Localizing Estimates of the Support of Solutions of some Nonlinear Schrödinger Equations – The Stationary Case

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Abstract

The main goal of this paper is to study the nature of the support of the solution of suitable nonlinear Schrödinger equations mainly the compactness of the support and its spatial localization. This question is very related with pure essence of the derivation of the Schrödinger equation since it is well-known that if the linear Schrödinger equation is perturbated with "regular" potentials then the corresponding solution never vanishes on a positive measured subset of the domain, which corresponds with the impossibility of localize the particle. Here we shall prove that if the perturbation involves suitable singular nonlinear terms then the support of the solution becomes a compact set and so any estimate on its spatial localization implies a very rich information on places which can not be. Our results are obtained by the application of some energy methods which connect the compactness of the support with the local vanishing of a suitable "energy function" which satisfies a nonlinear differential inequality with an exponent less than one. The results improve and extend a previous short presentation by the authors published in 2006.

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Contents

1 Introduction

This paper deals with the study of the following stationary nonlinear Schrödinger equation (SNLS) with a complex singular potential

$$-\mathrm{i}\Delta\boldsymbol{u}+\boldsymbol{a}|\boldsymbol{u}|^{-(1-m)}\boldsymbol{u}+\boldsymbol{b}\boldsymbol{u}=\boldsymbol{F}(x),\ \text{ in } \Omega. \tag{1.1}$$

Here, $\Omega\subseteq\mathbb{R}^N$ is an open subset, $0<m<1$ and $(\boldsymbol{a},\boldsymbol{b})\in\mathbb{C}^2$. Such stationary problem is motivated not only by its importance as asymptotic states, when $t\longrightarrow\infty$, of the associated evolution problem but also by the study of the so called standing waves of the evolution problem (1.2) below, with $\boldsymbol{b}\in\mathrm{i}\mathbb{R}$ in (1.1). Indeed, setting for any $(t,x)\in\mathbb{R}\times\Omega$, $\boldsymbol{u}(t,x)=\boldsymbol{\varphi}(x)e^{\mathrm{i}bt}$, if $\boldsymbol{\varphi}$ is a solution to the following stationary nonlinear Schrödinger equation,

$$\begin{cases}-\mathrm{i}\Delta\boldsymbol{\varphi}+\boldsymbol{a}|\boldsymbol{\varphi}|^{-(1-m)}\boldsymbol{\varphi}+\mathrm{i}b\boldsymbol{\varphi}=\boldsymbol{F}(x), \text{ in } \Omega,\\ \boldsymbol{\varphi}_{|\partial\Omega}=\boldsymbol{0}, \text{ on } \partial\Omega,\end{cases}$$

with $b\in\mathbb{R}$, then $\boldsymbol{u}$ is a solution to

$$\begin{cases}\mathrm{i}\dfrac{\partial\boldsymbol{u}}{\partial t}+\Delta\boldsymbol{u}+\mathrm{i}\boldsymbol{a}|\boldsymbol{u}|^{-(1-m)}\boldsymbol{u}=\mathrm{i}\boldsymbol{F}(x)e^{\mathrm{i}bt}, \text{ in } \mathbb{R}\times\Omega,\\ \boldsymbol{u}_{|\partial\Omega}=\boldsymbol{0}, \text{ on } \mathbb{R}\times\partial\Omega,\\ \boldsymbol{u}(0)=\boldsymbol{\varphi}, \text{ in } \Omega.\end{cases} \tag{1.2}$$

The main goal of this paper is to study the nature of the support of the solution of (1.1): mainly its compactness and localization. Let us mention that, in our opinion, this question is very related with pure essence of the derivation of the Schrödinger equation. Indeed, one of the main modifications introduced by Quantum Mechanics, with respect Classical Mechanics, is the impossibility to localize the state (position and velocity) of a particle. The solution $\boldsymbol{u}(t,x)$ is related with the probability of finding the position and momentum of particle (see, e.g. the presentation made in the text book by Strauss [24]. It is well-known that in most of the different versions of the Schrödinger equations the corresponding solution never vanishes on a positive measured subset of the domain, which corresponds with the mentioned impossibility of localize the particle. This is the case, for instance, of the linear Schrödinger equation and also of some nonlinear versions of it when the linear equation is perturbed with a nonlinear regular potential (see, for instance, the monographs Sulem and Sulem [25] and Cazenave [9]).

The main goal of this work is to show that if the linear Schrödinger equation is perturbed with suitable singular nonlinear potentials then the support of the solution becomes a compact set and so any estimate on its spatial localization implies a very rich information on places which can not be occupied by the particle.

We point out that complex potentials presenting some singularities of different types arise in many different situations (see, for instance, Brezis and Kato [7], LeMesurier [19] and Liskevitch and Stollmann [22], and the references therein). We also send the reader to the survey Belmonte-Beitia [6] in which the author supply many references connecting this type of equation with many other contexts such as: semiconductors, nonlinear optics, Bose-Einstein condensation, plasma physics, molecular dynamics. Special mention is made in this paper on the so called Gross-Pitaevskii (corresponding to $\boldsymbol{b} \neq \mathbf{0}$).

In this paper, we improve some of our previous results presented (in a short way) in Bégout and Díaz [4]. Moreover, we include here new estimates and generalizations. We are aware of very few other results in the literature dealing with the support of solutions of nonlinear Schrödinger equations. For instance, Rosenau and Schochet [23] propose a (one-dimensional) quasilinear Schrödinger equation in order to get solutions with compact support for each t fixed. That equation and the techniques used in that paper are very different to the ones of the present work. Analogously, in a paper dated from 2008 ([18]), Kashdan and Rosenau consider the question of the existence (with some numerical experiences) of some special solutions: an one-dimensional travelling wave solution of soliton type $\boldsymbol{u}(t,x) = A(x-\lambda t)\exp\big(\mathbf{i}(\ell(x-\lambda t)+\omega t)\big)$, for the special case of $\boldsymbol{a} = \mathbf{i}\gamma$ (in problem (1.2)) and assuming also that $m \in (0,1)$. They also consider the bidimensional case (now with changing propagation directions). A nonlinear term (of cubic type) is added in their equation. Those interesting results are independent of our study which also applies in the presence of some additional nonlinear terms as in the above mentioned reference.

A more closed point of view was taken in the paper Carles and Gallo [8] where the authors prove the finite time stabilization for a linear Schrödinger equations perturbated with a suitable singular nonlinear potential. In their case they prove also some kind of compactness of the support of the solution by means of a different energy method but in his case the compactness occurs merely in time and not in the spatial coordinates.

We also point out that different propagation effects have been intensively studied in the literature but most of them are related to singularities, spectral and other properties (see, for instance, Jensen [17]) but of a very different nature to the question of the compactness of the support considered here.

Before to state our main results we shall indicate here some of the notations we shall use in this paper. The bold symbol is used for complex mathematics objets. For a real number r, $r_+ = \max\{0, r\}$ is the positive part of r. We write $\mathbf{i}^2 = -1$. We denote by $\overline{\boldsymbol{z}}$ the conjugate of the complex number $\boldsymbol{z}$, by $\mathrm{Re}(\boldsymbol{z})$ its real part and by $\mathrm{Im}(\boldsymbol{z})$ its imaginary part. For $1 \leqslant p \leqslant \infty$, p' is the conjugate of p defined by $\frac{1}{p}+\frac{1}{p'}=1$. Let $j,k \in \mathbb{Z}$ with $j<k$. We then write $[\![j,k]\!] = [j,k] \cap \mathbb{Z}$. We denote by $\partial\Omega$ the boundary of a nonempty subset $\Omega \subseteq \mathbb{R}^N$, $\overline{\Omega}$ its closure, $\Omega^{\mathrm{c}} = \mathbb{R}^N \setminus \Omega$ its complementary and $\omega \Subset \Omega$ means that $\omega \subset \Omega$ and that ω is a compact subset of $\mathbb{R}^N$. For an open subset $\Omega \subseteq \mathbb{R}^N$, the usual Lebesgue and Sobolev spaces are respectively denoted by $\boldsymbol{L^p}(\Omega) = \boldsymbol{L^p}(\Omega;\mathbb{C})$ and $\boldsymbol{W^{m,p}}(\Omega) = \boldsymbol{W^{m,p}}(\Omega;\mathbb{C})$ ($1 \leqslant p \leqslant \infty$ and $m \in \mathbb{N}$), $\boldsymbol{H^m}(\Omega) = \boldsymbol{W^{m,2}}(\Omega;\mathbb{C})$, $\boldsymbol{H^m_0}(\Omega) = \boldsymbol{W^{m,2}_0}(\Omega;\mathbb{C})$ is the closure of $\mathscr{D}(\Omega) = \mathscr{D}(\Omega;\mathbb{C})$ for the $\boldsymbol{H^m}$-norm and $\boldsymbol{H^{-m}}(\Omega)$ is its topological dual. $\boldsymbol{H^1_c}(\Omega) = \left\{\boldsymbol{u} \in \boldsymbol{H^1}(\Omega); \operatorname{supp} \boldsymbol{u} \Subset \Omega\right\}$. $\boldsymbol{C}(\Omega) = \boldsymbol{C^0}(\Omega) = \boldsymbol{C}(\Omega;\mathbb{C}) = \boldsymbol{C^0}(\Omega;\mathbb{C})$ is the space of continuous functions from Ω to $\mathbb{C}$. For $k \in \mathbb{N}$, $\boldsymbol{C^k}(\Omega) = \boldsymbol{C^k}(\Omega;\mathbb{C})$ is the space of functions lying in $\boldsymbol{C}(\Omega;\mathbb{C})$ and having all derivatives of order lesser or equal than k belonging to $\boldsymbol{C}(\Omega;\mathbb{C})$. For $0<\alpha\leqslant 1$ and $k \in \mathbb{N}\cup\{0\}$, $\boldsymbol{C^{k,\alpha}_{\mathbf{loc}}}(\Omega) = \boldsymbol{C^{k,\alpha}_{\mathbf{loc}}}(\Omega;\mathbb{C}) = \left\{\boldsymbol{u} \in \boldsymbol{C^k}(\Omega;\mathbb{C}); \forall \omega \Subset \Omega,\ \sum\limits_{|\beta|=k} H^\alpha_\omega(D^\beta \boldsymbol{u}) < +\infty\right\}$, where $H^\alpha_\omega(\boldsymbol{u}) = \sup\limits_{\left\{\substack{(x,y)\in\omega^2\\ x\neq y}\right.} \frac{|\boldsymbol{u}(x)-\boldsymbol{u}(y)|}{|x-y|^\alpha}$. The Laplacian in Ω is written $\Delta = \sum\limits_{j=1}^N \frac{\partial^2}{\partial x_j^2}$. For a functional space $\boldsymbol{E} \subset \boldsymbol{L^1_{\mathbf{loc}}}(\Omega;\mathbb{C})$, we denote by $\boldsymbol{E_{\mathbf{rad}}}$ the space of functions $\boldsymbol{f} \in \boldsymbol{E}$ such that $\boldsymbol{f}$ is spherically symmetric. For a Banach space E, we denote by $E^\star$ its topological dual and by $\langle\, .\, ,\, .\,\rangle_{E^\star,E} \in \mathbb{R}$ the $E^\star - E$ duality product. In particular, for any $\boldsymbol{T} \in \boldsymbol{L^{p'}}(\Omega)$ and $\boldsymbol{\varphi} \in \boldsymbol{L^p}(\Omega)$ with $1 \leqslant p < \infty$, $\langle \boldsymbol{T}, \boldsymbol{\varphi}\rangle_{\boldsymbol{L^{p'}}(\Omega),\boldsymbol{L^p}(\Omega)} = \mathrm{Re}\int\limits_\Omega \boldsymbol{T}(x)\overline{\boldsymbol{\varphi}(x)}\mathrm{d}x$. For $x_0 \in \mathbb{R}^N$ and $r>0$, we denote by $B(x_0,r) = \{x \in \mathbb{R}^N;\ |x-x_0|<r\}$ the open ball of $\mathbb{R}^N$ of center x_0 and radius r, by $\mathbb{S}(x_0,r) = \{x \in \mathbb{R}^N;\ |x-x_0| = r\}$ its boundary and by $\overline{B}(x_0,r) = B(x_0,r)\cup\mathbb{S}(x_0,r)$ its closure. We also use the notation $B_\Omega(x_0,r) = \Omega \cap B(x_0,r)$. As usual, we denote by C auxiliary positive constants,

and sometimes, for positive parameters $a_1,\ldots,a_n$, write as $C(a_1,\ldots,a_n)$ to indicate that the constant C depends only on $a_1,\ldots,a_n$ and that dependence is continuous (we also will use this convention for constants which are not denoted by "C").

Let us return to equation (1.2). Note that no boundary condition is imposed since all the support compact results (which are due to Theorem 1.1 below) rest on the notion of local solution (Definition 1.3 below). If $\Omega \neq \mathbb{R}^N$, boundary condition is necessary to establish existence and uniqueness of global solutions of (1.1) and by worries of clarity we shall consider the Dirichlet case,

$$\boldsymbol{u}_{|\partial\Omega} = \boldsymbol{0}, \text{ on } \partial\Omega, \tag{1.3}$$

rather than Neumann boundary condition, mixed boundary condition or another one. The choice of the boundary condition is motivated to use integration by parts $\langle \Delta u, v\rangle = -\langle \nabla u, \nabla v\rangle$.

Compactness, existence and uniqueness results will follow from the below assumptions on $(\boldsymbol{a},\boldsymbol{b}) \in \mathbb{C}^2$. Let define the following subsets.

$$\begin{cases} \mathbb{A} = \mathbb{C} \setminus \big\{\boldsymbol{z} \in \mathbb{C}; \mathrm{Re}(\boldsymbol{z}) = 0 \text{ and } \mathrm{Im}(\boldsymbol{z}) \leqslant 0\big\}, \\ \mathbb{B} = \mathbb{A} \cup \big\{\boldsymbol{0}\big\}. \end{cases}$$

Existence assumption. Let $(\boldsymbol{a},\boldsymbol{b}) \in \mathbb{C}^2$ satisfies

$$(\boldsymbol{a},\boldsymbol{b}) \in \mathbb{A}\times\mathbb{B} \quad \text{and} \quad \begin{cases} \mathrm{Re}(\boldsymbol{a})\mathrm{Re}(\boldsymbol{b}) \geqslant 0, \\ \text{or} \\ \mathrm{Re}(\boldsymbol{a})\mathrm{Re}(\boldsymbol{b}) < 0 \text{ and } \mathrm{Im}(\boldsymbol{b}) > \dfrac{\mathrm{Re}(\boldsymbol{b})}{\mathrm{Re}(\boldsymbol{a})}\mathrm{Im}(\boldsymbol{a}). \end{cases} \tag{1.4}$$

Uniqueness assumption. Let $(\boldsymbol{a},\boldsymbol{b}) \in \mathbb{C}^2$ satisfies

$$\mathrm{Im}(\boldsymbol{a}) \geqslant 0 \quad \text{and} \quad \begin{cases} \boldsymbol{a} \neq \boldsymbol{0} \text{ and } \mathrm{Re}(\boldsymbol{a}\overline{\boldsymbol{b}}) \geqslant 0, \\ \text{or} \\ \boldsymbol{a} = \boldsymbol{0} \text{ and } \boldsymbol{b} \in \mathbb{B}. \end{cases} \tag{1.5}$$

For geometric explanation about these hypotheses, see Section 5. For $(\boldsymbol{a},\boldsymbol{b}) \in \mathbb{C}^2$ satisfying (1.4), it will be convenient to introduce the following constants. Let $\delta > 0$ be a parameter arbitrarily chosen.

$$A(\delta) = \frac{|\mathrm{Re}(\boldsymbol{a})| + |\mathrm{Im}(\boldsymbol{a})| + \delta}{|\mathrm{Re}(\boldsymbol{a})|}, \text{ if } \mathrm{Re}(\boldsymbol{a}) \neq 0, \tag{1.6}$$

$$B = \frac{|\mathrm{Re}(\boldsymbol{b})| + |\mathrm{Im}(\boldsymbol{b})|}{|\mathrm{Re}(\boldsymbol{b})|}, \text{ if } \mathrm{Re}(\boldsymbol{b}) \neq 0, \tag{1.7}$$

$$L=\begin{cases}\delta, & \text{if } \mathrm{Im}(\boldsymbol{a})<0 \ \text{ and } \ \mathrm{Re}(\boldsymbol{a})\mathrm{Re}(\boldsymbol{b})\geqslant 0,\\ |\mathrm{Re}(\boldsymbol{a})|, & \text{if } \mathrm{Im}(\boldsymbol{a})=0,\ \mathrm{Im}(\boldsymbol{b})\geqslant 0 \ \text{ and } \ \mathrm{Re}(\boldsymbol{a})\mathrm{Re}(\boldsymbol{b})\geqslant 0,\\ \mathrm{Im}(\boldsymbol{a}) & \text{if } \mathrm{Im}(\boldsymbol{a})>0 \ \text{ and } \ \mathrm{Im}(\boldsymbol{b})\geqslant 0,\\ \mathrm{Im}(\boldsymbol{a})-\dfrac{\mathrm{Re}(\boldsymbol{a})}{\mathrm{Re}(\boldsymbol{b})}\mathrm{Im}(\boldsymbol{b}), & \text{otherwise,}\end{cases} \tag{1.8}$$

$$M=\begin{cases}\max\big\{A(\delta),B\big\}, & \text{if } \mathrm{Im}(\boldsymbol{a})<0,\ \mathrm{Im}(\boldsymbol{b})<0 \ \text{ and } \ \mathrm{Re}(\boldsymbol{a})\mathrm{Re}(\boldsymbol{b})\geqslant 0,\\ A(\delta), & \text{if } \mathrm{Im}(\boldsymbol{a})<0,\ \mathrm{Im}(\boldsymbol{b})\geqslant 0 \ \text{ and } \ \mathrm{Re}(\boldsymbol{a})\mathrm{Re}(\boldsymbol{b})\geqslant 0,\\ 2 & \text{if } \mathrm{Im}(\boldsymbol{a})\geqslant 0,\ \mathrm{Im}(\boldsymbol{b})\geqslant 0 \ \text{ and } \ \big(\mathrm{Im}(\boldsymbol{a})>0 \ \text{ or } \ \mathrm{Re}(\boldsymbol{a})\mathrm{Re}(\boldsymbol{b})\geqslant 0\big),\\ B & \text{if } \big(\mathrm{Im}(\boldsymbol{a})\geqslant 0 \ \text{ and } \ \mathrm{Im}(\boldsymbol{b})<0\big) \ \text{ or } \ \mathrm{Re}(\boldsymbol{a})\mathrm{Re}(\boldsymbol{b})<0.\end{cases} \tag{1.9}$$

Under hypothesis (1.4), one easily checks that $A(\delta)$, B, L and M are well defined and positive.

Numerical computations about stationary solutions are done in Bégout and Torri [5], while the evolution case and self-similar solutions are respectively studied in Bégout and Díaz [2, 3]. In this paper, we prove the results stated in Bégout and Díaz [4] and add some generalizations. This paper is concerned by propagation of the support of $\boldsymbol{F}$ to the solution $\boldsymbol{u}$ and all these results are a consequence of the following theorem.

Theorem 1.1. *Let $\Omega\subseteq\mathbb{R}^N$ be a nonempty open subset, let $0<m<1$, let $(\boldsymbol{a},\boldsymbol{b})\in\mathbb{C}^2$ satisfying (1.4), let $L>0$ be given by (1.8) and let $M>0$ be given by (1.9). There exists $C=C(N,m)>0$ satisfying the following property. Let $\boldsymbol{F}\in\boldsymbol{L}^1_{\mathrm{loc}}(\Omega)$, let $\boldsymbol{u}\in\boldsymbol{H}^1_{\mathrm{loc}}(\Omega)$ be any local weak solution of (1.1) (see Definition 1.3 below), let $x_0\in\Omega$ and let $\rho_0>0$. If $\rho_0>\mathrm{dist}(x_0,\partial\Omega)$ then assume further that $\boldsymbol{u}\in\boldsymbol{H}^1_0(\Omega)$. If $\boldsymbol{F}_{|B_\Omega(x_0,\rho_0)}\equiv\boldsymbol{0}$ then $\boldsymbol{u}_{|B_\Omega(x_0,\rho_{\max})}\equiv\boldsymbol{0}$, where*

$$\rho_{\max}^\nu=\left(\rho_0^\nu-CM^2\max\left\{1,\frac{1}{L^2}\right\}\max\left\{\rho_0^{\nu-1},1\right\}\right.$$
$$\left.\times\min_{\tau\in\left(\frac{m+1}{2},1\right]}\left\{\frac{E(\rho_0)^{\gamma(\tau)}\max\{b(\rho_0)^{\mu(\tau)},b(\rho_0)^{\eta(\tau)}\}}{2\tau-(1+m)}\right\}\right)_+, \tag{1.10}$$

and where for any $\tau\in\left(\frac{m+1}{2},1\right]$,

$$E(\rho_0)=\|\nabla\boldsymbol{u}\|^2_{\boldsymbol{L^2}(B_\Omega(x_0,\rho_0))},\quad b(\rho_0)=\|\boldsymbol{u}\|^{m+1}_{\boldsymbol{L^{m+1}}(B_\Omega(x_0,\rho_0))},\quad \gamma(\tau)=\tfrac{2\tau-(1+m)}{k}\in(0,1),$$
$$\mu(\tau)=\tfrac{2(1-\tau)}{k},\qquad \eta(\tau)=\tfrac{1-m}{1+m}-\gamma(\tau)>0,\qquad k=2(1+m)+N(1-m),$$
$$\nu=\tfrac{k}{m+1}>2.$$

Remark 1.2. If the solution is too "large", it may happen that $\rho_{\max}=0$ and so the above result is not consistent. A sufficient condition to observe a localizing effect is that the solution is small enough, in a suitable sense. We give two results in this direction. The first one (Theorem 2.3) pertains to the size of the solution, while the second one is concerning to the size of the external source $\boldsymbol{F}$ (Theorem 2.5), which seems to be more natural. In addition, Theorem 2.5 says where the support of the solutions is localized with respect to the support of the external source $\boldsymbol{F}$.

