IDEAL DEPTH OF QF EXTENSIONS

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ABSTRACT. A minimum depth $d^I(S \to R)$ is assigned to a ring homomorphism $S \to R$ and a bimodule ${}_R I_R$. The recent notion of depth of a subring $d(S,R)$ in a paper by Boltje-Danz-Külshammer is recovered when $I = R$ and $S \to R$ is the inclusion mapping. Ideal depth gives lower bounds for $d(S,R)$ in case of group algebra pair or semisimple complex algebra extensions. If $R \,|\, S$ is a QF extension of finite depth, minimum left and right even depth are shown to coincide. If $R \supseteq S$ is moreover a Frobenius extension with R_S a generator, its subring depth is shown to coincide with its tower depth. In the process formulas for the ring, module, Frobenius and Temperley-Lieb structures are provided for the tensor product tower above a Frobenius extension. A depth 3 QF extension is embedded in a depth 2 QF extension; in turn certain depth n extensions embed in depth 3 extensions if they are Frobenius extensions or other special ring extensions with ring structures on their relative Hochschild bar resolution groups.

1. Introduction and Preliminaries

Algebras, coalgebras and Hopf algebras are some of the interesting objects with structure in representation categories of commutative rings. In the representation category of a noncommutative ring, these objects become ring extensions, corings and Hopf algebroids. Some basic algebras of interest are the cohomological dimension 0 and ∞ cases of separable algebra and Frobenius algebra; which become separable extensions and Frobenius extensions in noncommutative representation theory. Also, QF rings, semisimple rings, and Azumaya algebras generalize to ring extensions; however depth is not such a notion, originating as a tool of induced representation theory. Depth is essentially constant on (especially projective) algebras over a commutative ring, but gives different and interesting outcomes for ring extensions.

The depth of many subgroups are recently computed, both as induced complex representations [6] and as induced representations over general commutative rings of group algebras [1]. For example, the depth of the permutation groups $S_n \subset S_{n+1}$ is $2n-1$ over any ground ring and depends only on a combinatorial depth of subgroups defined in terms of bisets in [1]. The authors of [1] show that combinatorial depth $d_c(H,G)$ of a subgroup H in a finite group G satisfies $d_c(H,G) \leq 2n$ for $n \geq 1$ (respectively, $d_c(H,G) \leq 2n-1$ for $n > 1$) $\Leftrightarrow$ for any $x_1, \ldots, x_n \in G$, there is $y_1, \ldots, y_{n-1} \in G$ such that $H \cap_{i=1}^n x_i H x_i^{-1} = H \cap_{i=1}^{n-1} y_i H y_i^{-1}$ (respectively, the latter condition and additionally $x_1 h x_1^{-1} = y_1 h y_1^{-1}$, all $h \in H \cap_{i=1}^n x_i H x_i^{-1}$). All notions of depth ≤ 2 are the same and occur precisely if H is a normal subgroup. However, depth of subalgebras over base rings (of varying characteristic denoted by a subscript) for $R = k[G]$ and $S = k[H]$ and combinatorial depth diverge in a

In memory of Gerhard Hochschild.

string of inequalities given in [1] as follows:

$$d_0(H,G) \leq d_p(H,G) \leq d_{\mathbb{Z}}(H,G) \leq d_c(H,G) \leq 2[G : N_G(H)]. \tag{1}$$

Also $d_k(H,G) \leq d_c(H,G)$ showing that all extensions of finite dimensional group algebras have finite depth.

The authors begin in [1] with a new notion of subring depth $d(S,R)$, given below in (4). They show in an appendix how it is based on and equal to a previous notion where S and R are semisimple complex algebras given below in (5). Such a pair $R \supseteq S$ is a special case of (split separable) Frobenius extensions; in Theorem 5.3 below we show that subring depth is equal to tower depth of Frobenius extensions [15] satisfying a generator module condition. The authors of [1] define a left and right even depth and show these are the same on group algebra extensions; Theorem 3.4 below shows this equality holds for all QF extensions.

In this paper an obvious change is made to the definition of subring depth; we define an I-depth $d^I(S \to R)$ of a ring homomorphism $S \to R$ with R-bimodule I, which we use in place of R in the n-fold tensor products over S in the definition (4) of $d(S,R)$ (as well as a converse, automatic in the presence of units). When I is an ideal of a semisimple complex algebra R with semisimple subalgebra S the I-depth $d^I(S \to R)$ gives a lower bound, $d^I(S \to R) \leq d(S,R)$ discussed in Section 2 in terms of the part of the bipartite graph of the inclusion which is directly below the ideal I.

There are tantalizing similarities and intriguing relations between relative homological algebra and the subring depth definition and theory. For example, the depth two condition on a subring $S \subseteq R$ leads in [14] to an isomorphism of differential graded algebras between the relative Hochschild R-valued cochains with cup product and the Amitsur complex of a coring with grouplike element (on the endomorphism ring $\operatorname{End}{}_S R_S$ over the centralizer subring R^S). Also the paper [16] contains some relations between depth 2 and notions of relative homological algebra carried over to corings in [4]. The tower of iterated endomorphism rings above a ring extension becomes in the case of Frobenius extensions a tower of rings on the bar resolution groups $C_n(R,S)$ $(n = 0,1,2,\ldots)$ with Frobenius and Temperley-Lieb structures explicitly calculated from their more usual iterative definition in Section 4.1. At the same time Frobenius extensions of depth more than 2 are known to have depth 2 further out in the tower: we extend this observation in [15] with new proofs to include other ring extensions satisfying the hypotheses of Proposition 4.3.

1.1. **H-equivalent modules.** Let R be a ring. Two left R-modules, ${}_RN$ and ${}_RM$, are said to be *h-equivalent*, denoted by ${}_RM \overset{h}{\sim} {}_RN$ if two conditions are met. First, for some positive integer r, N is isomorphic to a direct summand in the direct sum of r copies of M, denoted by

$${}_RN \oplus * \cong {}_RM^r \quad \Leftrightarrow \quad N \,|\, M^r \;\Leftrightarrow \tag{2}$$

$$\exists f_i \in \operatorname{Hom}({}_RM, {}_RN),\; g_i \in \operatorname{Hom}({}_RN, {}_RM), i = 1, \ldots, r \;:\; \sum_{i=1}^r f_i \circ g_i = \mathrm{id}_N$$

Second, symmetrically there is $s \in \mathbb{Z}_+$ such that $M \,|\, N^s$. It is easy to extend this definition of h-equivalence (sometimes referred to as similarity) to h-equivalence of two objects in an abelian category, and to show that it is an equivalence relation.

If two modules are h-equivalent, ${}_RN \overset{h}{\sim} {}_RM$, then they have Morita equivalent endomorphism rings, $\mathcal{E}_N := \operatorname{End}{}_RN$ and $\mathcal{E}_M := \operatorname{End}{}_RM$. This is quite easy to see since a Morita context of bimodules are given by $H(M,N) := \operatorname{Hom}({}_RM, {}_RN)$, which is an $\mathcal{E}_N$-$\mathcal{E}_M$-bimodule via composition, and the bimodule ${}_{\mathcal{E}_M}H(N,M)_{\mathcal{E}_N}$; these are progenerator modules, by applying to (2) or its reverse, $M \mid N^s$, any of the four Hom-functors such as $\operatorname{Hom}({}_R-, {}_RM)$ from the category of left R-modules into the category of left E_M-modules showing that ${}_{\mathcal{E}_M}H(N,M)$ is finite projective; similarly, generator. Then the explicit conditions on mappings for h-equivalence show that $H(M,N) \otimes_{\mathcal{E}_M} H(N,M) \to \mathcal{E}_N$ and the reverse mapping given by composition are both bimodule isomorphisms as required. Since $\mathcal{E}_M$ and $\mathcal{E}_N$ are Morita equivalent rings, their centers are isomorphic:

$$\operatorname{End}{}_RM_{\mathcal{E}_M} \cong \operatorname{End}{}_RN_{\mathcal{E}_N}.$$

The theory of h-equivalent modules applies to bimodules ${}_TM_S \overset{h}{\sim} {}_TN_S$ by letting $R = T \otimes_{\mathbb{Z}} S^{\mathrm{op}}$ which sets up an equivalence of abelian categories between T-S-bimodules and left R-modules. Two additive functors $F, G : \mathcal{C} \hookrightarrow \mathcal{D}$ are h-equivalent if there are natural split epis $F(X)^n \to G(X)$ and $G(X)^m \to F(X)$ for all X in $\mathcal{C}$. We leave the proof of the lemma below as an elementary exercise.

