A GEOMETRIC CONSTRUCTION FOR INVARIANT JET DIFFERENTIALS

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1. Introduction

The action of the reparametrization group $\mathbb{G}_k$, consisting of k-jets of germs of biholomorphisms of $(\mathbb{C}, 0)$, on the bundle $J_k = J_k T^*X$ of k-jets at 0 of germs of holomorphic curves $f : \mathbb{C} \to X$ in a complex manifold X has been a focus of investigation since the work of Demailly [5] which built on that of Green and Griffiths [13]. Here $\mathbb{G}_k$ is a non-reductive complex algebraic group which is the semi-direct product $\mathbb{G}_k = \mathbb{U}_k \rtimes \mathbb{C}^*$ of its unipotent radical $\mathbb{U}_k$ with $\mathbb{C}^*$; it has the form

$$\mathbb{G}_k \cong \left\{ \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_k \\ 0 & \alpha_1^2 & \cdots & & \\ 0 & 0 & \alpha_1^3 & \cdots & \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & \alpha_1^k \end{pmatrix} : \alpha_1 \in \mathbb{C}^*, \alpha_2, \ldots, \alpha_k \in \mathbb{C} \right\}$$

where the entries above the leading diagonal are polynomials in $\alpha_1, \ldots, \alpha_k$, and $\mathbb{U}_k$ is the subgroup consisting of matrices of this form with $\alpha_1 = 1$. The bundle of Demailly-Semple jet differentials of order k over X has fibre at $x \in X$ given by the algebra $\mathcal{O}((J_k)_x)^{\mathbb{U}_k}$ of $\mathbb{U}_k$-invariant polynomial functions on the fibre $(J_k)_x = (J_k T^*X)_x$ of $J_k T^*X$. This bundle of algebras

$$\mathcal{O}(J_k)^{\mathbb{U}_k} = \bigoplus_{m \geq 0} E_{k,m}$$

is graded by the induced action of $\mathbb{C}^*$ which has weight m on $E_{k,m}$. For any positive integer ℓ we can consider the bundle of subalgebras

$$\mathcal{O}(J_k)_\ell^{\mathbb{U}_k} = \bigoplus_{m \geq 0} E_{k,m\ell}$$

spanned by the $\mathbb{U}_k$-invariant polynomial functions with weight divisible by ℓ; equivalently $\mathcal{O}(J_k)_\ell^{\mathbb{U}_k} = \mathcal{O}(J_k)^{\mathbb{U}_k \rtimes \mu_\ell}$ is given by the polynomial functions which are invariant under the semi-direct product $\mathbb{U}_k \rtimes \mu_\ell$ of $\mathbb{U}_k$ with the finite group μ_ℓ of ℓth roots of 1 in $\mathbb{C}$. We have a natural identification

$$\mathcal{O}(J_k)_\ell^{\mathbb{U}_k} \cong \mathcal{O}(J_k \times \mathbb{C})^{\mathbb{G}_k}$$

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where $\mathbb{U}_k$ acts trivially on $\mathbb{C}$ and $\mathbb{G}_k/\mathbb{U}_k \cong \mathbb{C}^*$ acts as multiplication by the character $t \mapsto t^\ell$. In particular when $\ell = 1 + \cdots + k = k(k+1)/2$ this action of $\mathbb{G}_k$ on $\mathbb{C}$ extends to the action of $GL(k)$ given by multiplication by the determinant.

More generally following [32] we can replace $\mathbb{C}$ with $\mathbb{C}^p$ for $p \geq 1$ and consider the bundle $J_{k,p}T^*X$ of k-jets at 0 of holomorphic maps $f : \mathbb{C}^p \to X$ and the reparametrization group $\mathbb{G}_{k,p}$ consisting of k-jets of germs of biholomorphisms of $(\mathbb{C}^p, 0)$; then $\mathbb{G}_{k,p}$ is the semi-direct product of its unipotent radical $\mathbb{U}_{k,p}$ and the complex reductive group $GL(p)$, while its subgroup $\mathbb{G}'_{k,p} = \mathbb{U}_{k,p} \rtimes SL(p)$ (which equals $\mathbb{U}_{k,p}$ when $p = 1$) fits into an exact sequence $1 \to \mathbb{G}'_{k,p} \to \mathbb{G}_{k,p} \to \mathbb{C}^* \to 1$. The generalized Demailly-Semple algebra is then $\mathcal{O}((J_{k,p})_x)^{\mathbb{G}'_{k,p}}$.

The Demailly-Semple algebras $\mathcal{O}(J_k)^{\mathbb{U}_k}$ and their generalizations have been studied for a long time. The invariant jet differentials play a crucial role in the strategy devised by Green, Griffiths [13], Bloch [4], Demailly [5], Siu [29, 30, 31] and others to prove Kobayashi's 1970 hyperbolicity conjecture [23] and the related conjecture of Green and Griffiths in the special case of hypersurfaces in projective space. This strategy has been recently used successfully by Diverio, Merker and Rousseau in [7] and then by the first author in [3] to give effective lower bounds for the degrees of generic hypersurfaces in $\mathbb{P}_n$ for which the Green-Griffiths conjecture holds.

In particular it has been a long-standing problem to determine whether the algebras of invariants $\mathcal{O}((J_{k,p})_x)^{\mathbb{G}'_{k,p}}$ and bi-invariants $\mathcal{O}((J_{k,p})_x)^{\mathbb{G}'_{k,p} \times U_{n,x}}$ (where $U_{n,x}$ is a maximal unipotent subgroup of $GL(T_xX) \cong GL(n)$) are finitely generated as graded complex algebras, and if so to provide explicit finite generating sets. In [24] Merker showed that when $p = 1$ and both k and $n = \dim X$ are small then these algebras are finitely generated, and for $p = 1$ and all k and n he provided an algorithm which produces finite sets of generators when they exist. In this paper we will use methods inspired by [3] and the approach of [9] to non-reductive geometric invariant theory to prove the finite generation of the subalgebra $\mathcal{O}(J_k)^{\mathbb{U}_k}_{k(k+1)/2}$ of $\mathcal{O}(J_k)^{\mathbb{U}_k}$ spanned by the $\mathbb{U}_k$-invariant polynomial functions with weight divisible by $k(k+1)/2$ for all n and $k \geq 4$ (from which the finite generation of the corresponding bi-invariants follows). We will use these methods to obtain a similar result for $p > 1$, and for $p \geq 1$ to study the geometric invariant theoretic quotients

$$((J_{k,p})_x \times \mathbb{C})//\mathbb{G}_{k,p} = \mathrm{Spec}(\mathcal{O}((J_{k,p})_x \times \mathbb{C})^{\mathbb{G}_{k,p}})$$

and give geometric descriptions for the invariants and bi-invariants. In particular when $k \geq 4$ we find an explicit finite set of generators for the subalgebra $\mathcal{O}(J_k)^{\mathbb{U}_k}_{k(k+1)/2}$ of $\mathcal{O}(J_k)^{\mathbb{U}_k}$ (and a similar result for $p > 1$ and all k). In fact we will show that if $k \geq 4$ then $\mathbb{G}_k$ is a Grosshans subgroup of $GL(k)$, so that every linear action of $\mathbb{G}_k$ which extends to a linear action of $GL(k)$ has finitely generated invariants; similarly if $p > 1$ then $\mathbb{G}_{k,p}$ is a Grosshans subgroup of $GL(\mathrm{sym}^{\leq k}(p))$ where $\mathrm{sym}^{\leq k}(p) = \sum_{i=1}^k \dim \mathrm{Sym}^i \mathbb{C}^p$.

The layout of this paper is as follows. §2 reviews the reparametrization groups $\mathbb{G}_k$ and $\mathbb{G}_{k,p}$ and their actions on jet bundles and jet differentials over a complex manifold

X. Next §3 reviews the results of [9] and [1] on non-reductive geometric invariant theory. In §4 we recall from [3] a geometric description of the quotients by $\mathbb{U}_k$ and $\mathbb{G}_k$ of open subsets of $(J_k)_x$, and in §5 this is used to find explicit affine and projective embeddings of these quotients. In §6 it is proved that the complements of these quotients in their closures for suitable embeddings in affine and projective spaces have codimension at least two, from which it follows that the relevant invariants on $(J_k)_x$ extend to these closures. In §7 this is used to prove that $\mathbb{G}_k$ is a Grosshans subgroup of $GL(p)$, and thus that $O(J_k)^{\mathbb{U}_k}_{k(k+1)/2}$ is finitely generated, and to provide a geometric description of the the invariants and bi-invariants. Finally §8 and §9 extend the results of §6 and §7 to the action of $\mathbb{G}_{k,p}$ on the jet bundle $J_{k,p} \to X$ of k-jets of germs of holomorphic maps from $\mathbb{C}^p$ to X for $p > 1$.

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2. Jets of curves and jet differentials

Let X be a complex n-dimensional manifold and let k be a positive integer. Green and Griffiths in [13] introduced the bundle $J_k \to X$ of k-jets of germs of parametrized curves in X; its fibre over $x \in X$ is the set of equivalence classes of germs of holomorphic maps $f : (\mathbb{C}, 0) \to (X, x)$, with the equivalence relation $f \sim g$ if and only if the derivatives $f^{(j)}(0) = g^{(j)}(0)$ are equal for $0 \le j \le k$. If we choose local holomorphic coordinates $(z_1, \ldots, z_n)$ on an open neighbourhood $\Omega \subset X$ around x, the elements of the fibre $J_{k,x}$ are represented by the Taylor expansions

$$f(t) = x + tf'(0) + \frac{t^2}{2!}f''(0) + \ldots + \frac{t^k}{k!}f^{(k)}(0) + O(t^{k+1})$$

up to order k at $t = 0$ of $\mathbb{C}^n$-valued maps

$$f = (f_1, f_2, \ldots, f_n)$$

on open neighbourhoods of 0 in $\mathbb{C}$. Thus in these coordinates the fibre is

$$J_{k,x} = \left\{(f'(0), \ldots, f^{(k)}(0)/k!)\right\} = (\mathbb{C}^n)^k,$$

which we identify with $\mathbb{C}^{nk}$. Note, however, that J_k is not a vector bundle over X, since the transition functions are polynomial, but not linear.

Let $\mathbb{G}_k$ be the group of k-jets at the origin of local reparametrizations of $(\mathbb{C}, 0)$

$$t \mapsto \varphi(t) = \alpha_1 t + \alpha_2 t^2 + \ldots + \alpha_k t^k, \quad \alpha_1 \in \mathbb{C}^*, \alpha_2, \ldots, \alpha_k \in \mathbb{C},$$

in which the composition law is taken modulo terms t^j for $j > k$. This group acts fibrewise on J_k by substitution. A short computation shows that this is a linear action on the fibre:

$$f\circ\varphi(t) = f'(0)\cdot(\alpha_1 t+\alpha_2 t^2+\ldots+\alpha_k t^k)+\frac{f''(0)}{2!}\cdot(\alpha_1 t+\alpha_2 t^2+\ldots+\alpha_k t^k)^2+\ldots$$
$$\ldots+\frac{f^{(k)}(0)}{k!}\cdot(\alpha_1 t+\alpha_2 t^2+\ldots+\alpha_k t^k)^k \text{ (modulo } t^{k+1})$$

so the linear action of φ on the k-jet $(f'(0), f''(0)/2!, \ldots, f^{(k)}(0)/k!)$ is given by the following matrix multiplication:

$$(1)\quad (f'(0), f''(0)/2!, \ldots, f^{(k)}(0)/k!)\cdot\begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_k \\ 0 & \alpha_1^2 & 2\alpha_1\alpha_2 & \cdots & \alpha_1\alpha_{k-1}+\ldots+\alpha_{k-1}\alpha_1 \\ 0 & 0 & \alpha_1^3 & \cdots & 3\alpha_1^2\alpha_{k-2}+\ldots \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & \alpha_1^k \end{pmatrix}$$

where the matrix has general entry

$$(\mathbb{G}_k)_{i,j} = \sum_{s_1\geq 1,\ldots,s_i\geq 1,\ s_1+\ldots+s_i=j} \alpha_{s_1}\ldots\alpha_{s_i}$$

for $i, j \leq k$.

