On the torsion homology of non-arithmetic hyperbolic tetrahedral groups

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Abstract

Numerical data concerning the growth of torsion in the first homology of non-arithmetic hyperbolic tetrahedral groups are collected. The data provide support the speculations of Bergeron and Venkatesh on the growth of torsion homology and the regulators for lattices in $SL_2(\mathbb{C})$.

1. Introduction

In this note, I report on my computations related to the torsion in the first homology of certain non-arithmetic Kleinian groups. The motivation for these computations is the recent paper of Bergeron and Venkatesh [1] on the size of the torsion in the homology of arithmetic uniform lattices in semisimple Lie groups. For $SL_2(\mathbb{C})$, they have the following result.

Theorem. *(Bergeron-Venkatesh) Let $\{\Gamma_n\}$ be a decreasing tower of cocompact arithmetic congruence subgroups of $SL_2(\mathbb{C})$ such that $\bigcap_n \Gamma_n = \{1\}$. Put $X_n = \Gamma_n \backslash \mathbb{H}$ where $\mathbb{H}$ is the hyperbolic 3-space. Then*

$$\lim_{n\to\infty} \frac{\log |H_1(\Gamma_n, E_{k,\ell})_{tor}|}{vol(X_n)} = \frac{1}{6\pi} \cdot c_{k,\ell}, \quad k \neq \ell$$

where $E_{k,\ell}$ is standard module $Sym^k(\mathbb{Z}^2) \otimes \overline{Sym^\ell(\mathbb{Z}^2)}$. Here $c_{k,\ell}$ is a positive integer depending only on the parameters k, ℓ, in particular $c_{0,0} = 1$.

The goal of this note is to explore the asymptotic behavior of torsion for non-arithmetic lattices. More precisely, I computationally investigated the limit

$$\lim_{n\to\infty} \frac{\log |H_1(\Gamma_n, \mathbb{Z})_{tor}|}{vol(X_n)} \tag{1}$$

for families of groups $\{\Gamma_n\}$ coming from projective covers of non-arithmetic hyperbolic tetrahedral groups (defined below). Note that $\mathbb{Z} \simeq E_{0,0}(\mathbb{Z})$ in the notation of the theorem. The data I collected show that if one considers only the projective covers with vanishing first Betti number (that is, $\dim H_1(\Gamma_n, \mathbb{Q}) = 0$), then the limit (1) tends to $1/(6\pi)$. In the case of positive first Betti numbers, the ratios "log(torsion)/volume" get much smaller than $1/(6\pi)$. These observations support the general philosophy of Bergeron and Venkatesh as discussed in Sections 4. and 6. Moreover, the data show that it is extremely rare that residue degree one prime level projective covers of non-arithmetic hyperbolic 3-folds have positive first Betti number. This has been first observed by Calegari and Dunfield in [4].

Essen and I gratefully acknowledge the wonderful hospitality and the state of the art clusters of this institute. I thank Akshay Venkatesh for our encouraging correspondence on the contents of this note. Last but certainly not least, I am grateful to Nicolas Bergeron for hosting me in Paris and explaining to me, among many other things, his joint work with Akshay Venkatesh.

2. Hyperbolic Tetrahedral Groups

A hyperbolic tetrahedral group is the index two subgroup consisting of orientation-preserving isometries in the discrete group generated by reflections in the faces of a hyperbolic tetrahedron whose dihedral angles are submultiples of π. Lannér [11] proved in 1950 that there are thirty two such hyperbolic tetrahedra.

Consider a tetrahedron in the hyperbolic 3-space $\mathbb{H}$ with vertices A, B, C, D with its dihedral angles submultiples of π. If the dihedral angles along the edges AB, AC, BC, DC, DB, DA are $\pi/\lambda_1, \pi/\lambda_2, \pi/\lambda_3$, $\pi/\mu_1, \pi/\mu_2, \pi/\mu_3$ respectively, then we denote the tetrahedron with $T(\lambda_1, \lambda_2, \lambda_3, \mu_1, \mu_2, \mu_3)$. A presentation for the tetrahedral group Γ associated to this tetrahedron is

$$\Gamma = \langle a, b, c, | a^{\mu_1} = b^{\mu_2} = c^{\lambda_3} = (ca)^{\lambda_2} = (cb^{-1})^{\lambda_1} = (ab)^{\mu_3} = 1 \rangle.$$

For my computations explicit realizations of these groups in $\mathrm{PSL}_2(\mathbb{C})$ are needed. Note that by Mostow's Rigidity Theorem, up to conjugation, such a realization will be unique. A general method to produce a realization is described to me by Grant Lakeland in [10].