Lemma 1.1. *Suppose two R-modules are h-equivalent, $M \overset{h}{\sim} N$ and two additive functors from R-modules to an abelian category are h-equivalent, $F \overset{h}{\sim} G$. Then $F(M) \overset{h}{\sim} G(N)$.*

For example, the following substitution in equations involving the $\overset{h}{\sim}$-equivalence relation follows from the lemma:

$$(3) \qquad {}_RP_T \overset{h}{\sim} {}_RQ_T \quad {}_TU_S \overset{h}{\sim} {}_TV_S \;\Rightarrow\; {}_RP \otimes_T U_S \overset{h}{\sim} {}_RQ \otimes_T V_S$$

Example 1.2. If R is a semisimple artinian ring with simples $\{P_1, \ldots, P_t\}$ (representatives from each isomorphism class), all finitely generated modules M_R and N_R have a unique factorization into simple components. Denote the simple constituents of M_R by $\operatorname{Simples}(M) = \{P_i \mid [P_i, M] \neq 0\}$ where $[P_i, M]$ is the number of factors in M isomorphic to P_i. Then $M \mid N^q$ for some positive q if $\operatorname{Simples}(M) \subseteq \operatorname{Simples}(N)$; and $M \overset{h}{\sim} N$ iff $\operatorname{Simples}(M) = \operatorname{Simples}(N)$.

Suppose R has central primitive idempotents $e_1, \ldots, e_t$ such that each $[P_i, e_iR] = n_i$, so that R decomposes into the product of matrix rings over each of the division rings $D_i := \operatorname{End}(P_i)_R$: $R \cong M_{n_1}(D_1) \times \cdots \times M_{n_t}(D_t)$. If M and N are h-equivalent f.g. R-modules, then the endomorphism rings E_M and E_N are explicitly Morita equivalent as they are both products of matrix rings over the same subset of division rings $D_1, \ldots, D_t$.

Example 1.3. Via some more category theory, we may see that positive integers n and m are h-equivalent if $n \mid m^r$ and $m \mid n^s$ for some positive integers r, s; whence there are primes $p_1, \ldots, p_k$ such that n and m lie in the same h-equivalence class $\{p_1^{r_1} \cdots p_k^{r_k} \mid r_1, \ldots, r_k \geq 1\}$. This explains the notation for eq. (2).

1.2. **Depth two.** A subring pair $S \subseteq R$ is said to have left depth 2 (or be a left depth two extension [13]) if $R \otimes_S R \overset{h}{\sim} R$ as natural S-R-bimodules. Right depth 2 is defined similarly in terms of h-equivalence of natural R-S-bimodules. In [13] it was noted that the left condition implies the right and conversely if R is a Frobenius extension of S. Also in [13] a Galois theory of Hopf algebroids was defined on the endomorphism ring $H := \mathrm{End}\,{}_S R_S$ as total ring and the centralizer $C := R^S$ as base ring. The antipode is the restriction of the natural anti-isomorphism stemming from following the arrows,

$$\mathrm{End}\, R_S \xrightarrow{\cong} R \otimes_S R \xrightarrow{\cong} (\mathrm{End}\,{}_S R)^{\mathrm{op}}.$$

The Galois properties may then be summarized by the invariants under the obvious action of H, $R^H = S$ if R_S is faithfully flat, and $\mathrm{End}\, R_S \cong R \# H$ a smash product product ring structure on $R \otimes_C H$: the details are in [13]. There is also a duality structure by going a step further along in the tower above $S \subseteq R \hookrightarrow \mathrm{End}\, R_S \hookrightarrow \mathrm{End}\, R \otimes_S R_R$, where the dual Hopf algebroid $H' := (R \otimes_S R)^S$ plays a role [13].

Conversely, Galois extensions have depth 2, which is most easily seen from the Galois map of an H-comodule algebra A with invariant subalgebra B and finite dimensional Hopf algebra H over a base field k, which is given by $A \otimes_B A \xrightarrow{\cong} A \otimes_k H$, $a' \otimes a \mapsto a' a_{(0)} \otimes a_{(1)}$, whence $A \otimes_B A \cong A^{\dim H}$ as A-B-bimodules. The Hopf subalgebras within a finite dimensional Hopf algebra which have depth 2 are precisely the normal Hopf subalgebras; if normal, it has depth 2 by applying the Hopf-Galois observation just made. The converse follows from an argument discovered by [2, Boltje-Külshammer] which divides the normality notion into right and left just like depth 2, where left normal is invariance under the left adjoint action. Note their argument given in the context of any augmented algebra A (such as a quasi-Hopf algebra) next. Let $\varepsilon : A \to k$ be the algebra homomorphism into a base ring k. Let A^+ denote $\ker \varepsilon$, and for a subalgebra $B \subseteq A$, let B^+ denote $\ker \varepsilon \cap B$. For example, it may be shown that if a (quasi-)Hopf algebra H has left normal (quasi-)Hopf subalgebra, then $HK^+ \subseteq K^+ H$.

Proposition 1.4. *Suppose $B \subseteq A$ is a subalgebra of an augmented algebra. If $B \subseteq A$ has right depth 2, then $AB^+ \subseteq B^+ A$.*

Proof. To $A \otimes_B A \,|\, A^q$ as A-B-bimodules, apply the additive functor $k_\varepsilon \otimes_A -$, which results in $A/B^+A \,|\, k^q$ as right B-modules. The annihilator of k^q restricted to B is of course B^+, which then also annihilates A/B^+A, so $AB^+ \subseteq B^+A$. □

The opposite inclusion is of course satisfied by a left depth 2 extension of augmented algebras.

Also subalgebra pairs of semisimple complex algebras have depth 2 exactly when they are normal in a classical sense of Rieffel. We note the theorem in [6] below and give a new proof in one direction along the lines of the previous proposition.

Theorem 1.5. [6, Theorem 4.6] *Suppose $B \subseteq A$ is a subalgebra pair of semisimple complex algebras. Then $B \subseteq A$ has depth 2 if and only if for every maximal ideal I in A, one has $A(I \cap B) = (I \cap B)A$.*

Proof. ($\Leftarrow$) See [6, Section 4]. ($\Rightarrow$) Given maximal ideal I in A, there is an ideal J with identity element 1_J such that $A = I \oplus J$, and algebra homomorphism $\varepsilon : A \to A/I \cong J$. Denote $I = A^+$, $I \cap B = B^+$, and note that the A-module $J_A = J_\varepsilon$. Given the right depth 2 condition ${}_A A \otimes_B A_B \,|\, A^q$, tensor from the left by J_A obtaining $J \otimes_B A_B \,|\, J^q$.

Note the B-module homomorphism $A/B^+A \to J \otimes_B A$ given by $a + B^+A \mapsto 1_J \otimes_B a$ (well-defined since $1_J \cdot B^+ = 0$) which we claim is monic. For suppose that $1_J \otimes a = 0$ for a in the projective module ${}_BA$, so $a = \sum_i f_i(a)e_i$ in some free module B^n. Then

$$0 = 1_J \otimes a = \sum_i \varepsilon(f_i(a)) \otimes e_i \Rightarrow \varepsilon(f_i(a)) = 0 \Rightarrow f_i(a) \in B^+, \forall i = 1, \dots, n$$

hence $a = \sum_i f_i(a)e_i \in B^+A$ so $a + B^+A = 0$ which proves the claim.

Since B^+ annihilates J^q_B, it annihilates $J \otimes_B A_B$ and therefore A/B^+A via the monomorphism. Thus $AB^+ \subseteq B^+A$. The opposite inclusion follows from a similar argument applied to the left depth 2 condition. □

2. Ideal depth of a ring homomorphism

Let S and R be unital associative rings and $S \to R$ a ring homomorphism where $1_S \mapsto 1_R$. Suppose ${}_RI_R$ is a bimodule. With no further ado, we will restrict I to bimodules ${}_SI_R$, ${}_RI_S$ or ${}_SI_S$ via the homomorphism $S \to R$. Note that the kernel of $S \to R$ is contained in the annihilator ideal in S of the left (or right) S-module I denoted by $\mathrm{ann}\,_SI$.

We let $C^I_0(S \to R) = S$, and for $n \geq 1$,

$$C^I_n(S \to R) = I \otimes_S \cdots \otimes_S I \quad (n \text{ times } I)$$

For $n \geq 1$, the $C^I_n(S \to R)$ has a natural R-R-bimodule (briefly R-bimodule) structure which restricts to S-R-, R-S- and S-bimodule structures occuring in the next definition.

Definition 2.1. *The ring homomorphism $S \to R$ has I-depth $2n+1 \geq 1$ if as S-bimodules $C^I_n(S \to R) \overset{h}{\sim} C^I_{n+1}(S \to R)$. The ring homomorphism $S \to R$ has left (right) I-depth $2n \geq 2$ if $C^I_n(S \to R) \overset{h}{\sim} C^I_{n+1}(S \to R)$ as S-R-bimodules (respectively, R-S-bimodules).*

It is clear that if $S \to R$ has either I-depth $2n$, it has I-depth $2n+1$ by restricting the h-equivalence condition to S-bimodules. If it has I-depth $2n+1$, it has I-depth $2n+2$ by tensoring the h-equivalence by $- \otimes_S I$ or $I \otimes_S -$. The *minimum I-depth* is denoted by $d^I(S \to R)$.