Now, we precise the notion of solution.

Definition 1.3. Let $\Omega \subseteq \mathbb{R}^N$ be an open subset, let $(\boldsymbol{a},\boldsymbol{b}) \in \mathbb{C}^2$, let $0<m<1$ and let $\boldsymbol{F} \in \boldsymbol{L}^1_{\mathrm{loc}}(\Omega)$. We say that $\boldsymbol{u}$ is a *local weak solution* of (1.1) if $\boldsymbol{u} \in \boldsymbol{H}^1_{\mathrm{loc}}(\Omega)$ and if $\boldsymbol{u}$ is a solution of (1.1) in $\mathscr{D}'(\Omega)$, that is

$$\langle -\mathrm{i}\Delta \boldsymbol{u} + \boldsymbol{a}|\boldsymbol{u}|^{-(1-m)}\boldsymbol{u} + \boldsymbol{b}\boldsymbol{u}, \boldsymbol{\varphi}\rangle_{\mathscr{D}'(\Omega),\mathscr{D}(\Omega)} = \langle \boldsymbol{F}, \boldsymbol{\varphi}\rangle_{\mathscr{D}'(\Omega),\mathscr{D}(\Omega)}, \tag{1.11}$$

for any $\boldsymbol{\varphi} \in \mathscr{D}(\Omega)$.
We say that $\boldsymbol{u}$ is a *global weak solution* of (1.1) and (1.3) if $\boldsymbol{u}$ is a local weak solution of (1.1) and if furthermore $\boldsymbol{u} \in \boldsymbol{H}^1_0(\Omega) \cap \boldsymbol{L}^{m+1}(\Omega)$.

Remark 1.4. Here are some comments about Definition 1.3.

1. For a global weak solution $\boldsymbol{u}$ of (1.1) and (1.3), the boundary condition $\boldsymbol{u}_{|\partial\Omega} = 0$ is included in the assumption $\boldsymbol{u} \in \boldsymbol{H}^1_0(\Omega)$. At the contrary, the notion of local weak solution does not consider any boundary condition.

2. When $\boldsymbol{u}$ is a local weak solution of (1.1), we have $\nabla \boldsymbol{u} \in \boldsymbol{L}^2_{\mathrm{loc}}(\Omega)$, $\boldsymbol{a}|\boldsymbol{u}|^{-(1-m)}\boldsymbol{u} \in \boldsymbol{L}^{\frac{m+1}{m}}_{\mathrm{loc}}(\Omega)$ and $\boldsymbol{b}\boldsymbol{u} \in \boldsymbol{L}^2_{\mathrm{loc}}(\Omega)$. Then $\Delta \boldsymbol{u} \in \boldsymbol{L}^1_{\mathrm{loc}}(\Omega)$ and equation (1.1) makes sense in $\boldsymbol{L}^1_{\mathrm{loc}}(\Omega)$. Furthermore, $\boldsymbol{L}^{\frac{m+1}{m}}_{\mathrm{loc}}(\Omega) \subset \boldsymbol{L}^2_{\mathrm{loc}}(\Omega)$ and $\mathscr{D}(\Omega)$ is dense in $\boldsymbol{H}^1_{\mathrm{c}}(\Omega)$. It follows from Sobolev's embedding that if $\boldsymbol{u}$ is a local weak solution of (1.1) then

$$\mathrm{Re}\int\limits_\Omega \mathrm{i}\nabla \boldsymbol{u}(x).\overline{\nabla \boldsymbol{\varphi}(x)}\mathrm{d}x + \mathrm{Re}\int\limits_\Omega \Big(\boldsymbol{a}|\boldsymbol{u}(x)|^{-(1-m)}\boldsymbol{u}(x) + \boldsymbol{b}\boldsymbol{u}(x)\Big)\overline{\boldsymbol{\varphi}(x)}\mathrm{d}x = \mathrm{Re}\int\limits_\Omega \boldsymbol{F}(x)\overline{\boldsymbol{\varphi}(x)}\mathrm{d}x, \tag{1.12}$$

for any $\boldsymbol{\varphi} \in \boldsymbol{H}^1_{\mathrm{c}}(\Omega)$ with either $\operatorname{supp}\boldsymbol{\varphi} \cap \operatorname{supp}\boldsymbol{F} = \emptyset$ or $\boldsymbol{F} \in \boldsymbol{L}^{\frac{p}{p-1}}_{\mathrm{loc}}(\Omega)$, for some $1 \leqslant p \leqslant \infty$ if $N=1$, $1 \leqslant p < \infty$ if $N=2$ or $1 \leqslant p \leqslant \frac{2N}{N-2}$, if $N \geqslant 3$. For example, $p = m+1$ is always an admissible value.

3. In the same way, by density of $\mathscr{D}(\Omega)$ in $\boldsymbol{H}^1_0(\Omega) \cap \boldsymbol{L}^{m+1}(\Omega) \cap \boldsymbol{L}^p(\Omega)$, for any $1 \leqslant p < \infty$, and in $\boldsymbol{H}^1_0(\Omega) \cap \boldsymbol{L}^{m+1}(\Omega)$, if u is a global weak solution of (1.1) and (1.3) then (1.12) holds for any

$\boldsymbol{\varphi} \in \boldsymbol{H_0^1}(\Omega) \cap \boldsymbol{L^{m+1}}(\Omega)$ with either $\operatorname{supp} \boldsymbol{\varphi} \cap \operatorname{supp} \boldsymbol{F} = \emptyset$ or $\boldsymbol{\varphi} \in \boldsymbol{L^p}(\Omega)$ and $\boldsymbol{F} \in \boldsymbol{L}^{\frac{p}{p-1}}(\Omega)$, for some $1 \leqslant p < \infty$. In particular, if p is as in 2. of this remark with additionally $p \geqslant m+1$, then $\boldsymbol{H_0^1}(\Omega) \cap \boldsymbol{L^{m+1}}(\Omega) \hookrightarrow \boldsymbol{L^p}(\Omega)$, equation (1.1) makes sense in $\boldsymbol{H^{-1}}(\Omega) + \boldsymbol{L}^{\frac{m+1}{m}}(\Omega)$ and (1.12) holds for any $\boldsymbol{\varphi} \in \boldsymbol{H_0^1}(\Omega) \cap \boldsymbol{L^{m+1}}(\Omega)$.

2 Spacial localization property

Theorem 2.1. *Let $\Omega \subseteq \mathbb{R}^N$ be a nonempty open subset, let $0 < m < 1$ and let $(\boldsymbol{a}, \boldsymbol{b}) \in \mathbb{C}^2$ satisfying (1.4). Let $\boldsymbol{F} \in \boldsymbol{L}^{\frac{m+1}{m}}(\Omega)$, let $\boldsymbol{u} \in \boldsymbol{H^1_{\mathrm{loc}}}(\Omega)$ be any local weak solution of (1.1) (Definition 1.3), let $x_0 \in \Omega$ and let $\rho_1 > 0$. If $\rho_1 > \operatorname{dist}(x_0, \partial\Omega)$ then assume further that $\boldsymbol{u} \in \boldsymbol{H_0^1}(\Omega)$. Then there exist $E_\star > 0$ and $\varepsilon_\star > 0$ satisfying the following property. Let $\rho_0 \in (0, \rho_1)$. If $\|\nabla \boldsymbol{u}\|^2_{\boldsymbol{L^2}(B(x_0,\rho_1))} < E_\star$ and if*

$$\forall \rho \in (0, \rho_1), \ \|\boldsymbol{F}\|^{\frac{m+1}{m}}_{\boldsymbol{L}^{\frac{m+1}{m}}(B(x_0,\rho))} < \varepsilon_\star \big((\rho - \rho_0)_+\big)^p, \tag{2.1}$$

where $p = \frac{2(1+m)+N(1-m)}{1-m} > N+2$, then $\boldsymbol{u}_{|B_\Omega(x_0,\rho_0)} \equiv \boldsymbol{0}$. In other words, with the notation of Theorem 1.1, $\rho_{\max} = \rho_0$.

Remark 2.2. We may estimate $E_\star$ and $\varepsilon_\star$ as

$$E_\star = E_\star\left(\|\boldsymbol{u}\|^{-1}_{\boldsymbol{L^{m+1}}(B(x_0,\rho_1))}, \rho_1, \frac{\rho_0}{\rho_1}, \frac{L}{M}, N, m\right),$$
$$\varepsilon_\star = \varepsilon_\star\left(\|\boldsymbol{u}\|^{-1}_{\boldsymbol{L^{m+1}}(B(x_0,\rho_1))}, \frac{\rho_0}{\rho_1}, \frac{L}{M}, N, m\right),$$

where $L > 0$ and $M > 0$ are given by (1.6) and (1.9), respectively. The dependence in $\frac{1}{\delta}$ means that for any value δ small enough, $E_\star$ and $\varepsilon_\star$ are bounded from below.
Note that $p = \frac{1}{\gamma(1)}$, where γ is the function defined in Theorem 1.1.

Theorem 2.3. *Let $\Omega \subseteq \mathbb{R}^N$ be a nonempty open subset, let $0 < m < 1$, let $(\boldsymbol{a}, \boldsymbol{b}) \in \mathbb{C}^2$ satisfying (1.4), let $L > 0$ be given by (1.8) and let $M > 0$ be given by (1.9). There exists $C = C(N, m) > 0$ satisfying the following property. Let $\boldsymbol{F} \in \boldsymbol{L^1_{\mathrm{loc}}}(\Omega)$, let $\boldsymbol{u} \in \boldsymbol{H^1_{\mathrm{loc}}}(\Omega)$ be any local weak solution of (1.1) (Definition 1.3), let $x_0 \in \Omega$ and let $\rho_0 > 0$. If $2\rho_0 > \operatorname{dist}(x_0, \partial\Omega)$ then assume further that $\boldsymbol{u} \in \boldsymbol{H_0^1}(\Omega)$. Finally, suppose $\boldsymbol{F}_{|B_\Omega(x_0,2\rho_0)} \equiv \boldsymbol{0}$, $\|\boldsymbol{u}\|_{\boldsymbol{L^{m+1}}(B_\Omega(x_0,2\rho_0))} \leqslant 1$ and one of the two estimates (2.2) or (2.3) below is satisfied.*

$$\|\nabla \boldsymbol{u}\|^{\frac{2(1-m)}{k}}_{\boldsymbol{L^2}(B_\Omega(x_0,2\rho_0))} \leqslant C(2^\nu - 1)(1-m)M^{-2} \min\left\{1, L^2\right\} \min\left\{\frac{1}{2}, \rho_0\right\}^{\nu-1} \rho_0, \tag{2.2}$$

$$\begin{cases} \|\nabla \boldsymbol{u}\|_{\boldsymbol{L^2}(B_\Omega(x_0,2\rho_0))} \leqslant 1, \\ \|\boldsymbol{u}\|^{\frac{2s(m+1)}{k}}_{\boldsymbol{L^{m+1}}(B_\Omega(x_0,2\rho_0))} \leqslant C(2^\nu - 1)(1-m-2s)M^{-2} \min\left\{1, L^2\right\} \min\left\{\frac{1}{2}, \rho_0\right\}^{\nu-1} \rho_0, \end{cases} \tag{2.3}$$

for some $s \in \left(0, \frac{1-m}{2}\right)$, *where the constants* $k > \nu > 2$ *are given in Theorem* 1.1. *Then* $\boldsymbol{u}_{|B_\Omega(x_0,\rho_0)} \equiv \mathbf{0}$.

Remark 2.4. Note that in estimate (2.2), $\frac{2(1-m)}{k} = \frac{2}{p}$, where $p > N+2$ is given in Theorem 2.1.

Theorem 2.5. *Let* $\Omega \subseteq \mathbb{R}^N$ *be a nonempty open subset, let* $0 < m < 1$, *let* $(\boldsymbol{a}, \boldsymbol{b}) \in \mathbb{C}^2$ *satisfying* (1.4), *let* $L > 0$ *be given by* (1.8) *and let* $M > 0$ *be given by* (1.9). *Then for any* $\varepsilon > 0$, *there exists* $\delta_0 = \delta_0(\varepsilon, N, m, L, M) > 0$ *satisfying the following property. Let* $\boldsymbol{F} \in \boldsymbol{L}^{\frac{m+1}{m}}(\Omega)$ *and let* $\boldsymbol{u} \in \boldsymbol{H_0^1}(\Omega) \cap \boldsymbol{L^{m+1}}(\Omega)$ *be any global weak solution of* (1.1) *and* (1.3). *If* $\operatorname{supp} \boldsymbol{F}$ *is a compact set and if* $\|\boldsymbol{F}\|_{\boldsymbol{L}^{\frac{m+1}{m}}(\Omega)} \leqslant \delta_0$ *then* $\operatorname{supp} \boldsymbol{u} \subset \overline{\Omega} \cap \mathcal{O}(\varepsilon)$, *where* $\mathcal{O}(\varepsilon)$ *is the open bounded set*

$$\mathcal{O}(\varepsilon) = \left\{x \in \mathbb{R}^N;\ \exists y \in \operatorname{supp} \boldsymbol{F} \ \text{ such that } \ |x-y| < \varepsilon\right\}.$$

In particular, if $\varepsilon > 0$ *is small enough then* $\operatorname{supp} \boldsymbol{u} \subset \mathcal{O}(\varepsilon) \subset \Omega$.

We see that localizing effect occurs under some smallness condition, either on the solution $\boldsymbol{u}$ (Theorem 2.3) or on the external source $\boldsymbol{F}$ (Theorem 2.5). When $\Omega = \mathbb{R}^N$, the phenomenon is simpler since localizing effect is always observed, without any condition of the size, neither on the solution nor on the external source, as show the following result.

Theorem 2.6. *Let* $0 < m < 1$, *let* $(\boldsymbol{a}, \boldsymbol{b}) \in \mathbb{C}^2$ *satisfying* (1.4), *let* $\boldsymbol{F} \in \boldsymbol{L^p}(\mathbb{R}^N)$, *for some* $1 \leqslant p \leqslant \infty$, *and let* $\boldsymbol{u} \in \boldsymbol{H^1}(\mathbb{R}^N) \cap \boldsymbol{L^{m+1}}(\mathbb{R}^N)$ *be any global weak solution of* (1.1). *If* $\operatorname{supp} \boldsymbol{F}$ *is a compact set then* $\operatorname{supp} \boldsymbol{u}$ *is also compact.*

3 Existence and smoothness

In this section, we give an existence result of solutions for equation (1.1) (Theorem 3.1), some *a priori* bounds for the solutions of equation (1.1) (Theorem 3.4), which will be useful to establish our existence result, and a smoothness result for equation (1.1) (Proposition 3.5).

Theorem 3.1. *Let* $\Omega \subseteq \mathbb{R}^N$ *be a nonempty open subset, let* $0 < m < 1$, *let* $(\boldsymbol{a}, \boldsymbol{b}) \in \mathbb{C}^2$ *satisfying* (1.4) *and let* $\boldsymbol{F} \in \boldsymbol{L}^{\frac{m+1}{m}}(\Omega)$. *Then equations* (1.1) *and* (1.3) *admits at least one global weak solution* $\boldsymbol{u} \in \boldsymbol{H_0^1}(\Omega) \cap \boldsymbol{L^{m+1}}(\Omega)$. *Furthermore, the following properties hold for any global weak solution* $\boldsymbol{u}$ *(except Property 3)).*

1) $\boldsymbol{u} \in \boldsymbol{W}^{\boldsymbol{2},\frac{m+1}{m}}_{\mathbf{loc}}(\Omega)$.
2) *Let* $\alpha \in (0, m]$. *If* $\boldsymbol{F} \in \boldsymbol{C}^{\boldsymbol{0},\boldsymbol{\alpha}}_{\mathbf{loc}}(\Omega)$ *then* $\boldsymbol{u} \in \boldsymbol{C}^{\boldsymbol{2},\boldsymbol{\alpha}}_{\mathbf{loc}}(\Omega)$.
3) *If* $\Omega = \left\{x \in \mathbb{R}^N;\ r < |x| < R\right\}$, *for some* $-\infty < r \leqslant r_+ < R \leqslant +\infty$, *and if* $\boldsymbol{F}$ *is spherically symmetric then there exists a spherically symmetric global weak solution* $\boldsymbol{u} \in \boldsymbol{H_0^1}(\Omega) \cap$

$\boldsymbol{L^{m+1}}(\Omega)$ of (1.1) and (1.3). For $N=1$, this means that if $\boldsymbol{F}$ is an even (respectively, an odd) function on $\Omega=(-R,-r)\cup(r,R)$ then $\boldsymbol{u}$ is also an even (respectively, an odd) function.

Remark 3.2. Assume $\boldsymbol{F}$ is spherically symmetric. Since we do not know, in general, if we have uniqueness of the solution, we are not able to show that any solution is radially symmetric. For a uniqueness result, see Theorem 4.2 below.

Remark 3.3. Assume $|\Omega|<\infty$. There exists $\varepsilon=\varepsilon(N)>0$ such that for any $(\boldsymbol{a},\boldsymbol{b})\in\mathbb{C}^2$, $0<m<1$ and $\boldsymbol{F}\in\boldsymbol{L^2}(\Omega)$, if $|\boldsymbol{b}||\Omega|^{\frac{2}{N}}<\varepsilon$ then equations (1.1) and (1.3) admits at least one global weak solution $\boldsymbol{u}\in\boldsymbol{H_0^1}(\Omega)$. In addition, $\boldsymbol{u}\in\boldsymbol{H^2_{\mathrm{loc}}}(\Omega)$. Finally, Properties 2) and 3) of Theorem 3.1 hold. For more details, see Bégout and Torri [5].

Theorem 3.4. *Let $\Omega\subseteq\mathbb{R}^N$ be a nonempty open subset, let $0<m<1$, let $(\boldsymbol{a},\boldsymbol{b})\in\mathbb{C}^2$ satisfying (1.4), let $L>0$ be given by (1.8), let $M>0$ be given by (1.9) and let $\boldsymbol{F}\in\boldsymbol{L^{\frac{m+1}{m}}}(\Omega)$. Let $\boldsymbol{u}\in\boldsymbol{H_0^1}(\Omega)\cap\boldsymbol{L^{m+1}}(\Omega)$ be any global weak solution of (1.1) and (1.3). Then we have the following estimates.*

$$\|\nabla\boldsymbol{u}\|^2_{\boldsymbol{L^2}(\Omega)}+\|\boldsymbol{u}\|^{m+1}_{\boldsymbol{L^{m+1}}(\Omega)}<M_0\|\boldsymbol{F}\|^{\frac{m+1}{m}}_{\boldsymbol{L^{\frac{m+1}{m}}}(\Omega)}, \tag{3.1}$$

$$\|\boldsymbol{u}\|^2_{\boldsymbol{H_0^1}(\Omega)}+\|\boldsymbol{u}\|^{m+1}_{\boldsymbol{L^{m+1}}(\Omega)}<C\widetilde{M_0}\left(1+\|\boldsymbol{F}\|^{\frac{\delta(m+1)}{m}}_{\boldsymbol{L^{\frac{m+1}{m}}}(\Omega)}\right)\|\boldsymbol{F}\|^{\frac{m+1}{m}}_{\boldsymbol{L^{\frac{m+1}{m}}}(\Omega)}, \tag{3.2}$$

where $M_0=M\left(\frac{2M}{L}\right)^{\frac{1}{m}}\max\left\{1,\frac{2}{L}\right\}$, $\delta=\frac{2(1-m)}{(N+2)-m(N-2)}$, $\widetilde{M_0}=M_0(1+M_0^\delta)$ and $C=C(N,m)$.

Proposition 3.5. *Let $\boldsymbol{a}\in\mathbb{C}$, let $0<m<1$, let $\boldsymbol{V}\in\boldsymbol{L^r_{\mathrm{loc}}}(\Omega;\mathbb{C})$, for any $1<r<\infty$, let $\boldsymbol{F}\in\boldsymbol{L^1_{\mathrm{loc}}}(\Omega;\mathbb{C})$ and, for some $\varepsilon>0$, let $\boldsymbol{u}\in\boldsymbol{L^{1+\varepsilon}_{\mathrm{loc}}}(\Omega;\mathbb{C})$ $\big(\boldsymbol{u}\in\boldsymbol{L^1_{\mathrm{loc}}}(\Omega;\mathbb{C})$ is enough if $\boldsymbol{V}\in\boldsymbol{L^\infty_{\mathrm{loc}}}(\Omega;\mathbb{C})\big)$ be a solution to*

$$-\Delta\boldsymbol{u}+\boldsymbol{V}\boldsymbol{u}+\boldsymbol{a}|\boldsymbol{u}|^{-(1-m)}\boldsymbol{u}=\boldsymbol{F}(x),\ \text{in } \mathscr{D}'(\Omega) \tag{3.3}$$

is enough if $\boldsymbol{V}\in\boldsymbol{L^\infty_{\mathrm{loc}}}(\Omega;\mathbb{C})$). Let $1<q<\infty$ and suppose $\boldsymbol{u}\in\boldsymbol{L^q_{\mathrm{loc}}}(\Omega)$. Then the following regularity results hold.

