A New Algorithm for Determinant Evaluation – The Reduction Method

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Abstract

We present here a new method for determinant evaluation – the reduction method. Firstly, in the section 2, we apply it to third order determinants and after, in the section 3, we generalize it to higher orders determinants. In the section 4 an alternative formulation of the reduction method is presented and, in the section 5, we give the respective mathematical demonstrations[1].

1 Introduction

As we know, determinants are very important in physics and mathematics. Effectively, in several circumstances, the solution of a given problem falls on the resolution of a determinant. If we need to solve a second or third order determinant we have no problem at all, since the calculations are really easy in these cases. Fourth order determinants can also be easily solved by the well-known Laplace method, so we are carefree in this case too. But, however, if the determinant is of fifth or higher order, then we begin to meet with hand-computational problems, since the resolution of such determinants, even with the Laplace method, could be a hard and boring job. The use of a computer machine is generally indispensable in these cases.

To attack this problem, we shall present here an new algorithm for determinant evaluation, which we believe be the simplest method for hand-evaluation of higher order determinants. Effectively, to use this method, the reader needs only to know how evaluate two order determinants. At each step of its application, the order of the determinant is reduced by one, which justifies the name of reduction.

We hope that the simplicity of this method may contribute to a better presentation of the theory of determinants, even in elementary levels. In fact, the exposition of the theory at this level usually stops on the fourth order determinants, due to the technical difficulties commented above. The reduction method might, thus, eliminate such difficulties.

The paper is organized as follows. In the section 2 we present the reduction method for third order determinants. In the next section we extend it to higher order determinants. In the section 4 an alternative formulation of the reduction method is presented. Finally, the mathematical demonstrations are shown in the last section.

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[1]The relationship between this work and that of the mathematicians C. L. Dodgson and F. Chiò is presented in the Addendum.

2 The Reduction Method for Third Order Determinants

Let A be a general third order determinant:

$$A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}. \tag{1}$$

The first step of the reduction method consists in evaluate the four minors b_{ij}, built up from the adjacent elements of A, namely

$$b_{11} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \qquad b_{12} = \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}, \qquad b_{21} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}, \qquad b_{22} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}. \tag{2}$$

This allow us to build the reduced determinant B, which are constructed by disposing the minors b_{ij} in a natural order:

$$B = \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix}. \tag{3}$$

Then, we argue that the evaluation of the determinant B divided by a_{22} (or be, divided by the central element of A), results to be the same as the determinant A, provided that the element a_{22} be different from zero:

$$A = B/a_{22}. \tag{4}$$

The proof is nothing but a matter of calculation. It can be done substituting the expression of the minors b_{ij} onto the expression of B and doing the calculi explicitly. We left this job to the reader.

Notice, however, that if the central element of A is null, the reduction method cannot be directly applied. Effectively, in this case we would meet with a division by zero. Fortunately, this is not a problem at all. We may, for instance, before applying the method, permutate cyclically the lines (or columns) of A in such a way that the new central element turns to be a non-null element. These operations, as we know, do not change the value of the determinant.

3 The Reduction Method for Higher Order Determinants

Let us now show how we can generalize the reduction method to higher order determinants. Let $A = |a_{ij}|$, $1 \leq i, j \leq n$, be a n-order determinant. As before, we proceed building up, from each group of four adjacent elements of A, a total of $n-1$ minors of second order, b_{ij}. That is, we construct the minors b_{ij} through the expression

$$b_{ij} = \begin{vmatrix} a_{ij} & a_{i,j+1} \\ a_{i+1,j} & a_{i+1,j+1} \end{vmatrix}, \qquad 1 \leq i, j \leq n-1. \tag{5}$$

With the values of these minors disposed in an adequate way, we build up the reduced determinant $B = |b_{ij}|$, whose order is $n-1$. Then, we proceed with the reduction of the determinant B itself. Notice, however, that the reduced minors c_{ij} must be defined now by the expression

$$c_{ij} = \frac{\begin{vmatrix} b_{ij} & b_{i,j+1} \\ b_{i+1,j} & b_{i+1,j+1} \end{vmatrix}}{a_{i+1,j+1}}, \qquad 1 \leq i, j \leq n-2. \tag{6}$$