There is an exact sequence of groups:

$$(2)\qquad 1\to\mathbb{U}_k\to\mathbb{G}_k\to\mathbb{C}^*\to 1,$$

where $\mathbb{G}_k\to\mathbb{C}^*$ is the morphism $\varphi\to\varphi'(0)=\alpha_1$ in the notation used above, and

$$\mathbb{G}_k = \mathbb{U}_k \rtimes \mathbb{C}^*$$

is a semi-direct product. With the above identification, $\mathbb{C}^*$ is the subgroup of diagonal matrices satisfying $\alpha_2=\ldots=\alpha_k=0$ and $\mathbb{U}_k$ is the unipotent radical of $\mathbb{G}_k$, consisting of matrices of the form above with $\alpha_1 = 1$. The action of $\lambda\in\mathbb{C}^*$ on k-jets is thus described by

$$\lambda\cdot(f'(0), f''(0)/2!, \ldots, f^{(k)}(0)/k!) = (\lambda f'(0), \lambda^2 f''(0)/2!, \ldots, \lambda^k f^{(k)}(0)/k!)$$

Let $\mathcal{E}^n_{k,m}$ denote the vector space of complex valued polynomial functions $Q(u_1, u_2, \ldots, u_k)$ of $u_1 = (u_{1,1}, \ldots, u_{1,n}), \ldots, u_k = (u_{k,1}, \ldots, u_{k,n})$ of weighted degree m with respect to this $\mathbb{C}^*$ action, where $u_i = f^{(i)}(0)/i!$; that is, such that

$$Q(\lambda u_1, \lambda^2 u_2, \ldots, \lambda^k u_k) = \lambda^m Q(u_1, u_2, \ldots, u_k).$$

Thus elements of $\mathcal{E}^n_{k,m}$ have the form

$$Q(u_1, u_2, \ldots, u_k) = \sum_{|i_1|+2|i_2|+\ldots+k|i_k|=m} u_1^{i_1}u_2^{i_2}\ldots u_k^{i_k},$$

where $i_1 = (i_{1,1}, \ldots, i_{1,n}), \ldots, i_k = (i_{k,1}, \ldots, i_{k,n})$ are multi-indices of length n. There is an induced action of $\mathbb{G}_k$ on the algebra $\bigoplus_{m\geq 0}\mathcal{E}^n_{k,m}$. Following Demailly (see [5]), we denote by $E^n_{k,m}$ (or $E_{k,m}$) the Demailly-Semple bundle whose fibre at x consists of the

$\mathbb{U}_k$-invariant polynomials on the fibre of J_k at x of weighted degree m, i.e those which satisfy

$$Q((f\circ\varphi)'(0),(f\circ\varphi)''(0)/2!,\ldots,(f\circ\varphi)^{(k)}(0)/k!) = \varphi'(0)^m\cdot Q(f'(0),f''(0)/2!,\ldots,f^{(k)}(0)/k!),$$

and we let $E_k^n = \oplus_m E_{k,m}^n$ denote the Demailly-Semple bundle of graded algebras of invariants.

We can also consider higher dimensional holomorphic surfaces in X, and therefore, we fix a parameter $1\le p\le n$, and study germs of maps $\mathbb{C}^p\to X$.

Again, we fix the degree k of our map, and introduce the bundle $J_{k,p}\to X$ of k-jets of maps $\mathbb{C}^p\to X$. The fibre over $x\in X$ is the set of equivalence classes of germs of holomorphic maps $f:(\mathbb{C}^p,0)\to(X,x)$, with the equivalence relation $f\sim g$ if and only if all derivatives $f^{(j)}(0)=g^{(j)}(0)$ are equal for $0\le j\le k$.

We need a description of the fibre $J_{k,p,x}$ in terms of local coordinates as in the case when $p=1$. Let $(z_1,\ldots,z_n)$ be local holomorphic coordinates on an open neighbourhood $\Omega\subset X$ around x, and let $(u_1,\ldots,u_p)$ be local coordinates on $\mathbb{C}^p$. The elements of the fibre $J_{k,p,x}$ are $\mathbb{C}^n$-valued maps

$$f=(f_1,f_2,\ldots,f_n)$$

on $\mathbb{C}^p$, and two maps represent the same jet if their Taylor expansions around $\mathbf{z}=0$

$$f(\mathbf{z}) = x+\mathbf{z}f'(0)+\frac{\mathbf{z}^2}{2!}f''(0)+\ldots+\frac{\mathbf{z}^k}{k!}f^{(k)}(0)+O(\mathbf{z}^{k+1})$$

coincide up to order k. Note that here

$$f^{(i)}(0)\in \operatorname{Hom}(\operatorname{Sym}^i\mathbb{C}^p,\mathbb{C}^n)$$

and in these coordinates the fibre is

$$J_{k,p,x}=\left\{(f'(0),\ldots,f^{(k)}(0)/k!)\right\}=\mathbb{C}^{n\binom{k+p-1}{k-1}}$$

which is a finite-dimensional vector space.

Let $\mathbb{G}_{k,p}$ be the group of k-jets of germs of biholomorphisms of $(\mathbb{C}^p,0)$. Elements of $\mathbb{G}_{k,p}$ are represented by holomorphic maps

$$\text{(3)}\quad \mathbf{u}\to\varphi(\mathbf{u})=\Phi_1\mathbf{u}+\Phi_2\mathbf{u}^2+\ldots+\Phi_k\mathbf{u}^k=\sum_{\mathbf{i}\in\mathbb{Z}^p\setminus 0}a_{i_1\ldots i_p}u_1^{i_1}\ldots u_p^{i_p},\ \ \Phi_1 \text{ is non-degenerate}$$

where $\Phi_i\in\operatorname{Hom}(\operatorname{Sym}^i\mathbb{C}^p,\mathbb{C}^p)$. The group $\mathbb{G}_{k,p}$ admits a natural fibrewise right action on $J_{k,p}$, by reparametrizing the k-jets of holomorphic p-discs. A computation similar to that in [3] shows that

$$f\circ\varphi(\mathbf{u}) = f'(0)\Phi_1\mathbf{u}+(f'(0)\Phi_2+\frac{f''(0)}{2!}\Phi_1^2)\mathbf{u}^2+\ldots+\sum_{i_1+\ldots+i_l=d}\frac{f^{(l)}(0)}{l!}\Phi_{i_1}\ldots\Phi_{i_l}\mathbf{u}^l.$$

This defines a linear action of $\mathbb{G}_{k,p}$ on the fibres $J_{k,p,x}$ of $J_{k,p}$ with the matrix representation given by

$$
\begin{pmatrix}
\Phi_1 & \Phi_2 & \Phi_3 & \dots & \Phi_k \\
0 & \Phi_1^2 & \Phi_1\Phi_2 & \dots & \\
0 & 0 & \Phi_1^3 & \dots & \\
. & . & . & . & . \\
 & & & & \Phi_1^k
\end{pmatrix}, \tag{4}
$$

where

- $\Phi_i \in \operatorname{Hom}(\operatorname{Sym}^i\mathbb{C}^p, \mathbb{C}^p)$ is a $p\times\dim(\operatorname{Sym}^i\mathbb{C}^p)$-matrix, the ith degree component of the map Φ, which is represented by a map $(\mathbb{C}^p)^{\otimes i} \to \mathbb{C}^p$;
- $\Phi_{i_1}\dots\Phi_{i_l}$ is the matrix of the map $\operatorname{Sym}^{i_1+\dots+i_l}(\mathbb{C}^p) \to \operatorname{Sym}^l\mathbb{C}^p$, which is represented by

$$\sum_{\sigma\in\mathcal{S}_l} \Phi_{i_1}\otimes\cdots\otimes\Phi_{i_l} : (\mathbb{C}^p)^{\otimes i_1}\otimes\cdots\otimes(\mathbb{C}^p)^{\otimes i_l} \to (\mathbb{C}^p)^{\otimes l};$$

- the (l,m) block of $\mathbb{G}_{k,p}$ is $\sum_{i_1+\dots+i_l=m}\phi_{i_1}\dots\Phi_{i_l}$. The entries in these boxes are indexed by pairs (τ,μ) where $\tau \in \binom{p+l-1}{l-1}, \mu \in \binom{p+m-1}{m-1}$ correspond to bases of $\operatorname{Sym}^l(\mathbb{C}^p)$ and $\operatorname{Sym}^m(\mathbb{C}^p)$.

Example 2.1. *For $p=2, k=3$ we get the following 9×9 matrix for a general element of $\mathbb{G}_{3,2}$, using the standard basis*

$$\left\{e_i, e_ie_j, e_ie_je_k : 1 \le i \le j \le k \le 2\right\}$$

of $(J_{3,2})_x$.

$$
\begin{pmatrix}
\alpha_{10} & \alpha_{01} & \alpha_{20} & \alpha_{11} & \alpha_{02} & \alpha_{30} & \alpha_{21} & \alpha_{12} & \alpha_{03} \\
\beta_{10} & \beta_{01} & \beta_{20} & \beta_{11} & \beta_{02} & \beta_{30} & \beta_{21} & \beta_{12} & \beta_{03} \\
0 & 0 & \alpha_{10}^2 & \alpha_{10}\alpha_{01} & \alpha_{01}^2 & \alpha_{10}\alpha_{20} & \alpha_{10}\alpha_{11}+\alpha_{01}\alpha_{20} & \alpha_{10}\alpha_{02}+\alpha_{11}\alpha_{01} & \alpha_{01}\alpha_{02} \\
0 & 0 & \alpha_{10}\beta_{10} & \alpha_{10}\beta_{01}+\alpha_{01}\beta_{10} & \alpha_{01}\beta_{01} & \alpha_{10}\beta_{20}+\alpha_{20}\beta_{10} & P & Q & \alpha_{01}\beta_{02}+\alpha_{02}\beta_{01} \\
0 & 0 & \beta_{10}^2 & \beta_{10}\beta_{01} & \beta_{01}^2 & \beta_{10}\beta_{20} & \beta_{10}\beta_{11}+\beta_{20}\beta_{01} & \beta_{01}\beta_{11}+\beta_{02}\beta_{10} & \beta_{01}\beta_{02} \\
0 & 0 & 0 & 0 & 0 & \alpha_{10}^3 & \alpha_{10}^2\alpha_{01} & \alpha_{10}\alpha_{01}^2 & \alpha_{01}^3 \\
0 & 0 & 0 & 0 & 0 & \alpha_{10}^2\beta_{10} & \alpha_{10}\alpha_{10}\beta_{01} & \alpha_{10}\alpha_{01}\beta_{01} & \alpha_{01}\beta_{01}^2 \\
0 & 0 & 0 & 0 & 0 & \alpha_{10}\beta_{10}^2 & \alpha_{10}\beta_{10}\beta_{01} & \alpha_{10}\beta_{01}\beta_{01} & \alpha_{01}\beta_{01}^2 \\
0 & 0 & 0 & 0 & 0 & \beta_{10}^3 & \beta_{10}^2\beta_{01} & \beta_{10}\beta_{01}^2 & \beta_{01}^3
\end{pmatrix} \tag{5}
$$

where

$$P = \alpha_{10}\beta_{11}+\alpha_{11}\beta_{10}+\alpha_{20}\beta_{01}+\alpha_{01}\beta_{20} \textit{ and } Q = \alpha_{01}\beta_{11}+\alpha_{11}\beta_{01}+\alpha_{02}\beta_{10}+\alpha_{10}\beta_{02}.$$

This is a subgroup of the parabolic $P_{2,3,4}\subset GL(9)$. The diagonal blocks are the representations $\operatorname{Sym}^i\mathbb{C}^2$ for $i=1,2,3$, where $\mathbb{C}^2$ is the standard representation of $GL(2)$.

In general the linear group $\mathbb{G}_{k,p}$ is generated along its first p rows; that is, the parameters in the first p rows are independent, and all the remaining entries are polynomials in these parameters. The assumption on the parameters is that the determinant of the smallest diagonal $p \times p$ block is nonzero; for the $p = 2, k = 3$ example above this means that

$$\det \begin{pmatrix} \alpha_{10} & \alpha_{01} \\ \beta_{10} & \beta_{01} \end{pmatrix} \neq 0.$$

The parameters in the $(1, m)$ block are indexed by a basis of $\mathrm{Sym}^m(\mathbb{C}^p) \times \mathbb{C}^p$, so they are of the form α^l_ν where $\nu \in \binom{p+m-1}{m-1}$ is an m-tuple and $1 \leq l \leq p$. An easy computation shows that:

Proposition 2.2. *The polynomial in the* (l, m) *block and entry indexed by*

$$\tau = (\tau[1], \ldots, \tau[l]) \in \binom{p+l-1}{l-1}$$

and $\nu \in \binom{p+m-1}{m-1}$ *is*

$$(\mathbb{G}_{k,p})_{\tau,\nu} = \sum_{\nu_1+\ldots+\nu_l=\nu} \alpha^{\tau[1]}_{\nu_1} \alpha^{\tau[2]}_{\nu_2} \ldots \alpha^{\tau[l]}_{\nu_l} \tag{6}$$

