On Nilpotent Multipliers of some Verbal Products of Groups

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Abstract

The paper is devoted to finding a homomorphic image for the c-nilpotent multiplier of the verbal product of a family of groups with respect to a variety $\mathcal{V}$ when $\mathcal{V} \subseteq \mathcal{N}_c$ or $\mathcal{N}_c \subseteq \mathcal{V}$. Also a structure of the c-nilpotent multiplier of a special case of the verbal product, the nilpotent product, of cyclic groups is given. In fact, we present an explicit formula for the c-nilpotent multiplier of the nth nilpotent product of the group $G = \mathbf{Z} \overset{n}{*} ... \overset{n}{*} \mathbf{Z} \overset{n}{*} \mathbf{Z}_{r_1} \overset{n}{*} ... \overset{n}{*} \mathbf{Z}_{r_t}$, where r_{i+1} divides r_i for all i, $1 \leq i \leq t-1$, and $(p, r_1) = 1$ for any prime p less than or equal to $n + c$, for all positive integers n, c.

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1 Introduction and Motivation

Let $G = F/R$ be a free presentation of a group G. Then the Baer invariant of G with respect to the variety $\mathcal{N}_c$ of nilpotent groups of class at most $c \geq 1$, denoted by $\mathcal{N}_c M(G)$, is defined to be

$$\mathcal{N}_c M(G) = \frac{R \cap \gamma_{c+1}(F)}{[R,\ {}_cF]}.$$

$\mathcal{N}_c M(G)$ is also called the c-nilpotent multiplier of G. Clearly if $c = 1$, then $\mathcal{N}_c = \mathcal{A}$ is the variety of all abelian groups and the Baer invariant of G with respect to this variety is

$$M(G) = \frac{R \cap F'}{[R, F]},$$

which is the well-known Schur multiplier of G.

It is important to find structures for the Schur multiplier and its generalization, the c-nilpotent multiplier, of some famous products of groups. Determining these Baer invariants of a given group is known to be very useful for the classification of groups into isoclinism classes (see [1]).

In 1907, Schur [17], using a representation method, found a structure for the Schur multiplier of a direct product of two groups. Also, Wiegold [19] obtained the same result by some properties of covering groups. In 1979 Moghaddam [13] found a formula for the c-nilpotent multiplier of a direct product of two groups, where $c + 1$ is a prime number or 4. Also, in 1998 Ellis [2] extended the formula for all $c \geq 1$. In 1997 the second author and Moghaddam [10] presented an explicit formula for the c-nilpotent multiplier of a finite abelian group for any $c \geq 1$. It is known that the direct product is a special case of the nilpotent product and we know that regular and verbal products are generalizations of the nilpotent product.

In 1972, Haebich [6] found a formula for the Schur multiplier of a regular product of a family of groups. Then the second author [8] extended the result

to find a homomorphic image with a structure similar to Haebich's type for the c-nilpotent multiplier of a nilpotent product of a family of groups.

In section two, we extend the above result and find a homomorphic image for the c-nilpotent multiplier of a verbal product of a family of groups with respect to a variety $\mathcal{V}$ when $\mathcal{V} \subseteq \mathcal{N}_c$ or $\mathcal{N}_c \subseteq \mathcal{V}$.

A special case of the verbal product of groups whose nilpotent multiplier has been studied more than others is the nilpotent product of cyclic groups. In 1992, Gupta and Moghaddam [5] calculated the c-nilpotent multiplier of the nilpotent dihedral group of class n, i.e. $G_n \cong \mathbf{Z}_2 \overset{n}{*} \mathbf{Z}_2$. (Note that in 2001 Ellis [3] remarked that there is a slip in the statement and gave the correct one.) In 2003, Moghaddam, the second author and Kayvanfar [14] extended the previous result and calculated the c-nilpotent multiplier of the nth nilpotent product of cyclic groups for n =2, 3, 4 under some conditions. Also, the second author and Parvizi [11, 12] presented structures for some Baer invariants of a free nilpotent group that is the nilpotent product of infinite cyclic groups. Finally the authors and Mohammadzadeh [9] obtained an explicit formula for the c-nilpotent multiplier of the nth nilpotent product of some cyclic groups $G = \mathbf{Z} \overset{n}{*} ... \overset{n}{*} \mathbf{Z} \overset{n}{*} \mathbf{Z}_{r_1} \overset{n}{*} ... \overset{n}{*} \mathbf{Z}_{r_t}$, where r_{i+1} divides r_i for all i, $1 \leq i \leq t-1$, for $c \geq n$ such that $(p, r_1) = 1$ for any prime p less than or equal to n.