The division by $a_{i+1,j+1}$ may easily be understood: we have seen in the precedent section that after the evaluation of the reduced determinant B was necessary a division by the central element of A. Here an analogous thing happens. After the second reduction, each minor c_{ij} it is the result of the reduction of a third order determinant. Consequently, each one of these reduced minors must be divided by the respective central element of the third order determinant whose it was derived.

With the elements c_{ij} evaluated by the expression above, we can build the second determinant reduced $C = |c_{ij}|$, now of $(n-2)$-order. We can carry on the reduction method as we did in the last step: we build up the $n-3$ minors d_{ij} from the adjacent elements of C, remembering that we must divide each one of them by the element $b_{i+1,j+1}$ also. This procedure must be repeated until the determinant A becomes a number – the value of the determinant A. Thus, if $S = |s_{ij}|$ is the determinant obtained from A when we apply the reduction method k times ($k \geq 2$), and $R = |r_{ij}|$ is the determinant whose reduction gives place to S, then, the elements t_{ij}, which comes from the reduction of S, must be given by

$$t_{ij} = \frac{\begin{vmatrix} s_{ij} & s_{i,j+1} \\ s_{i+1,j} & s_{i+1,j+1} \end{vmatrix}}{r_{i+1,j+1}}, \qquad 1 \leq i,j \leq n-k-2. \tag{7}$$

To illustrate the simplicity of the reduction method, we shall give at this point an example of a fifth order determinant. The reader can notice that the calculations are very easy and can really be made mentally.

$$\begin{vmatrix} 1 & 0 & -2 & 3 & 2 \\ -1 & -3 & 2 & -2 & 0 \\ -3 & -2 & 2 & -1 & 1 \\ -2 & 3 & -1 & 2 & 0 \\ 0 & -3 & 1 & -1 & -3 \end{vmatrix} \Rightarrow \begin{vmatrix} -3 & -6 & -2 & 4 \\ -7 & -2 & 2 & -2 \\ -13 & -4 & 3 & -2 \\ 6 & 0 & -1 & -6 \end{vmatrix} \Rightarrow \begin{vmatrix} 12 & -8 & 2 \\ -1 & 1 & -2 \\ 8 & -4 & -10 \end{vmatrix} \Rightarrow \begin{vmatrix} -2 & 7 \\ 1 & -6 \end{vmatrix} = 5. \tag{8}$$

It is necessarily comment, however, that the reduction method cannot be directly applied in some cases because it would lead to a division by zero. This will happen whenever some internal minor of A is singular[2]. In some cases, this problem can be repaired by a permutation of the lines (or columns) of the original determinant, as we did in the precedent section, but generally this is not enough to fix the problem. To overcome this difficult at all, we will proceed in another way. Suppose that the determinant A presents a singular minor M, whose order is $m < n$. We should notice that, if we substitute one (or more, if necessary) element of M, says m_{ij}, by a parameter ε, then M no more will be a singular minor (because now ε is parameter free). This solves the problem because with this substitution we eliminate the singularity. So, the reduction method is now applicable. Reducing the determinant A, at the end we will have found a expression that depends on ε; the value of A is simply obtained by restoring the original value $\varepsilon = m_{ij}$ (sometimes it is easier restore the value of ε soon after the division be done).