1) *If for some $p\in[q,\infty)$, $\boldsymbol{F}\in\boldsymbol{L^p_{\mathrm{loc}}}(\Omega)$ then $\boldsymbol{u}\in\boldsymbol{W^{2,p}_{\mathrm{loc}}}(\Omega)$.*
2) *Let $\alpha\in(0,m]$. If $(\boldsymbol{F},\boldsymbol{V})\in\boldsymbol{C^{0,\alpha}_{\mathrm{loc}}}(\Omega)\times\boldsymbol{C^{0,\alpha}_{\mathrm{loc}}}(\Omega)$ then $\boldsymbol{u}\in\boldsymbol{C^{2,\alpha}_{\mathrm{loc}}}(\Omega)$.*

Remark 3.6. Since $0<m<1$ and $\boldsymbol{u}\in\boldsymbol{L^1_{\mathrm{loc}}}(\Omega)$, one has $\boldsymbol{L^{\frac{1}{m}}_{\mathrm{loc}}}(\Omega)\subset\boldsymbol{L^1_{\mathrm{loc}}}(\Omega)$ and $|\boldsymbol{u}|^{-(1-m)}\boldsymbol{u}\in\boldsymbol{L^{\frac{1}{m}}_{\mathrm{loc}}}(\Omega)$. In addition, from Hölder's inequality, $\boldsymbol{V}\boldsymbol{u}\in\boldsymbol{L^1_{\mathrm{loc}}}(\Omega)$. It follows that $|\boldsymbol{u}|^{-(1-m)}\boldsymbol{u}\in\boldsymbol{L^1_{\mathrm{loc}}}(\Omega)$ and so $\Delta\boldsymbol{u}\in\boldsymbol{L^1_{\mathrm{loc}}}(\Omega)$. In conclusion, equation (3.3) makes senses in $\boldsymbol{L^1_{\mathrm{loc}}}(\Omega)$.

Remark 3.7. We only state a local smoothness result since we are interested by compactly supported solutions. In this case, global smoothness is immediate. Nevertheless, one may wonder what happens when a solution is not compactly supported. We use the notation of Proposition 3.5 and assume further that Ω is bounded[1] and has a $C^{1,1}$ boundary. Let the assumptions of Proposition 3.5 be fulfilled and let $\boldsymbol{u} \in \boldsymbol{L^q}(\Omega)$, for some $1 < q < \infty$, be a solution to (3.3) such that $\boldsymbol{u}_{|\partial\Omega} = \boldsymbol{0}$ in the sense of the trace[2].

1. If for some $p \in [q, \infty)$, $\boldsymbol{F} \in \boldsymbol{L^p}(\Omega)$ and $\boldsymbol{V} \in \boldsymbol{L^r}(\Omega)$, $\forall r \in (1, \infty)$, then $\boldsymbol{u} \in \boldsymbol{W^{2,p}}(\Omega) \cap \boldsymbol{W_0^{1,p}}(\Omega)$. Indeed, recalling that if for some $1 < p < \infty$, a function $\boldsymbol{v} \in \boldsymbol{L^p}(\Omega)$ satisfies $\Delta \boldsymbol{v} \in \boldsymbol{L^p}(\Omega)$ and $\boldsymbol{v}_{|\partial\Omega} = \boldsymbol{0}$ in the sense of the trace[2] then $\boldsymbol{v} \in \boldsymbol{W^{2,p}}(\Omega) \cap \boldsymbol{W_0^{1,p}}(\Omega)$ (Grisvard [15], Corollary 2.5.2.2 p.131). We then apply the bootstrap method of the proof of Proposition 3.5 to prove the result, where we use the embedding $\boldsymbol{L^r}(\Omega) \hookrightarrow \boldsymbol{L^s}(\Omega)$, which holds for any $r \geqslant s$ (since Ω is bounded) and the global regularity result of Grisvard [15] (Corollary 2.5.2.2 p.131) in place of a local regularity result (Cazenave [10], Theorem 4.1.2 p.101–102).

2. If Ω has a $C^{2,\alpha}$ boundary and $(\boldsymbol{F}, \boldsymbol{V}) \in \boldsymbol{C^{0,\alpha}}(\overline{\Omega}) \times \boldsymbol{C^{0,\alpha}}(\overline{\Omega})$ then $\boldsymbol{u} \in \boldsymbol{C^{2,\alpha}}(\overline{\Omega}) \cap \boldsymbol{C_0}(\Omega)$[3]. Indeed, it follows from the above remark that $\boldsymbol{u} \in \boldsymbol{W^{2,N+1}}(\Omega) \cap \boldsymbol{H_0^1}(\Omega)$ and by Sobolev's embedding, $\boldsymbol{u} \in \boldsymbol{C^{0,1}}(\overline{\Omega})$. Setting

$$\boldsymbol{f} = \boldsymbol{F}(x) - \boldsymbol{V}\boldsymbol{u} - \boldsymbol{a}|\boldsymbol{u}|^{-(1-m)}\boldsymbol{u},$$

it then follow from equation (3.3) and estimate (7.5) below that $\boldsymbol{f} \in \boldsymbol{C^{0,\alpha}}(\overline{\Omega})$. Let $\boldsymbol{v} \in \boldsymbol{\mathcal{C}} \stackrel{\text{def}}{=} \boldsymbol{C^{2,\alpha}}(\overline{\Omega}) \cap \boldsymbol{C_0}(\overline{\Omega})$ be a solution to

$$-\Delta \boldsymbol{w} = \boldsymbol{f}, \tag{3.4}$$

given by Gilbarg and Trudinger [14], Theorem 6.14 p.107. Since $\boldsymbol{u} \in \boldsymbol{H_0^1}(\Omega)$ is also a solution to (3.4), uniqueness for equation (3.4) holds in $\boldsymbol{H_0^1}(\Omega)$ (Lax-Milgram's Theorem) and $\boldsymbol{\mathcal{C}} \subset \boldsymbol{H_0^1}(\Omega)$, we conclude that $\boldsymbol{u} = \boldsymbol{v}$ and so $\boldsymbol{u} \in \boldsymbol{\mathcal{C}}$.

[1] Actually, assumptions on Ω we use in this remark are $\partial\Omega$ bounded and $|\Omega| < \infty$. But these two conditions imply that Ω is bounded.

[2] Let $\boldsymbol{T} : \boldsymbol{u} \longrightarrow \left\{\gamma \boldsymbol{u}, \gamma \frac{\partial \boldsymbol{u}}{\partial \nu}\right\}$ be the trace function defined on $\mathscr{D}(\overline{\Omega})$, let $1 < p < \infty$ and let $\boldsymbol{X_p}(\Omega) = \left\{\boldsymbol{u} \in \boldsymbol{L^p}(\Omega); \Delta \boldsymbol{u} \in \boldsymbol{L^p}(\Omega)\right\}$. By density of $\mathscr{D}(\overline{\Omega})$ in $\boldsymbol{X_p}(\Omega)$, $\boldsymbol{T}$ has a continuous and linear extension from $\boldsymbol{X_p}(\Omega)$ into $\boldsymbol{W^{-\frac{1}{p},p}}(\partial\Omega) \times \boldsymbol{W^{-1-\frac{1}{p},p}}(\partial\Omega)$ (Hörmander [16], Theorem 2 p.503; Lions and Magenes [20], Lemma 2.2 and Theorem 2.1 p.147; Lions and Magenes [21], Propositions 9.1, Proposition 9.2 and Theorem 9.1 p.82; Grisvard [15], p.54). Since $\boldsymbol{u} \in \boldsymbol{L^{m+1}}(\Omega)$, it follows from equation (3.3) and Hölder's inequality that $\boldsymbol{u} \in \boldsymbol{X_p}(\Omega)$, for any $1 < p < m+1$. Then "$\boldsymbol{u}_{|\partial\Omega} = \boldsymbol{0}$ in the sense of the trace" makes sense and means that $\gamma \boldsymbol{u} = \boldsymbol{0}$.

[3] For $k \in \mathbb{N} \cup \{0\}$ and $0 < \alpha \leqslant 1$, $\boldsymbol{C^{k,\alpha}}(\overline{\Omega}) = \left\{\boldsymbol{u} \in \boldsymbol{C^k}(\overline{\Omega}; \mathbb{C}); \sum\limits_{|\beta|=k} H_\Omega^\alpha(D^\beta \boldsymbol{u}) < +\infty\right\} \subset \boldsymbol{W^{k,\infty}}(\Omega)$ (since Ω is bounded) and $\boldsymbol{C_0}(\Omega) = \left\{\boldsymbol{u} \in \boldsymbol{C}(\overline{\Omega}); \forall x \in \partial\Omega,\ \boldsymbol{u}(x) = \boldsymbol{0}\right\}$.

We end this section by giving a result to the evolution equation (in a particular case).

Corollary 3.8. *Let* $0 < m < 1$, *let* $(\boldsymbol{\lambda}, b) \in \mathbb{C} \times \mathbb{R}$ *satisfying* $\boldsymbol{\lambda} \neq 0$ *and* $b \geqslant 0$. *If* $\mathrm{Im}(\boldsymbol{\lambda}) = 0$ *then assume further* $\mathrm{Re}(\boldsymbol{\lambda}) \leqslant 0$. *Finally, let* $\boldsymbol{F} \in \boldsymbol{C^{0,m}}(\mathbb{R}^N)$ *be compactly supported. Then there exists a solution* $\boldsymbol{u} \in \boldsymbol{C^\infty}\big(\mathbb{R}; \boldsymbol{C^{2,m}_{\mathrm{b}}}(\mathbb{R}^N)\big)$ *to*

$$\begin{cases} \mathrm{\mathbf{i}}\dfrac{\partial \boldsymbol{u}}{\partial t} + \Delta \boldsymbol{u} + \boldsymbol{\lambda}|\boldsymbol{u}|^{-(1-m)}\boldsymbol{u} = \boldsymbol{F}(x)e^{\mathrm{\mathbf{i}}bt}, & \text{in } \mathbb{R} \times \mathbb{R}^N, \\ \boldsymbol{u}(0) = \boldsymbol{\varphi}, & \text{in } \mathbb{R}^N. \end{cases} \tag{3.5}$$

given by

$$\forall (t,x) \in \mathbb{R} \times \mathbb{R}^N, \ \boldsymbol{u}(t,x) = \boldsymbol{\varphi}(x)e^{\mathrm{\mathbf{i}}bt}, \tag{3.6}$$

where $\boldsymbol{\varphi} \in \boldsymbol{C^{2,m}_{\mathrm{b}}}(\mathbb{R}^N)$ *is a solution compactly supported of*

$$-\Delta \boldsymbol{\varphi} - \boldsymbol{\lambda}|\boldsymbol{\varphi}|^{-(1-m)}\boldsymbol{\varphi} + b\boldsymbol{\varphi} = -\boldsymbol{F}(x), \text{ in } \mathbb{R}^N, \tag{3.7}$$

given by Theorem 3.1. Furthermore, for any $t \in \mathbb{R}$, $\operatorname{supp} \boldsymbol{u}(t)$ *is compact.*

4 Uniqueness

Theorem 4.1. *Let* $\Omega \subseteq \mathbb{R}^N$ *be a nonempty open subset, let* $0 < m < 1$, *let* $(\boldsymbol{a}, \boldsymbol{b}) \in \mathbb{C}^2 \setminus \{(\boldsymbol{0}, \boldsymbol{0})\}$ *satisfying* (1.5) *and let* $\boldsymbol{F_1}, \boldsymbol{F_2} \in \boldsymbol{L^1_{\mathrm{loc}}}(\Omega)$ *be such that* $\boldsymbol{F_1} - \boldsymbol{F_2} \in \boldsymbol{L^2}(\Omega)$. *Let* $\boldsymbol{u_1}, \boldsymbol{u_2} \in \boldsymbol{H^1_0}(\Omega) \cap \boldsymbol{L^{m+1}}(\Omega)$ *be two global weak solutions of*

$$-\mathrm{\mathbf{i}}\Delta \boldsymbol{u_1} + \boldsymbol{a}|\boldsymbol{u_1}|^{-(1-m)}\boldsymbol{u_1} + \boldsymbol{b}\boldsymbol{u_1} = \boldsymbol{F_1}(x), \text{ in } \Omega, \tag{4.1}$$

$$-\mathrm{\mathbf{i}}\Delta \boldsymbol{u_2} + \boldsymbol{a}|\boldsymbol{u_2}|^{-(1-m)}\boldsymbol{u_2} + \boldsymbol{b}\boldsymbol{u_2} = \boldsymbol{F_2}(x), \text{ in } \Omega, \tag{4.2}$$

respectively. We have the following estimates.

$$\begin{cases} \|\boldsymbol{u_1} - \boldsymbol{u_2}\|_{\boldsymbol{L^2}(\Omega)} \leqslant \dfrac{|\boldsymbol{a}|}{\mathrm{Re}\,(\boldsymbol{a}\overline{\boldsymbol{b}})}\|\boldsymbol{F_1} - \boldsymbol{F_2}\|_{\boldsymbol{L^2}(\Omega)}, & \text{if } \boldsymbol{a} \neq \boldsymbol{0} \text{ and } \mathrm{Re}\,(\boldsymbol{a}\overline{\boldsymbol{b}}) > 0, \\ \|\boldsymbol{u_1} - \boldsymbol{u_2}\|_{\boldsymbol{L^2}(\Omega)} \leqslant \dfrac{1}{b_0}\|\boldsymbol{F_1} - \boldsymbol{F_2}\|_{\boldsymbol{L^2}(\Omega)}, & \text{if } \boldsymbol{a} = \boldsymbol{0}, \end{cases} \tag{4.3}$$

where $b_0 = |\mathrm{Re}(\boldsymbol{b})|$, *if* $\mathrm{Re}(\boldsymbol{b}) \neq 0$ *and* $b_0 = \mathrm{Im}(\boldsymbol{b})$, *if* $\mathrm{Re}(\boldsymbol{b}) = 0$. *If* $\boldsymbol{a} \neq \boldsymbol{0}$ *and* $\mathrm{Re}\,(\boldsymbol{a}\overline{\boldsymbol{b}}) = 0$ *then assume further that* $\boldsymbol{u_1}, \boldsymbol{u_2} \in \boldsymbol{L^\infty}(\Omega)$. *Then there exists a positive constant* $C = C(N,m)$ *such that*

$$\|\boldsymbol{u_1} - \boldsymbol{u_2}\|_{\boldsymbol{L^2}(\Omega)} \leqslant C\frac{\big(\|\boldsymbol{u_1}\|_{\boldsymbol{L^\infty}(\Omega)} + \|\boldsymbol{u_2}\|_{\boldsymbol{L^\infty}(\Omega)}\big)^{1-m}}{|\boldsymbol{a}|}\|\boldsymbol{F_1} - \boldsymbol{F_2}\|_{\boldsymbol{L^2}(\Omega)}. \tag{4.4}$$

Theorem 4.2. *Let* $\Omega \subseteq \mathbb{R}^N$ *be a nonempty open subset, let* $0 < m < 1$, *let* $(\boldsymbol{a}, \boldsymbol{b}) \in \mathbb{C}^2$ *satisfying* (1.5) *and let* $\boldsymbol{F} \in \boldsymbol{L^1_{\mathrm{loc}}}(\Omega)$. *Then equations* (1.1) *and* (1.3) *admits at most one global weak solution* $\boldsymbol{u} \in \boldsymbol{H^1_0}(\Omega) \cap \boldsymbol{L^{m+1}}(\Omega)$.

Corollary 4.3. *Let $\Omega \subseteq \mathbb{R}^N$ be a nonempty open subset, let $0 < m < 1$, let $(\boldsymbol{a},\boldsymbol{b}) \in \mathbb{A}\times\mathbb{B}$ satisfying* (1.5) *and let $\boldsymbol{F} \in \boldsymbol{L}^{\frac{m+1}{m}}(\Omega)$. Then equations* (1.1) *and* (1.3) *admits a unique global weak solution $\boldsymbol{u} \in \boldsymbol{H_0^1}(\Omega) \cap \boldsymbol{L^{m+1}}(\Omega)$. Furthermore, this solution satisfies Properties $1) - 3)$ of Theorem* 3.1.

Corollary 4.4. *Let $\Omega \subseteq \mathbb{R}^N$ be a nonempty open subset, let $0 < m < 1$ and let $(\boldsymbol{a},\boldsymbol{b}) \in \mathbb{C}^2$ satisfying* (1.5)*. Then the problem*

$$\begin{cases} -\mathrm{i}\Delta \boldsymbol{u} + \boldsymbol{a}|\boldsymbol{u}|^{-(1-m)}\boldsymbol{u} + \boldsymbol{b}\boldsymbol{u} = \boldsymbol{0}, \text{ in } \Omega, \\ \boldsymbol{u} \in \boldsymbol{H_0^1}(\Omega) \cap \boldsymbol{L^{m+1}}(\Omega), \end{cases}$$

has for unique solution $\boldsymbol{u} \equiv \boldsymbol{0}$.

Corollary 4.5. *Let $0 < m < 1$, let $(\boldsymbol{a},\boldsymbol{b}) \in \mathbb{A}\times\mathbb{B}$ satisfying* (1.5) *and let $\boldsymbol{F} \in \boldsymbol{C^{0,m}}(\mathbb{R}^N)$ be compactly supported. Then there exists a unique solution $\boldsymbol{u} \in \boldsymbol{C_b^{2,m}}(\mathbb{R}^N)$ of* (1.1) *and* (1.3) *compactly supported. If furthermore $\boldsymbol{F}$ is spherically symmetric then $\boldsymbol{u}$ is also spherically symmetric. For $N = 1$, this means that if $\boldsymbol{F}$ is an even (respectively, an odd) function then $\boldsymbol{u}$ is also an even (respectively, an odd) function.*

5 Pictures

In this section, we give some geometric interpretation about the values of $\boldsymbol{a}$ and $\boldsymbol{b}$. For convenience, we repeat the hypotheses (1.4) and (1.5). We recall that,

$$\begin{cases} \mathbb{A} = \mathbb{C} \setminus \big\{ \boldsymbol{z} \in \mathbb{C}; \mathrm{Re}(\boldsymbol{z}) = 0 \text{ and } \mathrm{Im}(\boldsymbol{z}) \leqslant 0 \big\}, \\ \mathbb{B} = \mathbb{A} \cup \big\{ \boldsymbol{0} \big\}. \end{cases}$$

For existence of solutions to problem (1.1) and (1.3), we suppose $(\boldsymbol{a},\boldsymbol{b}) \in \mathbb{C}^2$ satisfies

$$(\boldsymbol{a},\boldsymbol{b}) \in \mathbb{A}\times\mathbb{B} \quad \text{and} \quad \begin{cases} \mathrm{Re}(\boldsymbol{a})\mathrm{Re}(\boldsymbol{b}) \geqslant 0, \\ \text{ or} \\ \mathrm{Re}(\boldsymbol{a})\mathrm{Re}(\boldsymbol{b}) < 0 \text{ and } \mathrm{Im}(\boldsymbol{b}) > \dfrac{\mathrm{Re}(\boldsymbol{b})}{\mathrm{Re}(\boldsymbol{a})}\mathrm{Im}(\boldsymbol{a}), \end{cases} \tag{5.1}$$

while for uniqueness, we assume

$$\mathrm{Im}(\boldsymbol{a}) \geqslant 0 \quad \text{and} \quad \begin{cases} \boldsymbol{a} \neq \boldsymbol{0} \text{ and } \mathrm{Re}(\boldsymbol{a}\overline{\boldsymbol{b}}) \geqslant 0, \\ \text{or} \\ \boldsymbol{a} = \boldsymbol{0} \text{ and } \boldsymbol{b} \in \mathbb{B}. \end{cases} \tag{5.2}$$

Existence. Condition (5.1) may easily be interpreted in this way: if $\boldsymbol{b} \neq \boldsymbol{0}$ then one requires that $[\boldsymbol{a},\boldsymbol{b}] \cap \mathscr{B} = \emptyset$, where $\mathscr{B}$ is the geometric representation of $\mathbb{B}$. See Figures 1 and 2 below.

Uniqueness. The second condition of (5.2) is trivial. Indeed, $\boldsymbol{b}$ can be chosen anywhere in the complex plane, except on the half-axis where $\mathrm{Im}(\boldsymbol{z}) < 0$. Let us consider the first condition. We first choose $\boldsymbol{a} \in \mathbb{C} \setminus \{\boldsymbol{0}\}$ such that $\mathrm{Im}(\boldsymbol{a}) \geqslant 0$, and we choose $\boldsymbol{b}$ with respect to $\boldsymbol{a}$. We see $\boldsymbol{a}$ and $\boldsymbol{b}$ as vectors of $\mathbb{R}^2$. Then we write, $\overrightarrow{a} = \begin{pmatrix} \mathrm{Re}(\boldsymbol{a}) \\ \mathrm{Im}(\boldsymbol{a}) \end{pmatrix}$, $\overrightarrow{b} = \begin{pmatrix} \mathrm{Re}(\boldsymbol{b}) \\ \mathrm{Im}(\boldsymbol{b}) \end{pmatrix}$ and we have

$$\mathrm{Re}\left(\boldsymbol{a}\overline{\boldsymbol{b}}\right) = \mathrm{Re}(\boldsymbol{a})\mathrm{Re}(\boldsymbol{b}) + \mathrm{Im}(\boldsymbol{a})\mathrm{Im}(\boldsymbol{b}) = \overrightarrow{a}.\overrightarrow{b}, \tag{5.3}$$

where . denotes the scalar product between two vectors of $\mathbb{R}^2$. Then the condition $\mathrm{Re}\left(\boldsymbol{a}\overline{\boldsymbol{b}}\right) \geqslant 0$ is equivalent to $\left|\angle(\overrightarrow{a}, \overrightarrow{b})\right| \leqslant \frac{\pi}{2}$rad (see Figure 3 below).

Remark 5.1. Let $(\boldsymbol{a}, \boldsymbol{b}) \in \mathbb{C}^2$. Thanks to (5.3), the following assertions are equivalent.

1) $(\boldsymbol{a}, \boldsymbol{b})$ satisfies (5.1)–(5.2) (or (1.4)–(1.5)).

2) $(\boldsymbol{a}, \boldsymbol{b}) \in \mathbb{A} \times \mathbb{B}$ satisfies (5.2) (or (1.5)).