In section three, we give an explicit formula for the c-nilpotent multiplier of the above group G when $(p, r_1) = 1$ for any prime p less than or equal to $n + c$, for all positive integers $c,\ n$.

2 Verbal products

A group G is said to be a *regular product* of its subgroups A_i, $i \in I$, where I is an ordered set, if the following two conditions hold:

i) $G = \langle A_i | i \in I \rangle$;
ii) $A_i \cap \hat{A}_i = 1$ for all $i \in I$, where $\hat{A}_i = \langle A_j | j \in I, j \neq i \rangle$.

Definition 2.1. Consider the map

$$\psi : \overset{*}{\prod_{i \in I}} A_i \to \overset{\times}{\prod_{i \in I}} A_i$$

$$a_1 a_2 \dots a_n \mapsto (a_1, a_2, \dots, a_n),$$

which is a natural map from the free product of $\{A_i\}_{i \in I}$ on to the direct product of $\{A_i\}_{i \in I}$. Clearly its kernel is the normal closure of

$$\langle [A_i, A_j] | i, j \in I, i \neq j \rangle$$

in the free product $A = \prod_{i \in I}^* A_i$. It is denoted by $[A_i^A]$ and called *the Cartesian subgroup* of the free product (see [16] for the properties of cartesian subgroups).

The following theorem gives a characterization of a regular product.

Theorem 2.2 (Golovin 1956 [4]). Suppose that a group G is generated by a family $\{A_i | i \in I\}$ of its subgroups, where I is an ordered set. Then G is a regular product of the A_i if and only if every element of G can be written uniquely as a product

$$a_1 a_2 ... a_n u,$$

where $1 \neq a_i \in A_{\lambda_i}$, $\lambda_1 < ... < \lambda_n$ and $u \in [A_i^G] = \langle [A_i^G, A_j^G] | i, j \in I, i \neq j \rangle$.

Definition 2.3. Let $\mathcal{V}$ be a variety of groups defined by a set of laws V. Then *the verbal product* of a family of groups $\{A_i\}_{i \in I}$ associated with the variety $\mathcal{V}$ is defined to be

$$\mathcal{V} \prod_{i \in I} A_i = \frac{\prod_{i \in I}^* A_i}{V(A) \cap [A_i^A]}.$$

The verbal product is also known as varietal product or simply $\mathcal{V}$-product. If $\mathcal{V}$ is the variety of all groups, then the corresponding verbal product is the free product; if $\mathcal{V} = \mathcal{A}$ is the variety of all abelian groups, then the verbal product is the direct product and if $\mathcal{V} = \mathcal{N}_c$ is the variety of all nilpotent groups of class at most c, then the verbal product will be the nilpotent product.

Let $\{A_i | i \in I\}$ be a family of groups and

$$1 \to R_i \to F_i \xrightarrow{\theta_i} A_i \to 1$$

be a free presentation for A_i. We denote by θ the natural homomorphism from the free product $F = \prod_{i\in I}^{*} F_i$ onto $A = \prod_{i\in I}^{*} A_i$ induced by the θ_i. Also we assume that the group G is the verbal product of $\{A_i\}_{i\in I}$ associated with the variety $\mathcal{V}$. If ψ is the natural homomorphism from A onto G induced by the identity map on each A_i, then we have the sequence

$$F = \prod_{i\in I}^{*} F_i \xrightarrow{\theta} A = \prod_{i\in I}^{*} A_i \xrightarrow{\psi_v} G = \mathcal{V}\prod_{i\in I} A_i \to 1.$$

The following notation will be used throughout this section.