In the sequence we shall present an example of this. Let A be the following fifth order determinant:

$$A = \begin{vmatrix} -1 & 0 & -1 & 0 & -2 \\ 2 & 1 & -2 & -1 & 0 \\ -1 & 2 & 1 & -2 & 1 \\ 1 & 3 & 1 & -2 & -1 \\ -1 & 1 & -2 & -2 & 0 \end{vmatrix}. \tag{9}$$

The determinant A has the value $A = 15$. Notice that the reduction method cannot be directly applied here because the minor $b_{33} = (a_{33}a_{44} - a_{34}a_{43})$ is singular (in fact, if we try to apply the method, then we would to meet us with a division by zero in the third reduction). To remove this singularity we may substitute, for instance, the element a_{33} by a parameter ε, which might a priori assume any value. After this we can apply the reduction method without problems. Doing so, we will find,

$$\begin{vmatrix} -1 & 0 & -1 & 0 & -2 \\ 2 & 1 & -2 & -1 & 0 \\ -1 & 2 & \varepsilon & -2 & 1 \\ 1 & 3 & 1 & -2 & -1 \\ -1 & 1 & -2 & -2 & 0 \end{vmatrix} \Rightarrow \begin{vmatrix} -1 & 1 & 1 & -2 \\ 5 & \varepsilon+4 & \varepsilon+4 & -1 \\ -5 & 2-3\varepsilon & 2-2\varepsilon & 4 \\ 4 & -7 & -6 & -2 \end{vmatrix} \Rightarrow \begin{vmatrix} -9-\varepsilon & 0 & -7-2\varepsilon \\ 15-5\varepsilon & \varepsilon+4 & -9-\varepsilon \\ 9+4\varepsilon & 2+4\varepsilon & -10-2\varepsilon \end{vmatrix} \Rightarrow$$

$$\Rightarrow \begin{vmatrix} -9-\varepsilon & 7+2\varepsilon \\ 8\varepsilon-3 & -11-\varepsilon \end{vmatrix} \Rightarrow 30 - 15\varepsilon. \tag{10}$$

[2]By singular we want mean that its determinant vanishes. Notice also that a simple element as a_{ij} should be considered as a minor of order 1.

Finally, restoring the value $\varepsilon = 1$, we get $A = 15$, which is the correct value of A. Notice that the substitution $\varepsilon = 1$ might be made soon after the third reduction as well. In this example the difference is insignificant, but to higher order determinants it may be most appreciable. The reader may see from this example that the introduction of the parameter ε makes the calculations more boring in general, but at least it is a solution to the problem. Anyway, in the next section we shall present an alternative form of the reduction method which solves the difficulty of the divisions by zero once and for all.

4 An Alternative Formulation of the Reduction Method

In the precedents sections we have seen as the reduction method of determinants works. We saw that the reduction is employed by the evaluation of the two order minors that are formed from the adjacent elements of the original determinant. This choice of the adjacent elements for the building it is, however, quite arbitrary. In fact, we could build the minors from non-adjacent elements as well, although the correct arrangements would be a hard job. That this is possible can be evidenced by the following argument: suppose that we change the lines and columns of the original determinant A, obtaining thus a new determinant A', but in such a way that its value do not change. Then, if we build the minors from the elements that were adjacent in the determinant A, nevertheless they are not more adjacent in the determinant A', it is evident that the reduction method will work as well. This shows that we can build minors from non-adjacent elements, although we should be careful with the configurations which are allowed.

Now we shall show one of these possible arrangements which came to be important, because of its simplicity and also because it eliminates at once the problem of the divisions by zero that we met on the precedent section. This alternative formulation of the reduction method is based on a different way of building the two-orders minors b_{ij} from the determinant A. Instead building them from the adjacent elements of A, we proceed as follows. We fix a specific element of A, says the element a_{rs}, and we build all minors only from this element. That is, we fix a element a_{rs} of A and, for each element a_{ij} with $i \neq r$ and $j \neq s$, we built the minors that contains the elements a_{rj} and a_{is}, so as the element a_{ij} stays in the diagonal with a_{rs} and the other two remain at the sides from them. Let us explain this in a better way. Consider the following n-order determinant

$$A = \begin{vmatrix} a_{11} & \dots & a_{1s} & \dots & a_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{r1} & \dots & a_{rs} & \dots & a_{rn} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{ns} & \dots & a_{nn} \end{vmatrix}. \tag{11}$$