Note that the minimum left and right minimum *even* I-depths may differ by 2 (in which case $d^I(S \to R)$ is the least of the two). In the next section we provide a general condition, which includes a Hopf subalgebra pair $S \subseteq R$ of symmetric Frobenius algebras with I an ideal in R, where the left and right minimum even I-depths coincide.

We also remark that once $S \to R$ has I-depth $2n+1$ the $C^I_{n+m}(S \to R)$'s stop growing as $m \to \infty$ in terms of adding new indecomposables in a category of modules with unique factorization, since $C^I_n(S \to R) \overset{h}{\sim} C^I_{n+m}(S \to R)$ for all $m \geq 0$ (see the example in the previous section). This corresponds well with the classical notion of finite depth in subfactor theory.

Lemma 2.2. *Let $S \to R$ have kernel K, $\overline{S} := S/K$ and $\overline{S} \hookrightarrow R$ be the induced ring monomorphism. Then the left or right minimum depth $d^I(S \to R) = d^I(\overline{S} \hookrightarrow R)$ unless $d^I(\overline{S} \hookrightarrow R) = 1$, in which case equality holds if the quotient homomorphism $p : S \to \overline{S}$ has a section.*

Proof. Note that if $M_R \,|\, N^q_R$, then $\mathrm{ann}\, N_R \subseteq \mathrm{ann}\, M_R$. Since K is in $\mathrm{ann}\, C^I_n(R, S)$ for all $n \geq 1$ and $C^I_n(R, S) \cong C^I_n(R, \overline{S})$ as $\overline{S}$-modules, it follows that $C^I_n(R, S) \overset{h}{\sim}$

$C^I_{n+1}(R,S)$ implies $C^I_n(R,\overline{S}) \overset{h}{\sim} C^I_{n+1}(R,\overline{S})$ for the bimodules at issue. The converse is easy by pullback along p.

$S \to R$ has I-depth 1 iff there are central elements $w_j, z_i \in I^S$ and mappings $f_j, g_i \in \mathrm{Hom}\,({}_SI_S, {}_SS_S)$ such that $x = \sum_i z_i g_i(x)$ for all $x \in I$ and $\sum_j f_j(w_j) = 1_S$. By composing with the quotient homomorphism $S \to \overline{S}$, we obtain $\tilde{f}_j, \tilde{g}_i \in \mathrm{Hom}\,({}_{\overline{S}}I_{\overline{S}}, {}_{\overline{S}}\overline{S}_{\overline{S}})$ and $z_i \in I^{\overline{S}}$ such that $x = \sum_i \tilde{g}_i(x) z_i$ and $\sum_j \tilde{f}_j(w_j) = 1_{\overline{S}}$. The converse may be proven with the extra hypothesis in the lemma, since all mappings in $\mathrm{Hom}\,({}_{\overline{S}}I_{\overline{S}}, {}_{\overline{S}}\overline{S}_{\overline{S}})$ have a lifting to $\mathrm{Hom}\,({}_SI_S, {}_SS_S)$ along p via a section $\sigma : \overline{S} \hookrightarrow S$ satisfying $p \circ \sigma = \mathrm{id}_{\overline{S}}$. □

Example 2.3. Suppose S is a subring of R (where $1_S = 1_R$). Let $S \to R$ be the inclusion monomorphism and $I = R$, the natural R-bimodule. The minimum depth of the subring $S \subseteq R$ as defined in [1, Boltje-Danz-Külshammer] is denoted by $d(S,R)$. We note that $d(S,R) = d^R(S \hookrightarrow R)$. In fact, $C^R_n(S \to R) = R \otimes_S \cdots \otimes_S R := C_n(R,S)$ (n times R) for $n > 0$, and the depth $2n+1$ condition in [1] is that

$$C_{n+1}(R,S) \,|\, C_n(R,S)^q \tag{4}$$

as S-bimodules (some $q \in \mathbb{Z}_+$). The left depth $2n$ condition in [1] is (4) more strongly as natural S-R-bimodules (and as R-S-bimodules for the right depth $2n$ condition). But (using a pair of classical face and degeneracy maps of homological algebra) we always have $C_n(R,S) \,|\, C_{n+1}(R,S)$ as R-S-, S-R- or S-bimodules, so that the depth $2n$ as well as $2n+1$ conditions coincide in the case of subring with the I-depth $2n$ and $2n+1$ conditions above where $I = R$. (Note though that R-depth 1 is slightly stronger than subring depth 1 since R is not just centrally projective over S (i.e., $R \,|\, S^q$ as S-bimodules) but also R is a split extension of S as S-bimodules since $S \,|\, R^q$ implies $S \,|\, R$; the split extension condition is satisfied by all group algebra extensions and subfactor examples of finite depth.)

Example 2.4. Let $S \subseteq R$ be a subring pair of semisimple complex algebras. Then the minimum depth $d(S,R)$ may be computed from the inclusion matrix M, alternatively an n by m induction-restriction table of n S-simples induced to non-negative integer linear combination of m R-simples along rows, and by Frobenius reciprocity, columns show restriction of R-simples in terms of S-simples). The procedure to obtain $d(S,R)$ given in the paper [6] is the following: let $M^{[2n]} = (MM^t)^n$ and $M^{[2n+1]} = M^{[2n]}M$ (and $M^{[0]} = I_n$), then the matrix M has depth $n \geq 1$ if for some $q \in \mathbb{Z}_+$

$$M^{[n+1]} \leq qM^{[n-1]} \tag{5}$$

The minimum depth of M is equal to $d(S,R)$ by [1, appendix] (or Theorem 5.3 below combined with [5, 6]).

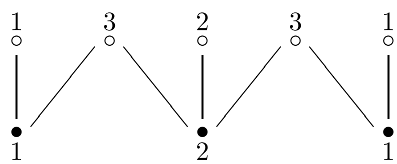
In terms of the bipartite graph of the inclusion $S \subseteq R$, $d(S,R)$ is the lesser of the minimum odd depth and the minimum even depth [6]. The matrix M is an incidence matrix of this bipartite graph if all entries greater than 1 are changed to 1, while zero entries are retained as 0: let the S-simples be represented by n white dots in a bottow row of the graph, and R-simples by m black dots in a top row, connected by edges joining black and white dots (or not) according to the 0-1-matrix entries obtained from M. The minimum odd depth of the bipartite graph is 1 plus the diameter in edges of the row of white dots (indeed an odd number), while the minimum even depth is 2 plus the largest of the diameters of the bottom row where a subset of black dots under one white dot is identified together.

Now suppose I is an ideal in R. Let the primitive central idempotents of R be given by $e_1, \ldots, e_m$ and those of S by $f_1, \ldots, f_n$. Then I is itself a semisimple complex algebra with unit $e = e_1 + \cdots + e_r$ (assumed with no loss of generality). Now suppose $f_i e_j = 0$ for $i > s$ and all $j \leq r$, while $f_i e_j \neq 0$ for $i \leq s$ and some $j \leq r$. Let $J = f_1 S \oplus \cdots \oplus f_s S$, a semisimple subalgebra of S: this ideal satisfies $J \oplus \mathrm{ann}\,({}_S I) = S$. Then it is not hard to see that I-depth of $S \subseteq R$ is computed as the depth of the subring pair of semisimple algebras $J \hookrightarrow I$ via $s \mapsto es$:

$$d^I(S,R) = d(J,I), \tag{6}$$

the minimum depth of the $s \times r$ submatrix M_1 in the upper lefthand corner of M. This follows from the lemma where $S/K = J$ and the realization that $I \otimes_J \cdots \otimes_J I$ is induction and restriction n times of I-simples as explained in the appendix of [1].

Example 2.5. As a sub-example of the previous example, let $R = \mathbb{C}\, S_4$, the complex group algebra of the permutation group on four letters, and $S = \mathbb{C}\, S_3$. The inclusion diagram pictured below with the degrees of the irreducible representations, is determined from the character tables of S_3 and S_4 or the branching rule (for the Young diagrams labelled by the partitions of n and representing the irreducibles of S_n).



This graph has minimum odd depth 5 and minimum even depth 6, whence $d(S,R) = 5$. Alternatively, the inclusion matrix M is given by

$$M = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

whose bracketed powers defined above satisfy a depth 5 inequality (5).