Notation 2.4.

i) $D_1 = \prod_{i\neq j}[R_i, F_j]^F$;

ii) $E_c = D_1 \cap \gamma_{c+1}(F)$;

iii) $D_c = \prod_{\exists j \ s.t. \ \mu_j \neq i}[R_i, F_{\mu_1}, ..., F_{\mu_c}]^F$;

iv) $K_v = V(F) \cap [F_i^F]$;

v) $K_c = \gamma_{c+1}(F) \cap [F_i^F]$.

Let H_v be the kernel of ψ_v and R be the kernel of $\psi_v \circ \theta$. It is clear that R is actually the inverse image of H_v in F under θ, where $H_v = V(A) \cap [A_i^A]$ by the definition of the verbal product. Put $H_c = \gamma_{c+1}(A) \cap [A_i^A]$, then an immediate consequence is the following lemma.

Lemma 2.5. With the above notation we have
i) $\theta(K_v) = H_v$ and $\theta(K_c) = H_c$;
ii) $G = F/R$ and $R = \prod_{i\in I} R_i^F K_v = (\prod_{i\in I} R_i)D_1K_v$.

Proof. (i) This follows from the definition of θ.
(ii) It is easy to see that $\ker\theta = \prod_{i\in I} R_i^F$. On the other hand, since $\theta(K_v) = \ker\psi_v$, we have $R = (\ker\theta)K_v = \prod_{i\in I} R_i^F K_v$. Also for all $r \in R_i$ and $f \in F$, $r^f = r[r, f]$. This implies that $\prod_{i\in I} R_i^F = \prod_{i\in I} R_i[R_i, F]$. Since $[R_i, F_i] \subseteq R_i$, $\prod_{i\in I} R_i^F = \prod_{i\in I} R_iD_1$.

We now prove some lemmas to compute the c-nilpotent multiplier of G.
Lemma 2.6. Keeping the above notation we have
i) $[R,\ {}_cF] = (\prod_{i\in I}[R_i,\ {}_cF_i])D_c[K_v,\ {}_cF]$.
ii) If $V(F) \subseteq \gamma_{c+1}(F)$, then $R \cap \gamma_{c+1}(F) = \prod_{i\in I}(R_i \cap \gamma_{c+1}(F_i))E_cK_v$.
iii) If $\gamma_{c+1}(F) \subseteq V(F)$, then $R \cap \gamma_{c+1}(F) = \prod_{i\in I}(R_i \cap \gamma_{c+1}(F_i))E_cK_c$.

Proof. i)

$$\begin{aligned}
[R,\ {}_cF] &= [\prod_{i\in I} R_i^F K_v,\ {}_cF] \\
&= \prod_{i\in I}[R_i,\ {}_cF]^F[K_v,\ {}_cF] \\
&= (\prod_{i\in I}[R_i,\ {}_cF_i])D_c[K_v,\ {}_cF].
\end{aligned}$$

ii) Let $g \in R\cap\gamma_{c+1}(F)$. Then $g = r_{\lambda_1}...r_{\lambda_t}dk$ by Lemma 2.5, where $r_{\lambda_i} \in R_{\lambda_i}$, $d \in D_1$ and $k \in K_v$. Now consider the natural homomorphism

$$\varphi : F = \overset{*}{\prod_{i\in I}} F_i \to \overset{\times}{\prod_{i\in I}} F_i.$$

Since $g \in \gamma_{c+1}(F)$, $\varphi(g) = (r_{\lambda_1}, ..., r_{\lambda_t}) \in \gamma_{c+1}(\prod_{i\in I}^{\times} F_i) = \prod_{i\in I}^{\times}\gamma_{c+1}(F_i)$. Therefore $r_{\lambda_i} \in \gamma_{c+1}(F_{\lambda_i}) \cap R_{\lambda_i}$ and then $dk \in \gamma_{c+1}(F) \cap [F_i^F]$. Now since

$k \in V(F) \subseteq \gamma_{c+1}(F)$, we have $d \in \gamma_{c+1}(F) \cap D_1 = E_c$ and so the result follows.
iii) Since $K_c \subseteq K_v$, $\prod_{i\in I}(R_i \cap \gamma_{c+1}(F_i))E_cK_c \subseteq R \cap \gamma_{c+1}(F)$. For the reverse inclusion, similar to part (i), $dk \in \gamma_{c+1}(F) \cap [F_i^F]$. Therefore $R \cap \gamma_{c+1}(F) \subseteq \prod_{i\in I}(R_i \cap \gamma_{c+1}(F_i))K_c$. Now the inclusion $E_c \subseteq K_c$ shows that the equality (iii) holds. □