If we fix the non-null element a_{rs} of A, the two order minors that we must build will be, respectively,

$$\begin{array}{ccc} b_{11} = \begin{vmatrix} a_{11} & a_{1s} \\ a_{r1} & a_{rs} \end{vmatrix}, & \dots & b_{1,n-1} = \begin{vmatrix} a_{1s} & a_{1n} \\ a_{rs} & a_{rn} \end{vmatrix}, \\ \vdots & \ddots & \vdots \\ b_{n-1,1} = \begin{vmatrix} a_{r1} & a_{rs} \\ a_{n1} & a_{ns} \end{vmatrix}, & \dots & b_{n-1,n-1} = \begin{vmatrix} a_{rs} & a_{rn} \\ a_{ns} & a_{nn} \end{vmatrix}. \end{array} \tag{12}$$

In a general way, these minors can be written in a more compact form as

$$b_{ij} = \sigma_{ij}(a_{ij}a_{rs} - a_{is}a_{rj}), \qquad 1 \leq i, j \leq n-1, \tag{13}$$

where $\sigma_{ij} = \frac{(r-i)(s-j)}{|(r-i)(s-j)|}$ it is just a function that provides the correct sign for each minor b_{ij}.

With those minors thus defined, we can reduce the determinant A to obtain the reduced determinant B, of $(n-1)$-order:

$$B = \begin{vmatrix} b_{11} & \dots & b_{1,n-1} \\ \vdots & \ddots & \vdots \\ b_{n-1,1} & \dots & b_{n-1,n-1} \end{vmatrix}. \tag{14}$$

We can continue with the reduction process by reducing at its turn the determinant B. For this, we choose a non-null element b_{rs} of B and build the minors c_{ij} as before, but now we must to divide each minors thus constructed by the element a_{rs} which we have fixed in the reduction of the determinant A. But since all minors should be divided by the same element, a_{rs}, and since the determinant C will have $n-2$ lines and columns, these divisions are the same as to divide the whole determinant by $(a_{rs})^{n-2}$. That is, after the second reduction, we should define

$$C = \frac{1}{(a_{rs})^{n-2}} \begin{vmatrix} c_{11} & \dots & c_{1,n-2} \\ \vdots & \ddots & \vdots \\ c_{n-2,1} & \dots & c_{n-2,n-2} \end{vmatrix}. \tag{15}$$

Then we can continue by reducing the determinant C itself: we fix a non-null element c_{rs}, build the $n-3$ minors $d_{ij} = \sigma_{ij}(c_{ij}c_{rs} - c_{is}c_{rj})$ and divide the resultant determinant by $(b_{rs})^{n-3}$, obtaining thus the reduced determinant D. This process must be repeated until the value of the determinant A be evaluated. In short, we found that the determinant A will be related to the last reduced determinant Z (of order 1), by the expression

$$A = \frac{Z}{(a_{rs})^{n-2}(b_{rs})^{n-3}\dots(x_{rs})^{1}}, \tag{16}$$

where x_{rs} it is the element fixed at the reduced determinant X, of order 3. We can also notice that, since we should have for the determinant B an analogous expression, namely,

$$B = \frac{Z}{(b_{rs})^{n-3}(c_{rs})^{n-4}\dots(x_{rs})^{1}}, \tag{17}$$

follows that,

$$A = \frac{B}{(a_{rs})^{n-2}}. \tag{18}$$

Hence, the reduction method presented here can be considered as a recursive application of this last formula.