It is well known that there are seven non-arithmetic tetrahedral groups and only of them is cocompact. Moreover, one of the non-cocompact ones is an index two subgroup of another. As nature of the experiment is insensitive to commensurability, I consider only the larger of these two tetrahedral groups. Thus I work with only six groups. Perhaps it should be noted that all the non-cocompact arithmetic tetrahedral groups are commensurable with the Bianchi groups $\mathrm{PSL}_2(\mathbb{Z}[\omega])$ with ω a 4th or 6th root of unity. These two Bianchi groups are covered with the computations in [13].

I will now go over each of the six groups, starting with the cocompact one. The volumes of the associated tetrahedra can be found in section 10.4 of the book [5] or the article [8]. Note that the volume of the 3-fold given by the tetrahedral group is twice the volume of the associated tetrahedron.

2.1. Descriptions of the Groups

2.1.1. $H(1)$

Let $H(1)$ be the tetrahedral group attached to the Coxeter symbol
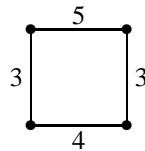
The tetrahedron associated to the Coxeter symbol can be described as $T(5, 3, 2; 4, 3, 2)$. The volume of the tetrahedron is $\simeq 0.3586534401$. A presentation can be given as (see [6])

$$H(1) = \langle a, b, c \mid a^3 = b^2 = c^5 = (bc^{-1})^3 = (ac^{-1})^2 = (ab)^4 = 1 \rangle.$$

where

$$a := \begin{pmatrix} \frac{2t^3+t^2+t+2}{5} & 1 \\ \frac{-t^3+t^2-2}{5} & \frac{-2t^3-t^2-t+3}{5} \end{pmatrix}, \quad b := \begin{pmatrix} \frac{-3t^3+t^2-4t+2}{5} & \beta \\ \alpha & \frac{3t^3-t^2+4t-2}{5} \end{pmatrix}, \quad c := \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}$$

where t is a primitive 10-th root of unity and α is one of the two complex roots of the polynomial
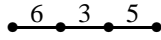
$$x^4 + \frac{-6t^3 + 6t^2 + 8}{5}x^3 + \frac{-t^3 + t^2 - 3}{5}x^2 + \frac{-4t^3 + 4t^2 + 2}{25}x + \frac{3t^3 - 3t^2 + 2}{25}.$$

Moreover,

$$\beta = (-20t^3 + 20t^2 + 35)\alpha^3 + (-50t^3 + 50t^2 + 80)\alpha^2 + (9t^3 - 9t^2 - 17)\alpha - 4t^3 + 4t^2 + 6.$$

2.1.2. $H(2)$

Let $H(2)$ be the tetrahedral group attached to the Coxeter symbol

$$\bullet \overset{6}{\rule{1cm}{0.4pt}} \bullet \overset{3}{\rule{1cm}{0.4pt}} \bullet \overset{5}{\rule{1cm}{0.4pt}} \bullet$$

The tetrahedron associated to the Coxeter symbol can be described as $T(5, 2, 2 \,; 6, 2, 3)$. It has one ideal vertex. The volume of the tetrahedron is $\simeq 0.1715016613$. A presentation is:
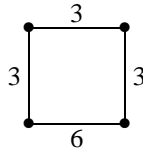
$$H(2) = \langle a, b, c, \,|\, a^6 = b^2 = c^2 = (ca)^2 = (cb^{-1})^5 = (ab)^3 = 1 \rangle,$$

where

$$a := \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}, \quad b := \begin{pmatrix} i & -(\frac{1+\sqrt{5}}{2})i \\ 0 & -i \end{pmatrix}, \quad c := \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

2.1.3. $H(3)$

Let $H(3)$ be the tetrahedral group given by the Coxeter symbol

The tetrahedron associated to the Coxeter symbol can be described as $T(3, 3, 2 \,; 6, 3, 2)$. It has two ideal vertices. The volume of the tetrahedron is $\simeq 0.3641071004$. A presentation is

$$H(3) = \langle a, b, c, \,|\, a^6 = b^3 = c^2 = (ca)^3 = (cb^{-1})^3 = (ab)^2 = 1 \rangle,$$
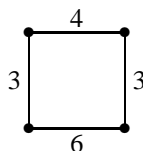
where

$$a := \begin{pmatrix} \zeta & i \\ 0 & \zeta^{-1} \end{pmatrix}, \quad b := \begin{pmatrix} \zeta^2 & -i \\ 0 & \zeta^{-2} \end{pmatrix}, \quad c := \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Here $\zeta = e^{i(\pi/6)}$.