Lemma 2.7. With the above notation, let $\varphi_c : F \to F/E_c$ be the natural homomorphism. Then $\varphi_c(\prod_{i\in I}(R_i \cap \gamma_{c+1}(F_i))K_v)$ is the direct product of its subgroups $\varphi_c(K_v)$ and $\varphi_c(R_i \cap \gamma_{c+1}(F_i))$, $i \in I$.

Proof. The Three Subgroups Lemma shows that

$$[R_i \cap \gamma_{c+1}(F_i), K_v] \subseteq E_c \quad for\ all\ i \in I$$

and

$$[R_i \cap \gamma_{c+1}(F_i), R_j \cap \gamma_{c+1}(F_j)] \subseteq E_c \quad for\ all\ i, j \in I, i \neq j.$$

So we have

$$[\varphi_c(R_i \cap \gamma_{c+1}(F_i)), \varphi_c(K_v)] = 1 \quad for\ all\ i \in I$$

and

$$[\varphi_c(R_i \cap \gamma_{c+1}(F_i)), \varphi_c(R_j \cap \gamma_{c+1}(F_j))] = 1 \quad for\ all\ i, j \in I, i \neq j.$$

Moreover, by Theorem 2.2 we conclude that

$$\varphi_c(R_i \cap \gamma_{c+1}(F_i)) \cap (\prod_{i\neq j} \varphi_c(R_j \cap \gamma_{c+1}(F_j))\varphi_c(K_v)) = 1.$$

Now the result follows by the definition of the direct product. □

Lemma 2.8. With the previous notation,
i) If $V(F) \subseteq \gamma_{c+1}(F)$, then $\varphi_c(K_v)/\varphi_c([K_v,\ {}_cF]) \cong H_v/[H_v,\ {}_cA]$.
ii) If $\gamma_{c+1}(F) \subseteq V(F)$, then $\varphi_c(K_c)/\varphi_c([K_c,\ {}_cF] \cong H_c/[H_v,\ {}_cA]$.

Proof. i) If $V(F) \subseteq \gamma_{c+1}(F)$, then

$$\frac{\varphi_c(K_v)}{\varphi_c([K_v,\ {}_cF])} \cong \frac{K_vE_c}{[K_v,\ {}_cF]E_c} \cong \frac{K_v}{K_v \cap [K_v,\ {}_cF]E_c}.$$

On the other hand

$$\frac{\theta(K_v)}{\theta([K_v,\ {}_cF])} \cong \frac{K_v \ker\theta}{[K_v,\ {}_cF]\ker\theta} \cong \frac{K_v}{K_v \cap [K_v,\ {}_cF]\ker\theta} \cong \frac{K_v}{K_v \cap [K_v,\ {}_cF]D_1\prod_{i\in I} R_i}.$$

Now Theorem 2.2 and definition of E_c imply that

$$\frac{\theta(K_v)}{\theta([K_v,\ {}_cF])} \cong \frac{K_v}{K_v \cap [K_v,\ {}_cF]E_c}.$$

Therefore by Lemma 2.5, we conclude that

$$\frac{\varphi_c(K_v)}{\varphi_c([K_v,\ {}_cF])} \cong \frac{\theta(K_v)}{\theta([K_v,\ {}_cF])} \cong \frac{H_v}{[H_v,\ {}_cA]}.$$

ii) The proof is similar to (i).

Now we are ready to state and prove the main result of this section.