3) $\Big((\boldsymbol{a}, \boldsymbol{b})$ satisfies (5.2)$\Big)$, $\Big(\boldsymbol{a} \neq \boldsymbol{0}\Big)$ and $\Big(\mathrm{Im}(\boldsymbol{a}) = \mathrm{Re}(\boldsymbol{b}) = 0 \implies \mathrm{Im}(\boldsymbol{b}) \geqslant 0\Big)$.

In other words, when $\mathrm{Im}(\boldsymbol{a}) \neq 0$, uniqueness hypothesis (5.2) implies existence hypothesis (5.1) (see Figure 4 below).

6 Proofs of the localization properties

In this Section, we prove Theorems 1.1, 2.1, 2.3, 3.4, 2.5 and 2.6.

We recall some useful Gagliardo-Nirenberg's and Young inequalities.

Proposition 6.1. *Let $\Omega \subseteq \mathbb{R}^N$ be a nonempty open subset and let $0 \leqslant p \leqslant 1$. Then, there exists a positive constant $C = C(N)$ such that*

$$\forall \boldsymbol{u} \in \boldsymbol{H_0^1}(\Omega) \cap \boldsymbol{L^{p+1}}(\Omega),\ \|\boldsymbol{u}\|_{\boldsymbol{L^2}(\Omega)} \leqslant C\|\nabla\boldsymbol{u}\|_{\boldsymbol{L^2}(\Omega)}^{\frac{N(1-p)}{(N+2)-p(N-2)}} \|\boldsymbol{u}\|_{\boldsymbol{L^{p+1}}(\Omega)}^{\frac{2(1+p)}{(N+2)-p(N-2)}}, \tag{6.1}$$

$$\forall \boldsymbol{u} \in \boldsymbol{H_0^1}(\Omega) \cap \boldsymbol{L^1}(\Omega),\ \|\boldsymbol{u}\|_{\boldsymbol{L^{p+1}}(\Omega)}^{p+1} \leqslant C\|\nabla\boldsymbol{u}\|_{\boldsymbol{L^2}(\Omega)}^{\frac{2pN}{N+2}} \|\boldsymbol{u}\|_{\boldsymbol{L^1}(\Omega)}^{\frac{(N+2)-p(N-2)}{N+2}}. \tag{6.2}$$

Note that C does not depend on Ω.

Lemma 6.2. *For any real $x \geqslant 0$, $y \geqslant 0$, $\varepsilon > 0$ and $p > 1$, one has*

$$xy \leqslant \frac{1}{p'}\varepsilon^{p'}x^{p'} + \frac{1}{p}\varepsilon^{-p}y^p. \tag{6.3}$$

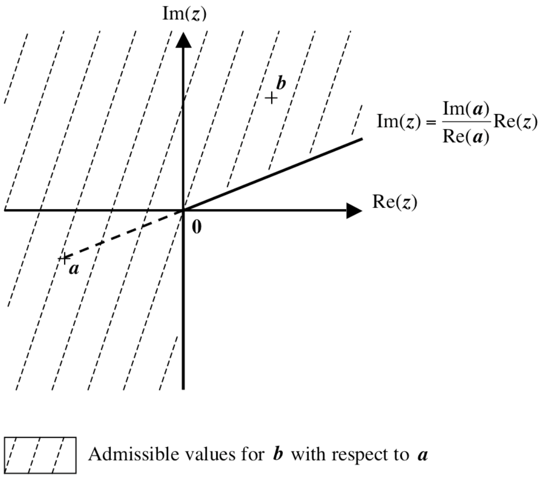

Figure 1: Existence, choice of $\boldsymbol{b}$

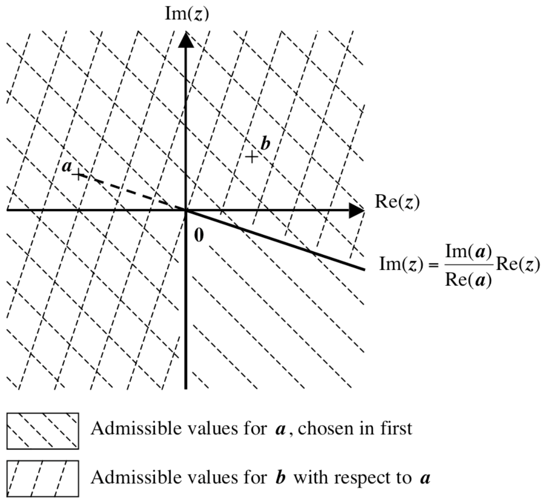

Figure 2: Existence, choice of $\boldsymbol{a}$ and $\boldsymbol{b}$

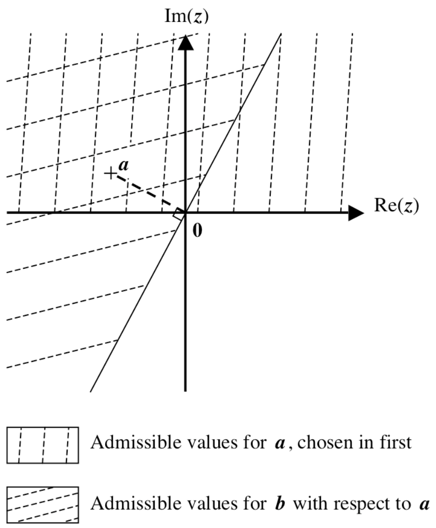

Figure 3: Uniqueness

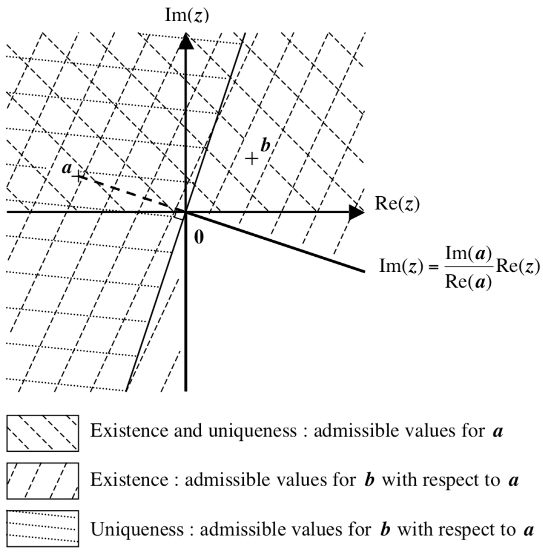

Figure 4: Uniqueness implies existence

Lemma 6.3. *Let* $(\boldsymbol{a}, \boldsymbol{b}) \in \mathbb{C}^2$ *satisfying* (1.4) *and let* C_0, C_1, C_2, C_3 *be four nonnegative real numbers satisfying*

$$\left|C_1 + \mathrm{Im}(\boldsymbol{a})C_2 + \mathrm{Im}(\boldsymbol{b})C_3\right| \leqslant C_0, \tag{6.4}$$

$$\left|\mathrm{Re}(\boldsymbol{a})C_2 + \mathrm{Re}(\boldsymbol{b})C_3\right| \leqslant C_0. \tag{6.5}$$

Then one has

$$0 \leqslant C_1 + LC_2 \leqslant MC_0, \tag{6.6}$$

where the positive constants L and M are defined by (1.8) *and* (1.9)*, respectively.*

Proof. We split the proof in 6 cases. Let $\delta > 0$.

Case 1. $\mathrm{Im}(\boldsymbol{a}) > 0$ and $\mathrm{Im}(\boldsymbol{b}) \geqslant 0$.

Then (6.6) follows from (6.4).

Case 2. $\mathrm{Im}(\boldsymbol{a}) = 0$, $\mathrm{Im}(\boldsymbol{b}) \geqslant 0$ and $\mathrm{Re}(\boldsymbol{a})\mathrm{Re}(\boldsymbol{b}) \geqslant 0$.

We compute $(6.4) + \mathrm{sign}(\mathrm{Re}(\boldsymbol{a}))(6.5)$ and then obtain (6.6).

Case 3. $\mathrm{Im}(\boldsymbol{a}) \geqslant 0$, $\mathrm{Im}(\boldsymbol{b}) < 0$ and $\mathrm{Re}(\boldsymbol{a})\mathrm{Re}(\boldsymbol{b}) \geqslant 0$.

We compute $(6.4) + \frac{|\mathrm{Im}(\boldsymbol{b})|}{\mathrm{Re}(\boldsymbol{b})}(6.5)$ and then obtain (6.6).

Case 4. $\mathrm{Re}(\boldsymbol{a})\mathrm{Re}(\boldsymbol{b}) < 0$.

If $\mathrm{Im}(\boldsymbol{b}) = 0$ then (1.4) implies $\mathrm{Im}(\boldsymbol{a}) > 0$, which fall into the scope of Case 1. So we may assume $\mathrm{Im}(\boldsymbol{b}) \neq 0$. We compute $(6.4) - \frac{\mathrm{Im}(\boldsymbol{b})}{\mathrm{Re}(\boldsymbol{b})}(6.5)$ and then obtain (6.6).

Case 5. $\mathrm{Im}(\boldsymbol{a}) < 0$, $\mathrm{Im}(\boldsymbol{b}) \geqslant 0$ and $\mathrm{Re}(\boldsymbol{a})\mathrm{Re}(\boldsymbol{b}) \geqslant 0$.

We compute $(6.4) + \frac{|\mathrm{Im}(\boldsymbol{a})|+\delta}{\mathrm{Re}(\boldsymbol{a})}(6.5)$ and then obtain (6.6).

Case 6. $\mathrm{Im}(\boldsymbol{a}) < 0$, $\mathrm{Im}(\boldsymbol{b}) < 0$ and $\mathrm{Re}(\boldsymbol{a})\mathrm{Re}(\boldsymbol{b}) > 0$.

We compute $(6.4) + \max\left\{\frac{|\mathrm{Im}(\boldsymbol{a})|+\delta}{\mathrm{Re}(\boldsymbol{a})}, \frac{|\mathrm{Im}(\boldsymbol{b})|}{\mathrm{Re}(\boldsymbol{b})}\right\}(6.5)$ and then obtain (6.6).

This ends the proof. □

Proof of Theorems 1.1 and 2.1. In order to establish our result in all cases of (1.4), we will follow the method of the proofs of Theorem 2.1 p.12–18 and Theorem 3.2 p.28–30 of Antontsev, Díaz and Shmarev [1], which has to be adapted. We denote by σ the surface measure on a sphere, $\rho_2 = \rho_0$, if we are concerned by Theorem 1.1 and $\rho_2 = \rho_1$, if we are concerned by Theorem 2.1. Assume we have either $\rho_2 < \mathrm{dist}(x_0, \partial\Omega)$ $\big(\iff \overline{B}(x_0,\rho_2) \subset \Omega\big)$ or $\rho_2 > \mathrm{dist}(x_0, \partial\Omega)$. The remaining case $\rho_2 = \mathrm{dist}(x_0, \partial\Omega)$ $\big(\iff B(x_0,\rho_2) \subset \Omega$ and $\partial\Omega \cap \mathbb{S}(x_0,\rho_2) \neq \emptyset\big)$, will be treated at the end of the proof[4]. If $\rho_2 > \mathrm{dist}(x_0, \partial\Omega)$, we have $\boldsymbol{u} \in \boldsymbol{H}^1_0(\Omega)$. So we may define $\widetilde{\boldsymbol{u}} \in \boldsymbol{H}^1_0\big(\Omega \cup B(x_0,\rho_2)\big)$ satisfying $\widetilde{\boldsymbol{u}}_{|\Omega} \in \boldsymbol{H}^1_0(\Omega)$, by setting $\widetilde{\boldsymbol{u}} = \boldsymbol{u}$, in Ω and $\widetilde{\boldsymbol{u}} = \boldsymbol{0}$, in $\Omega^{\mathrm{c}} \cap B(x_0,\rho_2)$. Then $\nabla\widetilde{\boldsymbol{u}} = \nabla\boldsymbol{u}$, almost everywhere in Ω and $\nabla\widetilde{\boldsymbol{u}} = \boldsymbol{0}$, almost everywhere in $\Omega^{\mathrm{c}} \cap B(x_0,\rho_2)$. Still if $\rho_2 > \mathrm{dist}(x_0, \Omega)$, we denote by $\widetilde{\boldsymbol{F}}$ the extension of $\boldsymbol{F}$ by $\boldsymbol{0}$ in $\Omega^{\mathrm{c}} \cap B(x_0,\rho_2)$. We now proceed to the proof in 7 steps.

Step 1. Let L and M be the constants defined by (1.8) and (1.9), respectively. For almost every

[4] We implicitly assume that $\partial\Omega \neq \emptyset$. Otherwise, we have $\Omega = \mathbb{R}^N$ and we only have to treat the first case: $\overline{B}(x_0,\rho_2) \subset \Omega$.

$\rho \in (0, \rho_2)$,

$$\|\nabla \widetilde{\boldsymbol{u}}\|^2_{\boldsymbol{L^2}(B(x_0,\rho))} + L\|\widetilde{\boldsymbol{u}}\|^{m+1}_{\boldsymbol{L^{m+1}}(B(x_0,\rho))} \leqslant MI(\rho) + MJ(\rho), \tag{6.7}$$

where $I(\rho) = \left|\int_{\mathbb{S}(x_0,\rho)} \widetilde{\boldsymbol{u}}\nabla\overline{\widetilde{\boldsymbol{u}}}.\dfrac{x-x_0}{|x-x_0|}\mathrm{d}\sigma\right|$ and $J(\rho) = \int_{B(x_0,\rho)} |\widetilde{\boldsymbol{F}}(x)\overline{\widetilde{\boldsymbol{u}}(x)}|\mathrm{d}x$. Moreover, $I, J \in L^1(0,\rho_2)$. From Hölder's inequality, the above discussion and Sobolev's embedding,

$$\|I\|_{L^1(0,\rho_2)} \leqslant \|\widetilde{\boldsymbol{u}}\|^2_{\boldsymbol{H^1}(B(x_0,\rho_2))} < \infty,$$
$$\|J\|_{L^1(0,\rho_2)} \leqslant \|\widetilde{\boldsymbol{F}}\|_{\boldsymbol{L}^{\frac{m+1}{m}}(B(x_0,\rho_2))}\|\widetilde{\boldsymbol{u}}\|_{\boldsymbol{L^m}(B(x_0,\rho_2))} < \infty.$$

Let $\rho \in (0,\rho_2)$ For any $n \in \mathbb{N}$, $n > \frac{1}{\rho}$, we define the cutoff function $\psi_n \in W^{1,\infty}(\mathbb{R})$ by

$$\forall t \in \mathbb{R},\ \psi_n(t) = \begin{cases} 1, & \text{if } |t| \in \left[0, \rho - \frac{1}{n}\right], \\ n(\rho - |t|), & \text{if } |t| \in \left(\rho - \frac{1}{n}, \rho\right), \\ 0, & \text{if } |t| \in [\rho, \infty), \end{cases}$$

and we set for almost every $x \in \Omega \cup B(x_0,\rho_2)$, $\boldsymbol{\varphi_n}(x) = \psi_n(|x-x_0|)\widetilde{\boldsymbol{u}}(x)$. If $\rho_2 < \operatorname{dist}(x_0,\partial\Omega)$ then $\operatorname{supp}\boldsymbol{\varphi_n} \subseteq \overline{B}(x_0,\rho) \subset \Omega$ and so $\boldsymbol{\varphi_n} \in \boldsymbol{H^1_c}(\Omega)$. If $\rho_2 > \operatorname{dist}(x_0,\partial\Omega)$ then $\boldsymbol{\varphi_n}_{|\Omega} \in \boldsymbol{H^1_0}(\Omega)$ and $\operatorname{supp}\boldsymbol{\varphi_n} \subseteq \overline{\Omega} \cap \overline{B}(x_0,\rho)$. It follows from Definition 1.3 and Remark 1.4, 2. and 3., that $\boldsymbol{\varphi} = \mathbf{i}\boldsymbol{\varphi_n}_{|\Omega}$ is an admissible test function and so

$$\operatorname{Re}\int\limits_{B(x_0,\rho)} \psi_n(|x-x_0|)\left(|\nabla\widetilde{\boldsymbol{u}}|^2 - \mathbf{i}\boldsymbol{a}|\widetilde{\boldsymbol{u}}|^{m+1} - \mathbf{i}\boldsymbol{b}|\widetilde{\boldsymbol{u}}|^2\right)\mathrm{d}x$$
$$= -\operatorname{Re}\int\limits_{B(x_0,\rho)} \psi_n'(|x-x_0|)\overline{\widetilde{\boldsymbol{u}}}\nabla\widetilde{\boldsymbol{u}}.\frac{x-x_0}{|x-x_0|}\mathrm{d}x + \operatorname{Im}\int\limits_{B(x_0,\rho)} \psi_n(|x-x_0|)\widetilde{\boldsymbol{F}}\overline{\widetilde{\boldsymbol{u}}}\mathrm{d}x.$$

Introducing the spherical coordinates (r,σ), we get

$$\left|\operatorname{Re}\int\limits_{B(x_0,\rho)} \psi_n(|x-x_0|)\left(|\nabla\widetilde{\boldsymbol{u}}|^2 - \mathbf{i}\boldsymbol{a}|\widetilde{\boldsymbol{u}}|^{m+1} - \mathbf{i}\boldsymbol{b}|\widetilde{\boldsymbol{u}}|^2\right)\mathrm{d}x\right|$$
$$= \left|\operatorname{Re}\left(n\int_{\rho-\frac{1}{n}}^{\rho}\left(\int\limits_{\mathbb{S}(x_0,r)} \overline{\widetilde{\boldsymbol{u}}}\nabla\widetilde{\boldsymbol{u}}.\frac{x-x_0}{|x-x_0|}\mathrm{d}\sigma\right)\mathrm{d}r\right) - \operatorname{Im}\int\limits_{B(x_0,\rho)} \psi_n(|x-x_0|)\widetilde{\boldsymbol{F}}\overline{\widetilde{\boldsymbol{u}}}\mathrm{d}x\right|$$
$$\leqslant n\int_{\rho-\frac{1}{n}}^{\rho} I(r)\mathrm{d}r + \int\limits_{B(x_0,\rho)} \psi_n(|x-x_0|)|\widetilde{\boldsymbol{F}}(x)\overline{\widetilde{\boldsymbol{u}}(x)}|\mathrm{d}x.$$

We now let $n \nearrow \infty$. Using the Lebesgue's dominated convergence Theorem and recalling that $I \in L^1(0,\rho_2)$, we obtain

$$\left|\|\nabla\widetilde{\boldsymbol{u}}\|^2_{\boldsymbol{L^2}(B(x_0,\rho))} + \operatorname{Im}(\boldsymbol{a})\|\widetilde{\boldsymbol{u}}\|^{m+1}_{\boldsymbol{L^{m+1}}(B(x_0,\rho))} + \operatorname{Im}(\boldsymbol{b})\|\widetilde{\boldsymbol{u}}\|^2_{\boldsymbol{L^2}(B(x_0,\rho))}\right| \leqslant I(\rho) + J(\rho). \tag{6.8}$$

Proceeding as above with $\boldsymbol{\varphi} = \boldsymbol{\varphi}_{\boldsymbol{n}|\Omega}$, we get

$$\left|\mathrm{Re}(\boldsymbol{a})\|\widetilde{\boldsymbol{u}}\|_{\boldsymbol{L}^{\boldsymbol{m+1}}(B(x_0,\rho))}^{m+1} + \mathrm{Re}(\boldsymbol{b})\|\widetilde{\boldsymbol{u}}\|_{\boldsymbol{L}^{\boldsymbol{2}}(B(x_0,\rho))}^{2}\right| \leqslant I(\rho) + J(\rho). \tag{6.9}$$

Then Step 1 follows from (6.8), (6.9) and Lemma 6.3.

Let us recall and introduce some notations. Let $\tau \in \left(\frac{m+1}{2},1\right]$ and let $\rho \in (0,\rho_2)$. We set

$$E(\rho) = \|\nabla\widetilde{\boldsymbol{u}}\|_{\boldsymbol{L}^{\boldsymbol{2}}(B(x_0,\rho))}^2, \qquad b(\rho) = \|\widetilde{\boldsymbol{u}}\|_{\boldsymbol{L}^{\boldsymbol{m+1}}(B(x_0,\rho))}^{m+1}, \qquad \delta = \tfrac{k}{2(1+m)},$$
$$\theta = \tfrac{(1+m)+N(1-m)}{k} \in (0,1), \qquad \ell = \tfrac{1}{\theta(1+m)}, \qquad \gamma(\tau) = \tfrac{2\tau-(1+m)}{k} \in (0,1),$$
$$\mu(\tau) = \tfrac{2(1-\tau)}{k}, \qquad \eta(\tau) = \tfrac{1-m}{1+m} - \gamma(\tau) > 0.$$

Step 2. $E \in W^{1,1}(0,\rho_2)$, for a.e. $\rho \in (0,\rho_2)$, $E'(\rho) = \|\nabla\widetilde{\boldsymbol{u}}\|_{\boldsymbol{L}^{\boldsymbol{2}}(\mathbb{S}(x_0,\rho))}^2$ and

$$\begin{aligned}0 \leqslant E(\rho) + b(\rho) \leqslant CL_1ME'(\rho)^{\frac12}\left(E(\rho)^{\frac12} + \rho^{-\delta}b(\rho)^{\frac{1}{m+1}}\right)^{\theta} b(\rho)^{\frac{1-\theta}{m+1}} \\ + (2L_1M)^{\frac{m+1}{m}}\|\widetilde{\boldsymbol{F}}\|_{\boldsymbol{L}^{\frac{\boldsymbol{m+1}}{\boldsymbol{m}}}(B(x_0,\rho))}^{\frac{m+1}{m}},\end{aligned} \tag{6.10}$$

where $C = C(N,m)$ and $L_1 = \max\left\{1, \frac{1}{L}\right\}$.

