 \to X \to X^0 \to X^1 \to 0$$

be an exact sequence in $\operatorname{Mod} R$. If

$$\cdots \to C_i^j \to \cdots \to C_1^j \to C_0^j \to X^j \to 0$$

is a (strongly) proper $\mathscr{C}$-resolution of X^j for $j = 0, 1$, then

$$\cdots \to C_{i+1}^1 \bigoplus C_i^0 \to \cdots \to C_2^1 \bigoplus C_1^0 \to C \to X \to 0$$

is also a (strongly) proper $\mathscr{C}$-resolution of X, and

$$0 \to C \to C_1^1 \bigoplus C_0^0 \to C_0^1 \to 0$$

is exact.

Theorem 1.2. *Let $\mathscr{C}$ be a full subcategory of $\operatorname{Mod} R$ closed under finite direct sums, and let*

$$0 \to X_1 \to X_0 \to X \to 0 \tag{1.1}$$

be an exact sequence in $\operatorname{Mod} R$ and

$$C_j^n \to \cdots \to C_j^1 \to C_j^0 \to X_j \to 0$$

a (strongly) coproper $\mathscr{C}$-coresolution of X_j for $j = 0, 1$. If (1.1) is (strongly) $\mathrm{Hom}_R(\mathscr{C}, -)$*-exact, then*

$$C_0^n \bigoplus C_1^{n-1} \to \cdots \to C_0^2 \bigoplus C_1^1 \to C_0^1 \bigoplus C_1^0 \to C_0^0 \to X \to 0$$

is also a (strongly) coproper $\mathscr{C}$-coresolution of X.

Let R be a left Noetherian ring and $n, k \geq 0$, and let $\{M_i\}_{i \in I}$ be a family of left R-modules and $M = \varinjlim_{i \in I} M_i$, where I is a directed index set. By using some techniques of direct limits and transfinite induction, we prove in Section 4 that if the flat dimension of the $(n+1)$st term in a minimal injective coresolution of M_i is at most k for any $i \in I$, then the flat dimension of the $(n+1)$st term in a minimal injective coresolution of M is also at most k.

For any $m, n \geq 0$, we introduce in Section 5 the notion of modules satisfying the Auslander-type conditions $G_n(m)$; in particular, a left R-module M for any ring R is said to satisfy the *Auslander condition* if the flat dimension of the ith term in a minimal injective coresolution of ${}_RM$ is at most $i-1$ for any $i \geq 1$. By using results obtained in the former sections, we will investigate the homological behavior of modules satisfying the Auslander-type conditions in terms of the relation between the flat (resp. injective) dimensions of modules and their injective envelopes (resp. flat covers). In particular, we get the following

Theorem 1.3. *Let R be a left and right Noetherian ring. Then the following statements are equivalent.*

(1) ${}_RR$ satisfies the Auslander condition.

(2) Every flat left R-module satisfies the Auslander condition.

(3) The flat dimension of the ith term in a minimal injective coresolution of any left R-module M is at most the flat dimension of M plus $i-1$ for any $i \geq 1$.

(4) The flat dimension of the injective envelope of any left R-module M is at most the flat dimension of M.

(5) The injective dimension of the ith term in a minimal flat resolution of any injective left R-module is at most $i-1$ for any $i \geq 1$.

(6) The injective dimension of the ith term in a minimal flat resolution of any left R-module M is at most the injective dimension of M plus $i-1$ for any $i \geq 1$.

(7) The injective dimension of the flat cover of any left R-module M is at most the injective dimension of M.

$(i)^{op}$ The opposite version of (i) $(1 \leq i \leq 7)$.

As applications of this theorem, we obtain some equivalent characterizations of Auslander-Gorenstein rings and Auslander-regular rings, respectively.

Note that a commutative Noetherian ring satisfies the Auslander condition if and only if it is Gorenstein ([B2]). Auslander and Reiten conjectured in [AR3] that an Artinian algebra satisfying the Auslander condition is Gorenstein. This conjecture is situated between the well known Nakayama conjecture and the finitistic dimension conjecture. The Nakayama conjecture states that an Artinian algebra R is selfinjective if all terms in a minimal injective coresolution of ${}_RR$ are projective; and the finitistic dimension conjecture states that the supremum of the projective dimensions of all finitely generated left R-modules with finite projective dimension for an Artinian algebra R is finite. All of these conjectures remains still open. In Section 6, we first obtain the approximation presentations of a given module relative to the subcategory of modules satisfying the Auslander condition and that of modules with finite injective dimension respectively. Then we establish the connection between the Auslander and Reiten conjecture mentioned above with the contravariant finiteness of some certain subcategories as follows.

Theorem 1.4. *Let R be an Artinian algebra satisfying the Auslander condition. Then the following statements are equivalent.*

(1) R is Gorenstein.

(2) The subcategory consisting of finitely generated modules satisfying the Auslander condition is contravariantly finite.

(3) The subcategory consisting of finitely generated modules which are n-syzygy for any $n \geq 1$ is contravariantly finite.

As a consequence, we get that an Artinian algebra is Auslander-regular if and only if the subcategory consisting of projective modules and that consisting of modules satisfying the Auslander condition coincide.

2. Preliminaries

Throughout this paper, R is an associative ring with identity, $\mathrm{Mod}\, R$ is the category of left R-modules and $\mathrm{mod}\, R$ is the category of finitely generated left R-modules. We use $\mathrm{gl.dim}\, R$ to denote the global dimension of R. In this section, we give some terminology and some preliminary results.

Definition 2.1. ([E]) Let $\mathscr{C} \subseteq \mathscr{D}$ be full subcategories of $\mathrm{Mod}\, R$. The homomorphism $f : C \to D$ in $\mathrm{Mod}\, R$ with $C \in \mathscr{C}$ and $D \in \mathscr{D}$ is said to be a *$\mathscr{C}$-precover* of D if for any homomorphism $g : C' \to D$ in $\mathrm{Mod}\, R$ with $C' \in \mathscr{C}$, there exists a homomorphism $h : C' \to C$

such that the following diagram commutes:

$$\begin{array}{ccc} & & C' \\ & {\scriptstyle h} \swarrow & \downarrow {\scriptstyle g} \\ C & \xrightarrow{f} & D \end{array}$$

The homomorphism $f : C \to D$ is said to be *right minimal* if an endomorphism $h : C \to C$ is an automorphism whenever $f = fh$. A $\mathscr{C}$-precover $f : C \to D$ is called a *$\mathscr{C}$-cover* if f is right minimal. Dually, the notions of a *$\mathscr{C}$-preenvelope*, a *left minimal homomorphism* and *a $\mathscr{C}$-envelope* are defined. Following Auslander and Reiten's terminology in [AR1], for a module over an Artinian algebra, a $\mathscr{C}$-(pre)cover and a $\mathscr{C}$-(pre)envelope are called a *(minimal) right $\mathscr{C}$-approximation* and a *(minimal) left $\mathscr{C}$-approximation*, respectively. If each module in $\mathscr{D}$ has a right (resp. left) $\mathscr{C}$-approximation, then $\mathscr{C}$ is called *contravariantly finite* (resp. *covariantly finite*) in $\mathscr{D}$.

Lemma 2.2. *([X, Theorem 1.2.9]) Let $\mathscr{C}$ be a full subcategory of $\operatorname{Mod} R$ closed under direct products. If $f_i : C_i \to M_i$ is a $\mathscr{C}$-precover of M_i in $\operatorname{Mod} R$ for any $i \in I$, where I is an index set, then $\prod_{i\in I} f_i : \prod_{i\in I} C_i \to \prod_{i\in I} M_i$ is a $\mathscr{C}$-precover of $\prod_{i\in I} M_i$.*

We use $\mathscr{F}^0(\operatorname{Mod} R)$ and $\mathscr{I}^0(\operatorname{Mod} R)$ to denote the subcategories of $\operatorname{Mod} R$ consisting of flat modules and injective modules, respectively. Recall that an $\mathscr{F}^0(\operatorname{Mod} R)$-(pre)cover and an $\mathscr{I}^0(\operatorname{Mod} R)$-(pre)envelope are called a *flat (pre)cover* and an *injective (pre)envelope*, respectively.

Bican, El Bashir and Enochs proved in [BEE, Theorem 3] that every R-module has a flat cover. For an R-module M, we call an exact sequence $\cdots \to F_i \xrightarrow{\pi_i} \cdots \xrightarrow{\pi_2} F_1 \xrightarrow{\pi_1} F_0 \xrightarrow{\pi_0} M \to 0$ a *proper flat resolution* of M if $\pi_i : F_i \to \operatorname{Im} \pi_i$ is a flat precover of $\operatorname{Im} \pi_i$ for any $i \geq 0$. Furthermore, we call the following exact sequence:

$$\cdots \to F_i(M) \xrightarrow{\pi_i(M)} \cdots \xrightarrow{\pi_2(M)} F_1(M) \xrightarrow{\pi_1(M)} F_0(M) \xrightarrow{\pi_0(M)} M \to 0$$

a *minimal flat resolution* of M, where $\pi_i(M) : F_i(M) \to \operatorname{Im} \pi_i(M)$ is a flat precover of $\operatorname{Im} \pi_i(M)$ for any $i \geq 0$. It is easy to verify that the flat dimension of M is at most n if and only if $F_{n+1}(M) = 0$. In addition, we use

$$0 \to M \to E^0(M) \to E^1(M) \to \cdots \to E^i(M) \to \cdots$$

to denote a minimal injective coresolution of M.

We denote by $(-)^+ = \operatorname{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$, where $\mathbb{Z}$ is the additive group of integers and $\mathbb{Q}$ is the additive group of rational numbers.

Lemma 2.3. *([EH, Theorem 3.7]) The following statements are equivalent.*

(1) R is a left Noetherian ring.

(2) A monomorphism $f : A \rightarrowtail E$ in $\operatorname{Mod} R$ is an injective preenvelope of A if and only if $f^+ : E^+ \twoheadrightarrow A^+$ is a flat precover of A^+ in $\operatorname{Mod} R^{op}$.

Let $M \in \operatorname{Mod} R$. We use $\operatorname{fd}_R M$, $\operatorname{pd}_R M$ and $\operatorname{id}_R M$ to denote the flat, projective and injective dimensions of M, respectively.

Lemma 2.4. *(1) ([F, Theorem 2.1]) For any $M \in \operatorname{Mod} R$, $\operatorname{fd}_R M = \operatorname{id}_{R^{op}} M^+$.*

(2) ([F, Theorem 2.2]) If R is a right Noetherian ring, then $\operatorname{fd}_R N^+ = \operatorname{id}_{R^{op}} N$ for any $N \in \operatorname{Mod} R^{op}$.

Recall that $\operatorname{Fin.dim} R = \sup\{\operatorname{pd}_R M \mid M \in \operatorname{Mod} R \text{ with } \operatorname{pd}_R M < \infty\}$. Observe that the first assertion in the following result was proved by Bass in [B1, Corollary 5.5] when R is a commutative Noetherian ring.

Lemma 2.5. *(1) For a left Noetherian ring R, $\operatorname{id}_R R \geq \sup\{\operatorname{fd}_R M \mid M \in \operatorname{Mod} R$ with $\operatorname{fd}_R M < \infty\}$.*

(2) For a left and right Noetherian ring R, $\operatorname{id}_R R \geq \sup\{\operatorname{id}_{R^{op}} N \mid N \in \operatorname{Mod} R^{op}$ with $\operatorname{id}_{R^{op}} N < \infty\}$.

Proof. (1) Without loss of generality, assume that $\operatorname{id}_R R = n < \infty$. Then $\operatorname{Fin.dim} R \leq n$ by [B1, Proposition 4.3]. It follows from [J1, Proposition 6] that the projective dimension of any flat left R-module is finite. So, if $M \in \operatorname{Mod} R$ with $\operatorname{fd}_R M < \infty$, then $\operatorname{pd}_R M < \infty$ and $\operatorname{pd}_R M \leq n$. Thus we have $\operatorname{fd}_R M (\leq \operatorname{pd}_R M) \leq n$.

(2) By [B1, Proposition 4.1], we have $\sup\{\operatorname{fd}_R M \mid M \in \operatorname{Mod} R \text{ with } \operatorname{fd}_R M < \infty\} = \sup\{\operatorname{id}_{R^{op}} N \mid N \in \operatorname{Mod} R^{op} \text{ with } \operatorname{id}_{R^{op}} N < \infty\}$. So the assertion follows from (1). □

3. The constructions of (strongly) proper resolutions and coproper coresolutions

In this section, we introduce the notion of strongly (co)proper (co)resolutions of modules. Then we give a method to construct a (strongly) proper resolution (resp. coproper coresolution) of the first (resp. last) term in a short exact sequence from that of the other two terms, as well as give a method to construct a (strongly) proper resolution (resp. coproper coresolution) of the last (resp. first) term in a short exact sequence from that of the other two terms.

We first give the following easy observation, which is a generalization of the horseshoe lemma.

Lemma 3.1. *Let* $0 \to A \xrightarrow{f} A' \xrightarrow{g} A'' \to 0$ *be an exact sequence in* $\operatorname{Mod} R$.

(1) If there exist homomorphisms $\alpha : C \to A$, $\alpha'' : C'' \to A''$ *and* $h : C'' \to A'$ *in* $\operatorname{Mod} R$ *such that* $\alpha'' = gh$, *then we have the following commutative diagram with exact rows:*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C & \xrightarrow{\binom{1_C}{0}} & C \bigoplus C'' & \xrightarrow{(0,1_{C''})} & C'' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\alpha'} & & \downarrow{\scriptstyle\alpha''} & & \\
0 & \longrightarrow & A & \xrightarrow{f} & A' & \xrightarrow{g} & A'' & \longrightarrow & 0
\end{array}$$

where $\alpha' = (f\alpha, h)$.

(2) If there exist homomorphisms $\beta : A \to D$, $\beta'' : A'' \to D''$ *and* $k : A' \to D$ *in* $\operatorname{Mod} R$ *such that* $\beta = kf$, *then we have the following commutative diagram with exact rows:*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{f} & A' & \xrightarrow{g} & A'' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\beta'} & & \downarrow{\scriptstyle\beta''} & & \\
0 & \longrightarrow & D & \xrightarrow{\binom{1_D}{0}} & D \bigoplus D'' & \xrightarrow{(0,1_{D''})} & D'' & \longrightarrow & 0
\end{array}$$

where $\beta' = \binom{k}{\beta'' g}$.

The following observation is useful in the rest of this section.

Lemma 3.2. *Let*

$$\begin{array}{ccc}
M & \xrightarrow{f_1} & N \\
\downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle g} \\
X & \xrightarrow{f} & Y
\end{array}$$

be a commutative diagram in $\operatorname{Mod} R$ *and* $C \in \operatorname{Mod} R$.

(1) If this diagram is a pull-back diagram of f *and* g *and* $\operatorname{Hom}_R(C, g)$ *is epic, then* $\operatorname{Hom}_R(C, g_1)$ *is also epic.*

(2) If this diagram is a push-out diagram of f_1 *and* g_1 *and* $\operatorname{Hom}_R(g_1, C)$ *is epic, then* $\operatorname{Hom}_R(g, C)$ *is also epic.*

Proof. Assume that the given diagram is a pull-back diagram of f and g and $\operatorname{Hom}_R(C, g)$ is epic. Let $\alpha \in \operatorname{Hom}_R(C, X)$. Then there exists $\beta \in \operatorname{Hom}_R(C, N)$ such that $f\alpha = \operatorname{Hom}_R(C, g)(\beta) = g\beta$. By the universal property of a pull-back diagram, there exists $\gamma \in \operatorname{Hom}_R(C, M)$ such that $\alpha = g_1\gamma = \operatorname{Hom}_R(C, g_1)(\gamma)$. So $\operatorname{Hom}_R(C, g_1)$ is epic and the assertion (1) follows.