Theorem 2.9. With the above notation,
i) If $\mathcal{N}_c \subseteq \mathcal{V}$, then $\prod_{i\in I}^{\times} \mathcal{N}_cM(A_i) \times H_v/[H_v,\ {}_cA]$ is a homomorphic image of $\mathcal{N}_cM(\mathcal{V}\prod_{i\in I} A_i)$, and if $\mathcal{V}\prod_{i\in I} A_i$ is finite, then the above structure is isomorphic to a subgroup of $\mathcal{N}_cM(\mathcal{V}\prod_{i\in I} A_i)$.
ii) If $\mathcal{V} \subseteq \mathcal{N}_c$, then $\prod_{i\in I}^{\times} \mathcal{N}_cM(A_i) \times H_c/[H_v,\ {}_cA]$ is a homomorphic image of $\mathcal{N}_cM(\mathcal{V}\prod_{i\in I} A_i)$, and if $\mathcal{V}\prod_{i\in I} A_i$ is finite, then the above structure is isomorphic to a subgroup of $\mathcal{N}_cM(\mathcal{V}\prod_{i\in I} A_i)$.

Proof. i) By Lemma 2.6 (i),(ii)

$$\mathcal{N}_cM(\mathcal{V}\prod_{i\in I}A_i)\cong\frac{R\cap\gamma_{c+1}(F)}{[R,\ {}_cF]}\cong\frac{\prod_{i\in I}(R_i\cap\gamma_{c+1}(F_i))E_cK_v}{\prod_{i\in I}[R_i,\ {}_cF_i]D_c[K_v,\ {}_cF]}.$$

Therefore there is a natural epimorphism from $\mathcal{N}_cM(\mathcal{V}\prod_{i\in I}A_i)$ to

$$\frac{\prod_{i\in I}(R_i\cap\gamma_{c+1}(F_i))E_cK_v}{\prod_{i\in I}[R_i,\ {}_cF_i]E_c[K_v,\ {}_cF]}\cong\frac{\varphi_c(\prod_{i\in I}(R_i\cap\gamma_{c+1}(F_i))K_v)}{\varphi_c(\prod_{i\in I}[R_i,\ {}_cF_i][K_v,\ {}_cF])}.$$

Lemma 2.7 and the fact that $\varphi_c([K_v,\ {}_cF])\subseteq\varphi_c(K_v)$ and $\varphi_c([R_i,\ {}_cF_i])\subseteq\varphi_c(R_i\cap\gamma_{c+1}(F_i))$ imply that

$$\frac{\varphi_c(\prod_{i\in I}(R_i\cap\gamma_{c+1}(F_i))K_v)}{\varphi_c(\prod_{i\in I}[R_i,\ {}_cF_i][K_v,\ {}_cF])}\cong\prod_{i\in I}^{\times}\frac{\varphi_c(R_i\cap\gamma_{c+1}(F_i))}{\varphi_c([R_i,\ {}_cF_i])}\times\frac{\varphi_c(K_v)}{\varphi_c([K_v,\ {}_cF])}.$$

It is straightforward to see that

$$\frac{\varphi_c(R_i\cap\gamma_{c+1}(F_i))}{\varphi_c([R_i,\ {}_cF_i])}\cong\frac{R_i\cap\gamma_{c+1}(F_i)}{[R_i,\ {}_cF_i]}$$

by Theorem 2.2. Therefore, the result holds by Lemma 2.8 (i).
ii) By an argument similar to (i), we obtain the result. □

We need the following lemma whose proof is straightforward.

Lemma 2.10. Let $\{A_i|i\in I\}$ be a family of groups. Put $A=\prod_{i\in I}^{*}A_i$. Then for all integers $m\geq 2$,

$$\gamma_m(A)=\prod_{i\in I}\gamma_m(A_i)(\gamma_m(A)\cap[A_i^A]).$$

In particular if the A_i are cyclic, then $\gamma_m(A)=\gamma_m(A)\cap[A_i^A]$.

The following corollary is an interesting consequence of Theorem 2.9 for cyclic groups.

Corollary 2.11. Let $\{A_i|i\in I\}$ be a family of cyclic groups. Then

i) If $\mathcal{N}_c \subseteq \mathcal{V}$, then $\mathcal{N}_c M(\mathcal{V}\prod_{i\in I} A_i) \cong H_v/[H_v,\ {}_cA]$. Moreover if $\mathcal{V} \subseteq \mathcal{N}_{2c}$, then $V(\prod^{2c}_{i\in I} A_i)$ is a homomorphic image of $\mathcal{N}_c M(\mathcal{V}\prod_{i\in I} A_i)$.
ii) If $\mathcal{V} \subseteq \mathcal{N}_c$, then $\mathcal{N}_c M(\mathcal{V}\prod_{i\in I} A_i) \cong H_c/[H_v,\ {}_cA]$. Moreover if $\mathcal{N}_m \subseteq \mathcal{V}$, then $\gamma_{c+1}(\prod^{m+c}_{i\in I} A_i)$ is a homomorphic image of $\mathcal{N}_c M(\mathcal{V}\prod_{i\in I} A_i)$.