2.1.4. $H(4)$

Let $H(4)$ be the tetrahedral group given by the Coxeter symbol

The tetrahedron associated to the Coxeter symbol can be described as $T(4, 3, 2 ; 6, 3, 2)$. It has two ideal vertices. The volume of the tetrahedron is $\simeq 0.5258402692$. A presentation is

$$H(4) = \langle a, b, c, | \, a^6 = b^3 = c^2 = (ca)^3 = (cb^{-1})^4 = (ab)^2 = 1 \rangle,$$
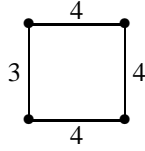
where

$$a := \begin{pmatrix} \zeta & i \\ 0 & \zeta^{-1} \end{pmatrix}, \quad b := \begin{pmatrix} \zeta^2 & -\sqrt{-2} \\ 0 & \zeta^{-2} \end{pmatrix}, \quad c := \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Here $\zeta = e^{i(\pi/6)}$.

2.1.5. $H(5)$

Let $H(5)$ be the tetrahedral group given by the Coxeter symbol

The tetrahedron associated to the Coxeter symbol can be described as $T(3, 4, 2 ; 4, 4, 2)$. It has two ideal vertices. The volume of the tetrahedron is $\simeq 0.5562821156$. A presentation is

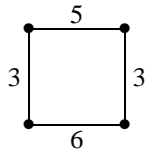$$H(5) = \langle a, b, c, | \, a^4 = b^4 = c^2 = (ca)^4 = (cb^{-1})^3 = (ab)^2 = 1 \rangle,$$

where

$$a := \begin{pmatrix} \zeta & \sqrt{-2} \\ 0 & \zeta^{-1} \end{pmatrix}, \quad b := \begin{pmatrix} \zeta & -i \\ 0 & \zeta^{-1} \end{pmatrix}, \quad c := \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Here $\zeta = e^{i(\pi/8)}$.

2.1.6. $H(6)$

Let $H(6)$ be the tetrahedral group given by the Coxeter symbol

The tetrahedron associated to the Coxeter symbol can be described as $T(5, 3, 2 ; 6, 3, 2)$. It has two ideal vertices. The volume of the tetrahedron is $\simeq 0.6729858045$. A presentation is

$$H(6) = \langle a, b, c, | \, a^6 = b^3 = c^2 = (ca)^3 = (cb^{-1})^5 = (ab)^2 = 1 \rangle,$$

where

$$a := \begin{pmatrix} \zeta & i \\ 0 & \zeta^{-1} \end{pmatrix}, \quad b := \begin{pmatrix} \zeta^2 & -(\frac{1+\sqrt{5}}{2})i \\ 0 & \zeta^{-2} \end{pmatrix}, \quad c := \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Here $\zeta = e^{i(\pi/6)}$.

3. Projective Covers

Let me describe the kind of covers that will be used for the experiment. Let Γ be a tetrahedral group as discussed above. As it has finite covolume, there is a number field K with ring of integers O and a finite set S of finite primes of O such that there is an embedding

$$\Gamma \hookrightarrow \mathrm{PSL}_2(O_S)$$

where O_S is the localization of O by the primes in S. Let $\mathfrak{p}$ be a prime ideal of O not in S and over the rational prime p. Let $\mathfrak{P}$ be the prime ideal of O_S corresponding to $\mathfrak{p}$. As the ideal is prime, the residue ring $\mathbb{F}$ is a finite field of characteristic p. We say that $\mathfrak{p}$ is of *residue degree a* if $\mathbb{F} \simeq \mathbb{F}_{p^a}$. Composing the above embedding with the reduction modulo $\mathfrak{P}$ map gives us a homomorphism

$$\phi : \ \Gamma \longrightarrow \mathrm{PSL}_2(\mathbb{F}).$$

By Strong Approximation Theorem (see Theorem 3.2. of [9]), this map is a surjection for almost all primes $\mathfrak{p}$. Let B denote the Borel subgroup of upper triangular elements in $\mathrm{PSL}_2(\mathbb{F})$. The elements in Γ which land in B under ϕ form a subgroup that we shall denote with $\Gamma_0(\mathfrak{p})$. Note that by rigidity, $\Gamma_0(\mathfrak{p})$ does not depend on ϕ.