Dually, we get the assertion (2). □

Let $\mathscr{C}$ be a full subcategory of $\operatorname{Mod} R$ and $M \in \operatorname{Mod} R$. Recall that a sequence in $\operatorname{Mod} R$ is called $\operatorname{Hom}_R(\mathscr{C}, -)$-*exact exact* if it is exact and remains still exact after applying the functor $\mathrm{Hom}_R(\mathscr{C}, -)$; and an exact sequence:

$$\cdots \to C_i \to \cdots \to C_1 \to C_0 \to M \to 0$$

in $\mathrm{Mod}\, R$ with each $C_i \in \mathscr{C}$ is called a *$\mathscr{C}$-resolution* of M. Avramov and Martsinkovsky called in [AM] the above exact sequence a *proper $\mathscr{C}$-resolution* of M if it is a $\mathscr{C}$-resolution of M and is $\mathrm{Hom}_R(\mathscr{C}, -)$-exact. Dually, the notions of a $\mathrm{Hom}_R(-, \mathscr{C})$*-exact exact sequence*, a *$\mathscr{C}$-coresolution* and a *coproper $\mathscr{C}$-coresolution* of M are defined.

We now introduce the notion of strongly (co)proper (co)resolutions of modules as follows.

Definition 3.3. Let $\mathscr{C}$ be a full subcategory of $\mathrm{Mod}\, R$ and $M \in \mathrm{Mod}\, R$.

(1) A sequence:

$$\cdots \to X_i \to \cdots \to X_1 \to X_0 \to M \to 0$$

in $\mathrm{Mod}\, R$ is called *strongly* $\mathrm{Hom}_R(\mathscr{C}, -)$*-exact exact* if it is exact and $\mathrm{Ext}^1_R(\mathscr{C}, K_i) = 0$ for any $i \geq 1$, where $K_i = \mathrm{Im}(X_i \to X_{i-1})$. Dually, the notion of a *strongly* $\mathrm{Hom}_R(-, \mathscr{C})$*-exact exact sequence* is defined.

(2) An exact sequence:

$$\cdots \to C_i \to \cdots \to C_1 \to C_0 \to M \to 0$$

in $\mathrm{Mod}\, R$ is called a *strongly proper $\mathscr{C}$-resolution* of M if it is a $\mathscr{C}$-resolution of M and is strongly $\mathrm{Hom}_R(\mathscr{C}, -)$-exact. Dually, the notion of a *strongly coproper $\mathscr{C}$-coresolution* of M is defined.

It is easy to see that a strongly (co)proper $\mathscr{C}$-(co)resolution is a (co)proper $\mathscr{C}$-(co)resolution. But the converse does not hold true in general. For example, let $\mathscr{C}$ be a full subcategory of $\mathrm{Mod}\, R$ such that there exists a module $M \in \mathscr{C}$ with $\mathrm{Ext}^1_R(M, M) \neq 0$. Then the exact sequence:

$$0 \to M \xrightarrow{\binom{1_M}{0}} M \bigoplus M \xrightarrow{(0, 1_M)} M \to 0$$

is both a proper $\mathscr{C}$-resolution and a coproper $\mathscr{C}$-coresolution of M, but it neither a strongly proper $\mathscr{C}$-resolution nor a strongly coproper $\mathscr{C}$-coresolution of M.

The following result contains Theorem 1.1, which gives a method to construct a (strongly) proper resolution of the first term in a short exact sequence from that of the last two terms.

Theorem 3.4. *Let $\mathscr{C}$ be a full subcategory of* $\mathrm{Mod}\, R$ *and* $0 \to X \to X^0 \to X^1 \to 0$ *an exact sequence in* $\mathrm{Mod}\, R$. *Let*

$$\cdots \to C_i^0 \to \cdots \to C_1^0 \to C_0^0 \to X^0 \to 0 \tag{3.1}$$

be a $\mathscr{C}$-resolution of X^0, and let

$$\cdots \to C_i^1 \to \cdots \to C_1^1 \to C_0^1 \to X^1 \to 0 \tag{3.2}$$

be a $\mathrm{Hom}_R(\mathscr{C}, -)$*-exact exact sequence in* $\mathrm{Mod}\, R$. *Then*

(1) We get the following exact sequences:

$$\cdots \to C_{i+1}^1 \bigoplus C_i^0 \to \cdots \to C_2^1 \bigoplus C_1^0 \to C \to X \to 0 \tag{3.3}$$

and

$$0 \to C \to C_1^1 \bigoplus C_0^0 \to C_0^1 \to 0 \tag{3.4}$$

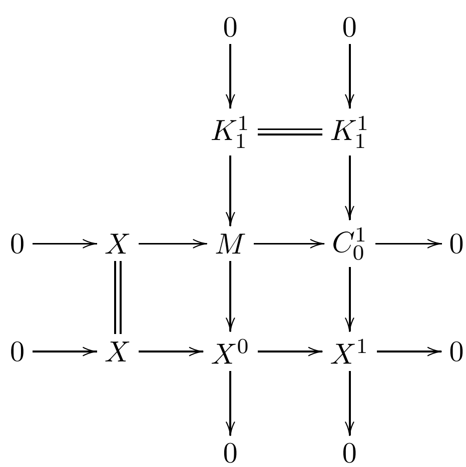
Assume that $\mathscr{C}$ is closed under finite direct sums and under kernels of epimorphisms. Then we have

(2) If the exact sequence (3.2) is a $\mathscr{C}$-resolution of X^1, then the exact sequence (3.3) is a $\mathscr{C}$-resolution of X.

(3) If both the exact sequences (3.1) and (3.2) are strongly proper $\mathscr{C}$-resolutions of X^0 and X^1 respectively, then the exact sequence (3.3) is a strongly proper $\mathscr{C}$-resolution of X.

(4) If both the exact sequences (3.1) and (3.2) are proper $\mathscr{C}$-resolutions of X^0 and X^1 respectively, then the exact sequence (3.3) is a proper $\mathscr{C}$-resolution of X.

Proof. (1) Put $K_i^0 = \mathrm{Im}(C_i^0 \to C_{i-1}^0)$ and $K_i^1 = \mathrm{Im}(C_i^1 \to C_{i-1}^1)$ for any $i \geq 1$. Consider the following pull-back diagram:



Because the third column in the above diagram is $\mathrm{Hom}_R(\mathscr{C}, -)$-exact exact, so is the middle column by Lemma 3.2(1). Thus by Lemma 3.1(1) we get the following commutative diagram with exact columns and rows and the middle row splitting:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \dashrightarrow & K_2^1 & \dashrightarrow & W_1 & \dashrightarrow & K_1^0 & \dashrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C_1^1 & \longrightarrow & C_1^1 \bigoplus C_0^0 & \longrightarrow & C_0^0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_1^1 & \longrightarrow & M & \longrightarrow & X^0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

where $W_1 = \mathrm{Ker}(C_1^1 \bigoplus C_0^0 \to M)$. It is easy to verify the upper row in the above diagram is $\mathrm{Hom}_R(\mathscr{C}, -)$-exact exact.

On the one hand, we have the following pull-back diagram:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & W_1 & = & W_1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & C & \longrightarrow & C_1^1 \bigoplus C_0^0 & \longrightarrow & C_0^1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & X & \longrightarrow & M & \longrightarrow & C_0^1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

On the other hand, again by Lemma 3.1(1) we get the following commutative diagram with exact columns and rows and the middle row splitting:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \dashrightarrow & K_3^1 & \dashrightarrow & W_2 & \dashrightarrow & K_2^0 & \dashrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C_2^1 & \longrightarrow & C_2^1 \bigoplus C_1^0 & \longrightarrow & C_1^0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_2^1 & \longrightarrow & W_1 & \longrightarrow & K_1^0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

where $W_2 = \mathrm{Ker}(C_2^1 \bigoplus C_1^0 \to W_1)$ and the upper row in the above diagram is $\mathrm{Hom}_R(\mathscr{C}, -)$-exact exact. Continuing this process, we get the desired exact sequences (3.3) and (3.4) with $W_i = \mathrm{Im}(C_{i+1}^1 \bigoplus C_i^0 \to C_i^1 \bigoplus C_{i-1}^0)$ for any $i \geq 2$ and $W_1 = \mathrm{Im}(C_2^1 \bigoplus C_1^0 \to C)$.

(2) It follows from the assumption and the assertion (1).

(3) If both the exact sequences (3.1) and (3.2) are strongly proper $\mathscr{C}$-resolutions of X^0 and X^1 respectively, then $\mathrm{Ext}_R^1(\mathscr{C}, K_i^j) = 0$ for any $i \geq 1$ and $j = 0, 1$. By the proof of (1), we have an exact sequence:

$$0 \to K_{i+1}^1 \to W_i \to K_i^0 \to 0$$

for any $i \geq 1$. So $\mathrm{Ext}_R^1(\mathscr{C}, W_i) = 0$ for any $i \geq 1$, and hence the exact sequence (3.3) is a strongly proper $\mathscr{C}$-resolution of X.

(4) Assume that both the exact sequences (3.1) and (3.2) are proper $\mathscr{C}$-resolutions of X^0 and X^1 respectively. Then by the proof of (1) and [EJ, Lemma 8.2.1], we have that both the middle column in the second diagram and the first column in the third diagram are $\mathrm{Hom}_R(\mathscr{C}, -)$-exact exact; and in particular we have a $\mathrm{Hom}_R(\mathscr{C}, -)$-exact exact sequence:

$$\cdots \to C_{i+1}^1 \bigoplus C_i^0 \to \cdots \to C_2^1 \bigoplus C_1^0 \to W_1 \to 0.$$

Thus we get the desired proper $\mathscr{C}$-resolution of X. $\square$

Based on Theorem 3.4, by using induction on n it is not difficult to get the following

Corollary 3.5. *Let $\mathscr{C}$ be a full subcategory of $\mathrm{Mod}\, R$ closed under finite direct sums and under kernels of epimorphisms, and let $0 \to X \to X^0 \to X^1 \to \cdots \to X^n \to 0$ be an exact sequence in $\mathrm{Mod}\, R$. If*

$$\cdots \to C_i^j \to \cdots \to C_1^j \to C_0^j \to X^j \to 0$$

is a (strongly) proper $\mathscr{C}$-resolution of X^j for any $0 \leq j \leq n$, then

$$\cdots \to \bigoplus_{i=0}^{n} C_{i+3}^i \to \bigoplus_{i=0}^{n} C_{i+2}^i \to \bigoplus_{i=0}^{n} C_{i+1}^i \to C \to X \to 0$$

is a (strongly) proper $\mathscr{C}$-resolution of X, and there exists an exact sequence:

$$0 \to C \to \bigoplus_{i=0}^{n} C_i^i \to \bigoplus_{i=1}^{n} C_{i-1}^i \to \bigoplus_{i=2}^{n} C_{i-2}^i \to \cdots \to C_0^{n-1} \bigoplus C_1^n \to C_0^n \to 0.$$

Remark. 3.6. By Wakamatsu's lemma (see [X, Lemma 2.1.1]), if the full subcategory $\mathscr{C}$ is closed under extensions, then a minimal proper $\mathscr{C}$-resolution of a module M is a strongly proper $\mathscr{C}$-resolution of M.

Note that any projective resolution is just a strongly proper $\mathscr{P}^0(\operatorname{Mod} R)$-resolution, where $\mathscr{P}^0(\operatorname{Mod} R) = \{$projective left R-modules$\}$. So putting $\mathscr{C} = \mathscr{P}^0(\operatorname{Mod} R)$ in Corollary 3.5, we get the following

Corollary 3.7. *Let $0 \to X \to X^0 \to X^1 \to \cdots \to X^n \to 0$ be an exact sequence in* $\operatorname{Mod} R$. *If*

$$\cdots \to P_i^j \to \cdots \to P_1^j \to P_0^j \to X^j \to 0$$

is a projective resolution of X^j for any $0 \leq j \leq n$, then

$$\cdots \to \bigoplus_{i=0}^{n} P_{i+3}^i \to \bigoplus_{i=0}^{n} P_{i+2}^i \to \bigoplus_{i=0}^{n} P_{i+1}^i \to C \to X \to 0$$

is a projective resolution of X, and there exists an exact and split sequence:

$$0 \to C \to \bigoplus_{i=0}^{n} P_i^i \to \bigoplus_{i=1}^{n} P_{i-1}^i \to \bigoplus_{i=2}^{n} P_{i-2}^i \to \cdots \to P_0^{n-1} \bigoplus P_1^n \to P_0^n \to 0.$$

The following 3.8–3.11 are dual to 3.4–3.7 respectively. The following result gives a method to construct a (strongly) coproper coresolution of the last term in a short exact sequence from that of the first two terms.

Theorem 3.8. *Let $\mathscr{C}$ be a full subcategory of* $\operatorname{Mod} R$ *and $0 \to Y_1 \to Y_0 \to Y \to 0$ an exact sequence in* $\operatorname{Mod} R$. *Let*

$$0 \to Y_0 \to C_0^0 \to C_0^1 \to \cdots \to C_0^i \to \cdots \tag{3.5}$$

be a $\mathscr{C}$-coresolution of Y_0, and let

$$0 \to Y_1 \to C_1^0 \to C_1^1 \to \cdots \to C_1^i \to \cdots \tag{3.6}$$

be a $\mathrm{Hom}_R(-,\mathscr{C})$-exact exact sequence in $\mathrm{Mod}\,R$.

(1) We get the following exact sequences:

$$0 \to Y \to C \to C_0^1 \bigoplus C_1^2 \to \cdots \to C_0^i \bigoplus C_1^{i+1} \to \cdots \tag{3.7}$$

and

$$0 \to C_1^0 \to C_0^0 \bigoplus C_1^1 \to C \to 0 \tag{3.8}$$

Assume that $\mathscr{C}$ is closed under finite direct sums and under cokernels of monomorphisms. Then we have

(2) If the exact sequence (3.6) is a $\mathscr{C}$-coresolution of Y_1, then the exact sequence (3.7) is a $\mathscr{C}$-coresolution of Y.

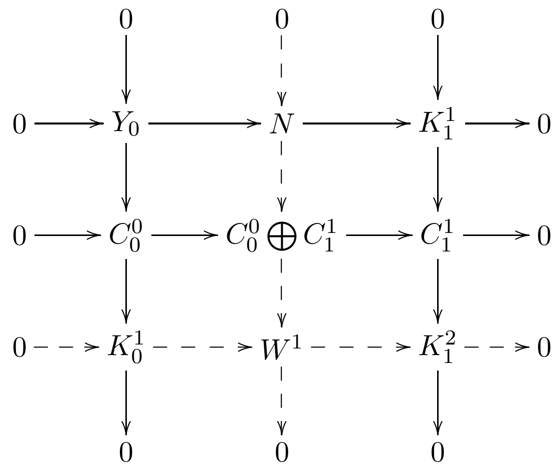
(3) If both the exact sequences (3.5) and (3.6) are strongly coproper $\mathscr{C}$-coresolutions of Y_0 and Y_1 respectively, then the exact sequence (3.7) is a strongly coproper $\mathscr{C}$-coresolution of Y.

(4) If both the exact sequences (3.5) and (3.6) are coproper $\mathscr{C}$-coresolutions of Y_0 and Y_1 respectively, then the exact sequence (3.7) is a coproper $\mathscr{C}$-coresolution of Y.

Proof. It is dual to the proof of Theorem 3.4, we give the proof here for the sake of completeness. Put $K_0^i = \mathrm{Im}(C_0^{i-1} \to C_0^i)$ and $K_1^i = \mathrm{Im}(C_1^{i-1} \to C_1^i)$ for any $i \geq 1$. Consider the following push-out diagram:

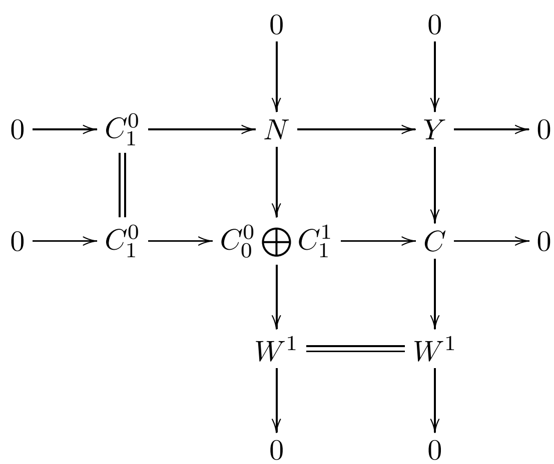
$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & Y_1 & \longrightarrow & Y_0 & \longrightarrow & Y & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & C_1^0 & \longrightarrow & N & \longrightarrow & Y & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & K_1^1 & = & K_1^1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Because the first column in the above diagram is $\mathrm{Hom}_R(-,\mathscr{C})$-exact exact, so is the middle column by Lemma 3.2(2). Then by Lemma 3.1(2) we get the following commutative diagram with exact columns and rows and the middle row splitting:



where $W^1 = \mathrm{Coker}(N \to C_0^0 \bigoplus C_1^1)$. It is easy to verify that the bottom row in the above diagram is $\mathrm{Hom}_R(-, \mathscr{C})$-exact exact.