To illustrate this alternative way of reduction, let us to apply it to the first example of the precedent section (the fixed elements in every determinant were written in bold type and the divisions were left to the final of the reduction process):

$$\begin{vmatrix} 1 & 0 & -2 & 3 & 2 \\ \mathbf{-1} & -3 & 2 & -2 & 0 \\ -3 & -2 & 2 & -1 & 1 \\ -2 & 3 & -1 & 2 & 0 \\ 0 & -3 & 1 & -1 & -3 \end{vmatrix} \Rightarrow \begin{vmatrix} -3 & 0 & 1 & 2 \\ -7 & 4 & -5 & -1 \\ -9 & 5 & -6 & 0 \\ 3 & -1 & 1 & \mathbf{3} \end{vmatrix} \Rightarrow \begin{vmatrix} -15 & 2 & 1 \\ -18 & \mathbf{11} & -14 \\ -27 & 15 & -18 \end{vmatrix} \Rightarrow \begin{vmatrix} -129 & 39 \\ 27 & 12 \end{vmatrix} \Rightarrow -495, \tag{19}$$

hence, we have,

$$A = \frac{-495}{(-1)^3(3)^2(11)^1} = 5. \tag{20}$$

5 Mathematical Demonstrations

Let us now to prove the statements of the precedents sections. As we know, a determinant A, which is composed by the elements a_{ij}, can be defined by the following expression:

$$A = \sum_{\alpha\beta\dots\nu=1}^{n} \varepsilon_{\alpha\beta\dots\nu} \left(a_{1\alpha}a_{2\beta}\dots a_{n\nu}\right), \tag{21}$$

where $\varepsilon_{\alpha\beta\ldots\nu}$ it is the well-known Levi-Civita symbol – a complete anti-symmetrical tensor with $\varepsilon_{12\ldots n} = 1$.

For a future use we also present two important theorems of the theory of determinants, the Laplace and Cauchy theorems. Let a_{ij} be an element of the determinant A and let A_{ij} be the minor obtained from A when we eliminate its line i and column j. We define thus the *cofactor* of a_{ij}, and we represent it by a^{ij}, by the expression

$$a^{ij} = (-1)^{i+j} A_{ij}. \tag{22}$$

With the concept of cofactor we can enunciate the Laplace theorem as follows: the summation of the elements of a given line (or column) of the determinant A, multiplied each one by its respective cofactor, equals the value of the determinant A. That is,

$$\sum_{\alpha=1}^{n} a_{\alpha j} a^{\alpha j} = \sum_{\beta=1}^{n} a_{i\beta} a^{i\beta} = A. \tag{23}$$

At the same foot, the Cauchy theorem establishes that the summation of a given line (or column) of the determinant A, each one multiplied by the cofactors associated with the elements of any other parallel line (or column) of A, it is zero. That is,

$$\sum_{\alpha=1}^{n} a_{\alpha k} a^{\alpha l} = \sum_{\beta=1}^{n} a_{k\beta} a^{l\beta} = 0, \qquad k \neq l. \tag{24}$$

The Laplace and Cauchy theorem can be written in a compact form with the definition of adjugate determinant, $A^{\dagger}$. This is just the determinant built up from the cofactors of A, namely,

$$A^{\dagger} = |a^{ij}| = |a^{ji}|. \tag{25}$$

From this definition, follows that the product of A by $A^{\dagger}$ – *et vice-versa* – results on a determinant whose all diagonal elements equals A (by the Laplace theorem) and whose all non-diagonal elements vanishes (by the Cauchy theorem). From which we obtain the important relation

$$AA^{\dagger} = A^{\dagger}A = A^{n}. \tag{26}$$

5.1 Demonstration for the First Form of the Reduction Method

Now we shall present the demonstration for the reduction method shown on the section 3. Before this, however, it is necessary to prove the following *Lemma*: Let A be a n-order determinant and let B be a 2-order determinant whose elements b_{ij} are obtained from A as below:

$$b_{11} = \begin{vmatrix} a_{11} & \cdots & a_{1,n-1} \\ \vdots & \ddots & \vdots \\ a_{n-1,1} & \cdots & a_{n-1,n-1} \end{vmatrix}, \qquad b_{12} = \begin{vmatrix} a_{12} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n-1,2} & \cdots & a_{n-1,n} \end{vmatrix},$$

$$b_{21} = \begin{vmatrix} a_{21} & \cdots & a_{2,n-1} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{n,n-1} \end{vmatrix}, \qquad b_{22} = \begin{vmatrix} a_{22} & \cdots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{n2} & \cdots & a_{nn} \end{vmatrix}, \tag{27}$$