Now let I be the ideal in R associated with the two-dimensional representation, the white dot labelled 2. Then $d(J,I)$ is the depth of the matrix (1), so $d^I(S,R) = 1$. If I is the ideal of R associated with the first three white dots in the diagram above, then J is the ideal in S associated to the first two black dots, and $d(J,I)$ is the minimum depth of the (upper-left hand corner) matrix

$$M' = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

which has minimum depth 3. If I is the ideal associated to the three white dots labelled 3,2, and 3, we similarly compute $d^I(S,R) = 4$. Finally, if I is ideal associated to the first four white dots in the diagram above, the $d^I(S,R) = 5$.

Proposition 2.6. *Suppose $S \subseteq R$ is a subring pair of semisimple complex algebras and $I \subseteq R$ is an ideal. Then $d^I(S,R) \leq d(S,R)$.*

Proof. This follows from the observation above that $d^I(S,R) = d(J,I)$ where $J \subseteq S$ and $I \subseteq R$ are both subring pairs of semisimple algebras. But $d(J,I)$ is the depth of a subgraph of the inclusion graph of $S \subseteq R$. By the description of depth of a bipartite graph as the minimum of the odd and even depths in terms of diameter of the row of black dots, it is clear that $d(J,I) \leq d(S,R)$. □

3. Even depth of QF extensions

A proper ring extension is taken to be a monomorphism $S \hookrightarrow R$; stretching this terminology slightly, a ring homomorphism $S \to R$ is referred to as a *ring extension*, denoted by $R \,|\, S$. A ring extension $R \,|\, S$ is a *left QF extension* if the induced module ${}_SR$ is finitely generated projective and the natural bimodules satisfy ${}_RR_S \,|\, {}_R\mathrm{Hom}\,({}_SR, {}_SS)_S{}^q$ for some positive integer q. A right QF extension is oppositely defined. A *QF extension* $R \,|\, S$ is both a left and right QF extension and may be characterized by both R_S and ${}_SR$ being finite projective, and two h-equivalences of bimodules given by ${}_RR_S \overset{h}{\sim} {}_R\mathrm{Hom}\,({}_SR, {}_SS)_S$ and ${}_SR_R \overset{h}{\sim} {}_S\mathrm{Hom}\,(R_S, S_S)_R$ [19, 20]. For example, a Frobenius extension $S \to R$ is a QF extension since it is left and right finite projective and satisfies the stronger conditions that R is *isomorphic* to its right S-dual R^* and its left S-dual ${}^*R$ as natural S-R-bimodules, respectively R-S-bimodules.

3.1. β-Frobenius extensions vs. QF extensions.

In Hopf algebras and quantum algebras, examples of Frobenius extensions often occur with a twist foreseen by Nakayama and Tzuzuku, their so-called beta-Frobenius extension. Let β be an automorphism of the ring S and $S \subseteq R$ a subring pair. We next denote the pullback module of a module ${}_SM$ along $\beta : S \to S$ by ${}_\beta M$. A proper ring extension $R \,|\, S$ is a *β-Frobenius extension* if R_S is finite projective and there is a bimodule isomorphism ${}_SR_R \cong {}_\beta\mathrm{Hom}\,(R_S, S_S)$. One shows that $R \,|\, S$ is a Frobenius extension if and only if β is an inner automorphism. A subring pair of Frobenius algebras $S \subseteq R$ is β-Frobenius extension so long as R_S is finite projective and the Nakayama automorphism η_R of R stabilizes S, in which case $\beta = \eta_S \circ \eta_R^{-1}$ [22]. For instance a finite dimensional Hopf algebra $R = H$ and $S = K$ a Hopf subalgebra of H are a pair of Frobenius algebras satisfying the conditions just given: the formula for β reduces to the following given in terms of the modular functions of H and K and the antipode S [11, 7.8]:

$$\beta(x) = \sum_{(x)} m_H(x_{(1)}) m_K(S(x_{(2)})) x_{(3)} \tag{7}$$

When a β-Frobenius extension is a QF extension is addressed in the next proposition.

Proposition 3.1. *A β-Frobenius extension $R \,|\, S$ is a left QF extension if and only if there are $u_i, v_i \in R$ $(i = 1, \ldots, n)$ such that $su_i = u_i\beta(s)$ and $v_is = \beta(s)v_i$ for all $i, s \in S$, and*

$$\beta^{-1}(s) = \sum_{i=1}^{n} u_i s v_i. \tag{8}$$

Proof. Suppose $R \,|\, S$ is β-Frobenius extension. Then the bimodule isomorphism given above applied to 1_R has value $E : R \to S$, a cyclic generator of ${}_\beta\mathrm{Hom}\,(R_S, S_S)_R$ satisfying $E(s_1rs_2) = \beta(s_1)E(r)s_2$ for all $s_1, s_2 \in S, r \in R$. If $x_1, \ldots, x_m \in R$ and $\phi_1, \ldots, \phi_m \in \mathrm{Hom}\,(R_S, S_S)$ are projective bases of R_S, and $E(y_j-) = \phi_j$ the equations $\sum_{j=1}^m x_jE(y_jr) = r$ and $\sum_{j=1}^m \beta^{-1}(E(rx_j))y_j = r$ hold for all $r \in R$. (Call

(E, x_j, y_j) a *β-Frobenius coordinate system* of $R \,|\, S$. Note that also ${}_S R$ is finite projective.)

Given the elements $u_i, v_i \in R$ satisfying the equations above, let $E_i = E(u_i -)$ which defines n mappings in (the untwisted) $\mathrm{Hom}\,({}_S R_S, {}_S S_S)$. Also define n mappings $\psi_i \in \mathrm{Hom}\,({}_R({}^*R)_S, {}_R R_S)$ by $\psi_i(g) = \sum_{j=1}^m x_j g(v_i y_j)$ where it is not hard to show using the β-Frobenius coordinate equations that $\sum_j x_j \otimes_S v_i y_j \in (R \otimes_S R)^R$ for each i (a Casimir element). It follows that $\sum_{i=1}^n \psi_i(E_i) = 1_R$ and that $R \,|\, {}^*R^n$ as natural R-S-bimodules, whence R is a left QF extension of S.

Conversely, assume the left QF condition ${}_S R^*{}_R \,|\, R^n$, equivalent to ${}_R R_S \,|\, {}^*R^n$ by applying the right S-dual functor and noting $({}^*R)^* \cong R$ as well ${}^*(R^*) \cong R$. Also assume the slightly rewritten β-Frobenius condition ${}_{\beta^{-1}} R_R \cong {}_S(R^*)_R$, which then implies ${}_{\beta^{-1}} R_R \,|\, R^n$. So there are n mappings $g_i \in \mathrm{Hom}\,({}_{\beta^{-1}} R_R, {}_S R_R)$ and n mappings $f_i \in \mathrm{Hom}\,({}_S R_R, {}_{\beta^{-1}} R_R)$ such that $\sum_{i=1}^n f_i \circ g_i = \mathrm{id}_R$. Equivalently, with $u_i := f(1_R)$ and $v_i := g(1_R)$, $\sum_{i=1}^n u_i v_i = 1_R$, and the equations in the proposition are satisfied. □

The following corollary weakens one of the equivalent conditions in [7]. It implies that a finite dimensional Hopf algebra that is QF over a Hopf subalgebra is necessarily Frobenius over it; nontrivial examples of QF extensions occur for weak Hopf algebras over their separable base algebra [10].

Corollary 3.2. *Let H be a finite dimensional Hopf algebra and K a Hopf subalgebra. In the notation of (7) the following are equivalent:*

1. *The automorphism, $\beta = \mathrm{id}_K$.*
2. *The algebra extension $H \,|\, K$ is a QF extension.*
3. *The modular functions $m_H(x) = m_K(x)$ for all $x \in K$.*

Proof. ($1 \Rightarrow 2$) A Frobenius extension is a QF extension. ($2 \Rightarrow 3$) Applying the counit ε to (8), one obtains $\varepsilon \circ \beta = \varepsilon$, since $\varepsilon(\sum_i u_i v_i) = 1$. Applied to (7) uniqueness of inverse in convolution algebra $\mathrm{Hom}\,(K, k)$ shows that $m_H = m_K$ on K. ($3 \Rightarrow 1$) This follows from (7). □

It is well-known that for a Frobenius extension $R \,|\, S$, coinduction of a module M_S (to the right R-module $\mathrm{Hom}\,(R_S, M_S)$) is naturally isomorphic to induction of M_S (to the right R-module $M \otimes_S R$). Similarly, a QF extension has h-equivalent coinduction and induction functors, which is seen from the naturality of the mappings in the next proof.