It is well known that the coset representatives of $\Gamma_0(\mathfrak{p})$ in Γ can be identified with the projective line $\mathbb{P}^1(\mathbb{F})$. With this in mind, we will call $\Gamma_0(\mathfrak{p})$ the *projective cover of level $\mathfrak{p}$* of Γ.

4. More Background for the Experiment

In this section I will *loosely* discuss some aspects of the work of Bergeron and Venkatesh. Any error is due solely to me.

In their recent preprint [1], Bergeron and Venkatesh study the growth of torsion in the homology of cocompact arithmetic groups. Their strongest result is in the case where the ambient Lie group is $\mathrm{SL}_2(\mathbb{C})$. In this case one only needs to consider the first homology. Let us specialize our discussion to trivial coefficients (see [1] p.45). Let M be a hyperbolic 3-manifold of finite volume. The principal object of interest is the product

$$|H_1(M,\mathbb{Z})_{tor}| \cdot R(M)$$

where $R(M)$ is the regulator. When M is compact, the regulator $R(M)$ can be given explicitly as $\left| \left(\int_{\gamma_i} \omega_j \right)_{i,j} \right|^{-2}$ where $\omega_1, \ldots, \omega_n$ is an L^2-basis of harmonic 1-forms and $\gamma_1, \ldots, \gamma_n$ is a basis of $H_1(M,\mathbb{Z})$. In the non-compact case, harmonic 2-forms and $H_2(M,\mathbb{Z})$ also get involved in $R(M)$.

The general philosophy is that given a tower of finite volume hyperbolic 3-manifolds M_n with "nice" properties, we should have

$$\frac{\log R(M_n)}{vol(M_n)} + \frac{\log|H_1(M_n,\mathbb{Z})_{tor}|}{vol(M_n)} \longrightarrow -\tau^{(2)}(\mathbb{Z}),$$

where $\tau^{(2)}(\mathbb{Z})$ is a certain quantity which only depends on the ambient Lie group and the coefficient module, which is $\mathbb{Z}$ in our limited discussion. This quantity can be explicitly computed and in fact it is equal to $-1/(6\pi)$ (see [1], p.29). It is a key fact that the product of $\tau^{(2)}$ with $vol(M_n)$ is equal to the L^2-analytic torsion of M_n.

Hence to study the growth of the torsion, we need to understand that of the regulator. When the first Betti number is zero, the regulator vanishes and according to the general philosophy, the ratio

Group	A	B	# of prime levels		Group	A	B	# of prime levels
$H(1)$					$H(4)$	66000	257	205
$H(2)$	150000	400	280		$H(5)$	63000	257	131
$H(3)$	94200	307	147		$H(6)$	55000	233	213

Table 1: description of the ranges of the second computations

"log(torsion) /volume" should converge to $1/(6\pi)$. When the first Betti number is positive, the regulator comes into the play and the problem of controlling the conductor is a hard one which has connections with the ABC conjecture (see Goldfeld [7], p.13). Nevertheless, extensive numerical data [13] show that in the case of arithmetic Kleinian groups, the ratio "log(torsion)/volume" converges to $1/(6\pi)$ regardless the first Betti numbers are zero or not. Bergeron and Venkatesh suspect that this has to do with the existence of Hecke operators acting on the homology (see the discussion on p.45 of [1]). In fact, they conjecture that the ratio "log(regulator)/volume" goes to zero in the arithmetic case.

5.   The Experiment

The method of my computations is essentially the same with the one I employed for the congruence subgroups of the Euclidean Bianchi groups in [13]. Namely, I compute the abelianization of the group instead of its first homology with trivial $\mathbb{Z}$-coefficients.

Let us fix one of our non-arithmetic tetrahedral groups above, call it Γ. Let K be the number field generated by the entries of the matrix realizations of the generators of Γ given in Section 2.1. and let O be the ring of integers of K. Let $\mathfrak{p}$ be a prime ideal of O which does not divide the ideal generated by the denominators of the entries of the matrices. Then we can reduce the matrices modulo $\mathfrak{p}$ as the reductions of the denominators are invertible in the residue field $\mathbb{F} := O/\mathfrak{p}$. The group $\Gamma_0(\mathfrak{p})$, defined in Section 3., has finite index in Γ and a set of coset representatives can be identified with the projective line $\mathbb{P}^1(\mathbb{F})$. The generators of Γ act as permutations on the set of coset representatives. The corresponding action on the projective line is the usual action of the matrices, after reducing the entries modulo $\mathfrak{p}$. Once the presentation of Γ is given to a symbolic algebra program (I used Magma [2]), next one computes the action of the generator matrices on the projective line. The resulting permutations uniquely determine the subgroup $\Gamma_0(\mathfrak{p})$. Now computing the abelian quotient invariants is a standard functionality.