We have the identity $E(\rho) = \displaystyle\int_0^\rho\left(\int_{\mathbb{S}(x_0,r)}|\nabla\widetilde{\boldsymbol{u}}|^2\mathrm{d}\sigma\right)dr$. So that the mapping $r \longmapsto \displaystyle\int_{\mathbb{S}(x_0,r)}|\nabla\widetilde{\boldsymbol{u}}|^2\mathrm{d}\sigma$ lies in $L^1(0,\rho_2)$, which means that E is absolutely continuous on $(0,\rho_2)$. We then get the first part of the claim and we only have to establish (6.10). Let $\rho \in (0,\rho_2)$. It follows from Cauchy-Schwarz's inequality that

$$I(\rho) \leqslant \|\nabla\widetilde{\boldsymbol{u}}\|_{\boldsymbol{L}^{\boldsymbol{2}}(\mathbb{S}(x_0,\rho))}\|\widetilde{\boldsymbol{u}}\|_{\boldsymbol{L}^{\boldsymbol{2}}(\mathbb{S}(x_0,\rho))} = E'(\rho)^{\frac12}\|\widetilde{\boldsymbol{u}}\|_{\boldsymbol{L}^{\boldsymbol{2}}(\mathbb{S}(x_0,\rho))}. \tag{6.11}$$

We recall the interpolation-trace inequality (see Corollary 2.1 in Díaz and Véron [12], where there is a misprint: δ has to be replaced with $-\delta$).

$$\|\widetilde{\boldsymbol{u}}\|_{\boldsymbol{L}^{\boldsymbol{2}}(\mathbb{S}(x_0,\rho))} \leqslant C\left(\|\nabla\widetilde{\boldsymbol{u}}\|_{\boldsymbol{L}^{\boldsymbol{2}}(B(x_0,\rho))} + \rho^{-\delta}\|\widetilde{\boldsymbol{u}}\|_{\boldsymbol{L}^{\boldsymbol{m+1}}(B(x_0,\rho))}\right)^{\theta}\|\widetilde{\boldsymbol{u}}\|_{\boldsymbol{L}^{\boldsymbol{m+1}}(B(x_0,\rho))}^{1-\theta}, \tag{6.12}$$

where $C = C(N,m)$. Putting together (6.7), (6.11) and (6.12), we obtain,

$$E(\rho) + b(\rho) \leqslant CL_1ME'(\rho)^{\frac12}\left(E(\rho)^{\frac12} + \rho^{-\delta}b(\rho)^{\frac{1}{m+1}}\right)^{\theta} b(\rho)^{\frac{1-\theta}{m+1}} + L_1M\int_{B(x_0,\rho)}|\widetilde{\boldsymbol{F}}(x)\overline{\widetilde{\boldsymbol{u}}(x)}|\mathrm{d}x. \tag{6.13}$$

Applying Young's inequalities (Lemma 6.2) with $x = \|\widetilde{\boldsymbol{F}}\|_{\boldsymbol{L}^{\frac{\boldsymbol{m+1}}{\boldsymbol{m}}}(B(x_0,\rho))}$, $y = \|\widetilde{\boldsymbol{u}}\|_{\boldsymbol{L}^{\boldsymbol{m+1}}(B(x_0,\rho))}$, $\varepsilon = \left(\frac{2L_1M}{m+1}\right)^{\frac{1}{m+1}}$ and $p = m+1$, we get

$$\int_{B(x_0,\rho)}|\widetilde{\boldsymbol{F}}(x)\overline{\widetilde{\boldsymbol{u}}(x)}|\mathrm{d}x \leqslant \frac{m}{m+1}\left(\frac{2L_1M}{m+1}\right)^{\frac1m}\|\widetilde{\boldsymbol{F}}\|_{\boldsymbol{L}^{\frac{\boldsymbol{m+1}}{\boldsymbol{m}}}(B(x_0,\rho))}^{\frac{m+1}{m}} + \frac{1}{2L_1M}b(\rho), \tag{6.14}$$

for any $\rho \in (0, \rho_2)$. Putting together (6.13) and (6.14), we obtain (6.10). Hence Step 2.

Step 3. Let C_0 be the constant in (6.10). For any $\tau \in \left(\frac{m+1}{2}, 1\right]$ and for a.e. $\rho \in (0, \rho_2)$,

$$C_0 L_1 M E'(\rho)^{\frac{1}{2}} \left(E(\rho)^{\frac{1}{2}} + \rho^{-\delta} b(\rho)^{\frac{1}{m+1}}\right)^{\theta} b(\rho)^{\frac{1-\theta}{m+1}} \leqslant \left(K_1(\tau)\rho^{-(\nu-1)} E'(\rho)\right)^{\frac{1}{2}} \left(E(\rho) + b(\rho)\right)^{\frac{\gamma(\tau)+1}{2}}, \quad (6.15)$$

where $K_1(\tau) = C L_1^2 M^2 \max\left\{\rho_2^{\nu-1}, 1\right\} \max\{b(\rho_2)^{\mu(\tau)}, b(\rho_2)^{\eta(\tau)}\}$ and $C = C(N, m)$.

Let $\tau \in \left(\frac{m+1}{2}, 1\right]$ and let $\rho \in (0, \rho_2)$. A straightforward calculation yields

$$\begin{aligned}
&\left(E(\rho)^{\frac{1}{2}} + \rho^{-\delta} b(\rho)^{\frac{1}{m+1}}\right) b(\rho)^{\frac{1-\theta}{\theta(m+1)}} \\
&= E(\rho)^{\frac{1}{2}} b(\rho)^{\frac{1-\theta}{\theta(m+1)}} + \rho^{-\delta} b(\rho)^{\frac{1}{\theta(m+1)}} \\
&= E(\rho)^{\frac{1}{2}} b(\rho)^{\tau(1-\theta)\ell} b(\rho)^{(1-\tau)(1-\theta)\ell} + \rho^{-\delta} b(\rho)^{\frac{1}{2}+\tau(1-\theta)\ell} b(\rho)^{\ell-\tau(1-\theta)\ell-\frac{1}{2}} \\
&\leqslant 2\rho^{-\delta} \max\left\{\rho_2^{\delta}, 1\right\} K_2(\tau)^{\frac{1}{\theta}} \left(E(\rho) + b(\rho)\right)^{\frac{1}{2}+\tau(1-\theta)\ell},
\end{aligned}$$

where $K_2^2(\tau) = \max\{b(\rho_2)^{\mu(\tau)}, b(\rho_2)^{\eta(\tau)}\}$. Hence (6.15) with $K_1(\tau) = 4C_0^2 L_1^2 M^2 K_2^2(\tau) \max\left\{\rho_2^{\nu-1}, 1\right\}$.

Step 4. For any $\tau \in \left(\frac{m+1}{2}, 1\right]$ and for a.e. $\rho \in (0, \rho_2)$,

$$0 \leqslant E(\rho)^{1-\gamma(\tau)} \leqslant K_1(\tau)\rho^{-(\nu-1)} E'(\rho) + (4L_1 M)^{\frac{(m+1)(1-\gamma(\tau))}{m}} \|\widetilde{\boldsymbol{F}}\|_{\boldsymbol{L}^{\frac{m+1}{m}}(B(x_0,\rho))}^{\frac{(m+1)(1-\gamma(\tau))}{m}}, \quad (6.16)$$

Putting together (6.10) and (6.15), and applying again Young's inequality (6.3) with $p = \frac{2}{\gamma(\tau)+1}$, $\varepsilon = (\gamma(\tau)+1)^{\frac{\gamma(\tau)+1}{2}}$, $x = \left(K_1(\tau)\rho^{-(\nu-1)} E'(\rho)\right)^{\frac{1}{2}}$ and $y = (E(\rho) + b(\rho))^{\frac{\gamma(\tau)+1}{2}}$, we obtain

$$\begin{aligned}
&E(\rho) + b(\rho) \\
&\leqslant \left(K_1(\tau)\rho^{-(\nu-1)} E'(\rho)\right)^{\frac{1}{2}} \left(E(\rho) + b(\rho)\right)^{\frac{\gamma(\tau)+1}{2}} + (2L_1 M)^{\frac{m+1}{m}} \|\widetilde{\boldsymbol{F}}\|_{\boldsymbol{L}^{\frac{m+1}{m}}(B(x_0,\rho))}^{\frac{m+1}{m}}, \\
&\leqslant C \left(K_1(\tau)\rho^{-(\nu-1)} E'(\rho)\right)^{\frac{1}{1-\gamma(\tau)}} + \frac{1}{2}(E(\rho) + b(\rho)) + (2L_1 M)^{\frac{m+1}{m}} \|\widetilde{\boldsymbol{F}}\|_{\boldsymbol{L}^{\frac{m+1}{m}}(B(x_0,\rho))}^{\frac{m+1}{m}},
\end{aligned}$$

where $C = \frac{p-1}{p} \varepsilon^{\frac{p}{p-1}} = C(N, m)$. Changing, if needed, the constant C in the definition of $K_1(\tau)$, we obtain

$$E(\rho) + b(\rho) \leqslant \left(K_1(\tau)\rho^{-(\nu-1)} E'(\rho)\right)^{\frac{1}{1-\gamma(\tau)}} + (4L_1 M)^{\frac{m+1}{m}} \|\widetilde{\boldsymbol{F}}\|_{\boldsymbol{L}^{\frac{m+1}{m}}(B(x_0,\rho))}^{\frac{m+1}{m}}.$$

Raising both sides of the above inequality to the power $1 - \gamma(\tau)$ and recalling that $\left(1 - \gamma(\tau)\right) \in (0, 1)$, we obtain (6.16).

Step 5. Let $\alpha \in (0, \rho_0]$. If $E(\alpha) = 0$ then $\boldsymbol{u}_{|B_\Omega(x_0,\alpha)} \equiv \boldsymbol{0}$.

From our hypothesis, $E' = 0$ on $(0, \alpha)$. Furthermore, $\|\widetilde{\boldsymbol{F}}\|_{\boldsymbol{L}^{\frac{m+1}{m}}(B(x_0,\alpha))} = 0$ (from assumption of Theorem 1.1 or (2.1)). It follows from Step 2 and continuity of b that $b(\alpha) = 0$. Hence Step 5 follows.

Step 6. Proof of Theorem 1.1.

Thus $\rho_2=\rho_0$ and $\|\widetilde{\boldsymbol{F}}\|_{\boldsymbol{L}^{\frac{m+1}{m}}(B(x_0,\rho_0))}=0$. For any $\tau\in\left(\frac{m+1}{2},1\right]$, set $\rho_1^\nu(\tau)=\left(\rho_2^\nu-\nu\frac{K_1(\tau)E(\rho_2)^{\gamma(\tau)}}{\gamma(\tau)}\right)_+$ and let $\rho_{\max}=\max\limits_{\tau\in\left(\frac{m+1}{2},1\right]}r(\tau)$. Note that definition of $\rho_{\max}$ coincides with (1.10). Let $\tau\in\left(\frac{m+1}{2},1\right]$. We claim that $E(r(\tau))=0$. Otherwise, $E(r(\tau))>0$ and so $E>0$ on $[r(\tau),\rho_0)$. From (6.16), one has (we recall that $\gamma(\tau)-1<0$),

$$\text{for a.e. } \rho\in(r(\tau),\rho_0),\ K_1(\tau)E'(\rho)E(\rho)^{\gamma(\tau)-1}\geqslant\rho^{\nu-1}. \tag{6.17}$$

We integrate this estimate between $r(\tau)$ and ρ_0. We obtain

$$\nu\frac{K_1(\tau)}{\gamma(\tau)}\left(E(\rho_0)^{\gamma(\tau)}-E(r(\tau))^{\gamma(\tau)}\right)\geqslant\rho_0^\nu-\rho_1^\nu(\tau).$$

By definition of $r(\tau)$, this gives $E(r(\tau))\leqslant 0$. A contradiction, hence the claim. In particular, $E(\rho_{\max})=0$. It follows from Step 5 that $\boldsymbol{u}_{|B_\Omega(x_0,\rho_{\max})}\equiv 0$, which is the desired result. It remains to treat the case where $\rho_0=\operatorname{dist}(x_0,\partial\Omega)$. We proceed as follows. Let $n\in\mathbb{N}$, $n\geqslant\frac{1}{\rho_0}$. We work on $B\left(x_0,\rho_0-\frac{1}{n}\right)$ instead of $B(x_0,\rho_0)$ and apply the above result. Thus $\boldsymbol{u}_{|B(x_0,\rho_{\max}^n)}\equiv\boldsymbol{0}$, where $\rho_{\max}^n$ is given by (1.10) with $\rho_0-\frac{1}{n}$ in place of ρ_0. We then let $n\nearrow\infty$ which leads to the result. This achieves the proof of Theorem 1.1.

Step 7. Proof of Theorem 2.1.

We have $\rho_2=\rho_1$. Let $\gamma=\gamma(1)$ and set for any $\rho\in[0,\rho_1]$, $F(\rho)=(4L_1M)^{\frac{m+1}{m}}\|\widetilde{\boldsymbol{F}}\|^{\frac{(m+1)(1-\gamma)}{m}}_{\boldsymbol{L}^{\frac{(m+1)(1-\gamma)}{m}}(B(x_0,\rho))}$ and $K=K_1(1)\rho_0^{-(\nu-1)}$. Let $E_\star=\left(\frac{\gamma}{2K}(\rho_1-\rho_0)\right)^{\frac{1}{\gamma}}$ and $\varepsilon_\star=\frac{1}{2^{p'}(4L_1M)^{\frac{m+1}{m}}}\left(\frac{\gamma}{2K}\right)^p$. Remark that $p=\frac{1}{\gamma}$. Assume now $E(\rho_1)<E_\star$. Applying Step 4 with $\tau=1$, one has for a.e. $\rho\in(\rho_0,\rho_1)$,

$$-KE'(\rho)+E(\rho)^{1-\gamma}\leqslant F(\rho). \tag{6.18}$$

Let define the function G by

$$\forall\rho\in[0,\rho_1],\ G(\rho)=\left(\frac{\gamma}{2K}(\rho-\rho_0)_+\right)^{\frac{1}{\gamma}}. \tag{6.19}$$

Then $G(\rho_1)=E_\star$ and G satisfies

$$\forall\rho\in[0,\rho_1],\ -KG'(\rho)+\frac{1}{2}G(\rho)^{1-\gamma}=0, \tag{6.20}$$

$$E(\rho_1)<G(\rho_1). \tag{6.21}$$

Finally and recalling that $\gamma=\frac{1}{p}$, from our hypothesis (2.1) and (6.19), one has

$$\forall\rho\in(0,\rho_1),\ F(\rho)<\frac{1}{2}\left(\frac{\gamma}{2K}(\rho-\rho_0)_+\right)^{\frac{1-\gamma}{\gamma}}=\frac{1}{2}G(\rho)^{1-\gamma}. \tag{6.22}$$

Putting together (6.18), (6.22) and (6.20), one obtains

$$\text{for a.e. } \rho \in (\rho_0,\rho_1),\ -KE'(\rho)+E(\rho)^{1-\gamma} < -KG'(\rho)+G(\rho)^{1-\gamma}. \tag{6.23}$$

Now, we claim that for any $\rho \in [\rho_0,\rho_1)$, $E(\rho) \leqslant G(\rho)$. Indeed, if the claim does not hold, it follows from (6.21) and continuity of E and G that there exist $\rho_\star \in (\rho_0,\rho_1)$ and $\delta \in (0,\rho_\star-\rho_0]$ such that

$$E(\rho_\star) = G(\rho_\star), \tag{6.24}$$

$$\forall \rho \in (\rho_\star-\delta,\rho_\star),\ E(\rho) > G(\rho). \tag{6.25}$$

It follows from (6.23) and (6.25) that for a.e. $\rho \in (\rho_\star-\delta,\rho_\star)$, $G'(\rho) < E'(\rho)$. But, with (6.24), this implies that for any $\rho \in (\rho_\star-\delta,\rho_\star)$, $G(\rho) > E(\rho)$, which contradicts (6.25), hence the claim. It follows that $0 \leqslant E(\rho_0) \leqslant G(\rho_0) = 0$. We deduce with help of Step 5 that $\boldsymbol{u}_{|B_\Omega(x_0,\rho_0)} \equiv \mathbf{0}$, which is the desired result. It remains to treat the case where $\rho_1 = \operatorname{dist}(x_0,\partial\Omega)$. We proceed as follows. Assume $E(\rho_1) < E_\star$. Then there exists $\varepsilon > 0$ small enough such that $\rho_0 < \rho_1-\varepsilon$ and $E(\rho_1) < E_\star(\varepsilon)$ where $E_\star(\varepsilon) = \left(\frac{\gamma}{2K}(\rho_1-\rho_0-\varepsilon)\right)^{\frac{1}{\gamma}}$. Since $\varepsilon_\star$ is a non decreasing function of ρ_1, we do not need to change its definition. Estimates (6.18)–(6.25) holding with $\rho_1-\varepsilon$ in place of ρ_1, it follows that $E(\rho_0) = 0$ and we conclude with help of Step 5. This ends the proof of Theorem 2.1. □

Proof of Theorem 2.3. Let $C_0 = C_0(N,m)$ be the constant in estimate (1.10) given by Theorem 1.1. We then choose $C = C_0^{-1}$ in (2.2) and (2.3). Using the notations of Theorem 1.1 and its proof, we define for any $\tau \in \left(\frac{m+1}{2},1\right]$,

$$r(\tau)^\nu = \left((2\rho_0)^\nu - C_0M^2\max\left\{1,\frac{1}{L^2}\right\}\max\left\{(2\rho_0)^{\nu-1},1\right\} \times \frac{E(2\rho_0)^{\gamma(\tau)}\max\{b(2\rho_0)^{\mu(\tau)},b(2\rho_0)^{\eta(\tau)}\}}{2\tau-(1+m)}\right)_+,$$

and recall that $\rho_{\max} = \max\limits_{\tau\in\left(\frac{m+1}{2},1\right]} r(\tau)$. Assume (2.2) holds. Then $\rho_{\max} \geqslant \rho_1(1) \geqslant \rho_0$ and it follows from (1.10) of Theorem 1.1 that $b(\rho_0) = 0$. Now assume (2.3) holds. Since $E(2\rho_0) \leqslant 1$, $b(2\rho_0) \leqslant 1$ and $0 < \mu(\tau) < \eta(\tau) < 1$, for any $\tau \in \left(\frac{m+1}{2},1\right)$, it follows from definitions of ρ_1 and $\rho_{\max}$, that

$$\rho_{\max}^\nu \geqslant \rho_1^\nu(1-s) \geqslant (2\rho_0)^\nu - C_0M^2\min\{1,L^2\}\frac{\max\{(2\rho_0)^{\nu-1},1\}}{1-m-2s}b(2\rho_0)^{\mu(1-s)} \geqslant \rho_0^\nu.$$

By (1.10) of Theorem 1.1, $b(\rho_0) = 0$. This concludes the proof. □

Proof of Theorem 3.4. By Definition 1.3 and of Remark 1.4, 3., we can choose $\boldsymbol{\varphi} = \mathbf{i}\boldsymbol{u}$ and $\boldsymbol{\varphi} = \boldsymbol{u}$

in (1.12). We then obtain,

$$\|\nabla \boldsymbol{u}\|^2_{\boldsymbol{L^2}(\Omega)} + \mathrm{Im}(\boldsymbol{a})\|\boldsymbol{u}\|^{m+1}_{\boldsymbol{L^{m+1}}(\Omega)} + \mathrm{Im}(\boldsymbol{b})\|\boldsymbol{u}\|^2_{\boldsymbol{L^2}(\Omega)} = \mathrm{Im}\int\limits_\Omega \boldsymbol{F}\overline{\boldsymbol{u}}\mathrm{d}x,$$

$$\mathrm{Re}(\boldsymbol{a})\|\boldsymbol{u}\|^{m+1}_{\boldsymbol{L^{m+1}}(\Omega)} + \mathrm{Re}(\boldsymbol{b})\|\boldsymbol{u}\|^2_{\boldsymbol{L^2}(\Omega)} = \mathrm{Re}\int\limits_\Omega \boldsymbol{F}\overline{\boldsymbol{u}}\mathrm{d}x.$$

Applying Lemma 6.3, these estimates yield,

$$\|\nabla \boldsymbol{u}\|^2_{\boldsymbol{L^2}(\Omega)} + L\|\boldsymbol{u}\|^{m+1}_{\boldsymbol{L^{m+1}}(\Omega)} \leqslant M\int\limits_\Omega |\boldsymbol{F}|\,|\boldsymbol{u}|\,\mathrm{d}x. \tag{6.26}$$

We apply Young's inequality (6.3) with $x = |\boldsymbol{F}|$, $y = |\boldsymbol{u}|$, $\varepsilon = \left(\frac{2M}{(m+1)L}\right)^{\frac{1}{m+1}}$ and $p = m+1$. With (6.26), we get

$$\|\nabla \boldsymbol{u}\|^2_{\boldsymbol{L^2}(\Omega)} + \frac{L}{2}\|\boldsymbol{u}\|^{m+1}_{\boldsymbol{L^{m+1}}(\Omega)} < M\left(\frac{2M}{L}\right)^{\frac{1}{m}}\|\boldsymbol{F}\|^{\frac{m+1}{m}}_{\boldsymbol{L^{\frac{m+1}{m}}}(\Omega)},$$

from which we deduce (3.1). Finally, applying Gagliardo-Nirenberg's inequality (6.1), with $p = m$, and Young's inequality (6.3), with $p = \frac{4+N(1-m)}{N(1-m)}$ and $\varepsilon = 1$, one obtains

$$\|\boldsymbol{u}\|^{2\frac{(N+2)-m(N-2)}{4+N(1-m)}}_{\boldsymbol{L^2}(\Omega)} \leqslant C\|\nabla \boldsymbol{u}\|^{\frac{2N(1-m)}{4+N(1-m)}}_{\boldsymbol{L^2}(\Omega)}\|\boldsymbol{u}\|^{\frac{4(1+m)}{4+N(1-m)}}_{\boldsymbol{L^{m+1}}(\Omega)} < C\left(\|\nabla \boldsymbol{u}\|^2_{\boldsymbol{L^2}(\Omega)} + \|\boldsymbol{u}\|^{m+1}_{\boldsymbol{L^{m+1}}(\Omega)}\right),$$

and finally

$$\|\boldsymbol{u}\|^2_{\boldsymbol{L^2}(\Omega)} < C\left(\|\nabla \boldsymbol{u}\|^2_{\boldsymbol{L^2}(\Omega)} + \|\boldsymbol{u}\|^{m+1}_{\boldsymbol{L^{m+1}}(\Omega)}\right)^{\delta+1}, \tag{6.27}$$

where $\delta = \frac{2(1-m)}{(N+2)-m(N-2)}$. Estimate (3.2) then follows from (3.1) and (6.27). $\square$