On the one hand, we have the following push-out diagram:



On the other hand, again by Lemma 3.1(2) we get the following commutative diagram with exact columns and rows and the middle row splitting:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & K_0^1 & \to & W^1 & \to & K_1^2 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & C_0^1 & \to & C_0^1 \bigoplus C_1^2 & \to & C_1^2 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \dashrightarrow & K_0^2 & \dashrightarrow & W^2 & \dashrightarrow & K_1^3 & \dashrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where $W^2 = \mathrm{Coker}(W^1 \to C_0^1 \bigoplus C_1^2)$ and the bottom row in the above diagram is $\mathrm{Hom}_R(-, \mathscr{C})$-exact exact. Continuing this process, we get the desired exact sequences (3.7) and (3.8) with $W^i = \mathrm{Im}(C_0^{i-1} \bigoplus C_1^i \to C_0^i \bigoplus C_1^{i+1})$ for any $i \geq 2$ and $W^1 = \mathrm{Im}(C \to C_0^1 \bigoplus C_1^2)$.

(2) It follows from the assumption and the assertion (1).

(3) If both the exact sequences (3.5) and (3.6) are strongly coproper $\mathscr{C}$-coresolutions of Y_0 and Y_1 respectively, then $\mathrm{Ext}_R^1(K_j^i, \mathscr{C}) = 0$ for any $i \geq 1$ and $j = 0, 1$. By the proof of (1), we have an exact sequence:

$$0 \to K_0^i \to W^i \to K_1^{i+1} \to 0$$

for any $i \geq 1$. So $\mathrm{Ext}_R^1(W^i, \mathscr{C}) = 0$ for any $i \geq 1$, and hence the exact sequence (3.7) is a strongly coproper $\mathscr{C}$-coresolution of Y.

(4) Assume that both the exact sequences (3.5) and (3.6) are coproper $\mathscr{C}$-coresolutions of X_0 and X_1 respectively. Then by the proof of (1) and the dual version of [EJ, Lemma 8.2.1], we have that both the middle column in the second diagram and the first column in the third diagram are $\mathrm{Hom}_R(-, \mathscr{C})$-exact exact; and in particular we have a $\mathrm{Hom}_R(-, \mathscr{C})$-exact exact sequence:

$$0 \to W^1 \to C_0^1 \bigoplus C_1^2 \to \cdots \to C_0^i \bigoplus C_1^{i+1} \to \cdots.$$

Thus we get the desired coproper $\mathscr{C}$-coresolution of X. □

Based on Theorem 3.8, by using induction on n it is not difficult to get the following

Corollary 3.9. *Let $\mathscr{C}$ be a full subcategory of $\mathrm{Mod}\, R$ closed under finite direct sums and under cokernels of monomorphisms and let $0 \to Y_n \to \cdots \to Y_1 \to Y_0 \to Y \to 0$ be an exact* *sequence in* $\operatorname{Mod} R$. *If*

$$0 \to Y_j \to C_j^0 \to C_j^1 \to \cdots C_j^i \to \cdots$$

is a (strongly) coproper $\mathscr{C}$-coresolution of Y_j for any $0 \leq j \leq n$, then

$$0 \to Y \to C \to \bigoplus_{i=0}^{n} C_i^{i+1} \to \bigoplus_{i=0}^{n} C_i^{i+2} \to \bigoplus_{i=0}^{n} C_i^{i+3} \to \cdots$$

is a (strongly) coproper $\mathscr{C}$-coresolution of Y, and there exists an exact sequence:

$$0 \to C_n^0 \to C_{n-1}^0 \bigoplus C_n^1 \to \cdots \to \bigoplus_{i=2}^{n} C_i^{i-2} \to \bigoplus_{i=1}^{n} C_i^{i-1} \to \bigoplus_{i=0}^{n} C_i^{i} \to C \to 0.$$

Remark. 3.10. By Wakamatsu's lemma (see [X, Lemma 2.1.1]), if the full subcategory $\mathscr{C}$ is closed under extensions, then a minimal coproper $\mathscr{C}$-coresolution of a module M is a strongly coproper $\mathscr{C}$-coresolution of M.

Note that any injective coresolution is just a strongly coproper $\mathscr{I}^0(\operatorname{Mod} R)$-coresolution. So putting $\mathscr{C} = \mathscr{I}^0(\operatorname{Mod} R)$ in Corollary 3.9, we get the following

Corollary 3.11. *([M, Corollary 1.3]) Let $0 \to Y_n \to \cdots \to Y_1 \to Y_0 \to Y \to 0$ be an exact sequence in* $\operatorname{Mod} R$. *If*

$$0 \to Y_j \to I_j^0 \to I_j^1 \to \cdots I_j^i \to \cdots$$

is an injective coresolution of Y_j for any $0 \leq j \leq n$, then

$$0 \to Y \to C \to \bigoplus_{i=0}^{n} I_i^{i+1} \to \bigoplus_{i=0}^{n} I_i^{i+2} \to \bigoplus_{i=0}^{n} I_i^{i+3} \to \cdots$$

is an injective coresolution of Y, and there exists an exact and split sequence:

$$0 \to I_n^0 \to I_{n-1}^0 \bigoplus I_n^1 \to \cdots \to \bigoplus_{i=2}^{n} I_i^{i-2} \to \bigoplus_{i=1}^{n} I_i^{i-1} \to \bigoplus_{i=0}^{n} I_i^{i} \to C \to 0.$$

The following result contains Theorem 1.2, which gives a method to construct a (strongly) proper resolution of the last term in a short exact sequence from that of the first two terms.

Theorem 3.12. *Let $\mathscr{C}$ be a full subcategory of* $\operatorname{Mod} R$ *and*

$$0 \to X_1 \to X_0 \to X \to 0 \tag{3.9}$$

an exact sequence in Mod R. *Let*

$$C_0^n \to \cdots \to C_0^1 \to C_0^0 \to X_0 \to 0 \tag{3.10}$$

be a $\mathrm{Hom}_R(\mathscr{C}, -)$*-exact exact sequence, and*

$$C_1^{n-1} \to \cdots \to C_1^1 \to C_1^0 \to X_1 \to 0 \tag{3.11}$$

a $\mathscr{C}$*-resolution of* X_1 *in* Mod R.

(1) We get the following exact sequences:

$$C_0^n \bigoplus C_1^{n-1} \to \cdots \to C_0^2 \bigoplus C_1^1 \to C_0^1 \bigoplus C_1^0 \to C_0^0 \to X \to 0 \tag{3.12}$$

Assume that $\mathscr{C}$ *is closed under finite direct sums. Then we have*

(2) If the exact sequence (3.10) is a $\mathscr{C}$*-resolution of* X_0*, then the exact sequence (3.12) is a* $\mathscr{C}$*-resolution of* X.

(3) If the exact sequence (3.9) is strongly $\mathrm{Hom}_R(\mathscr{C}, -)$*-exact and both the exact sequences (3.10) and (3.11) are strongly proper* $\mathscr{C}$*-resolutions of* X_0 *and* X_1 *respectively, then the exact sequence (3.12) is a strongly proper* $\mathscr{C}$*-resolution of* X.

(4) If the exact sequence (3.9) is $\mathrm{Hom}_R(\mathscr{C}, -)$*-exact and both the exact sequences (3.10) and (3.11) are proper* $\mathscr{C}$*-resolutions of* X_0 *and* X_1 *respectively, then the exact sequence (3.12) is a proper* $\mathscr{C}$*-resolution of* X.

Proof. (1) Put $K_j^i = \mathrm{Im}(C_j^i \to C_j^{i-1})$ for any $1 \leq i \leq n-j$ and $j = 0, 1$. Consider the following pull-back diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & K_0^1 & = & K_0^1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & W_1 & \to & C_0^0 & \to & X & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & X_1 & \to & X_0 & \to & X & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Note that the middle column in the above diagram is $\mathrm{Hom}_R(\mathscr{C}, -)$-exact exact. So by Lemma 3.2(1), the first column is also $\mathrm{Hom}_R(\mathscr{C}, -)$-exact exact. Then by Lemma 3.1(1) we get the following commutative diagram with exact columns and rows and the middle row splitting:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \dashrightarrow & K_0^2 & \dashrightarrow & W_2 & \dashrightarrow & K_1^1 & \dashrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C_0^1 & \longrightarrow & C_0^1 \bigoplus C_1^0 & \longrightarrow & C_1^0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_0^1 & \longrightarrow & W_1 & \longrightarrow & X_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

where $W_2 = \operatorname{Ker}(C_0^1 \bigoplus C_1^0 \to W_1)$. It is easy to check that the upper row in the above diagram is $\operatorname{Hom}_R(\mathscr{C}, -)$-exact exact. Then by using Lemma 3.1(1) iteratively we get the exact sequence (3.12) with $W_i = \operatorname{Im}(C_0^i \bigoplus C_1^{i-1} \to C_0^{i-1} \bigoplus C_1^{i-2})$ for any $2 \leq i \leq n$ and $W_1 = \operatorname{Im}(C_0^1 \bigoplus C_1^0 \to C_0^0)$.

(2) It follows from the assumption and the assertion (1).

(3) If the exact sequence (3.9) is strongly $\operatorname{Hom}_R(\mathscr{C}, -)$-exact and both the exact sequences (3.10) and (3.11) are strongly proper $\mathscr{C}$-resolutions of X_0 and X_1 respectively, then $\operatorname{Ext}_R^1(\mathscr{C}, K_j^i) = 0$ for any $1 \leq i \leq n-j$ and $j = 0, 1$. By the proof of (1), we have an exact sequence:

$$0 \to K_0^i \to W_i \to K_1^{i-1} \to 0$$

for any $1 \leq i \leq n$ (where $K_1^0 = X_1$). So $\operatorname{Ext}_R^1(\mathscr{C}, W_i) = 0$ for any $1 \leq i \leq n$, and hence the exact sequence (3.12) is a strongly proper $\mathscr{C}$-resolution of X.

(4) If the exact sequence (3.9) is $\operatorname{Hom}_R(\mathscr{C}, -)$-exact and both the exact sequences (3.10) and (3.11) are proper $\mathscr{C}$-resolutions of X_0 and X_1 respectively, then the middle row in the first diagram is $\operatorname{Hom}_R(\mathscr{C}, -)$-exact exact by Lemma 3.2(1). Thus by [EJ, Lemma 8.2.1], the exact sequence (3.12) is a proper $\mathscr{C}$-resolution of X. □

Based on Theorem 3.12, by using induction on n it is not difficult to get the following

Corollary 3.13. *Let $\mathscr{C}$ be a full subcategory of $\operatorname{Mod} R$ closed under finite direct sums, and let*

$$X_n \to \cdots \to X_1 \to X_0 \to X \to 0 \tag{3.13}$$

and

$$C_j^{n-j} \to \cdots \to C_j^1 \to C_j^0 \to X_j \to 0 \tag{3.14(j)}$$

be exact sequences in $\operatorname{Mod} R$ *for any* $0 \leq j \leq n$.

(1) If the exact sequence (3.13) is strongly $\operatorname{Hom}_R(\mathscr{C}, -)$*-exact, and if* $(3.14(j))$ *is a strongly proper* $\mathscr{C}$*-resolution of* X_j *for any* $0 \leq j \leq n$, *then*

$$\bigoplus_{i=0}^{n} C_i^{n-i} \to \bigoplus_{i=0}^{n-1} C_i^{(n-1)-i} \to \cdots \to C_0^1 \bigoplus C_1^0 \to C_0^0 \to X \to 0 \tag{3.15}$$

is a strongly proper $\mathscr{C}$*-resolution of* X.

(2) If the exact sequence (3.13) is $\operatorname{Hom}_R(\mathscr{C}, -)$*-exact, and if* $(3.14(j))$ *is a proper* $\mathscr{C}$*-resolution of* X_j *for any* $0 \leq j \leq n$, *then (3.15) is a proper* $\mathscr{C}$*-resolution of* X.

The following corollary is an immediate consequence of Corollary 3.13.

Corollary 3.14. *Let* $X_n \to \cdots \to X_1 \to X_0 \to X \to 0$ *be an exact sequence in* $\operatorname{Mod} R$. *If*

$$P_j^{n-j} \to \cdots \to P_j^1 \to P_j^0 \to X_j \to 0$$

is a projective resolution of X_j *for any* $0 \leq j \leq n$, *then*

$$\bigoplus_{i=0}^{n} P_i^{n-i} \to \bigoplus_{i=0}^{n-1} P_i^{(n-1)-i} \to \cdots \to P_0^1 \bigoplus P_1^0 \to P_0^0 \to X \to 0$$

is a projective resolution of X.

The following 3.15–3.17 are dual to 3.12–3.14 respectively. The following result gives a method to construct a strongly coproper coresolution of the first term in a short exact sequence from that of the last two terms.

Theorem 3.15. *Let* $\mathscr{C}$ *be a full subcategory of* $\operatorname{Mod} R$ *and*

$$0 \to Y \to Y^0 \to Y^1 \to 0 \tag{3.16}$$

an exact sequence in $\operatorname{Mod} R$. *Let*

$$0 \to Y^0 \to C_0^0 \to C_1^0 \to \cdots \to C_n^0 \tag{3.17}$$

be a $\operatorname{Hom}_R(-, \mathscr{C})$*-exact exact sequence, and*

$$0 \to Y^1 \to C_0^1 \to C_1^1 \to \cdots \to C_{n-1}^1 \tag{3.18}$$

a $\mathscr{C}$*-coresolution of* Y^1 *in* $\operatorname{Mod} R$.

(1) We get the following exact sequences:

$$0 \to Y \to C_0^0 \to C_0^1 \bigoplus C_1^0 \to C_1^1 \bigoplus C_2^0 \to \cdots \to C_{n-1}^1 \bigoplus C_n^0 \tag{3.19}$$

Assume that $\mathscr{C}$ is closed under finite direct sums. Then we have

(2) If the exact sequence (3.17) is a $\mathscr{C}$-coresolution of Y^0, then the exact sequence (3.19) is a $\mathscr{C}$-coresolution of X.

(3) If the exact sequence (3.16) is strongly $\mathrm{Hom}_R(-,\mathscr{C})$*-exact and both the exact sequences (3.17) and (3.18) are strongly coproper $\mathscr{C}$-coresolutions of Y^0 and Y^1 respectively, then the exact sequence (3.19) is a strongly coproper $\mathscr{C}$-coresolution of Y.*

(4) If the exact sequence (3.16) is $\mathrm{Hom}_R(-,\mathscr{C})$*-exact and both the exact sequences (3.17) and (3.18) are coproper $\mathscr{C}$-coresolutions of Y^0 and Y^1 respectively, then the exact sequence (3.19) is a coproper $\mathscr{C}$-coresolution of Y.*

Proof. It is dual to the proof of Theorem 3.12, we give the proof here for the sake of completeness. Put $K_i^j = \mathrm{Im}(C_{i-1}^j \to C_i^j)$ for any $1 \leq i \leq n-j$ and $j = 0, 1$. Consider the following push-out diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Y & \longrightarrow & Y^0 & \longrightarrow & Y^1 & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Y & \longrightarrow & C_0^0 & \longrightarrow & W^1 & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & K_1^0 & = & K_1^0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Note that the middle column in the above diagram is $\mathrm{Hom}_R(-,\mathscr{C})$-exact exact by assumption. So the third column is also $\mathrm{Hom}_R(-,\mathscr{C})$-exact exact by Lemma 3.2(2). Then by Lemma 3.1(2) we get the following commutative diagram with exact columns and rows and the middle row splitting:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Y^1 & \longrightarrow & W^1 & \longrightarrow & K_1^0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C_0^1 & \longrightarrow & C_0^1 \bigoplus C_1^0 & \longrightarrow & C_1^0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \dashrightarrow & K_1^1 & \dashrightarrow & W^2 & \dashrightarrow & K_2^0 & \dashrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where $W^2 = \mathrm{Coker}(W^1 \to C_0^1 \bigoplus C_1^0)$. It is easy to verify that the bottom row in the above diagram is $\mathrm{Hom}_R(-, \mathscr{C})$-exact exact. Then by using Lemma 3.1(2) iteratively we get the exact sequence (3.19) with $W^i = \mathrm{Im}(C_{i-2}^1 \bigoplus C_{i-1}^0 \to C_{i-1}^1 \bigoplus C_i^0)$ for any $2 \leq i \leq n$ and $W^1 = \mathrm{Im}(C_0^0 \to C_0^1 \bigoplus C_1^0)$.