Proof. i) Since the A_i are cyclic groups and the R_i have no commutators, it is concluded that $D_c = E_c$. So the epimorphism in the proof of Theorem 2.9, is actually an isomorphism. Also $\mathcal{N}_c M(A_i) = 1$, therefore $\mathcal{N}_c M(\mathcal{V}\prod_{i\in I} A_i) \cong H_v/[H_v,\ {}_cA]$. Now suppose $\mathcal{N}_c \subseteq \mathcal{V} \subseteq \mathcal{N}_{2c}$. The inclusion $V(A) \subseteq \gamma_{c+1}(A)$ and Lemma 2.10 imply that $V(A) \subseteq [A_i^A]$ and thus $H_v = V(A) \cap [A_i^A] = V(A)$. So we have $\mathcal{N}_c M(\mathcal{V}\prod_{i\in I} A_i) = V(A)/[V(A),\ {}_cA]$ and hence $V(A)/\gamma_{2c+1}(A)$ is a homomorphic image of $\mathcal{N}_c M(\mathcal{V}\prod_{i\in I} A_i)$. On the other hand since $\mathcal{V} \subseteq \mathcal{N}_{2c}$, we have $V(A)/\gamma_{2c+1}(A) = V(A/\gamma_{2c+1}(A)) = V(\prod^{2c}_{i\in I} A_i)$. This completes the proof.
ii) An argument similar to (i), shows that $\mathcal{N}_c M(\mathcal{V}\prod_{i\in I} A_i) \cong H_c/[H_v,\ {}_cA]$. Now since $\mathcal{N}_m \subseteq \mathcal{V} \subseteq \mathcal{N}_c$, $\gamma_{c+1}(A)/\gamma_{m+c+1}(A)$ is a homomorphic image of $\mathcal{N}_c M(\mathcal{V}\prod_{i\in I} A_i)$ and also

$$\frac{\gamma_{c+1}(A)}{\gamma_{m+c+1}(A)} = \gamma_{c+1}(\frac{A}{\gamma_{m+c+1}(A)}) = \gamma_{c+1}(\prod^{m+c}_{i\in I} A_i).$$

Hence the result follows. □

Remark 2.12. Let $\{A_i | i \in I\}$ be a family of groups.
i) If $\mathcal{V}$ is the variety of trivial groups, then Theorem 2.9 implies that $\prod^{\times}_{i\in I} \mathcal{N}_c M(A_i)$ is a homomorphic image of $\mathcal{N}_c M(\prod^{*}_{i\in I} A_i)$. In particular $M(\prod^{*}_{i\in I} A_i) = \prod^{\times}_{i\in I} M(A_i)$ which is a result of Miller [15].
ii) If $\mathcal{V}$ is the variety of nilpotent groups of class at most n, $\mathcal{N}_n$, then main results of the second author [8] are obtained by Theorem 2.9 and corollary 2.11.

3 Nilpotent Products of Cyclic Groups

In this section we use a result of the previous section and find a structure for the c-nilpotent multiplier of the group $G = \mathbf{Z} \overset{n}{*} ... \overset{n}{*} \mathbf{Z} \overset{n}{*} \mathbf{Z}_{r_1} \overset{n}{*} ... \overset{n}{*} \mathbf{Z}_{r_t}$, where r_{i+1} divides r_i for all i, $1 \leq i \leq t-1$, such that $(p, r_1) = 1$ for any prime p less than or equal to $n + c$. The proof relies on basic commutators [7] and related results. We recall that the number of basic commutators of weight c on n generators, denoted by $\chi_c(n)$, is determined by Witt formula [7]. Also, M. Hall proved that if F is the free group on free generators $x_1, x_2, ..., x_r$ and $c_1, ..., c_t$ are basic commutators of weight $1, 2, ..., n$, on $x_1, \ldots, x_r$, then an arbitrary element f of F has a unique representation,

$$f = c_1^{\beta_1} c_2^{\beta_2} ... c_t^{\beta_t} \quad mod \gamma_{n+1}(F).$$

In particular the basic commutators of weight n provide a basis for the free abelian group $\gamma_n(F)/\gamma_{n+1}(F)$ (see [7]).