Proof of Theorem 2.5. Let C be the constant given by Theorem 2.3 and let $\varepsilon > 0$. Set $K = \operatorname{supp} F$ and $K(\varepsilon) = \overline{\mathcal{O}(\varepsilon)}$. We would like to apply Theorem 2.3 with $\rho_0 = \frac{\varepsilon}{4}$. By (3.1) of Theorem 3.4, there exists $\delta_0 = \delta_0(\varepsilon, N, m, L, M) > 0$ such that if $\|\boldsymbol{F}\|_{\boldsymbol{L^{\frac{m+1}{m}}}(\Omega)} \leqslant \delta_0$ then $\|\boldsymbol{u}\|_{\boldsymbol{L^{m+1}}(\Omega)} \leqslant 1$ and

$$\|\nabla \boldsymbol{u}\|^{\frac{2(1-m)}{k}}_{\boldsymbol{L^2}(\Omega)} \leqslant C2^{-2\nu}(2^\nu - 1)(1-m)M^{-2}\min\{1, L^2\}\min\{2, \varepsilon\}^{\nu-1}\varepsilon. \tag{6.28}$$

We recall that the distance between two closed sets $\mathcal{A}$ and $\mathcal{B}$ of $\mathbb{R}^N$ with one of them compact is defined by

$$\mathrm{dist}(\mathcal{A}, \mathcal{B}) = \min_{(x,y)\in\mathcal{A}\times\mathcal{B}} |x - y|$$

and that

$$\mathrm{dist}(\mathcal{A}, \mathcal{B}) > 0 \iff \mathcal{A}\cap\mathcal{B} = \emptyset.$$

Let $x_0 \in \overline{K(\varepsilon)^{\mathrm{c}}}$. Let $y \in \overline{B}\left(x_0, \frac{\varepsilon}{2}\right)$ and let $z \in K$. By definition of $K(\varepsilon)$, $\mathrm{dist}(\overline{K(\varepsilon)^{\mathrm{c}}}, K) = \varepsilon$. We then have

$$\varepsilon = \mathrm{dist}(\overline{K(\varepsilon)^{\mathrm{c}}}, K) \leqslant |x_0 - z| \leqslant |x_0 - y| + |y - z| \leqslant \frac{\varepsilon}{2} + |y - z|.$$

Taking the minimum on $(y, z) \in \overline{B}\left(x_0, \frac{\varepsilon}{2}\right) \times K$, we get

$$\frac{\varepsilon}{2} \leqslant \mathrm{dist}\left(\overline{B}\left(x_0, \frac{\varepsilon}{2}\right), K\right),$$

which means that $\overline{B}\left(x_0, \frac{\varepsilon}{2}\right) \cap K = \emptyset$, for any $x_0 \in \overline{K(\varepsilon)^{\mathrm{c}}}$. By (6.28), $\boldsymbol{u}$ satisfies (2.2) with $\rho_0 = \frac{\varepsilon}{4}$ and we deduce that for any $x_0 \in \overline{K(\varepsilon)^{\mathrm{c}}}$, $\boldsymbol{u}_{|\Omega \cap B\left(x_0, \frac{\varepsilon}{4}\right)} \equiv \boldsymbol{0}$ (Theorem 2.3). Let $n \in \mathbb{N}$. By compactness, $\overline{K\left(\frac{7\varepsilon}{8}\right)^{\mathrm{c}}} \cap \overline{B}(0, n)$ may be covered by a finite number of balls $B\left(x_0, \frac{\varepsilon}{4}\right)$ with $x_0 \in \overline{K(\varepsilon)^{\mathrm{c}}}$. Thus for any $n \in \mathbb{N}$, $\boldsymbol{u}_{|\Omega \cap K\left(\frac{7\varepsilon}{8}\right)^{\mathrm{c}} \cap B(0,n)} \equiv \boldsymbol{0}$. It follows that $\boldsymbol{u} = \boldsymbol{0}$ almost everywhere on

$$\bigcup_{n \in \mathbb{N}} \left(\Omega \cap K\left(\frac{7\varepsilon}{8}\right)^{\mathrm{c}} \cap B(0, n)\right) = \Omega \cap K\left(\frac{7\varepsilon}{8}\right)^{\mathrm{c}}.$$

This means that $\mathrm{supp}\, \boldsymbol{u} \subset \overline{\Omega} \cap K\left(\frac{7\varepsilon}{8}\right) \subset \overline{\Omega} \cap \mathcal{O}(\varepsilon)$. Finally, since K is a compact set, Ω is open and $K \subset \Omega$, it follows that if ε is small enough then $\mathcal{O}(\varepsilon) \subset \Omega$. This ends the proof. $\square$

Proof of Theorem 2.6. Let L, M and C be the constants given by (1.8), (1.9) and Theorem 2.3, respectively. We would like to apply Theorem 2.3 with $\rho_0 = 1$. Since $\boldsymbol{F}$ is compactly supported and $\boldsymbol{u} \in \boldsymbol{H^1}(\mathbb{R}^N) \cap \boldsymbol{L}^{\frac{m+1}{m}}(\mathbb{R}^N)$, there exists $R > 1$ such that $\mathrm{supp}\, \boldsymbol{F} \subset B(0, R-1)$,

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^{m+1}(\{|x|>R-1\})} \leqslant 1 \quad \text{and} \quad \|\nabla \boldsymbol{u}\|_{\boldsymbol{L^2}(\{|x|>R-1\})}^{\frac{2(1-m)}{k}} \leqslant C 2^{1-\nu}(2^\nu - 1)(1-m) M^{-2} \min\{1, L^2\}.$$

Let $x_0 \in \mathbb{R}^N$ be such that $|x_0| \geqslant R + 1$. Then $\overline{B}(x_0, 2) \cap \mathrm{supp}\, \boldsymbol{F} = \emptyset$ and, with help of the above estimate, $\boldsymbol{u}$ satisfies (2.2) with $\rho_0 = 1$. It follows from Theorem 2.3 that $\boldsymbol{u}_{|B(x_0,1)} \equiv \boldsymbol{0}$. For each integer $n \geqslant 2$, define the compact set C_n by

$$C_n = \left\{x \in \mathbb{R}^N;\ R + \frac{1}{n} \leqslant |x| \leqslant R + n - \frac{1}{n}\right\}.$$

By compactness, C_n may be covered by a finite number of balls $B(x_0, 1)$, where $R+1 \leqslant |x_0| \leqslant R+1+n$. Thus for any $n \in \mathbb{N}$, $\boldsymbol{u}_{|C_n} \equiv \boldsymbol{0}$. It follows that $\boldsymbol{u} = \boldsymbol{0}$ almost everywhere on

$$\bigcup_{n \geqslant 2} C_n = \left\{x \in \mathbb{R}^N;\ |x| > R\right\}.$$

Then $\mathrm{supp}\, \boldsymbol{u} \subset \overline{B}(0, R)$, which is the desired result. $\square$

7 Proofs of the existence and smoothness results

In this Section, we prove Proposition 3.5, Theorem 3.1 and 3.8.

Proof of Proposition 3.5. By Remarks 3.6, equation (3.3) makes senses in $\boldsymbol{L^1_{\mathrm{loc}}}(\Omega)$.

Proof of Property 1). Let $1<q\leqslant p<\infty$. Assume $\boldsymbol{F}\in\boldsymbol{L^p_{\mathrm{loc}}}(\Omega)$ and $\boldsymbol{u}\in\boldsymbol{L^q_{\mathrm{loc}}}(\Omega)$ is a solution to (3.3). For $r\in(1,\infty)$, r^- denotes any real in $(1,r)$. Assume $\boldsymbol{v}\in\boldsymbol{L^{r^-}_{\mathrm{loc}}}(\Omega)$, for some $1<r<\infty$, is a solution of (3.3). It follows that $|\boldsymbol{v}|^{-(1-m)}\boldsymbol{v}\in\boldsymbol{L^{\frac{r^-}{m}}_{\mathrm{loc}}}(\Omega)$ and since $0<m<1$, $\boldsymbol{L^{\frac{r^-}{m}}_{\mathrm{loc}}}(\Omega)\subset\boldsymbol{L^r_{\mathrm{loc}}}(\Omega)$. So by (3.3) and Hölder's inequality, $\boldsymbol{V}\boldsymbol{u}\in\boldsymbol{L^{r^-}_{\mathrm{loc}}}(\Omega)$ and so $\Delta\boldsymbol{v}\in\boldsymbol{L^{\min\{r^-,p\}}_{\mathrm{loc}}}(\Omega)$. Furthermore, if for some $1<r<\infty$, $\boldsymbol{v}\in\boldsymbol{L^r_{\mathrm{loc}}}(\Omega;\mathbb{C})$ and $\Delta\boldsymbol{v}\in\boldsymbol{L^r_{\mathrm{loc}}}(\Omega;\mathbb{C})$ then $\boldsymbol{v}\in\boldsymbol{W^{2,r}_{\mathrm{loc}}}(\Omega;\mathbb{C})$ (see for instance Cazenave [10], Theorem 4.1.2 p.101–102). We then have shown the following property. Let $1<r<\infty$.

$$\boldsymbol{u}\in\boldsymbol{L^{r^-}_{\mathrm{loc}}}(\Omega)\implies\boldsymbol{u}\in\boldsymbol{W^{2,\min\{r^-,p\}}_{\mathrm{loc}}}(\Omega). \tag{7.1}$$

Now, we proceed to the proof of Property 1) in 2 cases.

Case 1. $\left(\frac{N}{2}\leqslant q\leqslant p\right)$ or $\left(q<\frac{N}{2}\text{ and }q\leqslant p\leqslant\frac{Nq}{N-2q}\right)$.

It follows from (7.1), applied with $r=q$, that $\boldsymbol{u}\in\boldsymbol{W^{2,q^-}_{\mathrm{loc}}}(\Omega)$. In one hand, if $q<\frac{N}{2}$ then $\boldsymbol{W^{2,q^-}_{\mathrm{loc}}}(\Omega)\subset\boldsymbol{L^{p^-}_{\mathrm{loc}}}(\Omega)$. It follows from (7.1) (applied with $r=p$) and Sobolev's embedding that $\boldsymbol{u}\in\boldsymbol{L^{p+\delta}_{\mathrm{loc}}}(\Omega)$, for $\delta\in(0,1)$ small enough. On the other hand, if $q\geqslant\frac{N}{2}$ then $\boldsymbol{W^{2,q^-}_{\mathrm{loc}}}(\Omega)\subset\boldsymbol{L^{p+1}_{\mathrm{loc}}}(\Omega)$. So in both cases, $\boldsymbol{u}\in\boldsymbol{L^{p+\delta}_{\mathrm{loc}}}(\Omega)$. Applying (7.1) with $r=p+\delta$, we then obtain $\boldsymbol{u}\in\boldsymbol{W^{2,p}_{\mathrm{loc}}}(\Omega)$.

Case 2. $1<q<p$, $q<\frac{N}{2}$ and $\frac{Nq}{N-2q}<p$.

We recall that if $1<r<\frac{N}{2}$ then Sobolev's embedding is

$$\boldsymbol{W^{2,r^-}_{\mathrm{loc}}}(\Omega)\subset\boldsymbol{L^{s^-}_{\mathrm{loc}}}(\Omega),\text{ for any }1\leqslant s<\infty\text{ such that }\frac{1}{s}\geqslant\frac{1}{r}-\frac{2}{N}. \tag{7.2}$$

Since $\frac{Nq}{N-2q}<p$, we may define the smallest integer $n_0\geqslant 2$ such that $\frac{1}{q}-\frac{2n_0}{N}<\frac{1}{p}$. We then set

$$\frac{1}{p_{n_0}}=\begin{cases}\frac{1}{p+1}, & \text{if }\frac{1}{q}-\frac{2n_0}{N}\leqslant 0,\\ \frac{1}{q}-\frac{2n_0}{N}, & \text{if }\frac{1}{q}-\frac{2n_0}{N}>0,\end{cases}$$

in order to have $p<p_{n_0}<\infty$. Finally, define the n_0 real $(p_n)_{n\in[\![0,n_0]\!]}$ by $p_0=q$ and

$$\forall n\in[\![0,n_0-1]\!],\ \frac{1}{p_n}=\frac{1}{p_0}-\frac{2n}{N}.$$

It follows that for any $n\in[\![1,n_0-1]\!]$, $q\leqslant p_{n-1}<p_n\leqslant p<p_{n_0}<\infty$ and

$$\forall n\in[\![1,n_0]\!],\ \frac{1}{p_n}\geqslant\frac{1}{p_{n-1}}-\frac{2}{N}. \tag{7.3}$$

From (7.1)–(7.3) applied n_0 times (and recalling that $p<p_{n_0}<\infty$), we then obtain $\boldsymbol{u}\in\boldsymbol{W^{2,p}_{\mathrm{loc}}}(\Omega)$. This ends the proof of Property 1).

Proof of Property 2). We recall the following Sobolev's embedding and estimate.

$$\boldsymbol{W}^{\mathbf{2},\boldsymbol{N}+\mathbf{1}}_{\mathbf{loc}}(\Omega)\subset\boldsymbol{C}^{\mathbf{1},\frac{\mathbf{1}}{\boldsymbol{N}+\mathbf{1}}}_{\mathbf{loc}}(\Omega)\subset\boldsymbol{C}^{\mathbf{0},\mathbf{1}}_{\mathbf{loc}}(\Omega), \tag{7.4}$$

$$\forall(\boldsymbol{z_1},\boldsymbol{z_2})\in\mathbb{C}^2,\ \left||\boldsymbol{z_1}|^{-(1-m)}\boldsymbol{z_1}-|\boldsymbol{z_2}|^{-(1-m)}\boldsymbol{z_2}\right|\leqslant 5|\boldsymbol{z_1}-\boldsymbol{z_2}|^m. \tag{7.5}$$

Assume further that $(\boldsymbol{F},\boldsymbol{V})\in\boldsymbol{C}^{\mathbf{0},\boldsymbol{\alpha}}_{\mathbf{loc}}(\Omega)\times\boldsymbol{C}^{\mathbf{0},\boldsymbol{\alpha}}_{\mathbf{loc}}(\Omega)$, for some $\alpha\in(0,m]$. In particular, $\boldsymbol{V}\in\boldsymbol{L}^{\boldsymbol{\infty}}_{\mathbf{loc}}(\Omega)$ and by Property 1), $\boldsymbol{u}\in\boldsymbol{W}^{\mathbf{2},\boldsymbol{N}+\mathbf{1}}_{\mathbf{loc}}(\Omega)$. It follows from (7.4), (7.5) and (3.3) that $|\boldsymbol{u}|^{-(1-m)}\boldsymbol{u}\in\boldsymbol{C}^{\mathbf{0},\boldsymbol{m}}_{\mathbf{loc}}(\Omega)$ and so $\Delta\boldsymbol{u}\in\boldsymbol{C}^{\mathbf{0},\boldsymbol{\alpha}}_{\mathbf{loc}}(\Omega)$. Thus $\boldsymbol{u}\in\boldsymbol{C}^{\mathbf{2},\boldsymbol{\alpha}}_{\mathbf{loc}}(\Omega)$ (Theorem 9.19 p.243–244 in Gilbarg and Trudinger [14]). This concludes the proof of the proposition. □

Proof of Theorem 3.1. Let L and M be the constants given by (1.8) and (1.9), respectively. We proceed in 4 steps.

Step 1. Let $\Omega\subset\mathbb{R}^N$ be an open bounded subset and let $\boldsymbol{g}\in\boldsymbol{L}^{\mathbf{2}}(\Omega)$. Then there exists a unique solution $\boldsymbol{u}\in\boldsymbol{H}^{\mathbf{1}}_{\mathbf{0}}(\Omega)$ of

$$-\Delta\boldsymbol{u}=\boldsymbol{g},\ \text{ in } \boldsymbol{L}^{\mathbf{2}}(\Omega). \tag{7.6}$$

Moreover, there exists a positive constant $C=C(|\Omega|,N)$ such that

$$\forall\boldsymbol{g}\in\boldsymbol{L}^{\mathbf{2}}(\Omega),\ \left\|(-\Delta)^{-1}\boldsymbol{g}\right\|_{\boldsymbol{H}^{\mathbf{1}}_{\mathbf{0}}(\Omega)}\leqslant C\|\boldsymbol{g}\|_{\boldsymbol{L}^{\mathbf{2}}(\Omega)}. \tag{7.7}$$

In particular, the mapping $(-\Delta)^{-1}:\boldsymbol{L}^{\mathbf{2}}(\Omega)\longrightarrow\boldsymbol{H}^{\mathbf{1}}_{\mathbf{0}}(\Omega)$ is linear continuous.
Existence and uniqueness come from Lax-Milgram's Theorem where the bounded coercive bilinear form a on $\boldsymbol{H}^{\mathbf{1}}_{\mathbf{0}}(\Omega)\times\boldsymbol{H}^{\mathbf{1}}_{\mathbf{0}}(\Omega)$ and the bounded linear functional L on $\boldsymbol{H}^{-\mathbf{1}}(\Omega)$ are defined by

$$a(u,v)=\mathrm{Re}\int\limits_\Omega\nabla\boldsymbol{u}(x).\overline{\nabla\boldsymbol{v}(x)}\mathrm{d}x\quad\text{ and }\quad\langle L,v\rangle_{\boldsymbol{H}^{-\mathbf{1}},\boldsymbol{H}^{\mathbf{1}}_{\mathbf{0}}}=\mathrm{Re}\int\limits_\Omega\boldsymbol{v}(x)\overline{\boldsymbol{g}(x)}\mathrm{d}x,$$

respectively. Note that a is coercive with help of Poincaré's inequality. Taking the $\boldsymbol{H}^{-\mathbf{1}}-\boldsymbol{H}^{\mathbf{1}}_{\mathbf{0}}$ duality product of equation (7.6) with $\boldsymbol{u}$ and applying Poincaré's inequality, we obtain estimate (7.7) and so continuity of $(-\Delta)^{-1}$.

Step 2. Let $\Omega\subset\mathbb{R}^N$ be an open bounded subset, let $0<m<1$, let $(\boldsymbol{a},\boldsymbol{b})\in\mathbb{C}^2$ and let $\boldsymbol{F}\in\boldsymbol{L}^{\mathbf{2}}(\Omega)$. For each $\ell\in\mathbb{N}$, define $\boldsymbol{f_\ell}=\boldsymbol{g_\ell}-\mathbf{i}\boldsymbol{F}$, where

$$\forall\boldsymbol{v}\in\boldsymbol{L}^{\mathbf{2}}(\Omega),\ \boldsymbol{g_\ell}(\boldsymbol{v})=\begin{cases}\mathbf{i}\boldsymbol{a}|\boldsymbol{v}|^{-(1-m)}\boldsymbol{v}+\mathbf{i}\boldsymbol{b}\boldsymbol{v}, & \text{if } |\boldsymbol{v}|\leqslant\ell,\\ \mathbf{i}\boldsymbol{a}\ell^m\dfrac{\boldsymbol{v}}{|\boldsymbol{v}|}+\mathbf{i}\boldsymbol{b}\ell\dfrac{\boldsymbol{v}}{|\boldsymbol{v}|}, & \text{if } |\boldsymbol{v}|>\ell.\end{cases} \tag{7.8}$$

Then for any $\ell\in\mathbb{N}$, there exists at least one solution $\boldsymbol{u_\ell}\in\boldsymbol{H}^{\mathbf{1}}_{\mathbf{0}}(\Omega)$ of

$$-\Delta\boldsymbol{u_\ell}=\boldsymbol{f_\ell}\left(\boldsymbol{u_\ell}\right),\ \text{ in } \boldsymbol{L}^{\mathbf{2}}(\Omega).$$

It is clear that $(\boldsymbol{f_\ell})_{\ell\in\mathbb{N}} \subset \boldsymbol{C(L^2(\Omega);L^2(\Omega))}$. With help of Step 1 and the continuous and compact embedding $\boldsymbol{i} : \boldsymbol{H_0^1}(\Omega) \hookrightarrow \boldsymbol{L^2}(\Omega)$, we may define a continuous and compact sequence of mappings $(\boldsymbol{T_\ell})_{\ell\in\mathbb{N}}$ of $\boldsymbol{H_0^1}(\Omega)$ as follows. For any $\ell \in \mathbb{N}$, set

$$\begin{array}{ccccccc} \boldsymbol{T_\ell} : \boldsymbol{H_0^1}(\Omega) & \overset{\boldsymbol{i}}{\hookrightarrow} & \boldsymbol{L^2}(\Omega) & \xrightarrow{\boldsymbol{f_\ell}} & \boldsymbol{L^2}(\Omega) & \xrightarrow{(-\Delta)^{-1}} & \boldsymbol{H_0^1}(\Omega) \\ \boldsymbol{v} & \longmapsto & \boldsymbol{i}(\boldsymbol{v}) = \boldsymbol{v} & \longmapsto & \boldsymbol{f_\ell}(\boldsymbol{v}) & \longmapsto & (-\Delta)^{-1}(\boldsymbol{f_\ell})(\boldsymbol{v}) \end{array}$$

Let $\ell \in \mathbb{N}$. Let C be the constant in (7.7) and set $R = C(|\boldsymbol{a}| + |\boldsymbol{b}| + 1)\Big(2\ell|\Omega|^{\frac{1}{2}} + \|\boldsymbol{F}\|_{\boldsymbol{L^2}(\Omega)}\Big)$. Let $\boldsymbol{v} \in \boldsymbol{H_0^1}(\Omega)$. It follows from (7.7) that

$$\begin{aligned} \|\boldsymbol{T_\ell}(\boldsymbol{v})\|_{\boldsymbol{H_0^1}(\Omega)} &= \big\|(-\Delta)^{-1}(\boldsymbol{f_\ell})(\boldsymbol{v})\big\|_{\boldsymbol{H_0^1}(\Omega)} \leqslant C\|\boldsymbol{f_\ell}(\boldsymbol{v})\|_{\boldsymbol{L^2}(\Omega)} \\ &\leqslant C(|\boldsymbol{a}| + |\boldsymbol{b}| + 1)\Big((\ell^m + \ell)|\Omega|^{\frac{1}{2}} + \|\boldsymbol{F}\|_{\boldsymbol{L^2}(\Omega)}\Big) \leqslant R. \end{aligned}$$

Hence, $\boldsymbol{T_\ell}\left(\boldsymbol{H_0^1}(\Omega)\right) \subset \overline{\boldsymbol{B}}_{\boldsymbol{H_0^1}}(0,R)$, where $\overline{\boldsymbol{B}}_{\boldsymbol{H_0^1}}(0,R) = \Big\{\boldsymbol{u} \in \boldsymbol{H_0^1}(\Omega); \|\boldsymbol{u}\|_{\boldsymbol{H_0^1}(\Omega)} \leqslant R\Big\}$. In a nutshell, $\boldsymbol{T_\ell}$ is a continuous and compact mapping from $\boldsymbol{H_0^1}(\Omega)$ into itself, $\overline{\boldsymbol{B}}_{\boldsymbol{H_0^1}}(0,R)$ is a closed convex subset of $\boldsymbol{H_0^1}(\Omega)$ and $\boldsymbol{T_\ell}\left(\overline{\boldsymbol{B}}_{\boldsymbol{H_0^1}}(0,R)\right) \subset \overline{\boldsymbol{B}}_{\boldsymbol{H_0^1}}(0,R)$. By the Schauder's fixed point Theorem, $\boldsymbol{T_\ell}$ admits at least one fixed point $\boldsymbol{u_\ell} \subset \overline{\boldsymbol{B}}_{\boldsymbol{H_0^1}}(0,R)$. Hence Step 2 follows.