(2) It follows from the assumption and the assertion (1).

(3) If the exact sequence (3.16) is strongly $\mathrm{Hom}_R(-, \mathscr{C})$-exact and both the exact sequences (3.17) and (3.18) are strongly coproper $\mathscr{C}$-coresolutions of Y^0 and Y^1 respectively, then $\mathrm{Ext}_R^1(K_i^j, \mathscr{C}) = 0$ for any $1 \leq i \leq n - j$ and $j = 0, 1$. By the proof of (1), we have an exact sequence:

$$0 \to K_{i-1}^1 \to W^i \to K_i^0 \to 0$$

for any $1 \leq i \leq n$ (where $K_0^1 = Y^1$). So $\mathrm{Ext}_R^1(W^i, \mathscr{C}) = 0$ for any $1 \leq i \leq n$, and hence the exact sequence (3.19) is a strongly coproper $\mathscr{C}$-coresolution of Y.

(4) If the exact sequence (3.16) is $\mathrm{Hom}_R(-, \mathscr{C})$-exact and both the exact sequences (3.17) and (3.18) are coproper $\mathscr{C}$-coresolutions of Y^0 and Y^1 respectively, then the middle row in the first diagram is $\mathrm{Hom}_R(-, \mathscr{C})$-exact exact by Lemma 3.2(2). Thus by the dual version of [EJ, Lemma 8.2.1], the exact sequence (3.19) is a coproper $\mathscr{C}$-coresolution of Y. □

Based on Theorem 3.15, by using induction on n it is not difficult to get the following

Corollary 3.16. *Let $\mathscr{C}$ be a full subcategory of* $\mathrm{Mod}\, R$ *closed under finite direct sums, and let*

$$0 \to Y \to Y^0 \to Y^1 \to \cdots \to Y^n \qquad (3.20)$$

and

$$0 \to Y^j \to C_0^j \to C_1^j \to \cdots C_{n-j}^j \qquad (3.21(j))$$

be exact sequences in $\operatorname{Mod} R$ *for any* $0 \leq j \leq n$.

(1) If the exact sequence (3.20) is strongly $\operatorname{Hom}_R(-,\mathscr{C})$*-exact, and if* $(3.21(j))$ *is a strongly coproper* $\mathscr{C}$*-coresolution of* Y^j *for any* $0 \leq j \leq n$*, then*

$$0 \to Y \to C_0^0 \to C_1^0 \bigoplus C_0^1 \cdots \to \bigoplus_{i=0}^{n-1} C^i_{(n-1)-i} \to \bigoplus_{i=0}^{n} C^i_{n-i} \tag{3.22}$$

is a strongly coproper $\mathscr{C}$*-coresolution of* Y.

(2) If the exact sequence (3.20) is $\operatorname{Hom}_R(-,\mathscr{C})$*-exact, and if* $(3.21(j))$ *is a coproper* $\mathscr{C}$*-coresolution of* Y^j *for any* $0 \leq j \leq n$*, then (3.22) is a coproper* $\mathscr{C}$*-coresolution of* Y.

The following corollary is an immediate consequence of Corollary 3.16.

Corollary 3.17. *Let* $0 \to Y \to Y^0 \to Y^1 \to \cdots \to Y^n$ *be an exact sequence in* $\operatorname{Mod} R$*. If*

$$0 \to Y^j \to I_0^j \to I_1^j \to \cdots I_{n-j}^j$$

is an injective coresolution of Y^j *for any* $0 \leq j \leq n$*, then*

$$0 \to Y \to I_0^0 \to I_1^0 \bigoplus I_0^1 \cdots \to \bigoplus_{i=0}^{n-1} I^i_{(n-1)-i} \to \bigoplus_{i=0}^{n} I^i_{n-i}$$

is an injective coresolution of Y.

4. Flat dimension of E^n of direct limits

In this section, R is a left Noetherian ring. The aim of this section is to prove the following

Theorem 4.1. *Let* $n, k \geq 0$ *and let* $\{M_i\}_{i\in I}$ *be a family of left* R*-modules, where* I *is a directed index set. If* $M = \varinjlim_{i\in I} M_i$ *and* $\operatorname{fd}_R E^n(M_i) \leq k$ *for any* $i \in I$*, then* $\operatorname{fd}_R E^n(M) \leq k$.

By [R, Theorem 5.40], every flat left R-module is a direct limit (over a directed index set) of finitely generated free left R-modules. So by Theorem 4.1, we have the following

Corollary 4.2. $\operatorname{fd}_R E^n({}_RR) = \sup\{\operatorname{fd}_R E^n(F) \mid F \in \operatorname{Mod} R$ *is flat*$\}$ *for any* $n \geq 0$.

Before giving the proof of Theorem 4.1, we need some preliminaries.

Definition 4.3. ([J2]) Let β be an ordinal number. A set S is called a *continuous union* of a family of subsets indexed by ordinals α with $\alpha < \beta$ if for each such α we have a subset $S_\alpha \subset S$ such that if $\alpha \leq \alpha'$ then $S_\alpha \subset S_{\alpha'}$, and such that if $\gamma < \beta$ is a limit ordinal then $S_\gamma = \bigcup_{\alpha<\gamma} S_\alpha$.

A main tool in our proof will be the next result.

Lemma 4.4. ([J2, Lemma 1.4]) *If I is an infinite directed index set, then for some ordinal β, I can be written as a continuous union $I = \bigcup_{\alpha<\beta} I_\alpha$, where each I_α is a directed index set with the order induced by that of I and where $|I_\alpha| < |I|$ for each $\alpha < \beta$.*

This result will be useful since it will allow us to rewrite a direct limit as a well-ordered direct limit. So if $M = \varinjlim_{i \in I} M_i$ with I infinite, then write $I = \bigcup_{\alpha<\beta} I_\alpha$ as above, and put $M_\alpha = \varinjlim_{i \in I_\alpha} M_i$. Hence if $\alpha \leq \alpha' < \beta$, since $I_\alpha \subset I_{\alpha'}$ we have an obvious map $M_\alpha \to M_{\alpha'}$. These maps then give us a directed system $\{M_\alpha\}_{\alpha<\beta}$. Clearly then $\varinjlim_{\alpha<\beta} M_\alpha = \varinjlim_{i \in I} M_i$.

Proposition 4.5. *Let β be an ordinal number and $\{M_\alpha\}$ a directed system of modules (indexed by $\alpha < \beta$). If*

$$\zeta_\alpha =: 0 \to M_\alpha \to E^0(M_\alpha) \to E^1(M_\alpha) \to \cdots$$

is a minimal injective coresolution of M_α for each α, then these exact sequences ζ_α are the members of a directed system indexed by $\alpha < \beta$ in such a way that if $\alpha \leq \alpha' < \beta$ the map from the sequence indexed by α into that indexed by α' agrees with the original map $M_\alpha \to M_{\alpha'}$.

Proof. Given an $\alpha + 1 < \beta$ we can form a commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M_\alpha & \longrightarrow & E^0(M_\alpha) & \longrightarrow & E^1(M_\alpha) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M_{\alpha+1} & \longrightarrow & E^0(M_{\alpha+1}) & \longrightarrow & E^1(M_{\alpha+1}) & \longrightarrow & \cdots
\end{array}$$

Using this observation we can successively get maps $\zeta_0 \to \zeta_1$, $\zeta_1 \to \zeta_2$, $\cdots$. So composing we get maps $\zeta_m \to \zeta_n$ whenever $m \leq n$. Since R is left Noetherian, any direct limit of injective left R-modules is injective by [B1, Theorem 1.1]. So $\varinjlim \zeta_n$ is in fact an injective coresolution of $\varinjlim M_n$. We have a map $\varinjlim M_n \to M_\omega$ given by the maps $M_n \to M_\omega$ (where ω is the least infinite ordinal). Then the above shows that this in turn gives a map $\varinjlim \zeta_n \to \zeta_\omega$. So these maps give maps $\zeta_n \to \zeta_\omega$ for any $n \geq 0$. Continuing this procedure we get the desired system. □

Note that this result gives that if ζ is an injective coresolution of M, then $\zeta \cong \varinjlim_{\alpha<\beta} \zeta_\alpha$. In particular, this gives that $E^n(M) \cong \varinjlim_{\alpha<\beta} E^n(M_\alpha)$. This then gives that if $\operatorname{fd}_R E^n(M_\alpha) \leq k$

for each α then $\operatorname{fd}_R E^n(M) \leq k$. In other words, Theorem 4.1 holds true when our direct system is over the well-ordered index set of $\alpha < \beta$ for some ordinal β.

Proof of Theorem 4.1. We proceed by transfinite induction on $|I|$. So to begin the induction we suppose that $|I| = \aleph_0$ (the first infinite cardinal number). Then I is countable, so we suppose $I = \{i_n | n \in \mathbb{N}\}$ with $\mathbb{N}$ the set of non-negative integers. We construct a sequence $j_0, j_1, j_2, \cdots$ of elements in I by letting $j_0 = i_0$. Then we choose j_1 so that $j_1 \geq j_0, i_1$. So in general we choose j_n so that $j_n \geq j_{n-1}, i_n$. Then let $J = \{j_n | n \in \mathbb{N}\}$. We have that J is well-ordered and is clearly a confinal subset of I. Hence $M = \varinjlim_{i \in I} M_i = \varinjlim_{j \in J} M_j$. Since J is well-ordered, $E^n(M) = \varinjlim_{j \in J} E^n(M_j)$. So the assumption that $\operatorname{fd}_R E^n(M_j) \leq k$ for each j gives that $\operatorname{fd}_R E^n(M) \leq k$.

Now we make the induction hypothesis and assume $|I| > \aleph_0$. We appeal to Lemma 4.4 and write $I = \bigcup_{\alpha < \beta} I_\alpha$ as in that lemma. Then $M = \varinjlim_{\alpha < \beta} M_\alpha$. We have M_α is the limit over I_α. But $|I_\alpha| < |I|$, so the assertion holds true for direct limits over I_α by the induction hypothesis. This means that we have $\operatorname{fd}_R M_\alpha \leq k$ for each α. Because the system $\{M_\alpha\}_{\alpha < \beta}$ is over a well-ordered index set of indices, we get that $\operatorname{fd}_R E^n(M_\alpha) \leq k$ for each α gives the assertion that $\operatorname{fd}_R E^n(M) \leq k$. □

Remark 4.6. The same techniques show that if for a given $n \geq 0$ we let

$$0 \to M_\alpha \to E^0(M_\alpha) \to E^1(M_\alpha) \to \cdots \to E^{n-1}(M_\alpha) \to C^n(M_\alpha) \to 0$$

be a partial minimal injective coresolution of M_α for each α. If $\operatorname{fd}_R C^n(M_\alpha) \leq k$ for each α, then we get that $\operatorname{fd}_R C^n(M) \leq k$, where

$$0 \to M \to E^0(M) \to E^1(M) \to \cdots \to E^{n-1}(M) \to C^n(M) \to 0$$

is a partial minimal injective coresolution of M.

5. Modules satisfying the Auslander-type conditions

As a generalization of rings satisfying the Auslander condition, Huang and Iyama introduced in [HI] the notion of rings satisfying the Auslander-type conditions. Now we introduce the notion of modules satisfying the Auslander-type conditions as follows.

Definition 5.1. Let $M \in \operatorname{Mod} R$ and let m and n be non-negative integers. M is said to be $G_n(m)$ if $\operatorname{fd}_R E^i(M) \leq m + i$ for any $0 \leq i \leq n-1$, and M is said to be $G_\infty(m)$ if it is $G_n(m)$ for all n.

Remark 5.2. Let R be a left and right Noetherian ring. Then we have

(1) ${}_RR$ is $G_n(m)$ if and only if R is $G_n(m)^{op}$ in the sense of Huang and Iyama in [HI].

(2) Recall from [FGR] that R is called *Auslander's n-Gorenstein* if $\operatorname{fd}_R E^i({}_RR) \leq i$ for any $0 \leq i \leq n-1$, and R is said to satisfy the *Auslander condition* if it is Auslander's n-Gorenstein for all n. So R is Auslander's n-Gorenstein if and only if ${}_RR$ is $G_n(0)$. Note that the notion of Auslander's n-Gorenstein rings (and hence that of the Auslander condition) is left-right symmetric ([FGR, Theorem 3.7]). So R satisfies the Auslander condition if and only if both ${}_RR$ and R_R are $G_\infty(0)$. However, in general, the notion of R being $G_n(m)$ is not left-right symmetric when $m \geq 1$ ([AR4] or [HI]).

The aim of this section is to study the homological behavior of modules (especially, ${}_RR$) satisfying certain Auslander-type conditions. We begin with the following

Lemma 5.3. *(1)* $\operatorname{fd}_R E^0(M) \leq \operatorname{fd}_R M$ *for any* $M \in \operatorname{Mod} R$ *if and only if* $\operatorname{fd}_R E^i(M) \leq \operatorname{fd}_R M + i$ *for any* $M \in \operatorname{Mod} R$ *and* $i \geq 0$.

(2) $\operatorname{id}_{R^{op}} F_0(N) \leq \operatorname{id}_{R^{op}} N$ *for any* $N \in \operatorname{Mod} R^{op}$ *if and only if* $\operatorname{id}_{R^{op}} F_i(N) \leq \operatorname{id}_{R^{op}} N + i$ *for any* $N \in \operatorname{Mod} R^{op}$ *and* $i \geq 0$.

Proof. (1) The necessity is trivial. We next prove the sufficiency. Without loss of generality, assume that $M \in \operatorname{Mod} R$ with $\operatorname{fd}_R M = s < \infty$. In a minimal injective coresolution

$$0 \to M \to E^0(M) \to E^1(M) \to \cdots \to E^i(M) \to \cdots$$

of M in $\operatorname{Mod} R$, putting $K_{i+1} = \operatorname{Im}(E^i(M) \to E^{i+1}(M))$ for any $i \geq 0$. By assumption, $\operatorname{fd}_R E^0(M) \leq \operatorname{fd}_R M = s$. So $\operatorname{fd}_R K_1 \leq s+1$ and hence $\operatorname{fd}_R E^1(M) = \operatorname{fd}_R E^0(K_1) \leq \operatorname{fd}_R K_1 \leq s+1$ again by assumption. Then $\operatorname{fd}_R K_2 \leq s+2$. Continuing this process, we get that $\operatorname{fd}_R E^i(M) \leq s+i$ for any $i \geq 0$.

(2) It is dual to (1). □

The following lemma plays an important role in the proof of the main result of this section.

Lemma 5.4. *For a left Noetherian ring R,* $\operatorname{id}_{R^{op}} F_i(E) \leq \operatorname{fd}_R E^i({}_RR)$ *for any injective right R-module E and $i \geq 0$.*

Proof. By Lemma 2.3, we have that

$$\cdots \to [E^i({}_RR)]^+ \xrightarrow{\pi_i} \cdots \xrightarrow{\pi_2} [E^1({}_RR)]^+ \xrightarrow{\pi_1} [E^0({}_RR)]^+ \xrightarrow{\pi_0} ({}_RR)^+ \to 0$$

is a proper flat resolution of $({}_RR)^+$ in $\operatorname{Mod} R^{op}$.