Proposition 3.3. *Suppose ${}_A M_S$ is a bimodule and $R \,|\, S$ is a QF extension. Then there is an h-equivalence of bimodules,*

$$ {}_A M \otimes_S R_R \overset{h}{\sim} {}_A \mathrm{Hom}\,(R_S, M_S)_R. \tag{9} $$

Proof. Since R_S is f.g. projective, it follows that there is an A-R-bimodule isomorphism

$$ M \otimes_S \mathrm{Hom}\,(R_S, S_S) \cong \mathrm{Hom}\,(R_S, M_S), \tag{10} $$

given by $m \otimes_S \phi \mapsto m\phi(-)$ with inverse constructed from projective bases for R_S. But the right S-dual of R is h-equivalent to ${}_S R_R$, so (9) holds by Lemma 1.1. □

The next theorem notes that minimum even depth of a QF extension is the same in its right and left versions given in Definition 2.1 (where $I = R$, $C_n(R, S) = R \otimes_S \cdots \otimes_S R$, n times R).

Theorem 3.4. *If $R\,|\,S$ is QF extension, then $R\,|\,S$ has left depth $2n$ if and only if $R\,|\,S$ has right depth $2n$.*

Proof. The left depth $2n$ condition on $S \to R$ recall is $C_{n+1}(R,S) \overset{h}{\sim} C_n(R,S)$ as S-R-bimodules. To this apply the additive functor $\mathrm{Hom}\,(-_R, R_R)$ (into the category of R-S-bimodules), noting that $\mathrm{Hom}\,(C_n(R,S)_R, R_R) \cong \mathrm{Hom}\,(C_{n-1}(R,S)_S, R_S)$ via $f \mapsto f(- \otimes_S \cdots - \otimes_S 1_R)$ for each integer $n \geq 1$. It follows (from Lemma 1.1) that there is an R-S-bimodule h-equivalence,

$$\mathrm{Hom}\,(C_n(R,S)_S, R_S) \overset{h}{\sim} \mathrm{Hom}\,(C_{n-1}(R,S)_S, R_S) \tag{11}$$

(Then in the depth two case, the left depth two condition is equivalent to $\mathrm{End}\, R_S \overset{h}{\sim} R$ as natural R-S-bimodules.)

Given bimodule ${}_RM_S$, we have ${}_RM \otimes_S R_R \overset{h}{\sim} {}_R\mathrm{Hom}\,(R_S, M_S)_R$ by the previous lemma: apply this to $C_{n+1}(R,S) = C_n(R,S) \otimes_S R$ using the hom-tensor adjoint relation: there are h-equivalences and isomorphisms of R-bimodules,

$$\begin{aligned} C_{n+1}(R,S) &\overset{h}{\sim} \mathrm{Hom}\,(R_S, C_n(R,S)_S) \\ &\overset{h}{\sim} \mathrm{Hom}\,(R_S, \mathrm{Hom}\,(R_S, C_{n-1}(R,S)_S)_S) \\ &\cong \mathrm{Hom}\,(R \otimes_S R_S, C_{n-1}(R,S)_S) \\ \cdots &\overset{h}{\sim} \mathrm{Hom}\,(C_p(R,S)_S, C_{n-p+1}(R,S)_S) \end{aligned} \tag{12}$$

for each $p = 1, 2, \ldots, n$ and $n = 1, 2, \ldots$. Compare (11) and (12) with $p = n$ to get ${}_RC_{n+1}(R,S)_S \overset{h}{\sim} {}_RC_n(R,S)_S$ which is the right depth $2n$ condition.

The converse is proven similarly from the symmetric conditions of the QF hypothesis. $\square$

The next proposition is an easy corollary of the proofs of Theorem 3.4 and of Proposition 3.3, therefore omitted. An R-bimodule I is said to be *QF relative to a subring* $S \subseteq R$ below if I_S and ${}_SI$ are f.g. projectives, ${}_SI_R \overset{h}{\sim} {}_S\mathrm{Hom}\,(I_S, S_S)_R$ and ${}_RI_S \overset{h}{\sim} {}_R\mathrm{Hom}\,({}_SI, {}_SS)_S$. We also suppose below an R-bimodule I is a ring with multiplication that is associative in all respects with the bimodule structure, such as $(x_1 \cdot r)x_2 = x_1(r \cdot x_2)$ for all $x_1, x_2 \in I, r \in R$. For example, an ideal I in a semisimple complex algebra R with semisimple subalgebra S satisfies this hypothesis.

Corollary 3.5. *Suppose I is a multiplicative R-bimodule with unit e and is QF relative to a subring $S \subseteq R$. Then $S \subseteq R$ has left I-depth $2n$ if and only if $S \subseteq R$ has right I-depth $2n$.*

4. Frobenius extensions

As noted above a Frobenius extension $R\,|\,S$ is characterized by any of the following four conditions [11]. First, that R_S is finite projective and ${}_SR_R \cong \mathrm{Hom}\,(R_S, S_S)$. Secondly, that ${}_SR$ is finite projective and ${}_RR_S \cong \mathrm{Hom}\,({}_SR, {}_SS)$. Thirdly, that coinduction and induction of right (or left) S-modules is naturally equivalent. Fourth, there is a Frobenius coordinate system $(E : R \to S; x_1, \ldots, x_m; y_1, \ldots, y_m)$, which satisfies

$$E \in \mathrm{Hom}\,({}_SR_S, {}_SS_S), \quad \sum_{i=1}^m E(rx_i)y_i = r = \sum_{i=1}^m x_iE(y_ir) \quad (\forall r \in R). \tag{13}$$

Lemma 4.1. *The natural module R_S is a generator iff ${}_SR$ is a generator iff there are elements $\{a_j\}_{j=1}^n$ and $\{c_j\}_{j=1}^n$ such that $\sum_{j=1}^n E(a_jc_j) = 1_S$.*

Proof. The bimodule isomorphism ${}_SR_R \xrightarrow{\cong} {}_S\mathrm{Hom}\,(R_S, S_S)_R$ is realized by $r \mapsto E(r-)$ (with inverse $\phi \mapsto \sum_{i=1}^m \phi(x_i)y_i$). If R_S is a generator, then there are elements $\{c_j\}_{j=1}^n$ of R and mappings $\{\phi_j\}_{j=1}^n$ of R^* such that $\sum_{j=1}^n \phi_j(c_j) = 1_S$. Let $Ea_j = \phi_j$. Then $\sum_{j=1}^n E(a_jc_j) = 1_S$.

Another bimodule isomorphism ${}_RR_S \xrightarrow{\cong} {}_R\mathrm{Hom}\,({}_SR, {}_SS)_S$ is realized by $r \mapsto E(-r) := rE$. Then writing the last equation as $\sum_j c_jE(a_j) = 1_S$ exhibits ${}_BA$ as a generator. □

A Frobenius (or QF) extension $R \,|\, S$ enjoys an *endomorphism ring theorem* [19, 18], which states that $R \,|\, \mathcal{E} := \mathrm{End}\, R_S$ is a Frobenius (respectively, QF) extension, where the default ring homomorphism $R \to \mathcal{E}$ is understood to be the left multiplication mapping $\lambda : r \mapsto \lambda_r$ where of course $\lambda_r(x) = rx$. It is worth noting that λ is a left split R-monomorphism (by evaluation at 1_R) so ${}_R\mathcal{E}$ is a generator.

The *tower* of a Frobenius (resp. QF) extension is obtained by iteration of the endomorphism ring and λ, obtaining a tower of Frobenius (resp. QF) extensions where occasionally we need the notation $S := \mathcal{E}_{-1}, R = \mathcal{E}_0$ and $\mathcal{E} = \mathcal{E}_1$

$$S \to R \hookrightarrow \mathcal{E}_1 \hookrightarrow \mathcal{E}_2 \hookrightarrow \cdots \hookrightarrow \mathcal{E}_n \hookrightarrow \cdots \tag{14}$$

so $\mathcal{E}_2 = \mathrm{End}\, \mathcal{E}_R$, etc. By transitivity ([22], resp. [19, Müller]), all sub-extensions $\mathcal{E}_m \hookrightarrow \mathcal{E}_{m+n}$ in the tower are also Frobenius (resp. QF) extensions.

The rings $\mathcal{E}_n$ are h-equivalent to $C_{n+1}(R, S) = R \otimes_S \cdots \otimes_S R$ as R-bimodules in case $R \,|\, S$ is a QF extension. This follows from noting the

$$\mathrm{End}\, R_S \cong R \otimes_S \mathrm{Hom}\,(R_S, S_S) \overset{h}{\sim} R \otimes_S R$$

also holding as natural $\mathcal{E}$-R-bimodules, obtained by substitution of $R^* \overset{h}{\sim} R$. This observation is then iterated followed by cancellations of the type $R \otimes_R M \cong M$.