I have performed two main sets of computations for each of the six tetrahedral groups above. In the first set, I computed the ratios "log(torsion)/volume" for projective covers with prime level $\mathfrak{p}$ over a rational prime $400 < p$ with norm $N(\mathfrak{p}) \leq 50000$. In this way, one captures only prime levels of residue degree one. In a second set, I aimed at prime levels with residue degree at least two. Table 1 gives the ranges A, B such that $N(\mathfrak{p}) \leq A$ and $p \leq B$ for this second set of computations for each group. In addition, I performed an extra set of computations for the groups $H(1)$ and $H(2)$ where I considered one prime level over each rational prime p which splits completely in the coefficient field attached to the given matrix realizations of these groups with $p \leq 90000$ and $p \leq 100000$ respectively.

6.   Results of the Experiment and Observations

The data collected by the experiments are provided on my website [14]. Table 2 gives sample data for projective covers of prime level $\mathfrak{p}$ of the tetrahedral group $H(2)$. Here p is the rational prime that $\mathfrak{p}$ is

p	$f(\mathfrak{p})$	$N(\mathfrak{p})$	rank	log(torsion)/volume
149	2	22201	35	0.0022345997342923510246048917312O
157	2	24649	26	0.0012780079774939007014661120604I
163	2	26569	27	0.00119388737659470181463861107528
179	2	32041	42	0.00284289451885127707642991787490
179	2	32041	42	0.00271675896640833678080894982568
191	2	36481	44	0.00287523607600174888464895189987
191	2	36481	44	0.00298602044559159299824708408103
193	2	37249	32	0.00093228187971271537570822454679
223	2	49729	37	0.0006339902340090512255204262310O
239	2	57121	56	0.00204858493468061135230951713718
239	2	57121	56	0.00197783049030932370565346415877
251	2	63001	58	0.00213916088532177764793377551638
251	2	63001	58	0.00207500997945566716013518089664
269	2	72361	63	0.00106809775602345757933959528322
277	2	76729	46	0.00042737846130476991778286827508
281	2	78961	65	0.00115206386564044163781991095396
283	2	80089	47	0.00046147256599894360457102615905
307	2	94249	51	0.00039717749821458448534705458744
311	2	96721	72	0.00161519518746068399522925506475
311	2	96721	72	0.00165698128960266506419274897754
313	2	97969	52	0.00036262142827441423794453634641
337	2	113569	56	0.00031660479790195204470659803953
359	2	128881	84	0.00127125170976407173402131001026
359	2	128881	84	0.00130261090167121697503469479815
367	2	134689	61	0.00025564809254642626286356822328
373	2	139129	62	0.00026269400067799622367417655168
383	2	146689	0	0.0531817473569159738385919121240O

Table 2: sample data on growth of torsion for projective covers of H(2)

over, $f(\mathfrak{p})$ is the residue degree of $\mathfrak{p}$, $N(\mathfrak{p})$ is the norm of $\mathfrak{p}$ (which is equal to $p^{f(\mathfrak{p})}$) and rank denotes the first Betti number of the associated 3-fold M (that is, the dimension of $H_1(M, \mathbb{Q})$).

It is interesting that %99.9 of the time the homology of a projective cover of residue degree one prime level has first Betti number equal to zero. For each of the six tetrahedral groups, I have computed approximately 5000 projective covers whose levels are prime with residue degree one. For each group, there were no more than 12 such covers with positive first Betti number, all of them with norm less than 200. Note that a similar paucity was observed by Calegari ad Dunfield [4] among the projective covers of twist knot orbifolds.

An immediate observation is that if we let Γ run only through the projective covers of G with first Betti number equal to *zero*, then the data strongly suggest that the limit (1) tends to $1/(6\pi) \simeq 0.053051647697298$. Such a growth was observed in [13] for the Euclidean Bianchi groups also, *but* regardless the first Betti number was zero or not. The convergence here seems to be faster than it was in [13].

We see that the behaviour is different in the cases of positive first Betti number. Here the ratios "log(torsion)/volume" get much smaller than $1/(6\pi)$. This suggests, under the general philosophy of Bergeron and Venkatesh, that the contribution of the regulator comes into play and decreases the contribution of the torsion. Remember that we are in the nonarithmetic case; there is no suitable family of operators acting on the homology. This fact and the numerically observed contribution of the regulator are in favor of the validity of the speculations of Bergeron and Venkatesh on the general behaviour of the regulator.

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