Let E be an injective right R-module. Because $({}_RR)^+$ is an injective cogenerator for $\operatorname{Mod} R^{op}$, E is isomorphic to a direct summand of $[({}_RR)^+]^I$ for some index set I. Because the subcategory of $\operatorname{Mod} R^{op}$ consisting of flat modules is closed under direct products by [C, Theorem 2.1], ${\pi_i}^I : ([E^i({}_RR)]^+)^I \to (\operatorname{Im}\pi_i)^I$ is a flat precover of $(\operatorname{Im}\pi_i)^I$ for any $i \geq 0$ by Lemma 2.2. Note that $F_i(E)$ is isomorphic to a direct summand of $([E^i({}_RR)]^+)^I$ for any $i \geq 0$. So by Lemma 2.4(1), we have that $\operatorname{id}_{R^{op}} F_i(E) \leq \operatorname{id}_{R^{op}}([E^i({}_RR)]^+)^I = \operatorname{id}_{R^{op}}[E^i({}_RR)]^+ = \operatorname{fd}_R E^i({}_RR)$ for any $i \geq 0$. □

As a consequence of Lemma 5.4 and Corollary 3.5, we get the following

Proposition 5.5. *Let R be a left Noetherian ring. If ${}_RR$ is $G_\infty(m)$ for a non-negative integer m, then $\operatorname{id}_{R^{op}} F_i(N) \leq \operatorname{id}_{R^{op}} N + m + i$ for any $N \in \operatorname{Mod} R^{op}$ and $i \geq 0$.*

Proof. Without loss of generality, assume that $\operatorname{id}_{R^{op}} N = s < \infty$. We will proceed by induction on s. Assume that ${}_RR$ is $G_\infty(m)$, that is, $\operatorname{fd}_R E^i({}_RR) \leq m + i$ for any $i \geq 0$. If $s = 0$, then the assertion follows from Lemma 5.4.

Now suppose $s \geq 1$. Then we have an exact sequence:

$$0 \to N \to E^0(N) \to N_1 \to 0$$

in $\operatorname{Mod} R^{op}$ with $\operatorname{id}_{R^{op}} N_1 = s - 1$. By the induction hypothesis, we have that $\operatorname{id}_{R^{op}} F_i(N_1) \leq (s-1) + m + i$ and $\operatorname{id}_{R^{op}} F_i(E^0(N)) \leq m + i$ for any $i \geq 0$. By Corollary 3.5 and Remark 3.6, we have that

$$\cdots \to F_{i+1}(N_1) \bigoplus F_i(E^0(N)) \to \cdots \to F_2(N_1) \bigoplus F_1(E^0(N)) \to F_0 \to N \to 0$$

is a strongly proper flat resolution of N, and

$$0 \to F_0 \to F_1(N_1) \bigoplus F_0(E^0(N)) \to F_0(N_1) \to 0$$

is exact. So $\operatorname{id}_{R^{op}} F_0 \leq s + m$, and $\operatorname{id}_{R^{op}} F_{i+1}(N_1) \bigoplus F_i(E^0(N)) \leq s + m + i$ for any $i \geq 1$. Notice that $F_0(N)$ is isomorphic to a direct summand of F_0 and $F_i(N)$ is isomorphic to a direct summand of $F_{i+1}(N_1) \bigoplus F_i(E^0(N))$ for any $i \geq 1$, thus we have $\operatorname{id}_{R^{op}} F_i(N) \leq s+m+i$ for any $i \geq 0$. □

Similarly, we get the following

Proposition 5.6. *For a non-negative integer m, $\operatorname{id}_{R^{op}} F_i(E) \leq m + i$ for any injective right R-module E and $i \geq 0$ if and only if $\operatorname{id}_{R^{op}} F_i(N) \leq \operatorname{id}_{R^{op}} N+m+i$ for any $N \in \operatorname{Mod} R^{op}$ and $i \geq 0$.*

As a consequence of Corollary 3.11, we get the following result. This result can be regarded as a dual version of Proposition 5.6.

Proposition 5.7. *For a non-negative integer m, any flat left R-module is $G_\infty(m)$ if and only if* $\mathrm{fd}_R E^i(M) \leq \mathrm{fd}_R M + m + i$ *for any left R-module M and $i \geq 0$.*

Proof. The sufficiency is trivial. We next prove the necessity. Without loss of generality, assume that $\mathrm{fd}_R M = s < \infty$. We will proceed by induction on s.

If $s = 0$, then the assertion follows from the assumption. Now suppose $s \geq 1$. Then we have an exact sequence:

$$0 \to M_1 \to F_0(M) \to M \to 0$$

in $\mathrm{Mod}\, R$ with $\mathrm{fd}_R M_1 = s - 1$. So by the induction hypothesis, we have that $\mathrm{fd}_R E^i(M_1) \leq (s-1) + m + i$ and $\mathrm{fd}_R E^i(F_0(M)) \leq m + i$ for any $i \geq 0$.

By Corollary 3.11, we have that

$$0 \to M \to I^0 \to E^1(F_0(M)) \bigoplus E^2(M_1) \to \cdots \to E^i(F_0(M)) \bigoplus E^{i+1}(M_1) \to \cdots$$

is an injective coresolution of M, and

$$0 \to E^0(M_1) \to E^0(F_0(M)) \bigoplus E^1(M_1) \to I^0 \to 0$$

is exact and split. So $\mathrm{fd}_R I^0 \leq s + m$ and $\mathrm{fd}_R E^i(F_0(M)) \bigoplus E^{i+1}(M_1) \leq s + m + i$ for any $i \geq 1$. Notice that $E^0(M)$ is isomorphic to a direct summand of I^0 and $E^i(M)$ is isomorphic to a direct summand of $E^i(F_0(M)) \bigoplus E^{i+1}(M_1)$ for any $i \geq 1$, thus we have $\mathrm{fd}_R E^i(M) \leq s + m + i$ for any $i \geq 0$. □

We also need the following

Lemma 5.8. *Let $M \in \mathrm{Mod}\, R$ and n be a non-negative integer.*

(1) If R is a right Noetherian ring and $\mathrm{id}_{R^{op}} F_0(M^+) \leq \mathrm{id}_{R^{op}} M^+ + n$, *then* $\mathrm{fd}_R E^0(M) \leq \mathrm{fd}_R M + n$.

(2) If R is a left Noetherian ring and $\mathrm{id}_{R^{op}} M^+ \leq \mathrm{id}_{R^{op}} F_0(M^+) + n$, *then* $\mathrm{fd}_R M \leq \mathrm{fd}_R E^0(M) + n$.

Proof. (1) Without loss of generality, assume that $\mathrm{fd}_R M = s < \infty$. Then $\mathrm{id}_{R^{op}} M^+ = s$ by Lemma 2.4(1). So $\mathrm{id}_{R^{op}} F_0(M^+) \leq \mathrm{id}_{R^{op}} M^+ = s + n$ by assumption, and hence we get an injective preenvelope $0 \to M^{++} \to [F_0(M^+)]^+$ of M^{++} with $\mathrm{fd}_R [F_0(M^+)]^+ = \mathrm{id}_{R^{op}} F_0(M^+) \leq s + n$ by Lemma 2.4. Notice that there exists an embedding $M \hookrightarrow M^{++}$ by [St, p.48, Exercise 41], thus $E^0(M)$ is isomorphic to a direct summand of $[F_0(M^+)]^+$ and therefore $\mathrm{fd}_R E^0(M) \leq s + n$.

(2) Without loss of generality, assume that $\mathrm{fd}_R E^0(M) = s < \infty$. By Lemmas 2.3 and 2.4(1), $[E^0(M)]^+ \twoheadrightarrow M^+$ is a flat precover of M^+ in $\mathrm{Mod}\, R^{op}$ with $\mathrm{id}_{R^{op}}[E^0(M)]^+ = s$. So $F_0(M^+)$ is isomorphic to a direct summand of $[E^0(M)]^+$ and $\mathrm{id}_{R^{op}} F_0(M^+) \leq s$. Then by assumption, we have that $\mathrm{id}_{R^{op}} M^+ \leq \mathrm{id}_{R^{op}} F_0(M^+) + n \leq s + n$. It follows from Lemma 2.4(1) that $\mathrm{fd}_R M \leq s + n$. □

We are now in a position to state the main result in this section, which is more general than Theorem 1.2.

Theorem 5.9. *For a left Noetherian ring R, consider the following conditions.*

(1) ${}_R R$ satisfies the Auslander condition.

(2) Any flat left R-module satisfies the Auslander condition.

(3) $\mathrm{fd}_R E^i(M) \leq \mathrm{fd}_R M + i$ *for any left R-module M and $i \geq 0$.*

(4) $\mathrm{fd}_R E^0(M) \leq \mathrm{fd}_R M$ *for any left R-module M.*

(5) $\mathrm{id}_{R^{op}} F_i(E) \leq i$ *for any injective right R-module E and $i \geq 0$.*

(6) $\mathrm{id}_{R^{op}} F_i(N) \leq \mathrm{id}_{R^{op}} N + i$ *for any right R-module N and $i \geq 0$.*

(7) $\mathrm{id}_{R^{op}} F_0(N) \leq \mathrm{id}_{R^{op}} N$ *for any right R-module N.*

We have (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) $\Rightarrow$ (5) $\Leftrightarrow$ (6) $\Leftrightarrow$ (7)*. If R is further right Noetherian, then all of the above and below conditions are equivalent.*

$(i)^{op}$ The opposite version of (i) $(1 \leq i \leq 7)$.

Proof. (2) $\Rightarrow$ (1) is trivial, and (1) $\Rightarrow$ (2) follows from Corollary 4.2. (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) follow from Proposition 5.7 and Lemma 5.3(1), and (5) $\Leftrightarrow$ (6) $\Leftrightarrow$ (7) follow from Proposition 5.6 and Lemma 5.3(2). By Proposition 5.5, we have (1) $\Rightarrow$ (5).

Assume that R is a left and right Noetherian ring. Then (1) $\Leftrightarrow$ $(1)^{op}$ follows from [FGR, Theorem 3.7], and (7) $\Rightarrow$ (4) follows from Lemma 5.8(1). □

Observe that Miyachi proved in [M, Theorem 4.1] that if R is a right coherent and left Noetherian projective K-algebra over a commutative ring K, then R satisfies the Auslander condition (that is, ${}_R R$ is $G_\infty(0)$) if and only if $\mathrm{fd}_R E^0(M) \leq \mathrm{fd}_R M$ for any left R-module M. Theorem 5.9 extends this result.

By Theorems 5.9, we immediately have the following

Corollary 5.10. *Let R be a left Noetherian ring such that ${}_R R$ satisfies the Auslander condition. If $M \in \mathrm{Mod}\, R$ with $\mathrm{fd}_R M \leq s(< \infty)$, then M is $G_\infty(s)$.*

Remark 5.11. By the dimension shifting, it is easy to verify that the converse of Corollary 5.10 holds true when $\mathrm{id}_R M < \infty$ even without the assumption "R is a left and right

Noetherian ring satisfying the Auslander condition". However, this converse does not hold true in general. For example, let R be a quasi Frobenius ring with infinite global dimension. Then R is a left and right Artinian ring satisfying the Auslander condition and every module in $\operatorname{Mod} R$ is $G_{\infty}(0)$, but there exists a module in $\operatorname{Mod} R$ which is not flat because $\operatorname{gl.dim} R$ is infinite.

For any $n, k \geq 0$, we use $\mathscr{G}_n(k)$ to denote the full subcategory of $\operatorname{Mod} R$ consisting of modules being $G_n(k)$, and denote by $\mathscr{G}_{\infty}(k) = \bigcap_{n \geq 0} \mathscr{G}_n(k)$. By Corollary 3.17, it is easy to get the following

Proposition 5.12. *Let $0 \to X \to X^0 \to X^1$ be an exact sequence in $\operatorname{Mod} R$, and let $s \geq 0$ and $n \geq 1$. If $X^0 \in \mathscr{G}_n(s)$ and $X^1 \in \mathscr{G}_{n-1}(s+1)$, then $X \in \mathscr{G}_n(s)$.*

For any $n \geq 0$, we use $\mathscr{F}^n(\operatorname{Mod} R)$ to denote the subcategory of $\operatorname{Mod} R$ consisting of modules with flat dimension at most n.

Corollary 5.13. *Let R be a left Noetherian ring such that ${}_RR$ satisfies the Auslander condition. Then we have*

(1) $\mathscr{G}_{\infty}(0) = \mathscr{F}^0(\operatorname{Mod} R)$ if and only if $\mathscr{G}_{\infty}(s) = \mathscr{F}^s(\operatorname{Mod} R)$ for any $s \geq 0$.

(2) $\mathscr{G}_{\infty}(0) \bigcap \operatorname{mod} R = \mathscr{F}^0(\operatorname{mod} R)$ if and only if $\mathscr{G}_{\infty}(s) \bigcap \operatorname{mod} R = \mathscr{F}^s(\operatorname{mod} R)$ for any $s \geq 0$.

Proof. (1) The sufficiency is trivial, so it suffices to prove the necessity. By Corollary 5.10, we have $\mathscr{F}^s(\operatorname{Mod} R) \subseteq \mathscr{G}_{\infty}(s)$ for any $s \geq 0$. In the following we will prove the converse inclusion by induction on s. The case for $s = 0$ follows from the assumption. Now suppose $s \geq 1$ and $M \in \mathscr{G}_{\infty}(s)$. Let $0 \to K \to F_0(M) \to M \to 0$ be an exact sequence in $\operatorname{Mod} R$. By assumption $F_0(M) \in \mathscr{G}_{\infty}(0)$. So $K \in \mathscr{G}_{\infty}(s-1)$ by Proposition 5.12, and hence $\operatorname{fd}_R K \leq s-1$ by the induction hypothesis. It follows that $\operatorname{fd}_R M \leq s$ and $M \in \mathscr{F}^s(\operatorname{Mod} R)$, which implies that $\mathscr{G}_{\infty}(s) \subseteq \mathscr{F}^s(\operatorname{Mod} R)$.

(2) It is an immediate consequence of (1). □

As applications of the results obtained above, in the rest of this section we will study the properties of rings satisfying the Auslander condition with finite certain homological dimension. In particular, we will get some equivalent characterizations of Auslander-Gorenstein rings and Auslander-regular rings.

For a module $M \in \operatorname{Mod} R$ and a non-negative integer t, we use $\Omega^t(M)$ to denote the tth syzygy of M (note: $\Omega^0(M) = M$). It is known that $\Omega^t(M)$ is unique up to projective equivalence for a given module M.

Lemma 5.14. *Let R be a left Noetherian ring. For a module $M \in \operatorname{Mod} R$ and non-negative integers t and n, if $\operatorname{fd}_R \Omega^t(M) \leq \operatorname{fd}_R E^0(\Omega^t(M)) + n$, then $\operatorname{fd}_R M \leq \operatorname{fd}_R E^0({}_R R) + n + t$.*

Proof. Let $M \in \operatorname{Mod} R$. Then there exist index sets $J_0, \cdots, J_{t-1}$ such that we have the following exact sequence:

$$0 \to \Omega^t(M) \to R^{(J_{t-1})} \to \cdots \to R^{(J_0)} \to M \to 0$$

in $\operatorname{Mod} R$. Because $E^0(R^{(J_{t-1})}) = [E^0({}_R R)]^{(J_{t-1})}$ by [B1, Theorem 1.1] and [AF, Proposition 18.12(4)], $\operatorname{fd}_R E^0(R^{(J_{t-1})}) = \operatorname{fd}_R E^0({}_R R)$. Notice that $E^0(\Omega^t(M))$ is isomorphic to a direct summand of $E^0(R^{(J_{t-1})})$, so $\operatorname{fd}_R E^0(\Omega^t(M)) \leq \operatorname{fd}_R E^0({}_R R)$. Thus by assumption we have that $\operatorname{fd}_R \Omega^t(M) \leq \operatorname{fd}_R E^0(\Omega^t(M)) + n \leq \operatorname{fd}_R E^0({}_R R) + n$ and $\operatorname{fd}_R M \leq \operatorname{fd}_R E^0({}_R R) + n + t$. $\square$

Recall from [Bj] that a left and right Noetherian ring R is called *Auslander-Gorenstein* (resp. *Auslander-regular*) if R satisfies the Auslander condition and $\operatorname{id}_R R = \operatorname{id}_{R^{op}} R$ (resp. $\operatorname{gl.dim} R$) $< \infty$. Also recall that $\operatorname{fin.dim} R = \sup\{\operatorname{pd}_R M \mid M \in \operatorname{mod} R \text{ with } \operatorname{pd}_R M < \infty\}$.