Step 3. Let be the hypotheses of the theorem. Assume further that Ω is bounded. Then equation (1.1) admits at least one solution $\boldsymbol{u} \in \boldsymbol{H_0^1}(\Omega)$.

In other words, we have to solve

$$-\Delta\boldsymbol{u} = \boldsymbol{f}(\boldsymbol{u}), \text{ in } \boldsymbol{L^2}(\Omega), \tag{7.9}$$

where $\boldsymbol{f} = \boldsymbol{g} - \mathrm{i}\boldsymbol{F}$ and for any $\boldsymbol{v} \in \boldsymbol{L^2}(\Omega)$, $\boldsymbol{g}(\boldsymbol{v}) = \mathrm{i}\boldsymbol{a}|\boldsymbol{v}|^{-(1-m)}\boldsymbol{v} + \mathrm{i}\boldsymbol{b}\boldsymbol{v}$. Let $(\boldsymbol{F^k})_{k\in\mathbb{N}} \subset \mathscr{D}(\Omega)$ be such that $\boldsymbol{F^k} \xrightarrow[k\to\infty]{\boldsymbol{L}^{\frac{m+1}{m}}(\Omega)} \boldsymbol{F}$ and for any $k \in \mathbb{N}$, $\|\boldsymbol{F^k}\|_{\boldsymbol{L}^{\frac{m+1}{m}}(\Omega)} \leqslant 2\|\boldsymbol{F}\|_{\boldsymbol{L}^{\frac{m+1}{m}}(\Omega)}$. Let $\boldsymbol{g_\ell}$ be defined by (7.8) and set for any $(k,\ell) \in \mathbb{N}^2$, $\boldsymbol{f_\ell^k} = \boldsymbol{g_\ell} - \mathrm{i}\boldsymbol{F^k}$. For any $(k,\ell) \in \mathbb{N}^2$, let $\boldsymbol{u_\ell^k} \in \boldsymbol{H_0^1}(\Omega)$ be a solution of

$$-\Delta\boldsymbol{u_\ell^k} = \boldsymbol{f_\ell}(\boldsymbol{u_\ell^k}), \text{ in } \boldsymbol{L^2}(\Omega), \tag{7.10}$$

given by Step 2. We take the $\boldsymbol{H^{-1}} - \boldsymbol{H_0^1}$ duality product of equation (7.10) with $\boldsymbol{u_\ell^k}$ first and $\mathrm{i}\boldsymbol{u_\ell^k}$ second. Applying Lemma 6.3, we then get for any $(k,\ell) \in \mathbb{N}^2$,

$$\begin{aligned} \|\nabla\boldsymbol{u_\ell^k}\|_{\boldsymbol{L^2}(\Omega)}^2 + L\|\boldsymbol{u_\ell^k}\|_{\boldsymbol{L^{m+1}}(\{|\boldsymbol{u_\ell^k}|\leqslant\ell\})}^{m+1} + L\ell^m\|\boldsymbol{u_\ell^k}\|_{\boldsymbol{L^1}(\{|\boldsymbol{u_\ell^k}|>\ell\})} \\ \leqslant M\int_\Omega|\boldsymbol{F^k}||\boldsymbol{u_\ell^k}|\left(\chi_{\{|\boldsymbol{u_\ell^k}|\leqslant\ell\}} + \chi_{\{|\boldsymbol{u_\ell^k}|>\ell\}}\right)\mathrm{d}x. \end{aligned}$$

Applying Young's inequality (6.3) to the first right-hand side member and Hölder's inequality to the second right-hand side member, we arise to the following estimate.

$$2\|\nabla \boldsymbol{u}_\ell^{\boldsymbol{k}}\|_{\boldsymbol{L}^2(\Omega)}^2 + L\|\boldsymbol{u}_\ell^{\boldsymbol{k}}\|_{\boldsymbol{L}^{m+1}(\{|\boldsymbol{u}_\ell^{\boldsymbol{k}}|\leqslant\ell\})}^{m+1} + 2\|\boldsymbol{u}_\ell^{\boldsymbol{k}}\|_{\boldsymbol{L}^1(\{|\boldsymbol{u}_\ell^{\boldsymbol{k}}|>\ell\})}\left(L\ell^m - M\|\boldsymbol{F}^{\boldsymbol{k}}\|_{\boldsymbol{L}^\infty(\Omega)}\right)$$
$$\leqslant M\left(\frac{2M}{L}\right)^{\frac{1}{m}}\|\boldsymbol{F}^{\boldsymbol{k}}\|_{\boldsymbol{L}^{\frac{m+1}{m}}(\Omega)}^{\frac{m+1}{m}} \leqslant C\|\boldsymbol{F}\|_{\boldsymbol{L}^{\frac{m+1}{m}}(\Omega)}^{\frac{m+1}{m}}. \tag{7.11}$$

For any $k\in\mathbb{N}$, there exists $\ell_k\in\mathbb{N}$ large enough such that $L\ell_k^m - M\|\boldsymbol{F}^{\boldsymbol{k}}\|_{\boldsymbol{L}^\infty(\Omega)}\geqslant 1$. Moreover, Ω being bounded, we have $\boldsymbol{L}^{\boldsymbol{m}+\boldsymbol{1}}(\Omega)\hookrightarrow\boldsymbol{L}^{\boldsymbol{1}}(\Omega)$. So $\left(\nabla \boldsymbol{u}_{\ell_k}^{\boldsymbol{k}}\right)_{k\in\mathbb{N}}$ and $(\boldsymbol{u}_{\ell_k}^{\boldsymbol{k}})_{k\in\mathbb{N}}$ are bounded in $\boldsymbol{L}^{\boldsymbol{2}}(\Omega)$ and $\boldsymbol{L}^{\boldsymbol{1}}(\Omega)$, respectively. It follows from Gagliardo-Nirenberg's inequality (6.2) (applied with $p=1$), that $(\boldsymbol{u}_{\ell_k}^{\boldsymbol{k}})_{k\in\mathbb{N}}$ is also bounded in $\boldsymbol{L}^{\boldsymbol{2}}(\Omega)$ and so in $\boldsymbol{H}_{\boldsymbol{0}}^{\boldsymbol{1}}(\Omega)$. Finally, by Rellich-Kondrachov's Theorem, there exists a subsequence $(\boldsymbol{u}_{\boldsymbol{\varphi(n)}}^{\boldsymbol{n}})_{n\in\mathbb{N}}$ of $(\boldsymbol{u}_{\ell_k}^{\boldsymbol{k}})_{k\in\mathbb{N}}$ and $h\in L^2(\Omega;\mathbb{R})$, such that

$$\boldsymbol{u}_{\boldsymbol{\varphi(n)}}^{\boldsymbol{n}} \xrightarrow[n\to\infty]{\boldsymbol{L}^2(\Omega)} \boldsymbol{u}, \tag{7.12}$$
$$\boldsymbol{u}_{\boldsymbol{\varphi(n)}}^{\boldsymbol{n}} \xrightarrow[n\to\infty]{\text{a.e. in }\Omega} \boldsymbol{u}, \tag{7.13}$$
$$\left|\boldsymbol{u}_{\boldsymbol{\varphi(n)}}^{\boldsymbol{n}}\right| \leqslant h, \text{ for any } n\in\mathbb{N}, \text{ a.e. in } \Omega, \tag{7.14}$$

By (7.13) and (7.14),

$$\boldsymbol{g}_{\boldsymbol{\varphi(n)}}\left(\boldsymbol{u}_{\boldsymbol{\varphi(n)}}^{\boldsymbol{n}}\right)\chi_{\left\{|\boldsymbol{u}_{\boldsymbol{\varphi(n)}}^{\boldsymbol{n}}|\leqslant\varphi(n)\right\}} \xrightarrow[n\to\infty]{\text{a.e. in }\Omega} \boldsymbol{g}(\boldsymbol{u}),$$
$$\forall n\in\mathbb{N},\ \left|\boldsymbol{g}_{\boldsymbol{\varphi(n)}}\left(\boldsymbol{u}_{\boldsymbol{\varphi(n)}}^{\boldsymbol{n}}\right)\right| \leqslant C(h^m+h)\in\boldsymbol{L}^{\boldsymbol{1}}(\Omega), \text{ a.e. in } \Omega.$$

It follows from the dominated convergence Theorem that

$$\boldsymbol{g}_{\boldsymbol{\varphi(n)}}\left(\boldsymbol{u}_{\boldsymbol{\varphi(n)}}^{\boldsymbol{n}}\right)\chi_{\left\{|\boldsymbol{u}_{\boldsymbol{\varphi(n)}}^{\boldsymbol{n}}|\leqslant\varphi(n)\right\}} \xrightarrow[n\to\infty]{\boldsymbol{L}^1(\Omega)} \boldsymbol{g}(\boldsymbol{u}). \tag{7.15}$$

In addition, by (7.12) and Hölder's inequality,

$$\left\|\boldsymbol{g}_{\boldsymbol{\varphi(n)}}\left(\boldsymbol{u}_{\boldsymbol{\varphi(n)}}^{\boldsymbol{n}}\right)\chi_{\left\{|\boldsymbol{u}_{\boldsymbol{\varphi(n)}}^{\boldsymbol{n}}|>\varphi(n)\right\}}\right\|_{\boldsymbol{L}^1(\Omega)} \leqslant \frac{C}{\varphi(n)}\left(\|\boldsymbol{u}_{\boldsymbol{\varphi(n)}}^{\boldsymbol{n}}\|_{\boldsymbol{L}^{m+1}(\Omega)}^{m+1} + \|\boldsymbol{u}_{\boldsymbol{\varphi(n)}}^{\boldsymbol{n}}\|_{\boldsymbol{L}^2(\Omega)}^2\right) \xrightarrow{n\to\infty} 0. \tag{7.16}$$

Putting together (7.15) and (7.16), we obtain

$$\boldsymbol{g}_{\boldsymbol{\varphi(n)}}\left(\boldsymbol{u}_{\boldsymbol{\varphi(n)}}^{\boldsymbol{n}}\right) \xrightarrow[n\to\infty]{\boldsymbol{L}^1(\Omega)} \boldsymbol{g}(\boldsymbol{u}). \tag{7.17}$$

Since $\boldsymbol{F}^{\boldsymbol{n}} \xrightarrow{n\to\infty} \boldsymbol{F}$ in $\boldsymbol{L}^{\frac{m+1}{m}}(\Omega)\hookrightarrow\boldsymbol{L}^{\boldsymbol{1}}(\Omega)$, we deduce with help of (7.12) and (7.17) that

$$\Delta\boldsymbol{u}_{\boldsymbol{\varphi(n)}}^{\boldsymbol{n}} \xrightarrow[n\to\infty]{\boldsymbol{H}^{-2}(\Omega)} \Delta\boldsymbol{u}, \tag{7.18}$$
$$\boldsymbol{f}_{\boldsymbol{\varphi(n)}}\left(\boldsymbol{u}_{\boldsymbol{\varphi(n)}}^{\boldsymbol{n}}\right) \xrightarrow[n\to\infty]{\boldsymbol{L}^1(\Omega)} \boldsymbol{f}(\boldsymbol{u}). \tag{7.19}$$

By (7.10), we have for any $n\in\mathbb{N}$, $-\Delta\boldsymbol{u}_{\boldsymbol{\varphi(n)}}^{\boldsymbol{n}} = \boldsymbol{f}_{\boldsymbol{\varphi(n)}}^{\boldsymbol{n}}\left(\boldsymbol{u}_{\boldsymbol{\varphi(n)}}^{\boldsymbol{n}}\right)$, in $\boldsymbol{L}^{\boldsymbol{2}}(\Omega)$. Estimates (7.18) and (7.19) allow to pass in the limit in this equation in the sense of $\mathscr{D}'(\Omega)$. This means that $\boldsymbol{u}\in\boldsymbol{H}_{\boldsymbol{0}}^{\boldsymbol{1}}(\Omega)$ is a solution of (7.9) and since $\boldsymbol{f}(\boldsymbol{u}) \in L^2(\Omega)$, equation (7.9) takes sense in $\boldsymbol{L^2}(\Omega)$.

Step 4. Conclusion. Under hypotheses of the theorem, equation (1.1) admits at least one solution $\boldsymbol{u} \in \boldsymbol{H_0^1}(\Omega) \cap \boldsymbol{L^{m+1}}(\Omega)$ and Properties 1)–5) of the theorem hold.

For any $n \in \mathbb{N}$, we write $\Omega_n = \Omega \cap B(0,n)$. Let $n_0 \in \mathbb{N}$ be large enough to have $\Omega_{n_0} \neq \emptyset$. For each $n > n_0$, let $\boldsymbol{u_n} \in \boldsymbol{H_0^1}(\Omega_n)$ be any solution of (1.1) in Ω_n given by Step 3, with the external source $\boldsymbol{F_n} = \boldsymbol{F}_{|\Omega_n}$. We define $\widetilde{\boldsymbol{u_n}} \in \boldsymbol{H_0^1}(\Omega)$ by extending $\boldsymbol{u_n}$ by $\boldsymbol{0}$ in $\Omega \cap B(0,n)^\mathrm{c}$. Then $\nabla\widetilde{\boldsymbol{u_n}} = \nabla\boldsymbol{u_n}$, almost everywhere in Ω_n and $\nabla\widetilde{\boldsymbol{u_n}} = \boldsymbol{0}$, almost everywhere in $\Omega \cap B(0,n)^\mathrm{c}$. It follows from (3.2) of Theorem 3.4 that $(\boldsymbol{u_n})_{n\in\mathbb{N}}$ is bounded in $\boldsymbol{H_0^1}(\Omega_n) \cap \boldsymbol{L^{m+1}}(\Omega_n)$, or equivalently, $(\widetilde{\boldsymbol{u_n}})_{n\in\mathbb{N}}$ is bounded in $\boldsymbol{H_0^1}(\Omega) \cap \boldsymbol{L^{m+1}}(\Omega)$. Up to a subsequence, that we still denote by $(\widetilde{\boldsymbol{u_n}})_{n\in\mathbb{N}}$, there exists $\boldsymbol{u} \in \boldsymbol{H_0^1}(\Omega) \cap \boldsymbol{L^{m+1}}(\Omega)$ such that $\widetilde{\boldsymbol{u_n}} \rightharpoonup \boldsymbol{u}$ in $\boldsymbol{H_{\mathrm{w}}^1}(\Omega)$, as $n \longrightarrow \infty$, and $\widetilde{\boldsymbol{u_n}} \xrightarrow[n\to\infty]{\boldsymbol{L^{m+1}_{\mathrm{loc}}}(\Omega)} \boldsymbol{u}$. Let $\boldsymbol{\varphi} \in \mathscr{D}(\Omega)$. Since $\widetilde{\boldsymbol{u_n}} \xrightarrow[n\to\infty]{\boldsymbol{L^{m+1}_{\mathrm{loc}}}(\Omega)} \boldsymbol{u}$, we have $|\widetilde{\boldsymbol{u_n}}|^{-(1-m)}\widetilde{\boldsymbol{u_n}} \xrightarrow[n\to\infty]{\boldsymbol{L^{\frac{m+1}{m}}_{\mathrm{loc}}}(\Omega)} |\boldsymbol{u}|^{-(1-m)}\boldsymbol{u}$, and in particular

$$\lim_{n\to\infty} \langle \boldsymbol{a}|\widetilde{\boldsymbol{u_n}}|^{-(1-m)}\widetilde{\boldsymbol{u_n}}, \boldsymbol{\varphi}\rangle_{\boldsymbol{L^{\frac{m+1}{m}}}(\Omega),\boldsymbol{L^{m+1}}(\Omega)} = \langle \boldsymbol{a}|\boldsymbol{u}|^{-(1-m)}\boldsymbol{u}, \boldsymbol{\varphi}\rangle_{\boldsymbol{L^{\frac{m+1}{m}}}(\Omega),\boldsymbol{L^{m+1}}(\Omega)}. \tag{7.20}$$

Recalling that $\boldsymbol{u} \in \boldsymbol{H_0^1}(\Omega)$ and $\widetilde{\boldsymbol{u_n}} \rightharpoonup \boldsymbol{u}$ in $\boldsymbol{H_{\mathrm{w}}^1}(\Omega)$, as $n \longrightarrow \infty$, we get with help of (7.20),

$$\begin{aligned} \lim_{n\to\infty} \Big(\langle \mathbf{i}\nabla\widetilde{\boldsymbol{u_n}}, \nabla\boldsymbol{\varphi}\rangle_{\boldsymbol{L^2}(\Omega),\boldsymbol{L^2}(\Omega)} + \langle \boldsymbol{a}|\widetilde{\boldsymbol{u_n}}|^{-(1-m)}\widetilde{\boldsymbol{u_n}}, \boldsymbol{\varphi}\rangle_{\boldsymbol{L^{\frac{m+1}{m}}}(\Omega),\boldsymbol{L^{m+1}}(\Omega)} \\ + \langle \boldsymbol{b}\widetilde{\boldsymbol{u_n}}, \boldsymbol{\varphi}\rangle_{\boldsymbol{L^2}(\Omega),\boldsymbol{L^2}(\Omega)} \Big) = \langle -\mathbf{i}\Delta\boldsymbol{u} + \boldsymbol{a}|\boldsymbol{u}|^{-(1-m)}\boldsymbol{u} + \boldsymbol{b}\boldsymbol{u}\rangle_{\mathscr{D}'(\Omega),\mathscr{D}(\Omega)}. \end{aligned} \tag{7.21}$$