4.1. **Tower above Frobenius extension.** Specialize now to $R \,|\, S$ a Frobenius extension with Frobenius coordinate system E and $\{x_i\}_{i=1}^m, \{y_i\}_{i=1}^m$. Then the h-equivalences above are replaced by isomorphisms, and $\mathcal{E}_n \cong C_{n+1}(R, S)$ for each $n \geq -1$ as ring isomorphisms with respect to a certain induced "E-multiplication." The E-multiplication on $R \otimes_S R$ is induced from the endomorphism ring $\mathrm{End}\, R_S \xrightarrow{\cong} R \otimes_S R$ given by $f \mapsto \sum_i f(x_i) \otimes_S y_i$ with inverse $r \otimes r' \mapsto \lambda_r \circ E \circ \lambda_{r'}$. The outcome is E-multiplication on $C_2(R, S)$ given by

$$(r_1 \otimes_S r_2)(r_3 \otimes_S r_4) = r_1E(r_2r_3) \otimes_S r_4 \tag{15}$$

with unity element $1_1 = \sum_{i=1}^m x_i \otimes_S y_i$. Note that the R-bimodule structure on $\mathcal{E}_1$ induced by $\lambda : R \hookrightarrow \mathcal{E}$ corresponds to the natural R-bimodule $R \otimes_S R$.

The E-multiplication is defined inductively on

$$\mathcal{E}_n \cong \mathcal{E}_{n-1} \otimes_{\mathcal{E}_{n-2}} \mathcal{E}_{n-1} \tag{16}$$

using the Frobenius homomorphism $E_{n-1} : \mathcal{E}_{n-1} \to \mathcal{E}_{n-2}$ obtained by iterating the following construction: a simple and natural Frobenius coordinate system on $\mathcal{E}_1 \cong R \otimes_S R$ is given by $E_1(r \otimes_S r') = rr'$ and $\{x_i \otimes_S 1_R\}_{i=1}^m$, $\{1_R \otimes_S y_i\}_{i=1}^m$ [21] as one checks.

The iterative E-multiplication on $C_n(R,S)$ clearly exists as an associative algebra, but it seems worthwhile (and not available in the literature) to compute it explicitly. The multiplication on $C_{2n}(R,S)$ is given by ($\otimes = \otimes_S, n \geq 1$)

$$(r_1 \otimes \cdots \otimes r_{2n})(t_1 \otimes \cdots \otimes t_{2n}) = \tag{17}$$
$$r_1 \otimes \cdots \otimes r_n E(r_{n+1} E(\cdots E(r_{2n-1} E(r_{2n} t_1) t_2) \cdots) t_{n-1}) t_n) \otimes t_{n+1} \otimes \cdots \otimes t_{2n}.$$

The identity on $C_{2n}(R,S)$ is in terms of the dual bases,

$$1_{2n-1} = \sum_{i_1,\ldots,i_n=1}^{m} x_{i_1} \otimes \cdots \otimes x_{i_n} \otimes y_{i_n} \otimes \cdots \otimes y_{i_1}. \tag{18}$$

The multiplication on $C_{2n+1}(R,S)$ is given by

$$(r_1 \otimes \cdots \otimes r_{2n+1})(t_1 \otimes \cdots \otimes t_{2n+1}) = \tag{19}$$
$$r_1 \otimes \cdots \otimes r_{n+1} E(r_{n+2} E(\cdots E(r_{2n} E(r_{2n+1} t_1) t_2) \cdots) t_n) t_{n+1} \otimes \cdots \otimes t_{2n+1}$$

with identity

$$1_{2n} = \sum_{i_1,\ldots,i_n=1}^{m} x_{i_1} \otimes \cdots \otimes x_{i_n} \otimes 1_R \otimes y_{i_n} \otimes \cdots \otimes y_{i_1}. \tag{20}$$

Let the rings $C_n(R,S) := R_n$ and distinguish them from the isomorphic rings $\mathcal{E}_{n-1}$ ($n = 0, 1, \ldots$).

The inclusions $R_n \hookrightarrow R_{n+1}$ are given by $r_{[n]} \mapsto r_{[n]} 1_n$, which works out in the odd and even cases to:

$$R_{2n-1} \hookrightarrow R_{2n},$$
$$r_1 \otimes \cdots \otimes r_{2n-1} \longmapsto \sum_i r_1 \otimes \cdots \otimes r_n x_i \otimes y_i \otimes r_{n+1} \otimes \cdots \otimes r_{2n-1} \tag{21}$$
$$R_{2n} \hookrightarrow R_{2n+1},$$
$$r_1 \otimes \cdots \otimes r_{2n} \longmapsto r_1 \otimes \cdots \otimes r_n \otimes 1_R \otimes r_{n+1} \otimes \cdots \otimes r_{2n} \tag{22}$$

Here the fact that $\sum_i x_i \otimes y_i \in (R \otimes_S R)^R$ is used.

The bimodule structure on R_n over a subalgebra R_m (with $m < n$ via composition of left multiplication mappings λ) is just given in terms of the multiplication in R_m as follows:

$$(r_1 \otimes \cdots \otimes r_m)(a_1 \otimes \cdots \otimes a_n) = \tag{23}$$
$$[(r_1 \otimes \cdots \otimes r_m)(a_1 \otimes \cdots \otimes a_m)] \otimes a_{m+1} \otimes \cdots \otimes a_n$$

with a similar formula for the right module structure.

The formulas for the successive Frobenius homomorphisms $E_m : R_{m+1} \to R_m$ are given in even degrees by

$$E_{2n}(r_1 \otimes \cdots \otimes r_{2n+1}) = r_1 \otimes \cdots \otimes r_n E(r_{n+1}) \otimes r_{n+2} \otimes \cdots \otimes r_{2n+1}. \tag{24}$$

for $n \geq 0$. The formulas in the odd case is

$$E_{2n+1}(r_1 \otimes \cdots \otimes r_{2n+2}) = r_1 \otimes \cdots \otimes r_n \otimes r_{n+1} r_{n+2} \otimes r_{n+3} \otimes \cdots \otimes r_{2n+2} \tag{25}$$

for $n \geq 0$.

The dual bases of E_n denoted by x_i^n and y_i^n are given by all-in-one formulas

$$x_i^n = x_i \otimes 1_{n-1} \tag{26}$$
$$y_i^n = 1_{n-1} \otimes y_i \tag{27}$$

for $n \geq 0$ (where $1_0 = 1_R$). Note that $\sum_i x_i^n \otimes_{R_n} y_i^n = 1_{n+1}$.

With another choice of Frobenius coordinate system (F, z_j, w_j) for $R \,|\, S$ there is in fact an invertible element d in the centralizer subring R^S of R such that $F = E(d-)$ and $\sum_i x_i \otimes_S y_i = \sum_j z_j \otimes_S d^{-1} w_j$ [11, 21]; whence an isomorphism of the E-multiplication onto the F-multiplication, both on $R \otimes_S R$, given by $r_1 \otimes r_2 \mapsto r_1 \otimes d^{-1} r_2$. If the tower with E-multiplication is denoted by R_n^E and the tower with F-multiplication by R_n^F, there is a sequence of ring isomorphisms

$$R_{2n}^E \xrightarrow{\cong} R_{2n}^F,$$

$$r_1 \otimes \cdots \otimes r_{2n} \longmapsto r_1 \otimes \cdots \otimes r_n \otimes d^{-1} r_{n+1} \otimes \cdots \otimes d^{-1} r_{2n} \tag{28}$$

$$R_{2n+1}^E \xrightarrow{\cong} R_{2n+1}^F,$$

$$r_1 \otimes \cdots r_{2n+1} \longmapsto r_1 \otimes \cdots \otimes r_{n+1} \otimes d^{-1} r_{n+2} \otimes \cdots \otimes d^{-1} r_{2n+1} \tag{29}$$

which commute with the inclusions $R_n^{E,F} \hookrightarrow R_{n+1}^{E,F}$.