As an application of Theorem 5.9, we get some equivalent characterizations of rings satisfying the Auslander condition with finite left self-injective dimension as follows, which generalizes [M, Proposition 4.4].

Theorem 5.15. *For a left and right Noetherian ring R and a positive integer n, the following statements are equivalent.*

(1) R satisfies the Auslander condition with $\operatorname{id}_R R \leq n$.

(2) $\operatorname{id}_{R^{op}} F_0(N) \leq \operatorname{id}_{R^{op}} N \leq \operatorname{id}_{R^{op}} F_0(N) + n - 1$ for any right R-module N with finite injective dimension.

(3) $\operatorname{fd}_R E^0(M) \leq \operatorname{fd}_R M \leq \operatorname{fd}_R E^0(M) + n - 1$ for any left R-module M with finite flat dimension.

Proof. (1) $\Rightarrow$ (2) Let $N \in \operatorname{Mod} R^{op}$ with finite injective dimension. By Theorem 5.9, we have $\operatorname{id}_{R^{op}} F_0(N) \leq \operatorname{id}_{R^{op}} N$. So we only need to prove the latter inequality. Because $\operatorname{id}_R R \leq n$, $\operatorname{id}_{R^{op}} N \leq n$ by Lemma 2.5(2). So if $\operatorname{id}_{R^{op}} F_0(N) \geq 1$, then the assertion holds true. Suppose $F_0(N)$ is injective. We have an exact sequence:

$$0 \to B \to F_0(N) \to N \to 0$$

in $\operatorname{Mod} R^{op}$ with $\operatorname{id}_{R^{op}} B < \infty$. If $\operatorname{id}_{R^{op}} N = n$, then $\operatorname{id}_{R^{op}} B = n+1$. It follows from Lemma 2.5(2) that $\operatorname{id}_R R \geq n+1$, which is a contradiction. Thus we have $\operatorname{id}_{R^{op}} N \leq n-1$.

(2) $\Rightarrow$ (3) Let $M \in \operatorname{Mod} R$ with finite flat dimension. Then $M^+ \in \operatorname{Mod} R^{op}$ with finite injective dimension by Lemma 2.4(1). Thus by Lemma 5.8, we get the assertion.

(3) $\Rightarrow$ (1) By (3) and Theorem 5.9, R satisfies the Auslander condition. Let $M \in \operatorname{mod} R$ with $\operatorname{pd}_R M (= \operatorname{fd}_R M) < \infty$. Then $\operatorname{fd}_R \Omega^1(M) < \infty$. By (3), we have $\operatorname{fd}_R \Omega^1(M) \leq \operatorname{fd}_R E^0(\Omega^1(M)) + n - 1$. So $\operatorname{pd}_R M = \operatorname{fd}_R M \leq \operatorname{fd}_R E^0({}_R R) + n = n$ by Lemma 5.14. Thus we have $\operatorname{fin.dim} R \leq n$. It follows from [HI, Corollary 5.3] that $\operatorname{id}_R R \leq n$. $\square$

In view of Theorem 5.15 it would be interesting to ask the following

Question 5.16. Let R be a left and right Noetherian ring satisfying the Auslander condition with $\operatorname{id}_R R < \infty$. Is then $\operatorname{id}_{R^{op}} R < \infty$? that is, is R Auslander-Gorenstein?

By [H, Proposition 4.6], the answer to Question 5.16 is positive if R is a left and right Artinian ring. It is a generalization of [AR3, Corollary 5.5(b)].

Putting $n = 1$ in Theorem 5.15, we have the following

Corollary 5.17. *For a left and right Noetherian ring R, the following statements are equivalent.*

(1) R satisfies the Auslander condition with $\operatorname{id}_R R \leq 1$.

(2) $\operatorname{id}_{R^{op}} F_0(N) = \operatorname{id}_{R^{op}} N$ *for any right R-module N with finite injective dimension.*

(3) $\operatorname{fd}_R E^0(M) = \operatorname{fd}_R M$ *for any left R-module M with finite flat dimension.*

As another application of Theorem 5.9, we get some equivalent characterizations of Auslander-regular rings as follows, which generalizes [M, Corollary 4.5].

Theorem 5.18. *For a left and right Noetherian ring R and a positive integer n, the following statements are equivalent.*

(1) R is an Auslander-regular ring with $\operatorname{gl.dim} R \leq n$.

(2) $\operatorname{id}_{R^{op}} F_0(N) \leq \operatorname{id}_{R^{op}} N \leq \operatorname{id}_{R^{op}} F_0(N) + n - 1$ *for any right R-module N.*

(3) $\operatorname{fd}_R E^0(M) \leq \operatorname{fd}_R M \leq \operatorname{fd}_R E^0(M) + n - 1$ *for any left R-module M.*

Proof. By Theorem 5.15 and Lemma 5.8, we have (1) $\Rightarrow$ (2) $\Rightarrow$ (3).

(3) $\Rightarrow$ (1) By (3) and Theorem 5.9, R satisfies the Auslander condition. Let $M \in \operatorname{mod} R$. By (3), we have $\operatorname{fd}_R \Omega^1(M) \leq \operatorname{fd}_R E^0(\Omega^1(M)) + n - 1$. So $\operatorname{pd}_R M = \operatorname{fd}_R M \leq \operatorname{fd}_R E^0({}_R R) + n = n$ by Lemma 5.14. Thus we have $\operatorname{gl.dim} R \leq n$. $\square$

Putting $n = 1$ in Theorem 5.18, we have the following

Corollary 5.19. *For a left and right Noetherian ring R, the following statements are equivalent.*

(1) R is an Auslander-regular ring with $\operatorname{gl.dim} R \leq 1$.

(2) $\mathrm{id}_{R^{op}} F_0(N) = \mathrm{id}_{R^{op}} N$ *for any right* R*-module* N*.*

(3) $\mathrm{fd}_R E^0(M) = \mathrm{fd}_R M$ *for any left* R*-module* M*.*

6. Approximation presentations and Gorenstein algebras

In this section, R is an Artinian algebra. We will establish the connection between the Auslander and Reiten's conjecture mentioned in the introduction and the contravariant finiteness of the full subcategory of $\mathrm{mod}\, R$ consisting of modules satisfying the Auslander condition. We begin with the following

Lemma 6.1. *Let* $X \in \mathrm{mod}\, R$ *and* $\{M_i\}_{i\in I}$ *be a family of left* R*-modules, where* I *is a directed index set. Then for any* $n \geq 0$ *we have*

$$\mathrm{Ext}^n_R(\varinjlim_{i\in I} M_i, X) \cong \varprojlim_{i\in I} \mathrm{Ext}^n_R(M_i, X).$$

Proof. Because R is an Artinian algebra, any module in $\mathrm{mod}\, R$ is pure-injective by [GT, Theorem 1.2.19]. Then the assertion follows from [GT, Lemma 3.3.4]. □

For $n \geq 0$, we use $\mathscr{I}^n(\mathrm{Mod}\, R)$ to denote the full subcategory of $\mathrm{Mod}\, R$ consisting of modules with injective dimension at most n. For a module $M \in \mathrm{Mod}\, R$, we denote by $\Omega^{-n}(M)$ the nth cosyzygy of M.

The following approximation theorem plays a crucial role in the rest of this section.

Theorem 6.2. *Let* ${}_RR \in \mathscr{G}_n(k)$ *and* $R_R \in \mathscr{G}_n(k)^{op}$ *with* $n, k \geq 0$*. Then for any* $M \in \mathrm{Mod}\, R$ *and* $1 \leq i \leq n-1$*, there exist the following commutative diagrams with exact rows:*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \longrightarrow & I_{i+1}(M) & \longrightarrow & G_{i+1}(M) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M & \longrightarrow & I_i(M) & \longrightarrow & G_i(M) & \longrightarrow & 0
\end{array}$$

and

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I^{i+1}(M) & \longrightarrow & G^{i+1}(M) & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & I^i(M) & \longrightarrow & G^i(M) & \longrightarrow & M & \longrightarrow & 0
\end{array}$$

with $G_j(M), G^j(M) \in \mathscr{G}_j(k)$*, and* $I_j(M), I^j(M) \in \mathscr{I}^{j+k}(\mathrm{Mod}\, R)$ *for* $j = i, i+1$*.*

Proof. By Corollary 3.14 and Lemma 3.1(1), we have the following commutative diagrams with exact columns and rows:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & M & \to & I_i(M) & \to & G_i(M) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & E^0(M) & \to & E^0(M)\bigoplus(\bigoplus_{j=0}^{i-1} P_j(E^{j+1}(M))) & \to & \bigoplus_{j=0}^{i-1} P_j(E^{j+1}(M)) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & E^1(M) & \to & E^1(M)\bigoplus(\bigoplus_{j=0}^{i-2} P_j(E^{j+2}(M))) & \to & \bigoplus_{j=0}^{i-2} P_j(E^{j+2}(M)) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & E^{i-2}(M) & \to & E^{i-2}(M)\bigoplus(P_1(E^i(M))\bigoplus P_0(E^{i-1}(M))) & \to & P_1(E^i(M))\bigoplus P_0(E^{i-1}(M)) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & E^{i-1}(M) & \to & E^{i-1}(M)\bigoplus P_0(E^i(M)) & \to & P_0(E^i(M)) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \Omega^{-i}(M) & \to & E^i(M) & \to & \Omega^{-(i+1)}(M) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

where $I_i(M) = \mathrm{Ker}(E^0(M)\bigoplus(\bigoplus_{j=0}^{i-1} P_j(E^{j+1}(M))) \to E^1(M)\bigoplus(\bigoplus_{j=0}^{i-2} P_j(E^{j+2}(M))))$ and $G_i(M) = \mathrm{Ker}(\bigoplus_{j=0}^{i-1} P_j(E^{j+1}(M)) \to \bigoplus_{j=0}^{i-2} P_j(E^{j+2}(M)))$ for any $i \geq 1$.

Consider the following pull-back diagram:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & \Omega^1(E^{i+1}(M)) & \to & X_{i+1} & \to & \Omega^{-(i+1)}(M) & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & \Omega^1(E^{i+1}(M)) & \to & P_0(E^{i+1}(M)) & \to & E^{i+1}(M) & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \Omega^{-(i+2)}(M) & = & \Omega^{-(i+2)}(M) & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

By Corollary 3.14 and Lemma 3.1(1) again, for any $i \geq 1$ we have the following commutative and exact columns and rows:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \Omega^{i+1}(E^{i+1}(M)) & \rightarrow & G_{i+1}(M) & \rightarrow & G_i(M) & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & P_i(E^{i+1}(M)) & \rightarrow & \bigoplus_{j=0}^{i} P_j(E^{j+1}(M)) & \rightarrow & \bigoplus_{j=0}^{i-1} P_j(E^{j+1}(M)) & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & P_{i-1}(E^{i+1}(M)) & \rightarrow & \bigoplus_{j=0}^{i-1} P_j(E^{j+2}(M)) & \rightarrow & \bigoplus_{j=0}^{i-2} P_j(E^{j+2}(M)) & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & P_2(E^{i+1}(M)) & \rightarrow & P_2(E^{i+1}(M)) \bigoplus (P_1(E^i(M)) \bigoplus P_0(E^{i-1}(M))) & \rightarrow & P_1(E^i(M)) \bigoplus P_0(E^{i-1}(M)) & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & P_1(E^{i+1}(M)) & \rightarrow & P_1(E^{i+1}(M)) \bigoplus P_0(E^i(M)) & \rightarrow & P_0(E^i(M)) & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \Omega^1(E^{i+1}(M)) & \rightarrow & X_{i+1} & \rightarrow & \Omega^{-(i+1)}(M) & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Then we get the following pull-back diagram:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \Omega^{i+1}(E^{i+1}(M)) & = & \Omega^{i+1}(E^{i+1}(M)) & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \rightarrow & M & \rightarrow & I_{i+1}(M) & \rightarrow & G_{i+1}(M) & \rightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \rightarrow & M & \rightarrow & I_i(M) & \rightarrow & G_i(M) & \rightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

Because $R_R \in \mathscr{G}_n(k)^{op}$, $\mathrm{id}_R\, P_j(E^t(M)) \leq j+k$ for any $0 \leq j \leq n-1$ and $t \geq 0$ by Lemma 5.4. So from the middle column in the first diagram we get $\mathrm{id}_R\, I_i(M) \leq i+k$ for any $1 \leq i \leq n$. Because ${}_RR \in \mathscr{G}_n(k)$, any projective module in $\mathrm{mod}\, R$ is also in $\mathscr{G}_n(k)$. So by Corollary 3.17

and the exactness of the rightmost column in the first diagram, we have $G_i(M) \in \mathscr{G}_i(k)$ for any $1 \leq i \leq n$. Thus the above diagram is the first desired one.

Put $I^i(M) = I_i(\Omega^1(M))$. Then we have the following push-out diagram:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \Omega^1(M) & \longrightarrow & P_0(M) & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & I^i(M) & \longrightarrow & G^i(M) & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & G_i(\Omega^1(M)) & = & G_i(\Omega^1(M)) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

Note that $P_0(M) \in \mathscr{G}_n(k)$. For any $1 \leq i \leq n$, because $G_i(\Omega^1(M)) \in \mathscr{G}_i(k)$ by the above argument, $G^i(M)$ is also in $\mathscr{G}_i(k)$ by the horseshoe lemma and the exactness of the middle column in the above diagram. By the above argument, we have the following pull-back diagram:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \Omega^{i+1}(E^{i+1}(\Omega^1(M))) & = & \Omega^{i+1}(E^{i+1}(\Omega^1(M))) & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P_0(M) & \longrightarrow & G^{i+1}(M) & \longrightarrow & G_{i+1}(\Omega^1(M)) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P_0(M) & \longrightarrow & G^i(M) & \longrightarrow & G_i(\Omega^1(M)) & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

Then the following pull-back diagram:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \Omega^{i+1}(E^{i+1}(\Omega^1(M))) & = & \Omega^{i+1}(E^{i+1}(\Omega^1(M))) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & I^{i+1}(M) & \longrightarrow & G^{i+1}(M) & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & I^{i}(M) & \longrightarrow & G^{i}(M) & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

is the second desired one. □

If R satisfies the Auslander condition, then the exact sequences

$$0 \to M \to I_i(M) \to G_i(M) \to 0$$

and

$$0 \to I^i(M) \to G^i(M) \to M \to 0$$

in Theorem 6.2 are a left $\mathscr{I}^i(\operatorname{Mod} R)$-approximation and a right $\mathscr{G}_i(0)$-approximation of M respectively for any $1 \leq i \leq n$.

Let $M \in \operatorname{Mod} R$ and $n, k \geq 0$, and let

$$\cdots \to P_i(M) \to \cdots \to P_1(M) \to P_0(M) \to M \to 0$$

be a minimal projective resolution of M. We use $\operatorname{Co}\mathscr{G}_n(k)$ to denote the full subcategory of $\operatorname{Mod} R$ consisting of the modules M satisfying $\operatorname{id}_R P_i(M) \leq i + k$ for any $0 \leq i \leq n-1$, and denote by $\operatorname{Co}\mathscr{G}_\infty(k) = \bigcap_{n \geq 0} \operatorname{Co}\mathscr{G}_n(k)$. We use $\mathscr{P}^n(\operatorname{mod} R)$ (resp. $\mathscr{I}^n(\operatorname{mod} R)$) to denote the full subcategory of $\operatorname{mod} R$ consisting of modules with projective (resp. injective) dimension at most n. As a consequence of Theorem 6.2, we get the following

Proposition 6.3. *Let R satisfy the Auslander condition and $M \in \operatorname{mod} R$. Then we have*

(1) There exists a countably generated left R-module $N \in \operatorname{Co}\mathscr{G}_\infty(0)$ and a monomorphism $\beta : M \rightarrowtail N$ in $\operatorname{Mod} R$ such that $\operatorname{Hom}_R(\beta, T)$ is epic for any $T \in \operatorname{Co}\mathscr{G}_\infty(0) \bigcap \operatorname{mod} R$.