Let $n_1 > n_0$ be large enough to have $\operatorname{supp}\boldsymbol{\varphi} \subset \Omega_{n_1}$. Using basic properties of $\widetilde{\boldsymbol{u_n}}$ described as above and the fact $\boldsymbol{u_n}$ is a solution of (1.1) in Ω_n, we obtain for any $n > n_1$, $\boldsymbol{\varphi}_{|\Omega_n} \in \mathscr{D}(\Omega_n)$ and

$$\begin{aligned} 0 &= \langle -\mathbf{i}\Delta\boldsymbol{u_n} + a|\boldsymbol{u_n}|^{-(1-m)}\boldsymbol{u_n} + \boldsymbol{b}\boldsymbol{u_n} - \boldsymbol{F_n}, \boldsymbol{\varphi}_{|\Omega_n}\rangle_{\mathscr{D}'(\Omega_n),\mathscr{D}(\Omega_n)} \\ &= \big\langle \mathbf{i}\nabla\boldsymbol{u_n}, \nabla\big(\boldsymbol{\varphi}_{|\Omega_n}\big)\big\rangle_{\boldsymbol{L^2}(\Omega_n),\boldsymbol{L^2}(\Omega_n)} + \big\langle \boldsymbol{a}|\boldsymbol{u_n}|^{-(1-m)}\boldsymbol{u_n}, \boldsymbol{\varphi}_{|\Omega_n}\big\rangle_{\boldsymbol{L^{\frac{m+1}{m}}}(\Omega_n),\boldsymbol{L^{m+1}}(\Omega_n)} \\ &\qquad + \big\langle \boldsymbol{b}\boldsymbol{u_n}, \boldsymbol{\varphi}_{|\Omega_n}\big\rangle_{\boldsymbol{L^2}(\Omega_n),\boldsymbol{L^2}(\Omega_n)} - \big\langle \boldsymbol{F_n}, \boldsymbol{\varphi}_{|\Omega_n}\big\rangle_{\boldsymbol{L^{\frac{m+1}{m}}}(\Omega_n),\boldsymbol{L^{m+1}}(\Omega_n)} \\ &= \langle \mathbf{i}\nabla\widetilde{\boldsymbol{u_n}}, \nabla\boldsymbol{\varphi}\rangle_{\boldsymbol{L^2}(\Omega),\boldsymbol{L^2}(\Omega)} + \big\langle \boldsymbol{a}|\widetilde{\boldsymbol{u_n}}|^{-(1-m)}\widetilde{\boldsymbol{u_n}}, \boldsymbol{\varphi}\big\rangle_{\boldsymbol{L^{\frac{m+1}{m}}}(\Omega),\boldsymbol{L^{m+1}}(\Omega)} \\ &\qquad + \langle \boldsymbol{b}\widetilde{\boldsymbol{u_n}}, \boldsymbol{\varphi}\rangle_{\boldsymbol{L^2}(\Omega),\boldsymbol{L^2}(\Omega)} - \langle \boldsymbol{F}, \boldsymbol{\varphi}\rangle_{\boldsymbol{L^{\frac{m+1}{m}}}(\Omega),\boldsymbol{L^{m+1}}(\Omega)}, \end{aligned}$$

from which we deduce

$$\begin{aligned} &\langle \mathbf{i}\nabla\widetilde{\boldsymbol{u_n}}, \nabla\boldsymbol{\varphi}\rangle_{\boldsymbol{L^2}(\Omega),\boldsymbol{L^2}(\Omega)} + \langle \boldsymbol{a}|\widetilde{\boldsymbol{u_n}}|^{-(1-m)}\widetilde{\boldsymbol{u_n}}, \boldsymbol{\varphi}\rangle_{\boldsymbol{L^{\frac{m+1}{m}}}(\Omega),\boldsymbol{L^{m+1}}(\Omega)} + \langle \boldsymbol{b}\widetilde{\boldsymbol{u_n}}, \boldsymbol{\varphi}\rangle_{\boldsymbol{L^2}(\Omega),\boldsymbol{L^2}(\Omega)} \\ &= \langle \boldsymbol{F}, \boldsymbol{\varphi}\rangle_{\boldsymbol{L^{\frac{m+1}{m}}}(\Omega),\boldsymbol{L^{m+1}}(\Omega)}, \end{aligned} \tag{7.22}$$

for any $n > n_1$. Passing to the limit in (7.22), we get with (7.21),

$$\forall\boldsymbol{\varphi} \in \mathscr{D}(\Omega),\ \langle -\mathbf{i}\Delta\boldsymbol{u} + \boldsymbol{a}|\boldsymbol{u}|^{-(1-m)}\boldsymbol{u} + \boldsymbol{b}\boldsymbol{u}, \boldsymbol{\varphi}\rangle_{\mathscr{D}'(\Omega),\mathscr{D}(\Omega)} = \langle \boldsymbol{F}, \boldsymbol{\varphi}\rangle_{\mathscr{D}'(\Omega),\mathscr{D}(\Omega)},$$

which is the desired result. Properties 1) and 2) come from Proposition 3.5. Finally, if $\boldsymbol{F}$ is spherically symmetric then $\boldsymbol{u}$, obtained as a limit, is also spherically symmetric. Indeed, we replace all the functional spaces $\boldsymbol{E}$ with $\boldsymbol{E}_{\mathbf{rad}}$ and we follow the above proof step by step. For $N=1$, this includes the case where $\boldsymbol{F}$ is an even function. Finally, if $\boldsymbol{F}$ is an odd function, it is sufficient to work with the space $\boldsymbol{E}_{\mathbf{odd}}=\{\boldsymbol{v}\in\boldsymbol{E};\ \boldsymbol{v}\text{ is odd}\}$ in place of $\boldsymbol{E}$. Hence Property 4). □

Proof of Corollary 3.8. Let the assumptions of the corollary be satisfied. Let $\boldsymbol{a}=-\mathrm{i}\boldsymbol{\lambda}$, $\boldsymbol{b}=\mathrm{i}b$ and $\boldsymbol{G}=-\mathrm{i}\boldsymbol{F}$. Then $(\boldsymbol{a},\boldsymbol{b})\in\mathbb{A}\times\mathbb{B}$ satisfies (1.4) and we may apply Theorem 3.1 and Theorem 2.6 to find a solution $\boldsymbol{\varphi}\in\boldsymbol{C}^{\mathbf{2},\boldsymbol{m}}_{\mathbf{b}}(\mathbb{R}^N)$ of (1.1) compactly supported for such $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{G}$. It follows that $\boldsymbol{\varphi}$ is a solution to (3.7). A straightforward calculation show that $\boldsymbol{u}$ defined by (3.6) is a solution to (3.5). This ends the proof. □

8 Proofs of the uniqueness results

In this Section, we prove Theorems 4.1 and 4.2, and Corollaries 4.3, 4.4 and 4.5. Let $0<m\leqslant 1$. Set for any $\boldsymbol{z}\in\mathbb{C}$, $\boldsymbol{f}(\boldsymbol{z})=|\boldsymbol{z}|^{-(1-m)}\boldsymbol{z}$, where it is understood that $\boldsymbol{f}(\mathbf{0})=\mathbf{0}$. The proof of Theorem 4.1 relies on the two following lemmas.

Lemma 8.1. *Let $0<m\leqslant 1$. Then there exists a positive constant C such that*

$$\forall(\boldsymbol{z_1},\boldsymbol{z_2})\in\mathbb{C}^2,\ \mathrm{Re}\Big(\big(\boldsymbol{f}(\boldsymbol{z_1})-\boldsymbol{f}(\boldsymbol{z_2})\big)(\overline{\boldsymbol{z_1}-\boldsymbol{z_2}})\Big)\geqslant C\frac{|\boldsymbol{z_1}-\boldsymbol{z_2}|^2}{(|\boldsymbol{z_1}|+|\boldsymbol{z_2}|)^{1-m}},$$

as soon as $|\boldsymbol{z_1}|+|\boldsymbol{z_2}|>0$.

Proof. We denote by $|\ .\ |_2$ the Euclidean norm in $\mathbb{R}^2$. From Lemma 4.10, p.264 of Díaz [11], there exists a positive constant C such that

$$\left(|X|_2^{-(1-m)}X-|Y|_2^{-(1-m)}Y\right).(X-Y)\geqslant C\frac{|X-Y|_2^2}{(|X|_2+|Y|_2)^{1-m}},$$

for any $(X,Y)\in\mathbb{R}^2\times\mathbb{R}^2$ satisfying $|X|_2+|Y|_2>0$. We apply this lemma with $X=\begin{pmatrix}\mathrm{Re}(\boldsymbol{z_1})\\ \mathrm{Im}(\boldsymbol{z_1})\end{pmatrix}$ and $Y=\begin{pmatrix}\mathrm{Re}(\boldsymbol{z_2})\\ \mathrm{Im}(\boldsymbol{z_2})\end{pmatrix}$. Note that $|X|_2=|\boldsymbol{z_1}|$, $|Y|_2=|\boldsymbol{z_2}|$ and $|X-Y|_2=|\boldsymbol{z_1}-\boldsymbol{z_2}|$. The result follows from a direct calculation. □

Corollary 8.2. *Let $0<m\leqslant 1$. Then,*

$$\mathrm{Re}\Big(\big(\boldsymbol{f}(\boldsymbol{z_1})-\boldsymbol{f}(\boldsymbol{z_2})\big)(\overline{\boldsymbol{z_1}-\boldsymbol{z_2}})\Big)\geqslant 0,$$

for any $(\boldsymbol{z_1},\boldsymbol{z_2})\in\mathbb{C}^2$.

Proof. The result is clear if $|\boldsymbol{z_1}|+|\boldsymbol{z_2}|=0$. Otherwise, apply Lemma 8.1. □

Remark 8.3. Corollary 8.2 still holds for any $m>0$ and can be directly obtained as follows. The mapping f (seen as a function from $\mathbb{R}^2$ onto $\mathbb{R}^2$) is the derivative of the convex function

$$F: \begin{array}{ccc} \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ (x,y) & \longmapsto & \frac{1}{m+1}(x^2+y^2)^{\frac{m+1}{2}}. \end{array}$$

It follows that f is a monotonic function (Proposition 5.5 p.25 of Ekeland and Temam [13]).

Lemma 8.4. *Let $\Omega\subseteq\mathbb{R}^N$ be an open subset, let $0<m<1$, let $(\boldsymbol{a},\boldsymbol{b})\in\mathbb{C}^2$ satisfying (1.5) and let $\boldsymbol{F_1},\boldsymbol{F_2}\in \boldsymbol{L^1_{\mathrm{loc}}}(\Omega)$ be such that $\boldsymbol{F_1}-\boldsymbol{F_2}\in\boldsymbol{L^2}(\Omega)$. Let $\boldsymbol{u_1},\boldsymbol{u_2}\in\boldsymbol{H^1_0}(\Omega)\cap\boldsymbol{L^{m+1}}(\Omega)$ be two solutions of (4.1) and (4.2), respectively. Then there exists a positive constant $C=C(N,m)$ satisfying the following property. If $\boldsymbol{a}\neq\boldsymbol{0}$ then*

$$\mathrm{Im}(\boldsymbol{a})\|\nabla\boldsymbol{u_1}-\nabla\boldsymbol{u_2}\|^2_{\boldsymbol{L^2}}+C|\boldsymbol{a}|^2\int\limits_\omega\frac{|\boldsymbol{u_1}-\boldsymbol{u_2}|^2}{(|\boldsymbol{u_1}(x)|+|\boldsymbol{u_2}(x)|)^{1-m}}\mathrm{d}x+\mathrm{Re}\left(\boldsymbol{a}\overline{\boldsymbol{b}}\right)\|\boldsymbol{u_1}-\boldsymbol{u_2}\|^2_{\boldsymbol{L^2}}$$
$$\leqslant\mathrm{Re}\int\limits_\Omega\overline{\boldsymbol{a}}\big(\boldsymbol{F_1}(x)-\boldsymbol{F_2}(x)\big)\overline{\big(\boldsymbol{u_1}(x)-\boldsymbol{u_2}(x)\big)}\mathrm{d}x,\quad(8.1)$$

where $\omega=\{x\in\Omega;|\boldsymbol{u_1}(x)|+|\boldsymbol{u_2}(x)|>0\}$. If $\boldsymbol{a}=\boldsymbol{0}$ then

$$\mathrm{Re}(\boldsymbol{b})\|\boldsymbol{u_1}-\boldsymbol{u_2}\|^2_{\boldsymbol{L^2}}=\mathrm{Re}\int\limits_\Omega\big(\boldsymbol{F_1}(x)-\boldsymbol{F_2}(x)\big)\overline{\big(\boldsymbol{u_1}(x)-\boldsymbol{u_2}(x)\big)}\mathrm{d}x,\qquad(8.2)$$

$$\|\nabla\boldsymbol{u_1}-\nabla\boldsymbol{u_2}\|^2_{\boldsymbol{L^2}}+\mathrm{Im}(\boldsymbol{b})\|\boldsymbol{u_1}-\boldsymbol{u_2}\|^2_{\boldsymbol{L^2}}=\mathrm{Im}\int\limits_\Omega\big(\boldsymbol{F_1}(x)-\boldsymbol{F_2}(x)\big)\overline{\big(\boldsymbol{u_1}(x)-\boldsymbol{u_2}(x)\big)}\mathrm{d}x.\qquad(8.3)$$

Proof. Let $\boldsymbol{u_1}$ and $\boldsymbol{u_2}$ be two solutions of (1.1) and (1.3) and set $\boldsymbol{u}=\boldsymbol{u_1}-\boldsymbol{u_2}$ and $\boldsymbol{F}=\boldsymbol{F_1}-\boldsymbol{F_2}$. Then $\boldsymbol{u}$ satisfies

$$-\mathbf{i}\Delta\boldsymbol{u}+\boldsymbol{a}\big(\boldsymbol{f}(\boldsymbol{u_1})-\boldsymbol{f}(\boldsymbol{u_2})\big)+\boldsymbol{b}\boldsymbol{u}=\boldsymbol{F},\ \text{in } \boldsymbol{H^{-1}}(\Omega)+\boldsymbol{L^{\frac{m+1}{m}}}(\Omega).\qquad(8.4)$$

Assume $\boldsymbol{a}\neq\boldsymbol{0}$. We take the $\boldsymbol{H^{-1}}+\boldsymbol{L^{\frac{m+1}{m}}}-\boldsymbol{H^1_0}\cap\boldsymbol{L^{m+1}}$ duality product of (8.4) with $\boldsymbol{a}\boldsymbol{u}$. We obtain,

$$\mathrm{Im}(\boldsymbol{a})\|\nabla\boldsymbol{u}\|^2_{\boldsymbol{L^2}}+|\boldsymbol{a}|^2\langle\boldsymbol{f}(\boldsymbol{u_1})-\boldsymbol{f}(\boldsymbol{u_2}),\boldsymbol{u}\rangle_{\boldsymbol{L^{\frac{m+1}{m}}},\boldsymbol{L^{m+1}}}+\mathrm{Re}\left(\boldsymbol{a}\overline{\boldsymbol{b}}\right)\|\boldsymbol{u}\|^2_{\boldsymbol{L^2}}=\langle\overline{\boldsymbol{a}}\boldsymbol{F},\boldsymbol{u}\rangle_{\boldsymbol{L^2},\boldsymbol{L^2}}.\qquad(8.5)$$

Applying Lemma 8.1, there exists a positive constant $C=C(N,m)$ such that

$$\langle\boldsymbol{f}(\boldsymbol{u_1})-\boldsymbol{f}(\boldsymbol{u_2}),\boldsymbol{u}\rangle_{\boldsymbol{L^{\frac{m+1}{m}}},\boldsymbol{L^{m+1}}}\geqslant C\int\limits_\omega\frac{|\boldsymbol{u}(x)|^2}{(|\boldsymbol{u_1}(x)|+|\boldsymbol{u_2}(x)|)^{1-m}}\mathrm{d}x.\qquad(8.6)$$

Then (8.1) follows from (8.5) and (8.6). We turn out the case $\boldsymbol{a}=\boldsymbol{0}$. Taking the $\boldsymbol{H^{-1}}+\boldsymbol{L^{\frac{m+1}{m}}}-\boldsymbol{H^1_0}\cap\boldsymbol{L^{m+1}}$ duality product of (8.4) with $\boldsymbol{u}$ and $\mathbf{i}\boldsymbol{u}$, one respectively obtains (8.2) and (8.3). □

Proof of Theorem 4.1. Note that since $(\boldsymbol{a},\boldsymbol{b})\in\mathbb{C}^2\setminus\{(\boldsymbol{0},\boldsymbol{0})\}$ satisfies (1.5), if $\boldsymbol{a}=\boldsymbol{0}$ and $\mathrm{Re}(\boldsymbol{b})=0$ then one necessarily has $\mathrm{Im}(\boldsymbol{b})>0$. We apply estimates (8.1)–(8.3) of Lemma 8.4, according to the different cases, and Cauchy-Schwarz's inequality. Estimates (4.3) and (4.4) follow. □

Proof of Theorem 4.2. Let $\boldsymbol{F}\in \boldsymbol{L^1_{\mathrm{loc}}}(\Omega)$ and let $\boldsymbol{u_1},\boldsymbol{u_2}\in \boldsymbol{H^1_0}(\Omega)\cap \boldsymbol{L^{m+1}}(\Omega)$ be two solutions of (1.1) and (1.3). By Lemma 8.4, (8.1)–(8.3) hold with $\boldsymbol{F_1}-\boldsymbol{F_2}=\boldsymbol{0}$. We first note that, since $\boldsymbol{u_1}-\boldsymbol{u_2}\in \boldsymbol{H^1_0}(\Omega)$, if $\|\nabla\boldsymbol{u_1}-\nabla\boldsymbol{u_2}\|_{\boldsymbol{L^2}}=0$ then $\boldsymbol{u_1}-\boldsymbol{u_2}=\boldsymbol{0}$, a.e. in Ω and uniqueness holds. It follows from hypotheses (1.5) and Lemma 8.4 that one necessarily has $\|\boldsymbol{u_1}-\boldsymbol{u_2}\|_{\boldsymbol{L^2}}=0$, $\|\nabla\boldsymbol{u_1}-\nabla\boldsymbol{u_2}\|_{\boldsymbol{L^2}}=0$ or $\int\limits_{\omega}\frac{|\boldsymbol{u_1}-\boldsymbol{u_2}|^2}{(|\boldsymbol{u_1}(x)|+|\boldsymbol{u_2}(x)|)^{1-m}}\mathrm{d}x$, where $\omega=\{x\in\Omega;|\boldsymbol{u_1}(x)|+|\boldsymbol{u_2}(x)|>0\}$. Those three cases imply that $\boldsymbol{u_1}=\boldsymbol{u_2}$, a.e. in Ω. This achieves the proof of the theorem. □

Proof of Corollary 4.3. Apply Theorem 3.1, Theorem 4.2 and Remark 5.1. □

Proof of Corollary 4.4. By uniqueness (Theorem 4.2), $\boldsymbol{u}\equiv\boldsymbol{0}$ is the unique solution. □

Proof of Corollary 4.5. Apply Theorem 2.6, Theorem 3.1, Proposition 3.5, Theorem 4.2 and Remark 5.1. □

References

[1] S. N. Antontsev, J. I. Díaz, and S. Shmarev. *Energy methods for free boundary problems*. Progress in Nonlinear Differential Equations and their Applications, 48. Birkhäuser Boston Inc., Boston, MA, 2002. Applications to nonlinear PDEs and fluid mechanics.

[2] P. Bégout and J. I. Díaz. Localizing estimates of the support of solutions of some nonlinear Schrödinger equations – The evolution case. In preparation.

[3] P. Bégout and J. I. Díaz. Self-similar solutions with compactly supported profile of some nonlinear Schrödinger equations. In preparation.

[4] P. Bégout and J. I. Díaz. On a nonlinear Schrödinger equation with a localizing effect. *C. R. Math. Acad. Sci. Paris*, 342(7):459–463, 2006.

[5] P. Bégout and V. Torri. Numerical computations of the support of solutions of some localizing stationary nonlinear Schrödinger equations. In preparation.

[6] J. A. Belmonte-Beitia. Varias cuestiones sobre la ecuación de Schrödinger non lineal con coeficientes dependientes del espacio. *Bol. Soc. Esp. Mat. Apl. S⃗eMA*, (52):97–128, 2010.

[7] H. Brezis and T. Kato. Remarks on the Schrödinger operator with singular complex potentials. *J. Math. Pures Appl. (9)*, 58(2):137–151, 1979.

[8] R. Carles and C. Gallo. Finite time stabilization of Schrödinger equation. arXiv:1007.0077v2 [math.AP].

[9] T. Cazenave. *Semilinear Schrödinger equations*, volume 10 of *Courant Lecture Notes in Mathematics*. New York University Courant Institute of Mathematical Sciences, New York, 2003.

[10] T. Cazenave. *An introduction to semilinear elliptic equations.* Editora do Instituto de Matemática, Universidade Federal do Rio de Janeiro, Rio de Janeiro, 2006.

[11] J. I. Díaz. *Nonlinear partial differential equations and free boundaries. Vol. I*, volume 106 of *Research Notes in Mathematics.* Pitman (Advanced Publishing Program), Boston, MA, 1985. Elliptic equations.

[12] J. I. Díaz and L. Véron. Local vanishing properties of solutions of elliptic and parabolic quasilinear equations. *Trans. Amer. Math. Soc.*, 290(2):787–814, 1985.

[13] I. Ekeland and R. Temam. *Convex analysis and variational problems*, volume 28 of *Classics in Applied Mathematics.* Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, english edition, 1999. Translated from the French.

[14] D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order.* Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.

[15] P. Grisvard. *Elliptic problems in nonsmooth domains*, volume 24 of *Monographs and Studies in Mathematics.* Pitman (Advanced Publishing Program), Boston, MA, 1985.

[16] L. Hörmander. Definitions of maximal differential operators. *Ark. Mat.*, 3:501–504, 1958.

[17] A. Jensen. Propagation estimates for Schrödinger-type operators. *Trans. Amer. Math. Soc.*, 291(1):129–144, 1985.

[18] E. Kashdan and P. Rosenau. Compactification of nonlinear patterns and waves. *Phys. Rev. Lett.*, 101(26):261602, 4, 2008.

[19] B. J. LeMesurier. Dissipation at singularities of the nonlinear Schrödinger equation through limits of regularisations. *Phys. D*, 138(3-4):334–343, 2000.

[20] J.-L. Lions and E. Magenes. Problèmes aux limites non homogènes. II. *Ann. Inst. Fourier (Grenoble)*, 11:137–178, 1961.

[21] J.-L. Lions and E. Magenes. Problemi ai limiti non omogenei. III. *Ann. Scuola Norm. Sup. Pisa (3)*, 15:41–103, 1961.

[22] V. Liskevich and P. Stollmann. Schrödinger operators with singular complex potentials as generators: existence and stability. *Semigroup Forum*, 60(3):337–343, 2000.

[23] P. Rosenau and S. Schochet. Compact and almost compact breathers: a bridge between an anharmonic lattice and its continuum limit. *Chaos*, 15(1):015111, 18, 2005.

[24] W. A. Strauss. *Partial differential equations.* John Wiley & Sons Inc., New York, 1992. An introduction.

[25] C. Sulem and P.-L. Sulem. *The nonlinear Schrödinger equation*, volume 139 of *Applied Mathematical Sciences.* Springer-Verlag, New York, 1999. Self-focusing and wave collapse.