Note that $\mathbb{G}_{k,p}$ is an extension of its unipotent radical by $GL(p)$; that is, we have an exact sequence

$$1 \to \mathbb{U}_{k,p} \to \mathbb{G}_{k,p} \to GL(p) \to 1,$$

and $\mathbb{G}_{k,p}$ is the semi-direct product $\mathbb{U}_{k,p} \rtimes GL(p)$. Here $\mathbb{G}_{k,p}$ has dimension $p \times \mathrm{sym}^{\leq k}(p)$ where $\mathrm{sym}^{\leq k}(p) = \dim(\oplus_{i=1}^k \mathrm{Sym}^i \mathbb{C}^p)$, and is a subgroup of the parabolic subgroup $P_{p,\mathrm{sym}^2(p),\ldots,\mathrm{sym}^k(p)}$ of $GL(\mathrm{sym}^{\leq k}(p))$ where $\mathrm{sym}^i(p) = \dim(\mathrm{Sym}^i \mathbb{C}^p)$. We define $\mathbb{G}'_{k,p}$ to be the subgroup of $\mathbb{G}_{k,p}$ which is the semi-direct product

$$\mathbb{G}'_{k,p} = \mathbb{U}_{k,p} \rtimes SL(p)$$

(so that $\mathbb{G}'_{k,p} = \mathbb{U}_{k,p}$ when $p = 1$) fitting into the exact sequence

$$1 \to \mathbb{U}_{k,p} \to \mathbb{G}'_{k,p} \to SL(p) \to 1.$$

The action of the maximal torus $(\mathbb{C}^*)^p \subset GL(p)$ of the Levi subgroup of $\mathbb{G}_{k,p}$ is

$$(\lambda_1, \ldots, \lambda_p) \cdot f^{(i)} = (\lambda_1^i \frac{\partial^i f}{\partial u_1^i}, \ldots, \lambda_1^{i_1} \cdots \lambda_p^{i_p} \frac{\partial^i f}{\partial u_1^{i_1} \cdots \partial u_p^{i_p}} \ldots \lambda_p^i \frac{\partial^i f}{\partial u_p^i}) \tag{7}$$

We introduce the *Green-Griffiths* vector bundle $E^{GG}_{k,p,m} \to X$, whose fibres are complex-valued polynomials $Q(f'(0), f''(0)/2!, \ldots, f^{(k)}(0)/k!)$ on the fibres of $J_{k,p}$, having weighted degree $(m, \ldots, m)$ with respect to the action (7) of $(\mathbb{C}^*)^p$. That is, for $Q \in E^{GG}_{k,p,m}$

$$Q(\lambda f'(0), \lambda f''(0)/2!, \ldots, \lambda f^{(k)}(0)/k!) = \lambda_1^m \cdots \lambda_p^m Q(f'(0), f''(0)/2!, \ldots, f^{(k)}(0)/k!)$$

for all $\lambda \in \mathbb{C}^p$ and $(f'(0), f''(0)/2!, \ldots, f^{(k)}(0)/k!) \in J_{k,p,m}$.

Definition 2.3. *The generalized Demailly-Semple bundle $E_{k,p,m} \to X$ over X has fibre consisting of the $\mathbb{G}'_{k,p}$-invariant jet differentials of order k and weighted degree $(m,\dots,m)$; that is the complex-valued polynomials $Q(f'(0), f''(0)/2!, \dots, f^{(k)}(0)/k!)$ on the fibres of $J_{k,p}$ which transform under any reparametrization $\phi \in \mathbb{G}_{k,p}$ of $(\mathbb{C}^p, 0)$ as*

$$Q(f \circ \phi) = (J_\phi)^m Q(f) \circ \phi,$$

where $J_\phi = \det \Phi_1$ denotes the Jacobian of ϕ at 0. The generalized Demailly-Semple bundle of algebras $E_{k,p} = \oplus_{m \ge 0} E_{k,p,m}$ is the associated graded algebra of $\mathbb{G}'_{k,p}$-invariants, whose fibre at $x \in X$ is the generalized Demailly-Semple algebra $\mathcal{O}((J_{k,p})_x)^{\mathbb{G}'_{k,p}}$.

The determination of a suitable generating set for the invariant jet differentials when $p = 1$ is important in the longstanding strategy to prove the Green-Griffiths conjecture. It has been suggested in a series of papers [13, 5, 34, 24, 7, 25] that the Schur decomposition of the Demailly-Semple algebra, together with good estimates of the higher Betti numbers of the Schur bundles and an asymptotic estimation of the Euler charactristic, should result in a positive lower bound for the global sections of the Demailly-Semple jet differential bundle.

3. Geometric invariant theory

Suppose now that Y is a complex quasi-projective variety on which a linear algebraic group G acts. For geometric invariant theory (GIT) we need a linearization of the action; that is, a line bundle L on Y and a lift $\mathcal{L}$ of the action of G to L. Usually L is ample, and hence (as it makes no difference for GIT if we replace L with $L^{\otimes k}$ for any integer $k > 0$) we can assume that for some projective embedding $Y \subseteq \mathbb{P}^n$ the action of G on Y extends to an action on $\mathbb{P}^n$ given by a representation $\rho : G \to GL(n+1)$, and take for L the hyperplane line bundle on $\mathbb{P}^n$.

For classical GIT developed by Mumford [27] we require the complex algebraic group G to be reductive. Let Y be a projective complex variety with an action of a complex reductive group G and linearization $\mathcal{L}$ with respect to an ample line bundle L on Y. Then $y \in Y$ is *semistable* for this linear action if there exists some $m > 0$ and $f \in H^0(Y, L^{\otimes m})^G$ not vanishing at y, and y is *stable* if also the action of G on the open subset

$$Y_f := \{x \in Y \mid f(x) \neq 0\}$$

is closed with all stabilizers finite. Y^{ss} has a projective categorical quotient $Y^{ss} \to Y/\!/G$, which restricts on the set of stable points to a geometric quotient $Y^s \to Y^s/G$ (see [27] Theorem 1.10). The morphism $Y^{ss} \to Y/\!/G$ is surjective, and identifies $x, y \in Y^{ss}$ if and only if the closures of the G-orbits of x and y meet in Y^{ss}. There is an induced action of G on the homogeneous coordinate ring

$$\hat{\mathcal{O}}_L(Y) = \bigoplus_{k \ge 0} H^0(Y, L^{\otimes k})$$

of Y. The subring $\hat{O}_L(Y)^G$ consisting of the elements of $\hat{O}_L(Y)$ left invariant by G is a finitely generated graded complex algebra because G is reductive, and the GIT quotient $Y//G$ is the projective variety $\mathrm{Proj}(\hat{O}_L(Y)^G)$ [27]. The subsets Y^{ss} and Y^s of Y are characterized by the following properties (see [27, Chapter 2] or [28]).

Proposition 3.1. *(Hilbert-Mumford criteria) (i) A point $x \in Y$ is semistable (respectively stable) for the action of G on Y if and only if for every $g \in G$ the point gx is semistable (respectively stable) for the action of a fixed maximal torus of G.*
(ii) A point $x \in Y$ with homogeneous coordinates $[x_0 : \ldots : x_n]$ in some coordinate system on $\mathbb{P}^n$ is semistable (respectively stable) for the action of a maximal torus of G acting diagonally on $\mathbb{P}^n$ with weights $\alpha_0, \ldots, \alpha_n$ if and only if the convex hull

$$\mathrm{Conv}\{\alpha_i : x_i \neq 0\}$$

contains 0 *(respectively contains* 0 *in its interior).*

Similarly if a complex reductive group G acts linearly on an affine variety Y then we have a GIT quotient

$$Y//G = \mathrm{Spec}(O(Y)^G)$$

which is the affine variety associated to the finitely generated algebra $O(Y)^G$ of G-invariant regular functions on Y. In this case $Y^{ss} = Y$ and the inclusion $O(Y)^G \hookrightarrow O(Y)$ induces a morphism of affine varieties $Y \to Y//G$.

Now suppose that H is any complex linear algebraic group, with unipotent radical $U \trianglelefteq H$ (so that $R = H/U$ is reductive and H is isomorphic to the semi-direct product $U \rtimes R$), acting linearly on a complex projective variety Y with respect to an ample line bundle L. Then $\mathrm{Proj}(\hat{O}_L(Y)^H)$ is not in general well-defined as a projective variety, since the ring of invariants

$$\hat{O}_L(Y)^H = \bigoplus_{k \geq 0} H^0(Y, L^{\otimes k})^H$$

is not necessarily finitely generated as a graded complex algebra. However in some cases it is known that $\hat{O}_L(Y)^U$ is finitely generated, which implies that

$$\hat{O}_L(Y)^H = \left(\bigoplus_{k \geq 0} H^0(Y, L^{\otimes k})^U \right)^{H/U}$$

is finitely generated and hence the *enveloping quotient* in the sense of [9]is given by the associated projective variety

$$Y//H = \mathrm{Proj}(\hat{O}_L(Y)^H).$$

Similarly if Y is affine and H acts linearly on Y with $O(Y)^H$ finitely generated, then we have the enveloping quotient

$$Y//H = \mathrm{Spec}(O(Y)^H).$$

There is a morphism
$$q : Y^{ss} \to Y//H,$$
from an open subset Y^{ss} of Y (where $Y^{ss} = Y$ when Y is affine), which restricts to a geometric quotient
$$q : Y^s \to Y^s/H$$
for an open subset $Y^s \subset Y^{ss}$. However in contrast with the reductive case, the morphism $q : Y^{ss} \to Y//H$ is not in general surjective; indeed the image of q is not in general a subvariety of $Y//H$, but is only a constructible subset.

Suppose that U is a unipotent group with a one-parameter group of automorphisms $\lambda : \mathbb{C}^* \to \mathrm{Aut}(U)$ such that the weights of the induced $\mathbb{C}^*$ action on the Lie algebra $\mathfrak{u}$ of U are all nonzero. Then we can form the semi-direct product
$$\hat{U} = \mathbb{C}^* \ltimes U$$
given by $\mathbb{C}^* \times U$ with group multiplication
$$(z_1, u_1).(z_2, u_2) = (z_1 z_2, (\lambda(z_2^{-1})(u_1))u_2).$$
Linear actions of such unipotent groups U which extend to the semi-direct product $\hat{U}$ are studied in [1], motivated by the actions of the groups $\mathbb{G}_k = \mathbb{U}_k \rtimes \mathbb{C}^*$ and $\mathbb{G}_{k,p} = \mathbb{U}_{k,p} \rtimes GL(p)$ on the fibres of the jet bundles J_k and $J_{k,p}$. In this paper we will use a different approach from that of [1] to study the Demailly-Semple algebras of invariant jet differentials E_k^n and $E_{k,p}^n$ and prove

Theorem 3.2. *The fibres* $\mathcal{O}((J_k)_x)_\ell^{\mathbb{U}_k} \cong \mathcal{O}((J_k)_x \times \mathbb{C})^{\mathbb{G}_k}$ *(when* $k \geq 4$*) and* $\mathcal{O}((J_{k,p})_x)_\ell^{\mathbb{G}'_{k,p}} \cong \mathcal{O}((J_{k,p})_x \times \mathbb{C})^{\mathbb{G}_{k,p}}$ *of the bundles* E_k^n *and* $E_{k,p}^n$ *are finitely generated graded complex algebras when* ℓ *is divisible by*
$$\sum_{i=1}^{k} i(\dim \mathrm{Sym}^i \mathbb{C}^p).$$

Thus we have non-reductive GIT quotients
$$((J_k)_x \times \mathbb{C})//\mathbb{G}_k = \mathrm{Spec}(\mathcal{O}((J_k)_x)_\ell^{\mathbb{U}_k})$$
and
$$((J_{k,p})_x \times \mathbb{C})//\mathbb{G}_{k,p} = \mathrm{Spec}(\mathcal{O}((J_{k,p})_x)_\ell^{\mathbb{G}'_{k,p}})$$
and we would like to understand them geometrically. There is a crucial difference here from the case of reductive group actions, even though the invariants are finitely generated: when H is a non-reductive group we cannot describe $Y//H$ geometrically as Y^{ss} modulo some equivalence relation. Instead our aim is to use methods inspired by [3] to study these geometric invariant theoretic quotients and the associated algebras of invariants.

Here the crucial ingredient is to be able to find an open subset W of $(J_{k,p})_x \times \mathbb{C}$ with a geometric quotient $W/\mathbb{G}_{k,p}$ embedded as an open subset of an affine variety Z such that the complement of $W/\mathbb{G}_{k,p}$ in Z has (complex) codimension at least two, and the complement of W in $(J_{k,p})_x \times \mathbb{C}$ has codimension at least two. For then we have

$$\mathcal{O}((J_{k,p})_x \times \mathbb{C}) = \mathcal{O}(W)$$

and

$$\mathcal{O}((J_{k,p})_x \times \mathbb{C})^{\mathbb{G}_{k,p}} = \mathcal{O}(W)^{\mathbb{G}_{k,p}} = \mathcal{O}(W/\mathbb{G}_{k,p}) = \mathcal{O}(Z),$$

and it follows that $\mathcal{O}((J_{k,p})_x \times \mathbb{C})^{\mathbb{G}_{k,p}}$ is finitely generated since Z is affine, and that

$$Z = \operatorname{Spec}(\mathcal{O}(Z)) = \operatorname{Spec}(\mathcal{O}((J_{k,p})_x \times \mathbb{C})^{\mathbb{G}_{k,p}}) = ((J_{k,p})_x \times \mathbb{C})//\mathbb{G}_{k,p}.$$

Similarly if we can find a complex reductive group G containing $\mathbb{G}_{k,p}$ as a subgroup, and an embedding of $G/\mathbb{G}_{k,p}$ as an open subset of an affine variety Z with complement of codimension at least two, then $\mathcal{O}(G)^{\mathbb{G}_{k,p}}$ is finitely generated (that is, $\mathbb{G}_{k,p}$ is a Grosshans subgroup of G) and so if Y is any affine variety on which G acts linearly then

$$\mathcal{O}(Y)^{\mathbb{G}_{k,p}} \cong (\mathcal{O}(Y) \otimes \mathcal{O}(G)^{\mathbb{G}_{k,p}})^G$$

is finitely generated.