(2) There exists a countably cogenerated left R-module $G \in \mathscr{G}_\infty(0)$ and an epimorphism $\alpha : G \twoheadrightarrow M$ in $\operatorname{Mod} R$ such that $\operatorname{Hom}_R(T', \alpha)$ is epic for any $T' \in \mathscr{G}_\infty(0) \bigcap \operatorname{mod} R$.

Proof. (1) Let R satisfy the Auslander condition. By Theorem 6.2, for any $M \in \operatorname{mod} R$ and $n \geq 1$, we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I^{n+1}(\mathbb{D}M) & \longrightarrow & G^{n+1}(\mathbb{D}M) & \longrightarrow & \mathbb{D}M & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & I^{n}(\mathbb{D}M) & \longrightarrow & G^{n}(\mathbb{D}M) & \longrightarrow & \mathbb{D}M & \longrightarrow & 0 \end{array}$$

with $G^i(\mathbb{D}M) \in \mathscr{G}_i(0)^{op} \bigcap \operatorname{mod} R^{op}$ and $I^i(\mathbb{D}M) \in \mathscr{I}^i(\operatorname{mod} R^{op})$ for $i = n, n+1$, where $\mathbb{D}$ is the ordinary Matlis duality between $\operatorname{mod} R$ and $\operatorname{mod} R^{op}$. Then we get the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \xrightarrow{\beta_n} & \mathbb{D}G^{n}(\mathbb{D}M) & \longrightarrow & \mathbb{D}I^{n}(\mathbb{D}M) & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M & \xrightarrow{\beta_{n+1}} & \mathbb{D}G^{n+1}(\mathbb{D}M) & \longrightarrow & \mathbb{D}I^{n+1}(\mathbb{D}M) & \longrightarrow & 0 \end{array}$$

with $\mathbb{D}G^i(\mathbb{D}M) \in \mathrm{Co}\mathscr{G}_i(0) \bigcap \operatorname{mod} R$ and $\mathbb{D}I^i(\mathbb{D}M) \in \mathscr{P}^i(\operatorname{mod} R)$ for $i = n, n+1$. Put $N_n = \mathbb{D}G^n(\mathbb{D}M)$ and $K_n = \mathbb{D}I^n(\mathbb{D}M)$ for any $n \geq 1$. Then we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccccccc} P_k(N_n) & \longrightarrow & P_{k-1}(N_n) & \longrightarrow & \cdots & \longrightarrow & P_1(N_n) & \longrightarrow & P_0(N_n) & \longrightarrow & N_n & \longrightarrow & 0 \\ \downarrow{\scriptstyle g^k_{n+1,n}} & & \downarrow{\scriptstyle g^{k-1}_{n+1,n}} & & & & \downarrow{\scriptstyle g^1_{n+1,n}} & & \downarrow{\scriptstyle g^0_{n+1,n}} & & \downarrow{\scriptstyle g_{n+1,n}} & & \\ P_k(N_{n+1}) & \longrightarrow & P_{k-1}(N_{n+1}) & \longrightarrow & \cdots & \longrightarrow & P_1(N_{n+1}) & \longrightarrow & P_0(N_{n+1}) & \longrightarrow & N_{n+1} & \longrightarrow & 0 \end{array}$$

If $n > m$, then put

$$g_{n,m} = g_{n,n-1} g_{n-1,n-2} \cdots g_{m+1,m}$$

and

$$g^k_{n,m} = g^k_{n,n-1} g^k_{n-1,n-2} \cdots g^k_{m+1,m}.$$

In this way, for any $k \geq 0$ we get direct systems: $\{N_n, g_{n,m}\}_{n \in \mathbb{Z}^+}$ and $\{P_k(N_n), g^k_{n,m}\}_{n \in \mathbb{Z}^+}$, where $\mathbb{Z}^+$ is the set of positive integers. Because each $g_{n,m} : N_m \to N_n$ is monic, we can identity $\varinjlim_{n \geq 1} N_n$ with the direct union. It follows that $\varinjlim_{n \geq 1} N_n = \varinjlim_{n \geq t} N_n$ for any $1 \leq t \leq n$. Put $N = \varinjlim_{n \geq 1} N_n$. Then N is countably generated.

Because $N_t \in \mathrm{Co}\mathscr{G}_t(0) \bigcap \operatorname{mod} R$, $\mathrm{id}_R\, P_k(N_t) \leq k$ for any $0 \leq k \leq t$. So $\varinjlim_{n \geq t} P_k(N_n)$ is projective and $\mathrm{id}_R \varinjlim_{n \geq t} P_k(N_n) \leq k$ for any $0 \leq k \leq t$ by [B1, Theorem 1.1]. On the other hand, we have an exact sequence:

$$\cdots \to \varinjlim_{n \geq t} P_t(N_n) \to \varinjlim_{n \geq t} P_{t-1}(N_n) \to \cdots \to \varinjlim_{n \geq t} P_0(N_n) \to \varinjlim_{n \geq t} N_n (= N) \to 0.$$

So $N \in \mathrm{Co}\mathscr{G}_\infty(0)$. Put $K = \varinjlim_{n\geq t} K_n$ and $\beta = \varinjlim_{n\geq t}\beta_n$. Then we get the following exact sequence:

$$0 \to M \xrightarrow{\beta} N \to K \to 0.$$

By Lemma 6.1, for any $T \in \mathrm{Co}\mathscr{G}_\infty(0) \bigcap \mathrm{mod}\, R$, we have $\mathrm{Ext}^1_R(K, T) \cong \mathrm{Ext}^1_R(\varinjlim_{n\geq t} K_n, T) \cong \varprojlim_{n\geq t} \mathrm{Ext}^1_R(K_n, T) = 0$, which implies that $\mathrm{Hom}_R(\beta, T)$ is epic.

(2) Let $M \in \mathrm{mod}\, R$ and $T' \in \mathscr{G}_\infty(0) \bigcap \mathrm{mod}\, R$. Then $\mathbb{D}M \in \mathrm{mod}\, R^{op}$ and $\mathbb{D}T' \in \mathrm{Co}\mathscr{G}_\infty(0)^{op} \bigcap \mathrm{mod}\, R^{op}$. By (1), there exists a monomorphism $\beta : \mathbb{D}M \rightarrowtail N$ in $\mathrm{Mod}\, R^{op}$ with N countably generated and $N \in \mathrm{Co}\mathscr{G}_\infty(0)$ such that $\mathrm{Hom}_{R^{op}}(\beta, \mathbb{D}T')$ is epic. Put $G = \mathbb{D}N$. Then G is countably cogenerated and $\mathbb{D}\beta : G \twoheadrightarrow M (\cong \mathbb{D}\mathbb{D}M)$ is epic in $\mathrm{Mod}\, R$ such that $\mathrm{Hom}_R(T', \mathbb{D}\beta) (\cong \mathrm{Hom}_R(\mathbb{D}\mathbb{D}T', \mathbb{D}\beta))$ is also epic. Because $N \in \mathrm{Co}\mathscr{G}_\infty(0)^{op}$, $\mathrm{id}_{R^{op}}\, P_i(N) \leq i$ for any $i \geq 0$. Note that $P_i(N) = \bigoplus_j P^i_j$ with all P^i_j projective in $\mathrm{mod}\, R$ for any $i \geq 0$ by [Wa, Theorem 1]. So we get an exact sequence:

$$0 \to G \to \prod_j \mathbb{D}P^0_j \to \prod_j \mathbb{D}P^1_j \to \cdots \to \prod_j \mathbb{D}P^i_j \to \cdots$$

in $\mathrm{Mod}\, R$ with $\prod_j \mathbb{D}P^i_j$ injective and $\mathrm{pd}_R \prod_j \mathbb{D}P^i_j \leq i$ for any $i \geq 0$. It implies that $G \in \mathscr{G}_\infty(0)$. $\square$

Following [AR2], for a full subcategory $\mathscr{X}$ of $\mathrm{mod}\, R$ we denote by

$$\mathrm{Rapp}(\mathscr{X}) = \{M \in \mathrm{mod}\, R \mid \text{there exists a right } \mathscr{X} - \text{approximation of } M\},$$

$$\mathrm{Lapp}(\mathscr{X}) = \{M \in \mathrm{mod}\, R \mid \text{there exists a left } \mathscr{X} - \text{approximation of } M\}.$$

We use $\mathscr{P}^\infty(\mathrm{mod}\, R)$ (resp. $\mathscr{I}^\infty(\mathrm{mod}\, R)$) to denote the full subcategory of $\mathrm{mod}\, R$ consisting of modules with finite projective (resp. injective) dimension.

Proposition 6.4. *Let R satisfy the Auslander condition. Then we have*

(1) $\mathrm{Lapp}(\mathrm{Co}\mathscr{G}_\infty(0) \bigcap \mathrm{mod}\, R) = \{M \in \mathrm{mod}\, R \mid$ *there exists an exact sequence* $0 \to M \to X \to Y \to 0$ *with* $X \in \mathrm{Co}\mathscr{G}_\infty(0) \bigcap \mathrm{mod}\, R$ *and* $Y \in \mathscr{P}^\infty(\mathrm{mod}\, R)\}$.

(2) $\mathrm{Rapp}(\mathscr{G}_\infty(0) \bigcap \mathrm{mod}\, R) = \{M \in \mathrm{mod}\, R \mid$ *there exists an exact sequence* $0 \to Y \to X \to M \to 0$ *with* $X \in \mathscr{G}_\infty(0) \bigcap \mathrm{mod}\, R$ *and* $Y \in \mathscr{I}^\infty(\mathrm{mod}\, R)\}$.

Proof. It is easy to see that $\mathrm{Lapp}(\mathrm{Co}\mathscr{G}_\infty(0) \bigcap \mathrm{mod}\, R) \supseteq \{M \in \mathrm{mod}\, R \mid$ there exists an exact sequence $0 \to M \to X \to Y \to 0$ with $X \in \mathrm{Co}\mathscr{G}_\infty(0) \bigcap \mathrm{mod}\, R$ and $Y \in \mathscr{P}^\infty(\mathrm{mod}\, R)\}$, and $\mathrm{Rapp}(\mathscr{G}_\infty(0) \bigcap \mathrm{mod}\, R) \supseteq \{M \in \mathrm{mod}\, R \mid$ there exists an exact sequence $0 \to Y \to X \to M \to 0$ with $X \in \mathscr{G}_\infty(0) \bigcap \mathrm{mod}\, R$ and $Y \in \mathscr{I}^\infty(\mathrm{mod}\, R)\}$. So it suffices to prove the converse inclusions.

(1) Let $M \in \mathrm{Lapp}(\mathrm{Co}\mathscr{G}_\infty(0)\bigcap \operatorname{mod} R)$. Because R satisfies the Auslander condition, the injective cogenerator $\mathbb{D}(R_R)$ for $\operatorname{Mod} R$ is in $\mathrm{Co}\mathscr{G}_\infty(0)\bigcap \operatorname{mod} R$. So we may assume that $0 \to M \xrightarrow{f} X^M \to Y^M \to 0$ is exact in $\operatorname{mod} R$ such that f is a minimal left $\mathrm{Co}\mathscr{G}_\infty(0)\bigcap \operatorname{mod} R$-approximation of M.

Let $0 \to M \overset{\beta}{\to} N \to K \to 0$ be an exact sequence in $\operatorname{Mod} R$ as in Proposition 6.3(1) such that $\mathrm{Hom}_R(\beta, T)$ is epic for any $T \in \mathrm{Co}\mathscr{G}_\infty(0)\bigcap \operatorname{mod} R$, where $N = \varinjlim_{n\geq 1} N_n (= \bigcup_{n\geq 1} N_n)$ and $K = \varinjlim_{n\geq 1} K_n (= \bigcup_{n\geq 1} K_n)$. Note that $\mathrm{Hom}_R(X^M, -)|_{\mathrm{Co}\mathscr{G}_\infty(0)\bigcap \operatorname{mod} R} \xrightarrow{\mathrm{Hom}_R(f,-)} \mathrm{Hom}_R(X^M, -)|_{\mathrm{Co}\mathscr{G}_\infty(0)\bigcap \operatorname{mod} R} \to 0$ is a projective cover of $\mathrm{Hom}_R(X^M, -)|_{\mathrm{Co}\mathscr{G}_\infty(0)\bigcap \operatorname{mod} R}$. Because $\mathrm{Hom}_R(N, -)|_{\mathrm{Co}\mathscr{G}_\infty(0)\bigcap \operatorname{mod} R}$ is a projective object in the category of functors from $\operatorname{Mod} R$ to Abelian groups, we have the following commutative diagram:

$$\begin{array}{ccccc}
\mathrm{Hom}_R(X^M, -)|_{\mathrm{Co}\mathscr{G}_\infty(0)\bigcap \operatorname{mod} R} & \xrightarrow{\mathrm{Hom}_R(f,-)} & \mathrm{Hom}_R(M, -)|_{\mathrm{Co}\mathscr{G}_\infty(0)\bigcap \operatorname{mod} R} & \longrightarrow & 0 \\
\Big\downarrow{\scriptstyle \mathrm{Hom}_R(s,-)} & & \Big\| & & \\
\mathrm{Hom}_R(N, -)|_{\mathrm{Co}\mathscr{G}_\infty(0)\bigcap \operatorname{mod} R} & \xrightarrow{\mathrm{Hom}_R(\beta,-)} & \mathrm{Hom}_R(M, -)|_{\mathrm{Co}\mathscr{G}_\infty(0)\bigcap \operatorname{mod} R} & \longrightarrow & 0 \\
\Big\downarrow{\scriptstyle \mathrm{Hom}_R(t,-)} & & \Big\| & & \\
\mathrm{Hom}_R(X^M, -)|_{\mathrm{Co}\mathscr{G}_\infty(0)\bigcap \operatorname{mod} R} & \xrightarrow{\mathrm{Hom}_R(f,-)} & \mathrm{Hom}_R(M, -)|_{\mathrm{Co}\mathscr{G}_\infty(0)\bigcap \operatorname{mod} R} & \longrightarrow & 0
\end{array}$$

where $s \in \mathrm{Hom}_R(N, X^M)$ and $t \in \mathrm{Hom}_R(X^M, N)$. Then $\mathrm{Hom}_R(st, -) = \mathrm{Hom}_R(t, -)\,\mathrm{Hom}_R(s, -)$ is an isomorphism. So there exist $s' \in \mathrm{Hom}_R(K, Y^M)$ and $t' \in \mathrm{Hom}_R(Y^M, K)$ such that the following diagram commutes:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \xrightarrow{f} & X^M & \longrightarrow & Y^M & \longrightarrow & 0 \\
 & & \Big\| & & \Big\downarrow{\scriptstyle t} & & \Big\downarrow{\scriptstyle t'} & & \\
0 & \longrightarrow & M & \xrightarrow{\beta} & N & \longrightarrow & K & \longrightarrow & 0 \\
 & & \Big\| & & \Big\downarrow{\scriptstyle s} & & \Big\downarrow{\scriptstyle s'} & & \\
0 & \longrightarrow & M & \xrightarrow{f} & X^M & \longrightarrow & Y^M & \longrightarrow & 0
\end{array}$$

By the minimality of f, we have that st is an isomorphism and so is $s't'$. It implies that $t' : Y^M \to K (= \varinjlim_{n\geq 1} K_n = \bigcup_{n\geq 1} K_n)$ is a split monomorphism. Because Y^M is finitely generated, $\operatorname{Im} t' \subseteq K_n$ for some n. So Y^M is isomorphic to a direct summand of K_n and hence $\mathrm{pd}_R Y^M \leq n$.