Also, let C be the central minor of A, whose order is $n-2$, namely,

$$C = \begin{vmatrix} a_{22} & \cdots & a_{2,n-1} \\ \vdots & \ddots & \vdots \\ a_{n-1,2} & \cdots & a_{n-1,n-1} \end{vmatrix}. \tag{28}$$

With these definitions, we affirm that, if $C \neq 0$, then $B = AC$. To prove this we start noticing that the elements of B are related to the cofactors of A by the expressions

$$b_{11} = a^{nn}, \qquad b_{12} = (-1)^{n+1} a^{n1}, \qquad b_{21} = (-1)^{n+1} a^{1n}, \qquad b_{22} = a^{11}. \tag{29}$$

Hence, we can write

$$B = \begin{vmatrix} a^{nn} & a^{n1} \\ a^{1n} & a^{11} \end{vmatrix} = \begin{vmatrix} a^{11} & a^{n1} \\ a^{1n} & a^{nn} \end{vmatrix}, \tag{30}$$

since the factor $(-1)^{n+1}$ does not matter in the calculation. Now, consider the determinant

$$D = \begin{vmatrix} a^{11} & 0 & \dots & 0 & a^{n1} \\ 0 & 1 & \vdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \vdots & 1 & 0 \\ a^{1n} & 0 & \dots & 0 & a^{nn} \end{vmatrix}, \tag{31}$$

which have the same value of B – as can be easily proof directly from the definition (21). The determinants D an A have the same order, hence we can perform the product $E = AD$. We will get,

$$E = \begin{vmatrix} (a_{11}a^{11} + a_{1n}a^{1n}) & a_{12} & \dots & a_{1,n-1} & (a_{11}a^{n1} + a_{1n}a^{nn}) \\ (a_{21}a^{11} + a_{2n}a^{1n}) & a_{22} & \dots & a_{2,n-1} & (a_{21}a^{n1} + a_{2n}a^{nn}) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ (a_{n-1,1}a^{11} + a_{n-1,n}a^{1n}) & a_{n-1,2} & \dots & a_{n-1,n-1} & (a_{n-1,1}a^{n1} + a_{n-1,n}a^{nn}) \\ (a_{n1}a^{11} + a_{nn}a^{1n}) & a_{n2} & \dots & a_{n,n-1} & (a_{n1}a^{n1} + a_{nn}a^{nn}) \end{vmatrix}. \tag{32}$$

Now, multiply the columns $2, 3, \dots, n-1$ of E respectively by $a^{12}, a^{13}, \dots, a^{1,n-1}$ and add the result to its first column. In the same way, multiply the columns $2, 3, \dots, n-1$ of E by $a^{n2}, a^{n3}, \dots, a^{n,n-1}$ and add the result to its last column. After doing this, we shall obtain a determinant F whose element f_{11} and f_{nn} are equal to the value of the determinant A (from the Laplace theorem), the others elements of these two columns being zero (from the Cauchy theorem). Therefore, we will have,

$$F = A^2 \begin{vmatrix} 1 & a_{12} & \dots & a_{1,n-1} & 0 \\ 0 & a_{22} & \dots & a_{2,n-1} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & a_{n-1,2} & \dots & a_{n-1,n-1} & 0 \\ 0 & a_{n2} & \dots & a_{n,n-1} & 1 \end{vmatrix} = A^2 \begin{vmatrix} a_{22} & \dots & a_{2,n-1} \\ \vdots & \ddots & \vdots \\ a_{n-1,2} & \dots & a_{n-1,n-1} \end{vmatrix} = A^2 C, \tag{33}$$

where we have used the Laplace method in the last passage, expanding F by its first and last column. Therefore, we have $F = A^2C$, but since F was obtained from E only from elementary operations (we had add to the first and last columns of E a linear combination of the others columns), the determinant F equals the determinant E. Thus, we have $A^2C = F = E = AD = AB$, that is, $B = AC$, which proves the *Lemma*.