We will use the ideas of [3] to find suitable affine varieties Z as above, and in particular to prove

Theorem 3.3. *If $p = 1$ and $k \geq 4$, or if $p > 1$, then $\mathbb{G}_{k,p}$ is a Grosshans subgroup of the general linear group $GL(\mathrm{sym}^{\leq k} p)$ where*

$$\mathrm{sym}^{\leq k} p = \sum_{i=1}^{k} \dim \mathrm{Sym}^i \mathbb{C}^p = \binom{k+p-1}{k-1},$$

so that every linear action of $\mathbb{G}_{k,p}$ which extends to a linear action of $GL(\mathrm{sym}^{\leq k} p)$ has finitely generated invariants.

Theorem 3.2 is an immediate consequence of this theorem, since the action of $\mathbb{G}_{k,p}$ on $(J_{k,p})_x$ extends to an action of the general linear group $GL(\mathrm{sym}^{\leq k} p)$, and the action of $\mathbb{G}_{k,p}$ on $(J_{k,p})_x \times \mathbb{C}$ with weight ℓ on $\mathbb{C}$ extends to an action of $GL(\mathrm{sym}^{\leq k} p)$ if ℓ is divisible by $\sum_{i=1}^k i(\dim \mathrm{Sym}^i \mathbb{C}^p)$ (which equals $k(k+1)/2$ when $p = 1$).

4. A description via test curves

In [3] the action of $\mathbb{G}_k$ on jet bundles is studied using an idea coming from global singularity theory. The construction goes as follows.

If u, v are positive integers, let $J_k(u, v)$ denote the vector space of k-jets of holomorphic maps $(\mathbb{C}^u, 0) \to (\mathbb{C}^v, 0)$ at the origin; that is, the set of equivalence classes of maps $f : (\mathbb{C}^u, 0) \to (\mathbb{C}^v, 0)$, where $f \sim g$ if and only if $f^{(j)}(0) = g^{(j)}(0)$ for all $j = 1, \ldots, k$.

With this notation, the fibres of J_k are isomorphic to $J_k(1, n)$, and the group $\mathbb{G}_k$ is simply $J_k(1, 1)$ with the composition action on itself.

If we fix local coordinates $z_1, \dots, z_u$ at $0 \in \mathbb{C}^u$ we can again identify the k-jet of f, using derivatives at the origin, with $(f'(0), f''(0)/2!, \dots, f^{(k)}(0)/k!)$, where $f^{(j)}(0) \in \mathrm{Hom}(\mathrm{Sym}^j\mathbb{C}^u, \mathbb{C}^v)$. This way we get an identification

$$J_k(u,v) = \oplus_{j=1}^k \mathrm{Hom}(\mathrm{Sym}^j\mathbb{C}^u, \mathbb{C}^v).$$

We can compose map-jets via substitution and elimination of terms of degree greater than k; this leads to the composition maps

$$J_k(v,w) \times J_k(u,v) \to J_k(u,w), \quad (\Psi_2, \Psi_1) \mapsto \Psi_2 \circ \Psi_1 \text{modulo terms of degree} > k\,. \tag{8}$$

When $k = 1$, $J_1(u,v)$ may be identified with u-by-v matrices, and (8) reduces to multiplication of matrices.

The k-jet of a curve $(\mathbb{C},0) \to (\mathbb{C}^n,0)$ is simply an element of $J_k(1,n)$. We call such a curve φ *regular* if $\varphi'(0) \neq 0$. Let us introduce the notation $J_k^{\mathrm{reg}}(1,n)$ for the set of regular curves:

$$J_k^{\mathrm{reg}}(1,n) = \{\gamma \in J_k(1,n); \gamma'(0) \neq 0\}\,.$$

Note that if $n > 1$ then the complement of $J_k^{\mathrm{reg}}(1,n)$ in $J_k(1,n)$ has codimension at least two. Let $N \geq n$ be any integer and define

$$\Upsilon_k = \left\{\Psi \in J_k(n,N) : \exists \gamma \in J_k^{\mathrm{reg}}(1,n) : \Psi \circ \gamma = 0\right\}$$

to be the set of those k-jets which take at least one regular curve to zero. By definition, Υ_k is the image of the closed subvariety of $J_k(n,N) \times J_k^{\mathrm{reg}}(1,n)$ defined by the algebraic equations $\Psi \circ \gamma = 0$, under the projection to the first factor. If $\Psi \circ \gamma = 0$, we call γ a *test curve* of Ψ.

This term originally comes from global singularity theory, where this is called the test curve model of A_k-singularities. In global singularity theory singularities of polynomial maps $f : (\mathbb{C}^n,0) \to (\mathbb{C}^m,0)$ are classified by their local algebras, and

$$\Sigma_k = \{f \in J_k(n,m) : \mathbb{C}[x_1, \dots, x_n]/\langle f_1, \dots, f_m \rangle \simeq \mathbb{C}[t]/t^{k+1}\}$$

is called a Morin singularity, or A_k-singularity. The test curve model of Gaffney [12] tells us that

$$\overline{\Sigma_k} = \overline{\Upsilon_k}$$

in $J_k(n,m)$.

A basic but crucial observation is the following. If γ is a test curve of $\Psi \in \Upsilon_k$, and $\varphi \in J_k^{\mathrm{reg}}(1,1) = G_k$ is a holomorphic reparametrization of $\mathbb{C}$, then $\gamma \circ \varphi$ is, again, a test curve of Ψ:

$$\mathbb{C} \xrightarrow{\varphi} \mathbb{C} \xrightarrow{\gamma} \mathbb{C}^n \xrightarrow{\Psi} \mathbb{C}^N \tag{9}$$

$$\Psi \circ \gamma = 0 \quad \Rightarrow \quad \Psi \circ (\gamma \circ \varphi) = 0.$$

In fact, we get all test curves of Ψ in this way from a single γ if the following open dense property holds: the linear part of Ψ has 1-dimensional kernel. Before stating this more precisely in Proposition 4.3 below, let us write down the equation $\Psi \circ \gamma = 0$ in coordinates in an illustrative case. Let $\gamma = (\gamma', \gamma'', \dots, \gamma^{(k)}) \in J_k^{\mathrm{reg}}(1,n)$ and $\Psi =$

$(\Psi', \Psi'', \ldots, \Psi^{(k)}) \in J_k(n, N)$ be the k-jets. Using the chain rule, the equation $\Psi \circ \gamma = 0$ reads as follows for $k = 4$:

$$\begin{gathered}
\Psi'(\gamma') = 0, \\
\tfrac{1}{2!}\Psi'(\gamma'') + \Psi''(\gamma', \gamma') = 0, \\
\tfrac{1}{3!}\Psi'(\gamma''') + \tfrac{2}{2!}\Psi''(\gamma', \gamma'') + \Psi'''(\gamma', \gamma', \gamma') = 0, \\
\tfrac{1}{4!}\Psi'(\gamma'''') + \tfrac{2}{3!}\Psi''(\gamma', \gamma''') + \tfrac{1}{2!2!}\Psi''(\gamma'', \gamma'') + \tfrac{3}{2!}\Psi'''(\gamma', \gamma', \gamma'') + \Psi''''(\gamma', \gamma', \gamma', \gamma') = 0.
\end{gathered} \tag{10}$$

Definition 4.1. To simplify our formulas we introduce the following notation for a partition $\tau = [i_1 \ldots i_l]$ of the integer $i_1 + \ldots + i_l$:

- the *length*: $|\tau| = l$,
- the *sum*: $\sum \tau = i_1 + \ldots + i_l$,
- the *number of permutations*: $\mathrm{perm}(\tau)$ is the number of different sequences consisting of the numbers $i_1, \ldots, i_l$ (e.g. $\mathrm{perm}([1, 1, 1, 3]) = 4$),
- $\gamma_\tau = \prod_{j=1}^{l} \gamma^{(i_j)} \in \mathrm{Sym}^l \mathbb{C}^n$ and $\Psi(\gamma_\tau) = \Psi^l(\gamma^{(i_1)}, \ldots, \gamma^{(i_l)}) \in \mathbb{C}^N$.

Lemma 4.2. *Let $\gamma = (\gamma', \gamma'', \ldots, \gamma^{(k)}) \in J_k^{\mathrm{reg}}(1, n)$ and $\Psi = (\Psi', \Psi'', \ldots, \Psi^{(k)}) \in J_k(n, N)$ be k-jets. Then the equation $\Psi \circ \gamma = 0$ is equivalent to the following system of k linear equations with values in $\mathbb{C}^N$:*

$$\sum_{\tau \in \Pi[m]} \frac{\mathrm{perm}(\tau)}{\prod_{i \in \tau} i!} \Psi(\gamma_\tau) = 0, \quad m = 1, 2, \ldots, k, \tag{11}$$

where $\Pi[m]$ denotes the set of all partitions of m.

For a given $\gamma \in J_k^{\mathrm{reg}}(1, n)$ let $\mathcal{S}_\gamma$ denote the set of solutions of (11); that is,

$$\mathcal{S}_\gamma = \{\Psi \in J_k(n, N); \Psi \circ \gamma = 0\}.$$

The equations (11) are linear in Ψ, hence

$$\mathcal{S}_\gamma \subset J_k(n, N)$$

is a linear subspace of codimension kN. Moreover, the following holds:

Proposition 4.3. *(*[3]*, Proposition 4.4)*

(i) *For $\gamma \in J_k^{\mathrm{reg}}(1, n)$, the set of solutions $\mathcal{S}_\gamma \subset J_k(n, N)$ is a linear subspace of codimension kN.*

(ii) *Set*

$$J_k^o(n, N) = \{\Psi \in J_k(n, N) |\, \dim \ker(\Psi') = 1\}.$$

For any $\gamma \in J_k^{\mathrm{reg}}(1, n)$, the subset $\mathcal{S}_\gamma \cap J_k^o(n, N)$ of $\mathcal{S}_\gamma$ is dense.

(iii) *If $\Psi \in J_k^o(n, N)$, then Ψ belongs to at most one of the spaces $\mathcal{S}_\gamma$. More precisely,*

$$\text{if } \gamma_1, \gamma_2 \in J_k^{\mathrm{reg}}(1, n), \quad \Psi \in J_k^o(n, N) \text{ and } \Psi \circ \gamma_1 = \Psi \circ \gamma_2 = 0,$$

then there exists $\varphi \in J_k^{\mathrm{reg}}(1, 1)$ such that $\gamma_1 = \gamma_2 \circ \varphi$.

(iv) *Given* $\gamma_1, \gamma_2 \in J_k^{\mathrm{reg}}(1,n)$*, we have* $\mathcal{S}_{\gamma_1} = \mathcal{S}_{\gamma_2}$ *if and only if there is some* $\varphi \in J_k^{\mathrm{reg}}(1,1)$ *such that* $\gamma_1 = \gamma_2 \circ \varphi$*.*

By the second part of Proposition 4.3 we have a well-defined map

$$\nu : J_k^{\mathrm{reg}}(1,n) \to \mathrm{Grass}(\mathrm{codim} = kN, J_k(n,N)), \quad \gamma \mapsto \mathcal{S}_\gamma$$

to the Grassmannian of codimension-kN subspaces in $J_k(n,N)$. From the last part of Proposition 4.3 it follows that:

Proposition 4.4. *(*[3]*)* ν *is* $\mathbb{G}_k$*-invariant on the* $J_k^{\mathrm{reg}}(1,1)$*-orbits, and the induced map on the orbits*

$$\bar{\nu} : J_k^{\mathrm{reg}}(1,n)/\mathbb{G}_k \hookrightarrow \mathrm{Grass}(\mathrm{codim} = kN, J_k(n,N)) \tag{12}$$

is injective.