Theorem 4.2. *The multiplication, module and Frobenius structures for the tower $R_n = R \otimes_S \cdots \otimes_S R$ (n times R) above a Frobenius extension $R \,|\, S$ are given by the formulas (15) to (29).*

Proof. First define Temperley-Lieb generators iteratively by $e_n = 1_{n-1} \otimes_{R_{n-2}} 1_{n-1} \in R_{n+1}$ for $n = 1, 2, \ldots$, which results in the explicit formulas,

$$\begin{aligned} e_{2n} &= \sum_{i_1,\ldots,i_{n+1}} x_{i_1} \otimes \cdots \otimes x_{i_n} \otimes y_{i_n} x_{i_{n+1}} \otimes y_{i_{n+1}} \otimes y_{i_{n-1}} \otimes \cdots \otimes y_{i_1} \\ e_{2n+1} &= \sum_{i_1,\ldots,i_n} x_{i_1} \otimes \cdots \otimes x_{i_n} \otimes 1_R \otimes 1_R \otimes y_{i_n} \otimes \cdots \otimes y_{i_1} \end{aligned} \tag{30}$$

These satisfy braid-like relations [13, p. 106]; namely,

$$e_i e_j = e_j e_i, \quad |i-j| \geq 2, \quad e_{i+1} e_i e_{i+1} = e_{i+1}, \quad e_i e_{i+1} e_i = e_i 1_{i+1}. \tag{31}$$

(The generators above fail to be idempotents to the extent that $E(1)$ differs from 1.) The proof that the formulas above are the correct outcomes of the inductive definitions may be given in terms of Temperley-Lieb generators, braid-like relations and important relations

$$e_n x e_n = e_n E_{n-1}(x), \quad \forall x \in R_n \tag{32}$$

$$y e_n = E_n(y e_n) e_n, \quad \forall y \in R_{n+1}, \quad E_n(e_n) = 1_{n-1} \tag{33}$$

$$x e_n = e_n x, \quad \forall x \in R_{n-1} \tag{34}$$

[13, p. 106] (for background see [8]) as well as the symmetric left-right relations. These relations and the Frobenius equations (13) may be checked to hold in terms of the equations above in a series of exercises left to the reader.

The formulas for the Frobenius bases follow from the iteratively apparent $x_i^n = x_i e_1 e_2 \cdots e_n$ and $y_i^n = e_n \cdots e_2 e_1 y_i$ and uniqueness of bases w.r.t. same Frobenius homomorphism. In fact $e_n \cdots e_2 e_1 r = 1_{n-1} \otimes r$ for any $r \in R, n = 1, 2, \ldots$ (and symmetrically) as well as $1_n = \sum_i x_i e_1 \cdots e_{n-1} e_n e_{n-1} \cdots e_1 y_i$.

Since the inductive definitions of the ring and modules structures on the R_n's also satisfy the relations listed above, and agree on and below R_2, the proof is finished with an induction argument based on expressing tensors as words in Temperley-Lieb generators and elements of R.

We note that

$$a_1 \otimes \cdots \otimes a_{n+1} = (a_1 \otimes \cdots \otimes a_n)(1_{n-1} \otimes a_{n+1}) \tag{35}$$
$$= (a_1 \otimes \cdots \otimes a_{n-1})(1_{n-2} \otimes a_n)(e_n \cdots e_1 a_{n+1})$$

$$= \cdots = a_1(e_1 a_2)(e_2 e_1 a_3) \cdots (e_{n-1} \cdots e_1 a_n)(e_n \cdots e_1 a_{n+1})$$

The formulas for multiplication (19), (17) and (23) follow from induction and applying the relations (31) through (34). □

For the next proposition the main point above is that given a Frobenius extension there is a ring structure on the $C_n(R,S)$'s satisfying the hypotheses below (for we compare with (23)). This is true as well if R is a commutative ring with S a subring, since the ordinary tensor algebra on $R \otimes_S R$ may be extended to any number of tensor products.

Proposition 4.3. *Let $R \,|\, S$ be a ring extension. Suppose that there is a ring structure on each $R_n := C_n(R,S)$ for each $n \geq 0$, a ring homomorphism $R_{n-1} \to R_n$ for each $n \geq 1$, and that the composite $R \to R_n$ induces the natural bimodule given by $r \cdot (r_1 \otimes \cdots r_n) \cdot r' = rr_1 \otimes r_2 \otimes \cdots \otimes r_n r'$. Then $R \,|\, S$ has depth $2n+1$ if and only if $R_n \,|\, S$ has depth 3.*

Proof. If $R \,|\, S$ has depth $2n+1$, then $R_n \overset{h}{\sim} R_{n+1}$ as S-bimodules. By induction of modules, also $R_n \overset{h}{\sim} R_{2n}$ as S-bimodules. But $R_{2n} \cong R_n \otimes_S R_n$. Then $R_n \,|\, S$ has depth three.

Conversely, if $R_n \,|\, S$ has depth 3, then $R_{2n} \overset{h}{\sim} R_n$ as S-bimodules. But $R_{n+1} \,|\, R_{2n}$ via the split S-bimodule epi $r_1 \otimes \cdots \otimes r_{2n} \mapsto r_1 \cdots r_n \otimes r_{n+1} \otimes \cdots \otimes r_{2n}$. Then $R_{n+1} \,|\, R_n^q$ for some $q \in \mathbb{Z}_+$. It follows that $R \,|\, S$ has depth $2n+1$. □

We may in turn embed a depth three extension into a ring extension having depth two. The proof requires the QF condition. Retain the notation for the endomorphism ring introduced earlier in this section.

Theorem 4.4. *Suppose $R \,|\, S$ is a QF extension. If $R \,|\, S$ has depth 3, then $\mathcal{E} \,|\, S$ has depth 2. Conversely, if $\mathcal{E} \,|\, S$ has depth 2, and R_S is a generator, then $R \,|\, S$ has depth 3.*

Proof. Since R is a QF extension of S, we have $\mathcal{E} \overset{h}{\sim} R \otimes_S R$ as $\mathcal{E}$-R-bimodules. Then $\mathcal{E} \otimes_S \mathcal{E} \overset{h}{\sim} R \otimes_S R \otimes_S R \otimes_S R$ as $\mathcal{E}$-S-bimodules. Given the depth 3 condition, $R \otimes_S R \overset{h}{\sim} R$ as S-bimodules, it follows by two substitutions that $\mathcal{E} \otimes_S \mathcal{E} \overset{h}{\sim} R \otimes_S R$ as $\mathcal{E}$-S-bimodules. Consequently, $\mathcal{E} \otimes_S \mathcal{E} \overset{h}{\sim} \mathcal{E}$ as $\mathcal{E}$-S-bimodules. Hence, $\mathcal{E} \,|\, S$ has right depth 2, and since it is a QF extension by the endomorphism ring theorem and transitivity, $\mathcal{E} \,|\, S$ also has left depth 2.

Conversely, we are given R_S a progenerator, so that $\mathcal{E}$ and S are Morita equivalent rings, where ${}_S\mathrm{Hom}\,(R_S, S_S)_{\mathcal{E}}$ and ${}_{\mathcal{E}}R_S$ are the context bimodules. If $\mathcal{E} \,|\, S$ has depth two, then $\mathcal{E} \otimes_S \mathcal{E} \overset{h}{\sim} \mathcal{E}$ as $\mathcal{E}$-S-bimodules. Then $R \otimes_S R \otimes_S R \otimes_S R \overset{h}{\sim} R \otimes_S R$ as $\mathcal{E}$-S-bimodules. Since $\mathrm{Hom}\,(R_S, S_S) \otimes_{\mathcal{E}} R \cong S$ as S-bimodules, a cancellation of the bimodules ${}_{\mathcal{E}}R_S$ follows, so $R \otimes_S R \otimes_S R \overset{h}{\sim} R$ as S-bimodules. Since $R \otimes_S R \,|\, R \otimes_S R \otimes_S R$, it follows that $R \otimes_S R \,|\, R^q$ for some $q \in \mathbb{Z}_+$. Then $R \,|\, S$ has depth 3. □

5. Tower depth vs. depth of subrings

In this section we review tower depth from [15] and find a general case when it is the same as subring depth defined in (4) and in [1]. We first require a generalization of left and right depth 2 to a tower of three rings. We say that a tower $R \,|\, S \,|\, T$ where $R \,|\, S$ and $S \,|\, T$ are ring extensions, has *generalized right depth* 2 if $R \otimes_S R \overset{h}{\sim} R$ as natural R-T-bimodules (where mappings $T \to S \to R$ are composed to induce the module R_T). (Note that if $T = S$, this is the definition of the ring extension $R \,|\, S$ having right depth 2.)

Throughout the section below we suppose $R \,|\, S$ a Frobenius extension and $\mathcal{E}_i \hookrightarrow \mathcal{E}_{i+1}$ its tower above it, as defined in (14) and the ensuing discussion in Section 4. Following [15] (with a small change in vocabulary), we say that $R \,|\, S$ has *right tower depth* $n \geq 2$ if the sub-tower of composite ring extensions $S \to \mathcal{E}_{n-3} \hookrightarrow \mathcal{E}_{n-2}$ has generalized right depth 2; i.e., as natural $\mathcal{E}_{n-2}$-S-bimodules,

$$\mathcal{E}_{n-2} \otimes_{\mathcal{E}_{n-3}} \mathcal{E}_{n-2} \ \oplus * \ \cong \ \mathcal{E}^q_{n-2} \tag{36}$$

for some positive integer q, since the reverse condition is always satisfied. Since $\mathcal{E}_{-1} = S$ and $\mathcal{E}_0 = R$, this recovers the right depth two condition on a subring S of R. To this definition we add that a Frobenius extension $R \,|\, S$ has depth 1 if it is a centrally projective ring extension; i.e., ${}_S R_S \,|\, S^q$ for some $q \in \mathbb{Z}_+$. Left tower depth n is just defined using (36) but as natural S-$\mathcal{E}_{n-2}$-bimodules. By [15, Theorem 2.7] the left and right tower depth n conditions are equivalent on Frobenius extensions.