With the aid of this *Lemma*, the first form of the reduction method can be proved from mathematical induction. To do this, let A be a $(n+1)$-order determinant. Suppose that the method is valid for n-order determinants, so that each n-order minor of A, that are built up from n adjacent lines and columns of A, can be reduced to a single number after $n-1$ reductions. Also, the central minor of A, whose order is $n-2$ (and will be called C), can be reduced to a number after $n-3$ reductions. The reduction of that n-orders minors will give place to four numbers b_{ij}, so that, by a adequate ordination of them, we can form the 2-order determinant $B = |b_{ij}|$. So, if the reduction method holds for the $(n+1)$-order determinant A – that is what we want to proof –, then the determinant A must be equal to the determinant B divided by the central element of the previous reduced determinant, that is, the minor C. Therefore, we must have $A = B/C$ (since we suppose that $C \neq 0$). But that is precisely what we have proved on the *Lemma* above. Hence, since the reduction method is valid for third order determinants, the method will holds for fourth order determinants too, and hence for fifth orders as well, and so on. Therefore, the reduction method will be valid for a general n-order determinant. This completes the proof.

5.2 Demonstration for the Second Form of the Reduction Method

The proof for this alternative form of the reduction method is, in fact, much more easy than the proof for the precedent one. Consider again a n-order determinant A and let a_{rs} be a non-zero element of it. If we fix the element a_{rs} from A and reduce it as we have done in the section 4, the reduced determinant B will assume the following form:

$$B = \begin{vmatrix} (a_{11}a_{rs} - a_{1s}a_{r1}) & \dots & -(a_{1n}a_{rs} - a_{1s}a_{rn}) \\ \vdots & \ddots & \vdots \\ -(a_{n1}a_{rs} - a_{ns}a_{r1}) & \dots & (a_{nn}a_{rs} - a_{ns}a_{rn}) \end{vmatrix}. \tag{34}$$

As commented on the section 4, it is enough to prove that $A = B/(a_{rs})^{n-2}$. To do this, we may multiply the $n-1$ lines of A, with $i \neq r$, by a_{rs} to obtain the determinant

$$C = \begin{vmatrix} a_{11}a_{rs} & \dots & a_{1s}a_{rs} & \dots & a_{1n}a_{rs} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{r1} & \dots & a_{rs} & \dots & a_{rn} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1}a_{rs} & \dots & a_{ns}a_{rs} & \dots & a_{nn}a_{rs} \end{vmatrix}. \tag{35}$$

We have, therefore, $A = (a_{rs})^{n-1}C$. Now, we add to each element of C, with $i \neq r$, the quantity $-a_{is}a_{rj}$, from which we will find

$$D = \begin{vmatrix} a_{11}a_{rs} - a_{1s}a_{r1} & \dots & 0 & \dots & a_{1n}a_{rs} - a_{1s}a_{rn} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{r1} & \dots & a_{rs} & \dots & a_{rn} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1}a_{rs} - a_{ns}a_{r1} & \dots & 0 & \dots & a_{nn}a_{rs} - a_{ns}a_{rn} \end{vmatrix}, \tag{36}$$

and since we have applied only elementary operations on D, we have $D = C$.