5. Embedding into the flag of equations

In this section we will recast the embedding (12) of $J_k^{\mathrm{reg}}(1,n)/\mathbb{G}_k$ given by Proposition 4.4 into a more useful form, still following [3]. Let us rewrite the linear system $\Psi \circ \gamma = 0$ associated to $\gamma \in J_k^{\mathrm{reg}}(1,n)$ in a dual form. The system is based on the standard composition map (8):

$$J_k(n,N) \times J_k(1,n) \longrightarrow J_k(1,N),$$

which, via the identification $J_k(n,N) = J_k(n,1) \otimes \mathbb{C}^N$, is derived from the map

$$J_k(n,1) \times J_k(1,n) \longrightarrow J_k(1,1)$$

via tensoring with $\mathbb{C}^N$. Observing that composition is linear in its first argument, and passing to linear duals, we may rewrite this correspondence in the form

$$\phi : J_k(1,n) \longrightarrow \mathrm{Hom}\,(J_k(1,1)^*, J_k(n,1)^*). \tag{13}$$

If $\gamma = (\gamma', \gamma'', \ldots, \gamma^{(k)}) \in J_k(1,n) = (\mathbb{C}^n)^k$ is the k-jet of a curve, we can put $\gamma^{(j)} \in \mathbb{C}^n$ into the jth column of an $n \times k$ matrix, and

- identify $J_k(1,n)$ with $\mathrm{Hom}\,(\mathbb{C}^k, \mathbb{C}^n)$;
- identify $J_k(n,1)^*$ with $\mathrm{Sym}^{\le k}\mathbb{C}^n = \oplus_{l=1}^k \mathrm{Sym}^l\mathbb{C}^n$;
- identify $J_k(1,1)^*$ with $\mathbb{C}^k$.

Using these identifications, we can recast the map ϕ in (13) as

$$\phi_k : \mathrm{Hom}\,(\mathbb{C}^k, \mathbb{C}^n) \longrightarrow \mathrm{Hom}\,(\mathbb{C}^k, \mathrm{Sym}^{\le k}\mathbb{C}^n), \tag{14}$$

which may be written out explicitly as follows

$$(\gamma', \gamma'', \ldots, \gamma^{(k)}) \longmapsto \left(\gamma', \gamma'' + (\gamma')^2, \ldots, \sum_{i_1+i_2+\ldots+i_s=d} \frac{1}{i_1! \ldots i_s!}\gamma^{(i_1)}\gamma^{(i_2)} \ldots \gamma^{(i_s)}\right).$$

The set of solutions $\mathcal{S}_\gamma$ is the linear subspace orthogonal to the image of $\phi_k(\gamma', \ldots \gamma^{(k)})$ tensored by $\mathbb{C}^N$; that is,

$$\mathcal{S}_\gamma = \mathrm{im}(\phi_k(\gamma))^\perp \otimes \mathbb{C}^N \subset J_k(n, N).$$

Consequently, it is straightforward to take $N = 1$ and define

$$\mathcal{S}_\gamma = \mathrm{im}(\phi_k(\gamma)) \in \mathrm{Grass}(k, \mathrm{Sym}^{\leq k}\mathbb{C}^n). \tag{15}$$

Moreover, let $B_k \subset GL(k)$ denote the Borel subgroup consisting of upper triangular matrices and let

$$\mathrm{Flag}_k(\mathbb{C}^n) = \mathrm{Hom}\,(\mathbb{C}^k, \mathrm{Sym}^{\leq k}\mathbb{C}^n)/B_k = \{0 = F_0 \subset F_1 \subset \cdots \subset F_k \subset \mathbb{C}^n,\ \dim F_l = l\}$$

denote the full flag of k-dimensional subspaces of $\mathrm{Sym}^{\leq k}\mathbb{C}^n$. In addition to (15) we can analogously define

$$\mathcal{F}_\gamma = (\mathrm{im}(\phi(\gamma^1)) \subset \mathrm{im}(\phi(\gamma^2)) \subset \ldots \subset \mathrm{im}(\phi(\gamma^k))) \in \mathrm{Flag}_k(\mathrm{Sym}^{\leq k}\mathbb{C}^n). \tag{16}$$

Using these definitions Proposition 4.3 implies the the following version of Proposition 4.4, which does not contain the parameter N.

Proposition 5.1. *The map ϕ in* (14) *is a $\mathbb{G}_k$-invariant algebraic morphism*

$$\phi : J_k^{\mathrm{reg}}(1, n) \to \mathrm{Hom}\,(\mathbb{C}^k, \mathrm{Sym}^{\leq k}\mathbb{C}^n),$$

which induces

- *an injective map on the $\mathbb{G}_k$-orbits to the Grassmannian:*

$$\phi^{Gr} : J_k^{\mathrm{reg}}(1, n)/\mathbb{G}_k \hookrightarrow \mathrm{Grass}(k, \mathrm{Sym}^{\leq k}\mathbb{C}^n)$$

  defined by $\phi^{Gr}(\gamma) = \mathcal{S}_\gamma$;
- *an injective map on the $\mathbb{G}_k$-orbits to the flag manifold:*

$$\phi^{Flag} : J_k^{\mathrm{reg}}(1, n)/G_k \hookrightarrow \mathrm{Flag}_k(\mathrm{Sym}^{\leq k}\mathbb{C}^n)$$

  defined by $\phi^{Flag}(\gamma) = \mathcal{F}_\gamma$.

In addition,

$$\phi^{Gr} = \phi^{Flag} \circ \pi_k$$

where π_k : $\mathrm{Flag}(k, \mathrm{Sym}^{\leq k}\mathbb{C}^n) \to \mathrm{Grass}_k(\mathrm{Sym}^{\leq k}\mathbb{C}^n)$ is the projection to the k-dimensional subspace.

Composing ϕ^{Gr} with the Plücker embedding

$$\mathrm{Grass}(k, \mathrm{Sym}^{\leq k}\mathbb{C}^n) \hookrightarrow \mathbb{P}(\wedge^k \mathrm{Sym}^{\leq k}\mathbb{C}^n)$$

we get an embedding

$$\phi^{\mathrm{Proj}} : J_k^{\mathrm{reg}}(1, n)/\mathbb{G}_k \hookrightarrow \mathbb{P}(\wedge^k(\mathrm{Sym}^{\leq k}\mathbb{C}^n)). \tag{17}$$

The image

$$\phi^{Gr}(J_k^{\mathrm{reg}}(1, n))/\mathbb{G}_k \subset \mathrm{Grass}(k, \mathrm{Sym}^{\leq k}\mathbb{C}^n)$$

is a $GL(n)$-orbit in $\mathrm{Grass}(k, \mathrm{Sym}^{\le k}\mathbb{C}^n)$, and therefore a nonsingular quasi-projective variety. Its closure is, however, a highly singular subvariety of $\mathrm{Grass}(k, \mathrm{Sym}^{\le k}\mathbb{C}^n)$, which when $k \le n$ is a finite union of $GL(n)$ orbits, with a nice orbit structure. We will return to describe the orbits in the next section.

Definition 5.2. We introduce the following notation

$$X_{n,k} = \phi^{\mathrm{Proj}}(J_k^{\mathrm{nondeg}}(1,n)), \quad Y_{n,k} = \phi^{\mathrm{Proj}}(J_k^{\mathrm{reg}}(1,n)) \subset \mathbb{P}(\wedge^k(\mathrm{Sym}^{\le k}\mathbb{C}^n)).$$

Then

$$\overline{X}_{n,k} = \overline{Y}_{n,k} = \overline{\phi^{\mathrm{Proj}}(J_k^{\mathrm{nondeg}}(1,n))} \subset \mathbb{P}(\wedge^k(\mathrm{Sym}^{\le k}\mathbb{C}^n)) \tag{18}$$

and

$$X_{n,k} \subset Y_{n,k} \subset \overline{X}_{n,k} \subset \mathbb{P}(\wedge^k(\mathrm{Sym}^{\le k}\mathbb{C}^n)).$$

6. Boundary components

In this section we study the boundary components of $X_{n,k}$ and $Y_{n,k}$ as defined in Definition 5.2 above. We will focus on the case when $k \le n$ first, and in §6.5 we will deal with the situation when $k > n$.

The main technical theorem which we are aiming to prove is Theorem 6.5 below, which tells us that the complement of $X_{n,k}$ in its closure in a subset $A_{n,k}$ of $\mathbb{P}(\wedge^k(\mathrm{Sym}^{\le k}\mathbb{C}^n))$ has codimension at least two. When $n = k$ this subset $A_{k,k}$ is affine, and as discussed at the end of §3, this result will be crucial in proving our finite generation result Theorem 3.2.

It is clear that $J_k^{\mathrm{nondeg}}(1,n)$ is an open subset of $J_k^{\mathrm{reg}}(1,n)$. If we identify the elements of $J_k(1,n)$ as $n \times k$ matrices whose columns are the derivatives of the map germs $f = (f', \ldots, f^{(n)}) : \mathbb{C} \to \mathbb{C}^n$, then $J_k^{\mathrm{nondeg}}(1,n)$ is the set of matrices of maximal rank k and $J_k^{\mathrm{reg}}(1,n)$ consists of the matrices with nonzero first column.

Definition 6.1. *Let $e_1, \ldots, e_n$ be the standard basis of $\mathbb{C}^n$; then*

$$\{e_{i_1,i_2,\ldots,i_s} = e_{i_1} \ldots e_{i_s} : 1 \le i_1 \le \ldots \le i_s \le n, 1 \le s \le k\}$$

is a basis of $\mathrm{Sym}^{\le k}\mathbb{C}^n$*, and*

$$\{e_{\varepsilon_1} \wedge \ldots \wedge e_{\varepsilon_n} : \varepsilon_l \in \Pi_{\le n}\}$$

is a basis of $\mathbb{P}(\wedge^n(\mathrm{Sym}^{\le k}\mathbb{C}^n))$*, where*

$$\Pi_{\le n} = \{(i_1, i_2, \ldots, i_s) : 1 \le i_1 \le \ldots \le i_s \le n, 1 \le s \le k\}.$$

The corresponding coordinates of $x \in \mathrm{Sym}^{\le k}\mathbb{C}^n$ will be denoted by $x_{\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_d}$. Let $A_{n,k} \subset \mathbb{P}(\wedge^k(\mathrm{Sym}^{\le k}\mathbb{C}^n))$ consist of the points whose projection to $\wedge^k(\mathbb{C}^n)$ is nonzero. This is the subset where $x_{i_1,i_2,\ldots,i_k} \ne 0$ for some $1 \le i_1 \le \ldots \le i_k \le n$.

Remark 6.2. If $n = k$ then $A_{n,n} \subset \mathbb{P}(\wedge^k(\mathrm{Sym}^{\le k}\mathbb{C}^n))$ is the affine chart where $x_{1,2,\ldots,n} \ne 0$.

Let us take a closer look at the space $\mathrm{Grass}(n, \mathrm{Sym}^{\leq k}\mathbb{C}^n)$. This has an induced right $GL(n)$ action coming from the $GL(n)$ action on $\mathrm{Sym}^{\leq k}\mathbb{C}^n$, and $GL(n)/\mathbb{G}_n$ has a left $GL(n)$ action induced by multiplication on the left. Since ϕ^{Proj} is a $GL(n)$-equivariant embedding, we conclude that

Lemma 6.3. (i) *For $k \leq n$ $X_{n,k}$ is the $GL(n)$ orbit of*

$$\mathbf{z} = \phi^{\mathrm{Proj}}(e_1, \ldots, e_k) = [e_1 \wedge (e_2 \oplus e_1^2) \wedge \ldots \wedge (\sum_{i_1+\ldots i_s=k} e_{i_1} \ldots e_{i_s})] \tag{19}$$

in $\mathbb{P}(\wedge^k(\mathrm{Sym}^{\leq k}\mathbb{C}^n))$. For arbitrary $g \in GL(n)$ with column vectors $v_1, \ldots, v_n$ the action is given by

$$g \cdot \mathbf{z} = \phi^{\mathrm{Proj}}(g) = \phi^{\mathrm{Proj}}(v_1, \ldots, v_n) = [v_1 \wedge (v_2 \oplus v_1^2) \wedge \ldots \wedge (\sum_{i_1+\ldots+i_s=n} v_{i_1} \ldots v_{i_s})].$$

(ii) *For $k \leq n$ $Y_{n,k}$ is the finite union of $GL(n)$ orbits.*
(iii) *For $k > n$ the images $X_{n,k}$ and $Y_{n,k}$ are $GL(n)$-invariant quasi-projective varieties, but they have no dense $GL(n)$ orbit.*

Similar statements hold for the closure of the image in the Grassmannian.