(2) Let $M \in \mathrm{Rapp}(\mathscr{G}_\infty(0)\bigcap \operatorname{mod} R)$. Then $\mathbb{D}M \in \mathrm{Lapp}(\mathrm{Co}\mathscr{G}_\infty(0)^{op}\bigcap \operatorname{mod} R^{op})$. By (1) there exists an exact sequence:

$$0 \to \mathbb{D}M \to X \to Y \to 0$$

with $X \in \mathrm{Co}\mathscr{G}_\infty(0)^{op} \bigcap \mathrm{mod}\, R^{op}$ and $Y \in \mathscr{P}^\infty(\mathrm{mod}\, R^{op})$. So we get an exact sequence:

$$0 \to \mathbb{D}Y \to \mathbb{D}X \to M \to 0$$

with $\mathbb{D}X \in \mathscr{G}_\infty(0) \bigcap \mathrm{mod}\, R$ and $\mathbb{D}Y \in \mathscr{I}^\infty(\mathrm{mod}\, R)$. $\square$

As a consequence of Proposition 6.4, we get the following

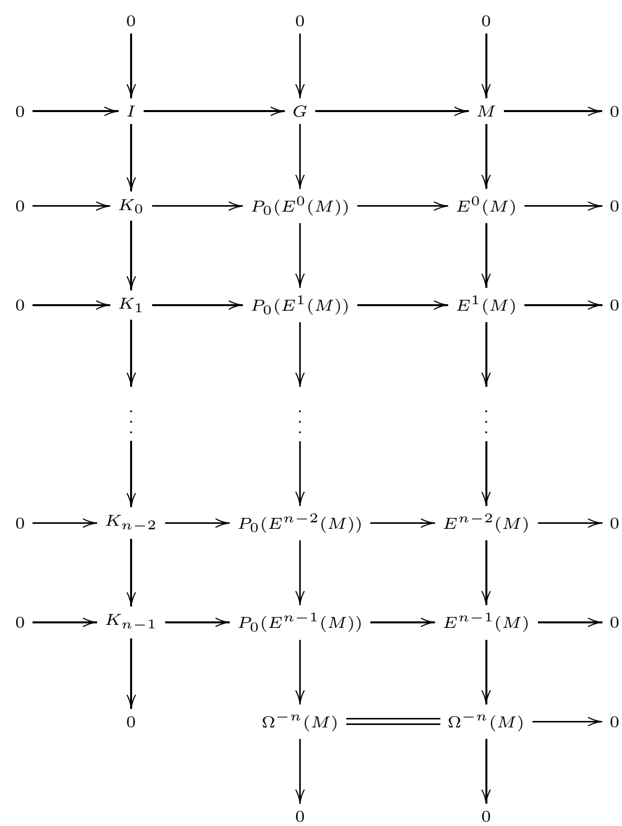
Proposition 6.5. *Let R satisfy the Auslander condition. Then we have*

(1) $\mathrm{Rapp}(\mathscr{G}_\infty(0) \bigcap \mathrm{mod}\, R) = \{M \in \mathrm{mod}\, R \mid$ *there exists a positive integer n such that* $\Omega^{-n}(M) \in \mathscr{G}_\infty(n) \bigcap \mathrm{mod}\, R\}$.

(2) $\mathrm{Lapp}(\mathrm{Co}\mathscr{G}_\infty(0) \bigcap \mathrm{mod}\, R) = \{M \in \mathrm{mod}\, R \mid$ *there exists a positive integer n such that* $\Omega^{n}(M) \in \mathrm{Co}\mathscr{G}_\infty(n) \bigcap \mathrm{mod}\, R\}$.

Proof. (1) Let $M \in \mathrm{Rapp}(\mathscr{G}_\infty(0) \bigcap \mathrm{mod}\, R)$. Then by Proposition 6.4(2), there exists an exact sequence $0 \to Y \to X \to M \to 0$ with $X \in \mathscr{G}_\infty(0) \bigcap \mathrm{mod}\, R$ and $Y \in \mathscr{I}^\infty(\mathrm{mod}\, R)$. Assume that $\mathrm{id}_R Y = k (< \infty)$. Then for any $n > k$, $\mathrm{Ext}^1_R(-, \Omega^{-n+1}(X)) \cong \mathrm{Ext}^n_R(-, X) \cong \mathrm{Ext}^n_R(-, M) \cong \mathrm{Ext}^1_R(-, \Omega^{-n+1}(M))$, which implies that $\Omega^{-n+1}(X)$ and $\Omega^{-n+1}(M)$ are injectively equivalent. Because $X \in \mathscr{G}_\infty(0)$, $\Omega^{-n+1}(X) \in \mathscr{G}_\infty(n-1)$. So $\Omega^{-n+1}(M) \in \mathscr{G}_\infty(n-1)$ and $\Omega^{-n}(M) \in \mathscr{G}_\infty(n)$.

Conversely, assume that $\Omega^{-n}(M) \in \mathscr{G}_\infty(n) \bigcap \mathrm{mod}\, R$. We have the following commutative diagrams with exact columns and rows:



where $G = \operatorname{Ker}(P_0(E^0(M)) \to P_0(E^1(M)))$ and $I = \operatorname{Ker}(K_0 \to K_1)$. Because R satisfies the Auslander condition, $P_0(E^i(M))$ is injective and satisfies the Auslander condition for any $0 \leq i \leq n-1$ by Theorem 5.9. So $\operatorname{id}_R K_i \leq 1$ for any $0 \leq i \leq n-1$, and hence $\operatorname{id}_R I \leq n$ by the exactness of the leftmost column in the above diagram. On the other hand, by Corollary 3.17 and the exactness of the middle column in the above diagram, we have that $G \in \mathscr{G}_\infty(0) \bigcap \operatorname{mod} R$. Thus the exact sequence $0 \to I \to G \to M \to 0$ is a right $\mathscr{G}_\infty(0) \bigcap \operatorname{mod} R$-approximation of M and $M \in \operatorname{Rapp}(\mathscr{G}_\infty(0) \bigcap \operatorname{mod} R)$.

(2) It is dual to the proof of (1), so we omit it. □

Corollary 6.6. *Let R satisfy the Auslander condition. Then we have*

(1) $\mathscr{G}_\infty(0) \bigcap \operatorname{mod} R$ is contravariantly finite in $\operatorname{mod} R$ if and only if there exists a positive integer n such that $\Omega^{-n}(M) \in \mathscr{G}_\infty(n) \bigcap \operatorname{mod} R$ for any $M \in \operatorname{mod} R$.

(2) $\operatorname{Co}\mathscr{G}_\infty(0) \bigcap \operatorname{mod} R$ is covariantly finite in $\operatorname{mod} R$ if and only if there exists a positive integer n such that $\Omega^{n}(M) \in \operatorname{Co}\mathscr{G}_\infty(n) \bigcap \operatorname{mod} R$ for any $M \in \operatorname{mod} R$.

Proof. (1) The sufficiency follows from Proposition 6.5(1).

Conversely, let $\mathscr{G}_\infty(0) \bigcap \operatorname{mod} R$ be contravariantly finite in $\operatorname{mod} R$ and $\{S_1, S_2, \cdots, S_t\}$ a complete set of non-isomorphic simple R-modules. By Proposition 6.5(1), there exists a positive integer n_i such that $\Omega^{-n_i}(S_i) \in \mathscr{G}_\infty(n_i)$ for any $1 \leq i \leq t$. Put $n = \max\{n_1, n_2, \cdots, n_t\}$. Then $\Omega^{-n}(S_i) \in \mathscr{G}_\infty(n)$ for any $1 \leq i \leq t$.

We will prove that $\Omega^{-n}(M) \in \mathscr{G}_\infty(n)$ for any $M \in \operatorname{mod} R$ by induction on $\operatorname{length}(M)$ (the length of M). If $\operatorname{length}(M) = 1$, then $M \cong S_i$ for some $1 \leq i \leq t$ and the assertion follows. Now suppose $\operatorname{length}(M) \geq 2$. Then there exists an exact sequence $0 \to S \to M \to M/S \to 0$ in $\operatorname{mod} R$ with S simple and $\operatorname{length}(M/S) < \operatorname{length}(M)$. By the induction hypothesis, both S and M/S are in $\mathscr{G}_\infty(n)$. Then M is also in $\mathscr{G}_\infty(n)$ by the horseshoe lemma.

(2) It is dual to the proof of (1), so we omit it. □

Let $M \in \operatorname{mod} R$ and $P_1(M) \to P_0(M) \to M \to 0$ be a minimal projective presentation of $M \in \operatorname{mod} R$. For a non-negative integer n, recall from [AR4] that M is called *n-torsionfree* if $\operatorname{Ext}^i_{R^{op}}(\operatorname{Tr} M, R) = 0$ for any $1 \leq i \leq n$, where $\operatorname{Tr} M = \operatorname{Coker}(P_0(M)^* \to P_1(M)^*)$ is the *transpose* of M and $(-)^* = \operatorname{Hom}_R(-, R)$. We use $\Omega^n(\operatorname{mod} R)$ (resp. $\mathscr{T}_n(\operatorname{mod} R)$) to denote the full subcategory of $\operatorname{mod} R$ consisting of n-syzygy (resp. n-torsionfree) modules. In general, we have $\Omega^n(\operatorname{mod} R) \supseteq \mathscr{T}_n(\operatorname{mod} R)$ for any $n \geq 0$ (cf. [AR4]).

Lemma 6.7. *If $R \in \mathscr{G}_n(0)$ with $n \geq 0$, then $\mathscr{G}_n(0) \bigcap \operatorname{mod} R = \Omega^n(\operatorname{mod} R) = \mathscr{T}_n(\operatorname{mod} R)$.*

Proof. $\mathscr{G}_n(0) \bigcap \operatorname{mod} R = \Omega^n(\operatorname{mod} R)$ by [AR3, Proposition 5.1], and $\Omega^n(\operatorname{mod} R) = \mathscr{T}_n(\operatorname{mod} R)$ by [AR4, Proposition 1.6 and Theorem 4.7]. □

Auslander and Reiten conjectured in [AR3] that R is Gorenstein (that is, $\mathrm{id}_R R = \mathrm{id}_{R^{op}} R < \infty$) if R satisfies the Auslander condition. It remains still open. For a full subcategory $\mathscr{C}$ of $\mathrm{mod}\, R$, we denote by $\mathscr{C}^{\perp_1} = \{M \in \mathrm{mod}\, R \mid \mathrm{Ext}^1_R(\mathscr{C}, M) = 0\}$. Put $\Omega^\infty(\mathrm{mod}\, R) = \bigcap_{n\geq 0} \Omega^n(\mathrm{mod}\, R)$. Now we are in a position to establish the connection between this conjecture and the contravariant finiteness of $\mathscr{G}_\infty(0)\bigcap \mathrm{mod}\, R$ and that of $\Omega^\infty(\mathrm{mod}\, R)$ as follows.

Theorem 6.8. *Let R satisfy the Auslander condition. Then the following statements are equivalent.*

(1) R is Gorenstein.

(2) $\mathscr{G}_\infty(0)\bigcap \mathrm{mod}\, R$ is contravariantly finite in $\mathrm{mod}\, R$.

(3) $\mathrm{Co}\mathscr{G}_\infty(0)\bigcap \mathrm{mod}\, R$ is covariantly finite in $\mathrm{mod}\, R$.

(4) $\Omega^\infty(\mathrm{mod}\, R)$ is contravariantly finite in $\mathrm{mod}\, R$.

Proof. Because R satisfies the Auslander condition if and only if so does R^{op}, we get (2) $\Leftrightarrow$ (3). By Lemma 6.7, we have (2) $\Leftrightarrow$ (4).

(1) $\Rightarrow$ (2) Assume that R is Gorenstein with $\mathrm{id}_R R = \mathrm{id}_{R^{op}} R = n$. By [I, Proposition 1], $\mathrm{pd}_R E \leq n$ for any injective left R-module E. So $\mathscr{G}_\infty(0)\bigcap \mathrm{mod}\, R = \mathscr{G}_n(0)\bigcap \mathrm{mod}\, R$, and hence $\mathscr{G}_\infty(0)\bigcap \mathrm{mod}\, R$ is contravariantly finite in $\mathrm{mod}\, R$ by Theorem 6.2.

(2) $\Rightarrow$ (1) Assume that $\mathscr{G}_\infty(0)\bigcap \mathrm{mod}\, R$ is contravariantly finite in $\mathrm{mod}\, R$. Then there exists a positive integer n such that $\Omega^{-n}(M) \in \mathscr{G}_\infty(n)\bigcap \mathrm{mod}\, R$ for any $M \in \mathrm{mod}\, R$ by Corollary 6.6, which implies that $\mathscr{G}_\infty(0)\bigcap \mathrm{mod}\, R = \mathscr{G}_n(0)\bigcap \mathrm{mod}\, R$. Because $\mathscr{G}_n(0)\bigcap \mathrm{mod}\, R = \mathscr{T}_n(\mathrm{mod}\, R)$ by Lemma 6.7, $(\mathscr{G}_\infty(0)\bigcap \mathrm{mod}\, R)^{\perp_1} = (\mathscr{G}_n(0)\bigcap \mathrm{mod}\, R)^{\perp_1} = \mathscr{T}_n(\mathrm{mod}\, R)^{\perp_1} = \mathscr{I}^n(\mathrm{mod}\, R)$ by [HI, Theorem 1.3]. On the other hand, it is easy to see that $\mathscr{I}^\infty(\mathrm{mod}\, R) \subseteq (\mathscr{G}_\infty(0)\bigcap \mathrm{mod}\, R)^{\perp_1}$. So $\mathscr{I}^\infty(\mathrm{mod}\, R) = \mathscr{I}^n(\mathrm{mod}\, R)$ and hence $\mathscr{P}^\infty(\mathrm{mod}\, R^{op}) = \mathscr{P}^n(\mathrm{mod}\, R^{op})$. Thus $\mathrm{id}_{R^{op}} R \leq n$ by [HI, Corollary 5.3], which implies that R is Gorenstein by [AR3, Corollary 5.5(b)]. □

As an application of Theorem 6.8, we obtain in the following result some equivalent characterizations of Auslander-regular algebras. Note that the converse of Corollary 5.10 does not hold true in general by Remark 5.11. The following result also shows when this converse holds true.

Theorem 6.9. *The following statements are equivalent.*

(1) R is Auslander-regular.

(2) $\mathscr{G}_\infty(0) = \mathscr{P}^0(\mathrm{Mod}\, R)$.

(3) $\mathscr{G}_\infty(0)\bigcap \mathrm{mod}\, R = \mathscr{P}^0(\mathrm{mod}\, R)$.

(4) $\mathscr{G}_\infty(s) = \mathscr{P}^s(\mathrm{Mod}\, R)$ *for any* $s \geq 0$.

(5) $\mathscr{G}_\infty(s) \bigcap \mathrm{mod}\, R = \mathscr{P}^s(\mathrm{mod}\, R)$ *for any* $s \geq 0$.

Proof. Both (2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (5) are trivial. By Corollary 5.13, we have (2) $\Leftrightarrow$ (4) and (3) $\Leftrightarrow$ (5).

(1) $\Rightarrow$ (2) By (1) and Corollary 5.10, we have $\mathscr{P}^0(\mathrm{Mod}\, R) \subseteq \mathscr{G}_\infty(0)$.

Assume that gl.dim $R = n(< \infty)$ and $M \in \mathscr{G}_\infty(0)$. Then in a minimal injective resolution $0 \to M \to E^0(M) \to E^1(M) \to \cdots \to E^n(M) \to 0$ of M in Mod R, $\mathrm{pd}_R\, E^i(M) \leq i$ for any $0 \leq i \leq n$. By the dimension shifting we have that M is projective. It implies that $\mathscr{G}_\infty(0) \subseteq \mathscr{P}^0(\mathrm{Mod}\, R)$.

(5) $\Rightarrow$ (1) By (5), R satisfies the Auslander condition and $\mathscr{G}_\infty(0) \bigcap \mathrm{mod}\, R = \mathscr{P}^0(\mathrm{mod}\, R)$ is contravariantly finite in mod R. So R is Gorenstein by Theorem 6.8. Assume that $\mathrm{id}_{R^{op}}\, R = \mathrm{id}_R\, R = n(< \infty)$. Then $\mathrm{pd}_R\, E \leq n$ for any injective left R-module E by [I, Proposition 1]. So for any $M \in \mathrm{mod}\, R$, $M \in \mathscr{G}_\infty(n) \bigcap \mathrm{mod}\, R$, and hence $\mathrm{pd}_R\, M \leq n$ by (5). It follows that gl.dim $R \leq n$. □

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