After, multiply that lines of D for which $i > r$, and also that columns for which $j > s$, by -1. We will find so the determinant

$$E = \begin{vmatrix} (a_{11}a_{rs} - a_{1s}a_{r1}) & \dots & 0 & \dots & -(a_{1n}a_{rs} - a_{1s}a_{rn}) \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{r1} & \dots & a_{rs} & \dots & -a_{rn} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ -(a_{n1}a_{rs} - a_{ns}a_{r1}) & \dots & 0 & \dots & (a_{nn}a_{rs} - a_{ns}a_{rn}) \end{vmatrix}, \tag{37}$$

and since we have multiplied $n-r$ lines, and $n-s$ columns, of D by -1, we get

$$E = (-1)^{n-r}(-1)^{n-s}D = (-1)^{r+s}D. \tag{38}$$

Now, if we apply the Laplace method for the column $j = s$ of the determinant E, we will obtain

$$E = (-1)^{r+s}a_{rs} \begin{vmatrix} (a_{11}a_{rs} - a_{1s}a_{r1}) & \dots & -(a_{1n}a_{rs} - a_{1s}a_{rn}) \\ \vdots & \ddots & \vdots \\ -(a_{n1}a_{rs} - a_{ns}a_{r1}) & \dots & (a_{nn}a_{rs} - a_{ns}a_{rn}) \end{vmatrix} = (-1)^{r+s}a_{rs}B, \tag{39}$$

but since $D = C = (a_{rs})^{n-1}A$, follows from (38) and (39) that $a_{rs}B = (a_{rs})^{n-1}A$, that is,

$$B = (a_{rs})^{n-2}A, \tag{40}$$

which proves our proposition. Finally, since the determinant A can be evaluated by a recursive use of this formula, this complete our proof.

Addendum

After the composition of this text I was informed that similar methods to this presented here was already developed by two mathematics of the XIX century. Due to this, I wish to comment something about.

The first mathematician commented above is Mr. Charles L. Dodgson[3], who developed essentially the same form of the reduction method that I had shown in the sections 2 and 3, and which he called it "*condensation method*". In his approach, nevertheless, it was not presented a general demonstration, but only a demonstration for third and fourth orders determinants. Moreover, the problem of the division by zero was only attacked by the procedure of the permutation of lines and columns of the original determinant, which is not enough in the general, as we have seen. Nevertheless, it should also be noted that Mr. Dodgson had used his method to develop a very interesting way of solving linear systems of equations. (see ref. [1]).

The other mathematician is Felice Chiò[4], who developed essentially the same form of the reductions method that I have shown in the section 4. Sometimes his method is cited as "*pivotal method*". Unfortunately, I did not have access to his original paper, so I have not comment about this.

It is of noticing, however, that the work of these two mathematicians seems to be independent one to another and, actually, they were regarded until now as independent of each other, while I showed here that these two approaches have the same origin and can, therefore, be considered as two different forms of a same technique.

With despite of my own researches, I which to inform that I begin my studies on the field about ten years ago, when I was just a student of secondary school. At this time I could develop only the reduction method for third order determinants, but, unfortunately, these questions were to me only a matter of curiosity, and I was discouraged from them soon after. I only returned to subject when I arrived at my graduating college, about five years ago, when I eventually browsed my old annotations. At this time I generalized the reduction method for higher order determinants as well as I found the other formulation of this method which was presented at the section 4. The mathematical demonstrations took a little more of time and an appropriated presentation of the subject was possible only recently. All work was made in a complete independent way and I did have no knowledge of the works of Mr. Dodgson and Mr. Chiò during this time.

Finally, I wish to point out that this technique of solving determinants seems to be an almost unknown technique. In fact, it is not even commented on the majority of books, specialized or not in the theory of determinant. It is unknown from the majority of the teachers as well. Thus I think that the present work may provide such a popularization of the reduction method, as well as be helpful to the teaching of the theory to students and also to the development of the theory of determinants itself.

References

[1] C. L. Dodgson,"*Condensation of Determinants, being a new and brief method for computing their arithmetical values*", Proceedings of the Royal Society of London, 1866, XV, pg. 150.

[2] F. Chiò, "*Mémoire sur les Fonctions Connues sous le nom des Résultants ou des Déterminants*", Turin, 1853.

[3]C. L. Dodgson is most known by the pseudonymous Lewis Carroll – the famous writer of "*Alice's Adventures in Wonderland*", among others histories.

[4]F. Chiò was a Italian mathematician. He was also disciple of Amadeo Avogadro.