Lemma 6.4. *Let $k \leq n$, then*

(i) *$A_{n,k}$ is invariant under the $GL(n)$ action on $\mathbb{P}(\wedge^k(\mathrm{Sym}^{\leq k}\mathbb{C}^n))$.*
(ii) *$X_{n,k} \subset A_{n,k}$; however, $Y_{n,k} \not\subseteq A_{n,k}$.*
(iii) *$\overline{X}_{n,k}$ is the union of finitely many $GL(n)$-orbits.*

Proof. To prove the first part take a lift

$$\tilde{z} = \tilde{z}^1 \oplus \tilde{z}^2 \in \mathrm{Hom}\,(\mathbb{C}^n, \mathrm{Sym}^{\leq k}\mathbb{C}^n)$$

of $z \in \mathrm{Grass}(n, \mathrm{Sym}^{\leq k}\mathbb{C}^n)$, where

$$z^1 \in \mathrm{Hom}\,(\mathbb{C}^n, \mathbb{C}^n) \text{ and } z^2 \in \mathrm{Hom}\,(\mathbb{C}^n, \oplus_{i=2}^n \mathrm{Sym}^i(\mathbb{C}^n))$$

Then $z \in A_{n,k}$ if and only if $x_{1,2,\ldots,n}(z) = \det(\tilde{z}^1) \neq 0$, which is preserved by the $GL(n)$ action. For the second part note that for $(v_1, \ldots, v_k) \in J_k^{\mathrm{nondeg}}(1, n)$ $v_1 \wedge \ldots, \wedge v_k \neq 0$ so by definition $\phi^{\mathrm{Proj}}(v_1, \ldots, v_k) \in A_{n,k}$. On the other hand

$$\phi^{\mathrm{Proj}}(e_1, 0, \ldots, 0) = e_1 \wedge e_1^2 \wedge \ldots \wedge e_1^k \in Y_{n,k} \setminus A_{n,k}.$$

The last part follows from the existense of a dense open $GL(n)$ orbit. □

The main technical theorem of this paper, which will allow us to prove Theorems 3.3 and 3.2, is the following:

Theorem 6.5. *Let $k \geq 4$. Then*

(i) *The boundary components of $Y_{n,k} \subset \mathbb{P}(\wedge^n(\mathrm{Sym}^{\leq k}\mathbb{C}^n))$ have codimension at least two in $\overline{Y}_{n,k}$.*

(ii) *The intersection with $A_{n,k}$ of the boundary components of $X_{n,k}$ have codimension at least two in $\overline{X}_{n,k} = \overline{Y}_{n,k}$.*

Remark 6.6. There is a codimension-one boundary component of $X_{n,k}$ for $k = n$. This is the closure of the image of the singular matrices. This component is, however, outside $A_{n,k}$, and the image of the singular matrices with $v_1 \neq 0$ is in $Y_{n,k}$, so this is not a boundary component of $Y_{n,k}$.

Remark 6.7. In fact it is not hard to see that Theorem 6.5 is true for $k \leq 2$, but it fails for $k = 3$ (see Example 7.5 below).

We devote the rest of this section to the proof of Theorem 6.5. We start with the proof of the case when $k \leq n$, and in §6.5 we study the case when $k > n$.

The strategy of the proof is the following: first we notice that the dimension of the stabilizer of any point in $X_{n,k}$ is $k + n(n-k)$, and next we prove that the dimension of the stabilizer of any point in $\overline{Y}_{n,k} \setminus Y_{n,k}$ and $(\overline{X}_{n,k} \setminus X_{n,k}) \cap A_{n,k}$ is at least $k + n(n-k) + 2$. The result will then follow from Lemmas 6.3 and 6.4.

The first half of this strategy is clear: the stabilizer of $\mathbf{z}$ in $GL(n)$ is

$$G_{\mathbf{z}} = \{ \begin{pmatrix} \mathbb{G}_k & * \\ 0 & GL(n-k) \end{pmatrix} \}, \tag{20}$$

where the entries $*$ are arbitrary, and the stabilizer of any point in $X_{n,k}$ is conjugate to $G_{\mathbf{z}}$. In order to execute the second step, we need to identify the boundary components of $X_{n,k}$ and $Y_{n,k}$. These boundary components are closures of $GL(n)$ orbits and fall into two groups: the ones in A_n and the rest, and the stabilizer subgroups are very different in these two cases.

6.1. **Orbit structure.** As we indicated before, we assume from now on until §6.5 that $k \leq n$. Let $Z_{n,k} \subset X_{n,k}$ be the torus orbit $T \cdot \mathbf{z} \subset \mathbb{P}(\wedge^n(\mathrm{Sym}^{\leq k}\mathbb{C}^n))$; then $\overline{Z}_{n,k}$ is, by definition, a toric variety.

Proposition 6.8. *Assume that $k \leq n$. Then every $GL(n)$-orbit in $\overline{X}_{n,k}$ intersects $\overline{Z}_{n,k}$. In other words*

$$(GL(n) \cdot z) \cap \overline{Z}_{n,k} \neq \emptyset \text{ for all } z \in \overline{X}_{n,k}.$$

In particular, any $GL(n)$-fixed points in $\overline{X}_{n,k}$ sit in $\overline{Z}_{n,k}$.

Proof. For the proof we make two observations; the first is a straightforward computation and the second is easy to check:

Lemma 6.9. *Let $(\tau_1, \ldots, \tau_k) \in \mathcal{F} \subset \overline{Z}^T$ be a fixed point of the $T \subset GL(n)$ action on $\overline{B_n \cdot \mathbf{z}}$, and $\lambda(t) = (t^{\lambda_1}, \ldots, t^{\lambda_n}) \subset T \subset GL(n)$ such that* $\lim_{t\to 0} \lambda(t) \cdot \mathbf{z} = e_{\tau_1} \wedge \ldots \wedge e_{\tau_k}$. *Then λ acts with positive weights on $A_{\tau_1 \wedge \ldots \wedge \tau_k} \cap \tilde{\mathbb{P}}$.*

Lemma 6.10. *Let S be a closed subset containing the origin of a vector space V on which $\mathbb{C}^*$ acts with all weights positive. Then $\mathbb{C}^* \cdot S$ is closed in V.*

Now we claim that

$$\overline{G \cdot \mathbf{z}} = G \cdot \overline{(T \cdot \mathbf{z})}, \tag{21}$$

where $G = GL(n)$; that is, $\overline{X}_{n,k} \subset \mathbb{P}(\wedge^k(\mathrm{Sym}^{\leq k}\mathbb{C}^n))$ is the union of G-orbits of the points of $\overline{Z}_{n,k}$, the closure of the torus orbit. This will imply Proposition 6.8. Since $\overline{G \cdot \mathbf{z}} \supset G \cdot \overline{(T \cdot \mathbf{z})}$ automatically holds, it is enough to prove that $\overline{G \cdot \mathbf{z}} \subset G \cdot \overline{(T \cdot \mathbf{z})}$, and this follows from the property

$$G \cdot \overline{(T \cdot \mathbf{z})} \text{ is closed in } \mathbb{P}(\wedge^k(\mathrm{Sym}^{\leq k}\mathbb{C}^n)) \tag{22}$$

To prove (22), let $B_n \subset GL(n)$ be the standard Borel subgroup of $GL(n)$ consisting of upper triangular matrices and $B_{n-1} \subset GL(n)$ (respectively $U_{n-1} \subset GL(n)$) be the standard Borel subgroup of $GL(n-1)$ (respectively the standard maximal unipotent of B_{n-1}) embedded as $A \mapsto \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}$. Since $GL(n)/B_n$ is projective and $B_n \cdot \mathbf{z} = U_{n-1}T \cdot \mathbf{z}$, it is enough to prove that

$$U_{n-1} \cdot \overline{(T \cdot \mathbf{z})} \text{ is closed in } \mathbb{P}(\wedge^k(\mathrm{Sym}^{\leq k}\mathbb{C}^n)) \tag{23}$$

Recall that $\mathbb{P}(\wedge^k(\mathrm{Sym}^{\leq k}\mathbb{C}^n))$ is the union of the affine charts $A_{\varepsilon_1\wedge\ldots\wedge\varepsilon_k}$ where the coordinate dual to $e_{\varepsilon_1} \wedge \ldots \wedge e_{\varepsilon_k}$ is nonzero. To prove (23) it is enough to show that $U_{n-1} \cdot \overline{(T \cdot \mathbf{z})}$ contains all its limit points in each of these affine charts. Indeed it is not necessary to consider all of these charts; we need only a cover of $\overline{Z}_{n,k}$. Let $\mathcal{F} = \overline{Z}_{n,k}^T$ be the set of fixed points in $\overline{Z}_{n,k}$ under the action of the maximal torus T of $GL(n)$ on $\mathbb{P}(\wedge^k(\mathrm{Sym}^{\leq k}\mathbb{C}^n)$. Elements of $\mathcal{F}$ are points of the form $e_{\varepsilon_1} \wedge \ldots \wedge e_{\varepsilon_k}$, where $\varepsilon_i \in \Pi_{\leq n}$ defined as in Definition 6.1, and

$$\overline{Z}_{n,k} \subset \bigcup_{(\varepsilon_1,\ldots,\varepsilon_k)\in\mathcal{F}} A_{\varepsilon_1\wedge\ldots\wedge\varepsilon_k}.$$

If $\Sigma(\varepsilon_i) > i$ for some $1 \leq i \leq k$, where $\Sigma(\varepsilon_i)$ is the sum of the partition ε_i defined as in Definition 4.1, then the coordinate dual to $e_{\varepsilon_1} \wedge \ldots \wedge e_{\varepsilon_k}$ is zero for any point in $B_n \cdot \mathbf{z}$. Therefore for any $(\tau_1, \ldots \tau_k) \in \mathcal{F}$ we have $\Sigma(\varepsilon_i) \leq i$, and $(\tau_1, \ldots \tau_k)$ lies in

$$\tilde{\mathbb{P}} = \{z \in \mathbb{P}(\wedge^k(\mathrm{Sym}^{\leq k}\mathbb{C}^n)) : x_{\varepsilon_1,\ldots,\varepsilon_k}(z) \neq 0 \Rightarrow \Sigma(\varepsilon_i) \leq i \text{ for } 1 \leq i \leq k\}.$$

Let $(\tau_1, \ldots \tau_k) \in \mathcal{F} \in \tilde{\mathbb{P}}$ be a fixed point. There is a 1-parameter subgroup $\lambda(t) = (t^{\lambda_1}, \ldots, t^{\lambda_n}) \in T \subset GL(n)$ such that $\lim_{t\to 0} \lambda(t) \cdot \mathbf{z} = e_{\tau_1} \wedge \ldots \wedge e_{\tau_k}$. According to Lemma 6.9 above, the weights of λ on the affine space $A_{\tau_1,\ldots,\tau_k} \cap \tilde{\mathbb{P}}$ are positive. Now

$$U_{n-1} = \bigcup_{t\in\mathbb{C}^*} \lambda(t)\{u \in U_{n-1} : \|u\| \leq 1\}\lambda(t^{-1}) \tag{24}$$

where $\|\cdot\|$ is any norm on $\mathrm{Lie}U_{n-1}$. Indeed, λ normalizes U_{n-1} and the induced conjugation action on $\mathrm{Lie}U_{n-1}$ has all weights > 0. Consequently,

$$U_{n-1}\overline{T \cdot \mathbf{z}} = \mathbb{C}^*\{u \in U_{n-1} : \|u\| \leq 1\}\overline{T \cdot \mathbf{z}}.$$

Since $\{u \in U_{n-1} : ||u|| \le 1\}$ is compact, the set $S = \{u \in U_{n-1} : ||u|| \le 1\}\overline{T \cdot \mathbf{z}}$ is closed. Now the Proposition follows from Lemma 6.10. □

Corollary 6.11. *The boundary components of $X_{n,k}$ are closures of orbits $GL(n) \cdot z$ where $z \in \overline{Z}_{n,k}$ is a boundary point of $Z_{n,k} = T \cdot \mathbf{z}$.*

In order to identify the boundary points of $Z_{n,k}$ we use the following standard result:

Lemma 6.12. *Let T be an algebraic torus acting on the projective variety Z, and $z \in Z$. Then $y \in \overline{Tz}$ if and only if there is $\tau \in T$ and a one-parameter subgroup $\lambda : \mathbb{C}^* \to T$ such that $y \in \overline{\lambda(\mathbb{C}^*)\tau z}$.*

Apply Lemma 6.12 with $Z = Z_{n,k}, z = \mathbf{z} = \phi^{\text{proj}}(e_1, \ldots, e_n)$ and T the maximal torus of diagonal elements in $GL(n)$. It is clear from Lemma 6.4 that $\overline{X}_{n,k}$ is the union of $GL(n)$-orbits. Choose a one-parameter subgroup:

$$\lambda(t) = \left\{ \begin{matrix} t^{\lambda_1} & & & \\ & \cdot & & \\ & & \cdot & \\ & & & t^{\lambda_n} \end{matrix} \right\}$$

Let $\tau = (\rho_1, \ldots, \rho_n) \in T$. We aim to compute the limit points

$$z_{\lambda,t} = \lim_{t \to 0} \lambda(t)(\tau \cdot \mathbf{z}) \in \text{Grass}(k, \text{Sym}^{\le k}\mathbb{C}^n).$$