From the definition of tower depth and a comparison of (16) and (2.1) with $I = R$, the following lemma is obtained:

Lemma 5.1. *Suppose $S \subseteq R$ is a subring such that R is a Frobenius extension of S. If $R \,|\, S$ has tower depth n, then $S \subseteq R$ has depth $2n - 2$ for each $n = 1, 2, \ldots$.*

Proof. From (36) we obtain $R_n \,|\, R^q_{n-1}$ as R-S-bimodules; the rest of the proof is sorting out notation and indices. □

From [15, Lemma 8.3], it follows that if $R \,|\, S$ has tower depth n, it has tower depth $n + 1$. Define $d_F(R, S)$ to be the minimum tower depth if $R \,|\, S$ has tower depth n for some integer n, $d_F(R, S) = \infty$ if the condition (36) is not satisfied for any $n \geq 2$ nor is it depth 1. Notice that if $S \subseteq R$ is a subring with R a Frobenius extension of S, then $d_F(R, S) = d(S, R)$ if one of $d(S, R) \leq 2$ or $d_F(R, S) \leq 2$. This is extended to $d_F(R, S) = d(S, R)$ if one of $d(S, R), d_F(R, S) \leq 3$ in the next lemma.

Notice that tower depth n makes sense for a QF extension $R \,|\, S$: by elementary considerations, it has right tower depth 3 if $S \to R \hookrightarrow \mathcal{E}$ satisfies $\mathcal{E} \otimes_R \mathcal{E} \overset{h}{\sim} \mathcal{E}$ as $\mathcal{E}$-S-bimodules. It has been noted elsewhere that a QF extension has right tower depth 3 if and only if it has left tower depth 3 by an argument essentially identical to that in [15, Th. 2.8] but replacing Frobenius isomorphisms with quasi-Frobenius h-equivalences.

Lemma 5.2. *A QF extension $R \,|\, S$ such that R_S is a generator has tower depth 3 if and only if S has depth 3 as a subring in R.*

Proof. Since R_S is a generator, $R \,|\, S$ is a proper extension by a short argument. Assume $S \subseteq R$.

($\Rightarrow$) By the QF property, $\mathcal{E} \overset{h}{\sim} R \otimes_S R$ as $\mathcal{E}$-S-bimodules. By the tower depth 3 condition, $\mathcal{E} \otimes_R \mathcal{E} \overset{h}{\sim} \mathcal{E}$ as $\mathcal{E}$-S-bimodules. Then $R \otimes_S R \otimes_S R \overset{h}{\sim} R \otimes_S R$ as $\mathcal{E}$-S-bimodules. Since R_S is a progenerator, we cancel bimodules ${}_{\mathcal{E}}R_S$ as in the proof of Theorem 4.4 to obtain $R \otimes_S R \overset{h}{\sim} R$ as S-bimodules. Hence $S \subseteq R$ has depth 3.

($\Leftarrow$) Given ${}_S R_S \overset{h}{\sim} {}_S R \otimes_S R_S$, by tensoring with ${}_{\mathcal{E}}R \otimes_S -$ we get $R \otimes_S R \overset{h}{\sim} R \otimes_S R \otimes_S R$ as $\mathcal{E}$-S-bimodules. By the QF property, $\mathcal{E} \otimes_R \mathcal{E} \overset{h}{\sim} \mathcal{E}$ as $\mathcal{E}$-S-bimodules follows, whence $R \,|\, S$ has tower depth 3. □

The theorem below proves that subring depth and tower depth coincide on Frobenius generator extensions. At a certain point in the proof, we use the following fundamental fact about the tower R_n above a Frobenius extension $R \,|\, S$: since the compositions of the Frobenius extensions remain Frobenius, the iterative constructions of E-multiplication on tensor-squares isomorphic to endomorphism rings applies, but gives isomorphic ring structures to those on the R_n. For example, the composite extension $S \to R_n$ is Frobenius with $\mathrm{End}\,(R_n)_S \cong R_n \otimes_S R_n \cong R_{2n}$, isomorphic in its $E \circ E_1 \circ \cdots \circ E_{n-1}$-multiplication or its E-multiplication given in (17) [12].

Theorem 5.3. *Let $S \subseteq R$ be a subring such that R is a Frobenius extension of S and R_S is a generator. Then $R \,|\, S$ has tower depth m for $m = 1, 2, \ldots$ if and only if the subring $S \subseteq R$ has depth m. Consequently, $d_F(R, S) = d(S, R)$.*

Proof. The cases $m = 1, 2, 3$ have been dealt with above. We divide the rest of the proof into odd m and even m. The proof for odd $m = 2n + 1$: ($\Rightarrow$) If $R \,|\, S$ has tower depth $2n + 1$, then $R_{2n} \otimes_{R_{2n-1}} R_{2n} \,|\, R_{2n}^q$ as R_{2n}-S-bimodules. Continuing with $R_{2n} \cong R_{2n-1} \otimes_{R_{2n-2}} R_{2n-1}$, iterating and performing standard cancellations, we obtain

$$R_{2n+1} \,|\, R_{2n}^q \tag{37}$$

as $\mathrm{End}\,(R_n)_S$-S-bimodules. But the module $(R_n)_S$ is a generator for all n by Lemma 4.1, the endomorphism ring theorem for Frobenius generator extensions and transitivity of generator property for modules (if M_R and R_S are generators, then restricted module M_S is clearly a generator). It follows that $(R_n)_S$ is a progenerator and cancellable as an $\mathrm{End}\,(R_n)_S$-S-bimodule (applying the Morita theorem as in the proof of Theorem 4.4). Then ${}_S(R_{n+1})_S \,|\, {}_S(R_n)_S$ after cancellation of R_n from (37), which is the depth $2n + 1$ condition in (4).

($\Leftarrow$) Suppose $R_{n+1} \oplus * \cong R_n$ as S-bimodules. Apply to this the additive functor $R_n \otimes_S -$ from category of S-bimodules into the category of $\mathrm{End}\,(R_n)_S$-S-bimodules. We obtain (37) which is equivalent to the tower depth $2n + 1$ condition of $R \,|\, S$.

The proof in the even case, $m = 2n$ does not need the generator condition (since even non-generator Frobenius extensions have endomorphism ring extensions that are generators):

($\Rightarrow$) Given the tower depth $2n$ condition $R_{2n-1} \otimes_{R_{2n-2}} R_{2n-1} \cong R_{2n}$ is isomorphic as R_{2n-1}-S-bimodules to a direct summand in R_{2n-1}^q for some positive integer q. Introduce a cancellable extra term in $R_{2n} \cong R_n \otimes_R R_{n+1}$ and in $R_{2n-1} \cong R_n \otimes_R R_n$. Now note that $R_{2n-1} \cong \mathrm{End}\,(R_n)_R$ which is Morita equivalent to R. After cancellation of the $\mathrm{End}\,(R_n)_R$-R-bimodule R_n, we obtain $R_{n+1} \,|\, R_n$ as R-S-bimodules as required by (4).

($\Leftarrow$) Given ${}_R(R_{n+1})_S \mid {}_R(R_n)_S$, we apply $\mathrm{End}\,{}_{(R_n)_R} R_n \otimes_R -$ obtaining $R_{2n} \mid R_{2n-1}$ as R_{2n-1}-S-bimodules, which is equivalent to the tower depth $2n$ condition. $\square$

A depth 2 extension $R \mid S$ often has easier equivalent conditions, e.g., a normality condition, to fulfill than the S-R-bimodule condition $R \otimes_S R \mid R^q$ [2]. Therefore the next corollary (or one like it stated more generally for Frobenius extensions) is interesting in pursuing questions of whether a special type of ring extension has finite depth (and placing finite depth ring extensions in the context of a Galois-normal extension). The corollary follows from [15, 8.6], Proposition 4.3 and Theorem 4.4.

Corollary 5.4. *Let $K \subseteq H$ be a Hopf subalgebra pair of finite dimensional unimodular Hopf algebras. Then K has finite depth in H if and only if there is a tower algebra H_m such that $K \subseteq H_m$ has depth 2.*

In practice, the depth is n or less if $m \geq n-1$ where H_m denotes $H \otimes_K \cdots \otimes_K H$ (m times H); cf. [6, Theorem 3.14]. In particular, when $H = k[G]$ and $K = k[J]$ are group algebras of a subgroup pair $G \geq J$, $K \subseteq H_m$ has depth 2 for some $m > 2[G : N_G(J)]$ [1].

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