The following theorem represents the elements of some nilpotent products of cyclic groups in terms of basic commutators.

Theorem 3.1 ([18]). Let $A_1, ..., A_t$ be cyclic groups of order $\alpha_1, ..., \alpha_t$ respectively, where if A_i is infinite cyclic, then $\alpha_i = 0$. Let a_i generate A_i and let $G = A_1 \overset{n}{*} ... \overset{n}{*} A_t$, where n is greater than or equal to 2. Suppose that all the primes appearing in the factorizations of the α_i are greater than or equal to n and $u_1, u_2, ...,$ are basic commutators of weight less than n, on the letters $a_1, \ldots, a_t$. Put $N_i = \alpha_{i_j}$ if $u_i = a_{i_j}$ of weight 1, and

$$N_i = gcd(\alpha_{i_1}, ..., \alpha_{i_k})$$

if a_{i_j}, $1 \leq j \leq k$, appears in u_i. Then every element g of G can be uniquely expressed as

$$g = \prod u_i^{m_i},$$

where the m_i are integers modulo N_i (by *gcd* we mean the greatest common divisor).

The following theorem is an interesting consequence of Corollary 2.11.

Theorem 3.2. Let $\{A_i | i \in I\}$ be a family of cyclic groups. Then
i) if $n \geq c$, then $\mathcal{N}_c M(\prod_{i\in I}^{n*} A_i) \cong \gamma_{n+1}(\prod_{i\in I}^{n+c*} A_i)$;
ii) if $c \geq n$, then $\mathcal{N}_c M(\prod_{i\in I}^{n*} A_i) \cong \gamma_{c+1}(\prod_{i\in I}^{n+c*} A_i)$.

Proof. i) Put $\mathcal{V} = \mathcal{N}_n$ in Corollary 2.11 and deduce that

$$\mathcal{N}_c M(\prod_{i\in I}^{n*} A_i) \cong H_n/[H_n,\ {}_cA].$$

On the other hand by Lemma 2.10, $H_n = \gamma_{n+1}(A) \cap [A_i^A] = \gamma_{n+1}(A)$. Therefore

$$\mathcal{N}_c M(\prod_{i\in I}^{n*} A_i) \cong \frac{\gamma_{n+1}(A)}{[\gamma_{n+1}(A),\ {}_cA]} = \gamma_{n+1}(\frac{A}{\gamma_{n+c+1}(A)}) = \gamma_{n+1}(\prod_{i\in I}^{n+c*} A_i).$$

ii) The result follows as for (i). □

Now, we are in a position to state and prove the main result of this section .

Theorem 3.3. Let $G = A_1 \overset{n}{*} ... \overset{n}{*} A_{m+t}$ be the nth nilpotent product of cyclic groups such that $A_i \cong \mathbf{Z}$ for $1 \leq i \leq m$ and $A_{m+j} \cong \mathbf{Z}_{r_j}$ and $r_{j+1} \mid r_j$ for all $1 \leq j \leq t-1$. If $(p, r_1) = 1$ for any prime p less than or equal to $n+c$, then
i) if $n \geq c$, then $\mathcal{N}_c M(G) \cong \mathbf{Z}^{(g_0)} \oplus \mathbf{Z}_{r_1}^{(g_1-g_0)} \oplus ... \oplus \mathbf{Z}_{r_t}^{(g_t-g_{t-1})}$;
ii) if $c \geq n$, then $\mathcal{N}_c M(G) \cong \mathbf{Z}^{(f_0)} \oplus \mathbf{Z}_{r_1}^{(f_1-f_0)} \oplus ... \oplus \mathbf{Z}_{r_t}^{(f_t-f_{t-1})}$,
where $f_k = \sum_{i=1}^{n} \chi_{c+i}(m+k)$ and $g_k = \sum_{i=1}^{c} \chi_{n+i}(m+k)$ for $0 \leq k \leq t$ and $\mathbf{Z}_r^{(d)}$ denotes the direct sum of d copies of the cyclic group $\mathbf{Z}_r$.