Notice that the last $n - k$ coordinates of λ are irrelevant for the action on $\mathbf{z}$ since $\mathbf{z} \in \text{Sym}^{\le k}\mathbb{C}^n$. By Lemma 6.21 the closure of the orbit $Tz_{\lambda,t}$ forms a boundary component of $Z_{n,k}$. Since

$$\tau \cdot \mathbf{z} = \rho_1 e_1 \wedge (\rho_2 e_2 \oplus \rho_1^2 e_1^2) \wedge \ldots \wedge (\bigoplus_{i_1 + \ldots + i_s = k} \rho_{i_1} \ldots \rho_{i_s} e_{i_1} \ldots e_{i_s})$$

we have

$$\lambda(t)\mathbf{z}^\tau = [t^{\lambda_1}\rho_1 e_1 \wedge (t^{\lambda_2}\rho_2 v_2 \oplus t^{2\lambda_1}\rho_1^2 v_1^2) \wedge \ldots \wedge (\bigoplus_{i_1 + \ldots + i_s = k} t^{\lambda_{i_1} + \ldots + \lambda_{i_s}} \rho_{i_1} \ldots \rho_{i_s} e_{i_1} \ldots e_{i_s})] =$$

$$= [t^{\lambda_1 + \ldots + \lambda_k}\rho_1 \ldots \rho_k(e_1 \wedge \ldots \wedge e_k) + t^{\lambda_1 + 2\lambda_1 + \lambda_3 + \ldots + \lambda_k}\rho_1\rho_2^2\rho_3 \ldots \rho_k(e_1 \wedge e_1^2 \wedge e_3 \wedge \ldots \wedge e_k) + \ldots].$$

The generic term of the last expression is

$$t^{\lambda_{\varepsilon_1} + \lambda_{\varepsilon_2} + \ldots \lambda_{\varepsilon_k}}\rho_{\varepsilon_1} \ldots \rho_{\varepsilon_k}(\mathbf{e}_{\varepsilon_1} \wedge \ldots \wedge \mathbf{e}_{\varepsilon_k}),\ \Sigma(\varepsilon_i) = i$$

where

$$\lambda_\tau = \sum_{i \in \tau} \lambda_i; \mathbf{e}_\tau = \Pi_{i \in \tau} e_i; \rho_\tau = \Pi_{i \in \tau}\rho_i \text{ if } \tau = (i_1, \ldots, i_s) \text{ is an ordered sequence.} \tag{25}$$

Definition 6.13. Let

- $m_\lambda = \min_{\substack{(\varepsilon_1, \ldots \varepsilon_k) \\ \Sigma(\varepsilon_i) = i}}(\lambda_{\varepsilon_1} + \lambda_{\varepsilon_2} + \ldots \lambda_{\varepsilon_k})$,
- $m_\lambda[i] = \min_{\Sigma(\varepsilon) = i} \lambda_\varepsilon$ for $1 \le i \le k$,

- $z_\lambda[i] = \sum_{\Sigma\varepsilon=i,\lambda_\varepsilon=m_\lambda[i]} \mathbf{e}_\varepsilon$.

Remark 6.14. We make three straightforward observations:

(i) $z_{\lambda,t_2} = [\sum_{\lambda_{\varepsilon_1}+\ldots\lambda_{\varepsilon_k}=m(\lambda)} \rho_{\varepsilon_1}\ldots\rho_{\varepsilon_k}\mathbf{e}_{\varepsilon_1}\wedge\ldots\wedge\mathbf{e}_{\varepsilon_k}]$;

(ii) $z_{\lambda,t_2} = t_2\cdot\mathbf{z}_\lambda$, where

$$\mathbf{z}_\lambda = \lim_{t\to 0}\lambda(t)\mathbf{z} = [\sum_{\lambda_{\varepsilon_1}+\ldots\lambda_{\varepsilon_k}=m(\lambda)} e_{\varepsilon_1}\wedge\ldots\wedge e_{\varepsilon_k}] = \wedge_{i=1}^k z_\lambda[i],$$

so the boundary components of $Z_{n,k}$ (respectively $X_{n,k}$) are $\overline{T\cdot\mathbf{z}_\lambda}$ (respectively $\overline{G\cdot\mathbf{z}_\lambda}$) for some one-parameter subgroups λ;

(iii) $\mathbf{z}_\lambda \in A_{n,k}$ if and only if $\lambda_1+\ldots+\lambda_k = m(\lambda)$.

From this we see that the $GL(n)$-orbits in the boundary of $X_{n,k}$ correspond to the combinatorial data in the description of the limit point. The proof of Theorem 6.5 now consists of three steps:

- In the first step we describe the maximal boundary components as closures of orbits of limit points $\mathbf{z}_\lambda$ for some 1-parameter subgroups $\lambda,\mu:\mathbb{C}^*\to T_\mathbb{C}$. There are $k-1$ maximal boundary components in $A_{n,k}$ indexed by λ^σ for $2\le\sigma\le k$, and $k-1$ maximal boundary components in $\mathbb{P}(\wedge^k(\mathrm{Sym}^{\le k}\mathbb{C}^n))\setminus A_{n,k}$ indexed by μ^σ, which are in $Y_{n,k}$.
- The next step is to compute the limit $\lim G_{\lambda^\sigma(t)\mathbf{z}}$ of the stabilizer subgroups as we move to the boundary component, and check that it has dimension $k+n(k-n)$.
- Finally, we study the stabilizer group $G_{\mathbf{z}_{\lambda^\sigma}}$ of the limit point, and find two extra dimensions in addition to $\lim G_{\lambda^\sigma(t)\mathbf{z}}$ in order to complete the proof of Theorem 6.5.

6.2. **The maximal boundary components.** The open dense orbit O_0 is the $GL(n)$-orbit of $\mathbf{z}=\mathbf{z}_0$ in the Grassmannian, where $\mathbf{z}$ is defined in (19). It is clear that the 1-parameter subgroup $\tilde\lambda(t) = (t,t^2,\ldots,t^k,1,\ldots,1)$ stabilizes $\mathbf{z}$, and therefore $\mathbf{z}=\mathbf{z}_{\tilde\lambda}$.

Let O_λ denote the $GL(k)$-orbit of $\mathbf{z}_\lambda$. Then $O_0 = O_{\tilde\lambda}$ by definition. If

$$\Lambda = \{\lambda : \lambda_1+\ldots+\lambda_k = m(\lambda)\}$$

is the set of one-parameter subgroups where $\lambda_1+\ldots+\lambda_k$ is minimal among the sums $\lambda_{\varepsilon_1}+\lambda_{\varepsilon_2}+\ldots\lambda_{\varepsilon_k}$, then the orbits in $A_{n,k}$ are $\{O_\lambda : \lambda\in\Lambda\}$.

We need a more precise description of the orbit structure. Since $\tilde\lambda_i = i\tilde\lambda_1$ for $i=1,\ldots,k$, for $\lambda\ne\tilde\lambda$ we have a smallest index $1<\sigma\le k$ with $\lambda_\sigma\ne\sigma\lambda_1$.

Definition 6.15. *We call $\sigma = Head(\lambda)$ the head of $\lambda = (\lambda_1,\ldots,\lambda_n)$ if*

$$\lambda_i = i\lambda_1 \text{ for } i<\sigma \text{ and } \lambda_\sigma\ne\sigma\lambda_1.$$

If $\lambda_\sigma<\sigma\lambda_1$ then we call λ regular *; otherwise we call λ* degenerate.

We will often identify a one-parameter subgroup λ with the orbit $GL(n) \cdot \mathbf{z}_\lambda \in \overline{X}_{n,k}$ and say that λ is *maximal* if $\overline{GL(k) \cdot \mathbf{z}_\lambda}$ is a maximal boundary component of $X_{n,k}$, in the sense that the orbit closure $\overline{GL(k) \cdot \mathbf{z}_\lambda}$ is contained in $\overline{X}_{n,k} \setminus X_{n,k}$ and is not contained in the closure of any other $GL(k)$-orbit in $\overline{X}_{n,k} \setminus X_{n,k}$.

Definition 6.16. *Fix $0 < \varepsilon << 1$ and $2 \leq \sigma \leq k$. Let $\lambda^\sigma = (\lambda^\sigma_1, \dots, \lambda^\sigma_n)$ and $\mu^\sigma = (\mu^\sigma_1, \dots, \mu^\sigma_n)$ be the following one-parameter subgroups of $GL(n)$:*

$$\lambda^\sigma_i = \begin{cases} i - \lfloor \frac{i}{\sigma} \rfloor \varepsilon \text{ for } 1 \leq i \leq k, \\ 0 \text{ for } k < i \leq n; \end{cases} \tag{26}$$

$$\mu^\sigma_i = \begin{cases} i \text{ for } i \neq \sigma, i \leq k, \\ \sigma + \varepsilon \text{ for } i = \sigma, \\ 0 \text{ for } k < i \leq n. \end{cases} \tag{27}$$

Here $\lfloor \frac{i}{\sigma} \rfloor$ denotes the largest integer m such that $m \leq \frac{i}{\sigma}$.

It is easy to see that $\mathrm{Head}(\lambda^\sigma) = \mathrm{Head}(\mu^\sigma) = \sigma$, and λ^σ is regular, whereas μ^σ is degenerate.

Definition 6.17. *Let λ be a 1-parameter subgroup. We call*

$$\sharp\{i : \mathbf{z}_\lambda[i] = e_i\}$$

the toral dimension of λ, or of the limit point $\mathbf{z}_\lambda$.

We will see that the dimension of a maximal torus of the stabilizer of $\mathbf{z}_\lambda$ in $GL(n)$ is equal to the toral dimension of λ, and so if the toral dimension of λ is at least three the orbit of $\mathbf{z}_\lambda$ will have codimension at least two, and we have to focus on those λ whose toral dimension is 1 or 2.

Lemma 6.18.
(i) *The maximal regular 1-parameter subgroups have toral dimension at least 2. Those with toral dimension 2 are $\lambda^2, \dots \lambda^k$; in other words for a regular λ with $Head(\lambda) = \sigma$ we have $O_\lambda \subset \overline{O_{\lambda^\sigma}}$. The regular boundary components lie in $A_{n,k}$.*
(ii) *The maximal degenerate 1-parameter subgroups are $\mu^2, \dots \mu^k$; in other words for a regular μ with $Head(\mu) = \sigma$ we have $O_\mu \subset \overline{O_{\mu^\sigma}}$. The degenerate boundary components lie outside $A_{n,k}$.*
(iii) *$\mathbf{z}_{\mu^\sigma} \in Y_{n,k}$ and therefore the degenerate boundary orbits are in $Y_{n,k}$, and they are not boundary orbits.*

Proof. Let λ be a regular 1-parameter subgroup with $\mathrm{Head}(\lambda) = \sigma$. Without loss of generality we can assume that

$$\lambda_i = i \text{ for } i < \sigma \text{ and } \lambda_\sigma = \sigma - \varepsilon.$$

We will call $d(i) = \lfloor \frac{i}{\sigma} \rfloor$ the defect of i and the defect of $\tau = (i_1, \ldots, i_s)$ is $d(\tau) = d(i_1) + \ldots + d_{i_s}$. Since

$$\lambda_{(j,\underbrace{\sigma,\ldots,\sigma}_{m})} = j + m(\sigma - \varepsilon) \text{ for } 1 \le j \le \sigma - 1, m \ge 0,$$

we have

$$m_\lambda[i] \le i - d(i)\varepsilon \text{ for } 1 \le i \le n. \tag{28}$$

If $\lambda_s < s - d(s)\varepsilon$ for $s > i$ and s is the smallest index with this property then $m_\lambda[s] = \lambda_s$ and $\mathbf{z}_\lambda[s] = e_s$, so the dimension of λ is at least 3. Indeed,

$$\mathbf{z}_\lambda[1] = e_1, \mathbf{z}_\lambda[\sigma] = e_\sigma, \mathbf{z}_\lambda[s] = e_s.$$

So we can assume that $\lambda_i \ge i - d(i)\varepsilon$ for $1 \le i \le k$, and therefore

$$m_\lambda[i] = i - d(i)\varepsilon \text{ for } 1 \le i \le k.$$

So

$$\mathbf{e}_\tau \notin \mathbf{z}_\lambda[i] \text{ if } d(\tau) > d(i). \tag{29}$$

On the other hand the distinguished 1-parameter subgroup λ^σ is defined so that $\lambda^\sigma_i = i - d(i)\varepsilon$, where $0 < \varepsilon << 1$, and therefore

$$\mathbf{z}_{\lambda^\sigma}[i] = \sum_{\Sigma(\tau)=i, d(\tau)=d(i)} \mathbf{e}_\tau. \tag{30}$$

Comparing (29) and (30) we conclude

$$\mathbf{z}_\lambda[i] \subset \mathbf{z}_{l^\sigma}[i] \text{ for } 1 \le i \le n$$

and the first part of Lemma 6.18 follows. To prove the second part let μ be a degenerate 1-parameter subgroup with $\mathrm{Head}(\mu) = \sigma$. Without loss of generality we can assume again that

$$\mu_i = i \text{ for } i < \sigma \text{ and } \mu_\sigma = \sigma + \varepsilon.$$

Since

$$\mu_{(\underbrace{1,\ldots 1}_{i})} = i \text{ for } 1 \le i \le k$$

we have

$$m_\mu[i] \le i. \tag{31}$$

Again, $\mu_s < s$ cannot happen for $s > \sigma$ since in that case $\mathbf{z}_\mu[s] = e_s$ would hold and the toral dimension would be at least 3. So $\mu_s \ge s$ and therefore $\mu_\tau \ge \Sigma(\tau)$ with strict inequality if $\sigma \in \tau$. Therefore

$$\mathbf{e}_\tau \notin \mathbf{z}_\mu[i] \text{ if } \sigma \in \tau \tag{32}$$

On the other hand μ^σ satisfies equality in (31), and

$$\mathbf{z}_{\mu^\sigma}[i] = \sum_{\Sigma(\tau)=i, \sigma\notin\tau} \mathbf{e}_\tau. \tag{33}$$

Comparing (32) and (33) we get

$$\mathbf{z}_\lambda[i] \subset \mathbf{z}_{l^\sigma}[i] \text{ for } 1 \le i \le k$$

and the second part of Lemma 6.18 follows. □

Remark 6.19. According to Lemma 6.18, the codimension-at-least-two property has to be proved only for the regular boundary components.