Proof. i) If $n \geq c$, then by Theorem 3.2, it is enough to find the structure of $\gamma_{n+1}(\prod_{i\in I}^{n+c*} A_i)$. Suppose that a_i generates A_i and F is the free group generated by $a_1, ..., a_{m+t}$. Let B be the set of all basic commutators of weight $1, 2, ..., c+n$ on the letters $a_1, ..., a_{m+t}$. Now define

$$D = \{u^{N_i} |\ u \in B \ and \ N_i = gcd(\alpha_{i_1}, ..., \alpha_{i_k}) \ if \ a_{i_j} \ appears \ in \ u \ for \ 1 \leq j \leq k\}.$$

Then Theorem 3.1 implies that $\prod_{i\in I}^{n+c*} A_i = F/\langle D\rangle\gamma_{c+n+1}(F)$ and so

$$\begin{aligned}
\gamma_{n+1}(\prod_{i\in I}^{n+c*} A_i) &= \gamma_{n+1}(\frac{F}{\langle D\rangle\gamma_{c+n+1}(F)}) \\
&= \frac{\gamma_{n+1}(F)}{\langle D\rangle\gamma_{c+n+1}(F) \cap \gamma_{n+1}(F)} \\
&\cong \frac{\gamma_{n+1}(F)/\gamma_{c+n+1}(F)}{(\langle D\rangle \cap \gamma_{n+1}(F))\gamma_{c+n+1}(F)/\gamma_{c+n+1}(F)}.
\end{aligned}$$

It can be deduced from Hall Theorem that $\gamma_{n+1}(F)/\gamma_{c+n+1}(F)$ is a free abelian group with a basis $\bar{B_1} = \{u\gamma_{c+n+1}(F)|\ u \in B_1\}$, where B_1 is the set of all basic commutators of weight $n+1, ..., c+n$ on $a_1, ..., a_{m+t}$. Also, the uniqueness of the presentation of elements implies that the abelian group $(\langle D\rangle \cap \gamma_{n+1}(F))\gamma_{c+n+1}(F)/\gamma_{c+n+1}(F)$ is free with a basis

$$\bar{E} = \{u\gamma_{c+n+1}(F) | u \in D \cap \bar{B_1} = \bigcup_{j=1}^{t} D_j\}\ ,$$

where D_j is the set of all u^{r_j}, such that u is a basic commutator of weight $n+1, ..., c+n$ on $a_1, ..., a_{m+j}$ such that a_{m+j} appears in u. Also we have

$$|D_j| = \sum_{i=1}^{c} \chi_{n+i}(m+j) - \chi_{n+i}(m+j-1) = g_j - g_{j-1}.$$

This completes the proof.

ii) The proof is similar to (i). □

Note that the authors with F. Mohammadzadeh [9] by a different method presented a similar structure for $\mathcal{N}_cM(G)$, for $c \geq n$ with a weaker condition $(p, r_1) = 1$ for any prime p less than or equal to n.

Remark 3.4. The condition $r_{j+1} \mid r_j$, in the above theorem, simplifies the structure of the c-nilpotent multiplier of G and gives a clear formula. One can use the above method and find the structure of $\mathcal{N}_cM(G)$ without the condition $r_{j+1} \mid r_j$, but with a more complex formula. For example, for a simple case if $G = \mathbf{Z}_r \overset{n}{*} \mathbf{Z}_s$ where $(p, r) = (p, s) = 1$ for any prime p less than or equal to $n + c$ and $(r, s) = d$, then
i) if $n \geq c$, then $\mathcal{N}_cM(G) \cong \mathbf{Z}_d^{(\sum_{i=1}^{c} \chi_{n+i}(2))}$;
ii) if $c \geq n$, then $\mathcal{N}_cM(G) \cong \mathbf{Z}_d^{(\sum_{i=1}^{n} \chi_{c+i}(2))}$.

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