We summarize our information about the maximal boundary components in

Proposition 6.20. *We have* $\mathbf{z}_{\lambda^\sigma} = \wedge_{i=1}^k \mathbf{z}_{\lambda^\sigma}[i]$, *where* $\mathbf{z}_{\lambda^\sigma}[i] = \oplus_{\Sigma(\tau)=i, d(\tau)=d(i)} \mathbf{e}_\tau$, *and* $\mathbf{z}_{\mu^\sigma} = \wedge_{i=1}^k \mathbf{z}_{\mu^\sigma}[i]$ *where* $\mathbf{z}_{\mu^\sigma}[i] = \oplus_{\Sigma(\tau)=i, \sigma\notin\tau} \mathbf{e}_\tau$.

Lemma 6.18 describes the boundary components of $X_{n,k}$. In section §6.5 we will need a bit more information about the boundary components of $Y_{n,k}$. We prove the following

Proposition 6.21. *Let $k \le n$. The boundary orbits of $Y_{n,k}$ lie in the closures of boundary orbits in $A_{n,k}$.*

Proof. Let $\mathbf{z}_\lambda$ be a boundary point corresponding to a 1-parameter subgroup λ.
Case 1. If $\lambda_i \ge i\lambda_1$ for $1 \le i \le k$ then $m_\lambda[i] = i$ since $\lambda_{(1,\ldots,1)} = i$. Let $I_\lambda = \{i : \lambda_i > i\lambda_1\}$ be the set of abundant indices; then

$$\mathbf{z}_\lambda[i] = \sum_{I\cap\tau=\emptyset} \mathbf{e}_\tau,$$

and therefore

$$\mathbf{z}_\lambda = \phi^{\mathrm{proj}}(\delta_{1,I}e_1, \ldots, \delta_{k,I}e_k) \text{ where } \delta_{i,I} = \begin{cases} 0 \text{ if } i \in I \\ 1 \text{ if } i \notin I \end{cases},$$

showing that these orbits lie in $Y_{n,k}$, so they are not boundary orbits.
Case 2. If $\lambda_\sigma > \sigma\lambda_1$ and $\lambda_\rho < \rho\lambda_1$ with some $1 < \sigma, \rho \le k$, then we claim that $O_\lambda \subset \overline{O}_{\tilde\lambda}$, where $\tilde\lambda = (\lambda_1, \ldots \lambda_{\sigma-1}, \sigma\lambda_1, \lambda_{\sigma+1}, \ldots, \lambda_k)$. Indeed,

$$\mathbf{z}_\lambda[i] = \sum_{\lambda(\tau)=m_\lambda[i]} \mathbf{e}_\tau,$$

and since we can replace in any partition τ which contains σ the partition $1^i = (\underbrace{1, \ldots, 1}_{\sigma})$, and $\lambda(\tau - \sigma + 1^i) < \lambda(\tau)$. So

$$m_\lambda[i] = \min_{\Sigma(\tau)=i} \lambda(\tau) = \min_{\Sigma(\tau)=i, \sigma\notin\tau} \lambda(\tau).$$

Therefore

$$\mathbf{z}_\lambda[i] \subset \mathbf{z}_{\tilde\lambda}[i] \text{ for } 1 \le i \le n,$$

and $\mathbf{z}_\lambda = \lim_{\theta\to 0}\lambda(t)\mathbf{z}_{\tilde\lambda}$. Repeating the same procedure, we can get rid of the indices where $\lambda_i > i\lambda_1$ and $O_\lambda \subset \overline{O}_{\tilde\lambda}$ where $\tilde\lambda_i \le i\tilde\lambda_1$ for $1 \le i \le k$.

Case 3. Finally, assume that $\lambda_i \le i\lambda_1$ for $1 \le i \le k$. Note that

$$\lambda_i \le m_\lambda[i]\ \forall i \Leftrightarrow \mathbf{z}_\lambda \in A_{n,k}.$$

Assume that

$$\lambda_\sigma > m_\lambda[\sigma], \tag{34}$$

and σ is the smallest index with this property, and furthermore μ is a partition with $m_\lambda[\sigma] = \lambda_\mu$. Define $\tilde\lambda = (\lambda_1,\dots,\lambda_{\sigma-1}, m_\lambda[\sigma], \lambda_{\sigma+1},\dots,\lambda_k)$. Then $\mathbf{z}_\lambda \ne \mathbf{z}_{\tilde\lambda}$ since $e_\sigma \in \mathbf{z}_{\tilde\lambda}[\sigma]$ but $e_\sigma \notin \mathbf{z}_\lambda[\sigma]$. We show that $\mathbf{z}_\lambda = \lim_{t\to 0}\lambda(t)\mathbf{z}_{\tilde\lambda}$, and $\tilde\lambda$ has fewer indices σ with property (34) than λ has, and then by induction we can prove that $\mathbf{z}_\lambda$ is in the closure of a maximal orbit O_μ in $A_{n,k}$. For by (34) in any partition τ which contains the index σ, we can replace σ with the maximal partition μ such that $\mathrm{Head}(\mu) = \sigma$ and then $\lambda(\tau - \sigma + \mu) < \lambda(\tau)$. So

$$m_\lambda[i] = \min_{\Sigma(\tau)=i}\lambda(\tau) = \min_{\Sigma(\tau)=i,\sigma\notin\tau}\lambda(\tau),$$

which implies

$$\mathbf{z}_\lambda[i] \subset \mathbf{z}_{\tilde\lambda}[i] \text{ for } 1 \le i \le k,$$

and therefore the result follows. □

6.3. **The limit of the stabilizers.** According to Remark 6.19, we have to prove that the boundary components of $X_{n,k}$ corresponding to the 1-parameter subgroups λ^σ have codimension at least 2 for $2 \le \sigma \le k$.

Recall that the second step in the proof of Theorem 6.5 according to our strategy is the study of the limits of the stabilizer groups, i.e. of $\lim G_{\lambda^\sigma(t)\mathbf{z}}$ and $\lim G_{\lambda^\sigma(t)\mathbf{z}}$ for the one-parameter subgroups λ^σ and μ^σ when $2 \le \sigma \le k$.

In this subsection we prove

Proposition 6.22. *$G^\sigma = \lim_{t\to 0} G_{\lambda^\sigma(t)\mathbf{z}} \subset GL(n)$ is a $k + n(n-k)$-dimensional subgroup of $G_{\mathbf{z}_{\lambda^\sigma}}$.*

Proof. Consider the stabilizer $G_{\lambda^\sigma(t)\mathbf{z}}$. Since $GL(n)$ acts on the right on $\mathrm{Grass}(k, \mathrm{Sym}^{\le 2}\mathbb{C}^n)$,

$$G_{\lambda^\sigma(t)\mathbf{z}} = \lambda^\sigma(t)^{-1}G_{\mathbf{z}}\lambda^\sigma(t).$$

Recall that (in shorthand) $G_{\mathbf{z}} = \{\begin{pmatrix} \mathbb{G}_k & * \\ 0 & GL(n-k)\end{pmatrix}\}$ where

$$\mathbb{G}_k = \{\begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \dots & \alpha_k \\ 0 & \alpha_1^2 & 2\alpha_1\alpha_2 & \dots & 2\alpha_1\alpha_{n-1} + \dots \\ 0 & 0 & \alpha_1^3 & \dots & 3\alpha_1^2\alpha_{k-2} + \dots \\ 0 & 0 & 0 & \dots & \cdot \\ \cdot & \cdot & \cdot & \dots & \alpha_1^d \end{pmatrix}\}$$

and the polynomial in the (i, j) entry is

$$p_{i,j}(\alpha) = \sum_{a_1+a_2+\ldots+a_i=j} \alpha_{a_1}\alpha_{a_2}\ldots\alpha_{a_i}.$$

Therefore, the (i, j) entry of the stabilizer of $\lambda^s(t)\mathbf{z}$ is

$$(G_{\lambda^\sigma(t)\mathbf{z}})_{i,j} = t^{\lambda_i^\sigma-\lambda_j^\sigma} p_{i,j}(\alpha). \tag{35}$$

If ε is small enough then $\lambda_1^\sigma < \lambda_2^\sigma < \ldots < \lambda_k^\sigma$, and we define the positive number

$$n_i^\sigma = \max_{1\le j\le n-i+1}(\lambda_{j+i-1}^\sigma - \lambda_j^\sigma),\ i = 1,\ldots,k. \tag{36}$$

Note that by definition $n_1^\sigma = 0$ for all σ.

Lemma 6.23. *Under the substitution*

$$\beta_i^\sigma = t^{-n_i^\sigma}\alpha_i^\sigma$$

we have

$$G_{\lambda^\sigma(t)\mathbf{z}}(\beta_1,\ldots,\beta_k) \in GL(\mathbb{C}[\beta_1,\ldots,\beta_k][t]),$$

so the entries are polynomials in t with coefficients in $\mathbb{C}[\beta_1,\ldots,\beta_k]$.

Proof. Compute the substitution as follows:

$$G_{\lambda^\sigma(t)\mathbf{z}})_{i,j} = t^{\lambda_i^\sigma-\lambda_j^\sigma} \sum_{a_1+a_2+\ldots+a_i=j} \alpha_{a_1}\alpha_{a_2}\ldots\alpha_{a_i} = \tag{37}$$

$$= \sum_{a_1+\ldots a_i=j} t^{\lambda_i^\sigma-\lambda_j^\sigma} t^{n_{a_1}^\sigma+n_{a_2}^\sigma+\ldots+n_{a_i}^\sigma}\beta_{a_1}\beta_{a_2}\ldots\beta_{a_i}. \tag{38}$$

By definition

$$n_{a_1}^\sigma \ge \lambda_{i+a_1-1}^\sigma - \lambda_i^\sigma;\ n_{a_2}^\sigma \ge \lambda_{i+a_1+a_2-2}^\sigma - \lambda_{i+a_1-1}^\sigma;\ \ldots\ ;\ n_{a_j}^\sigma \ge \lambda_{i+a_1+\ldots+a_i-i}^\sigma - \lambda_{i+a_1+\ldots+a_{i-1}-(i-1)}^\sigma.$$

Adding up these inequilites and using $a_1 + \ldots + a_i = j$ we get an alternating sum on the left cancelling up to

$$n_{a_1}^\sigma + \ldots + n_{a_i}^\sigma \ge \lambda_j^\sigma - \lambda_i^\sigma.$$

Substituting this into (37) we get

$$(G_{\lambda^\sigma(t)\mathbf{z}})_{i,j} = \sum_{a_1+\ldots a_i=j} t^{\lambda_i^\sigma-\lambda_j^\sigma} t^{n_{a_1}^\sigma+n_{a_2}^\sigma+\ldots+n_{a_i}^\sigma}\beta_{a_1}\beta_{a_2}\ldots\beta_{a_i} \in \mathbb{C}[\beta_1,\ldots,\beta_k][t]. \tag{39}$$

This proves Lemma 6.23. □

As a corollary we get the existence of

$$G^\sigma = \lim_{t\to 0} G_{\lambda^\sigma(t)\mathbf{z}}(\beta_1,\ldots,\beta_k) \in GL(\mathbb{C}[\beta_1,\ldots,\beta_k]).$$

To prove that $\dim G^\sigma = k + n(n-k)$ and complete the proof of Proposition 6.22, for $1 \le i \le k$ choose $\theta(i)$ such that

$$n_i^\sigma = \lambda_{\theta(i)+i-1} - \lambda_{\theta(i)} \tag{40}$$

holds. Then

$$p_{\theta(i),\theta(i)+i-1}(\beta_1,\ldots,\beta_k)=\sum_{a_1+\ldots+a_{\theta(i)}=\theta(i)+i-1} t^{n^\sigma_{a_1}+\ldots+n^\sigma_{a_{\theta(i)}}}\beta_{a_1}\ldots\beta_{a_{\theta(i)}} \tag{41}$$

so

$$(G^\sigma)_{\theta(i),\theta(i)+i-1}=\lim_{t\to 0} t^{-n_i^\sigma} p_{\theta(i),\theta(i)+i-1}(\beta_1,\ldots,\beta_k)=\lim_{t\to 0}(t^{n_i^\sigma}\beta_1^{\theta(i)-1}\beta_i+\ldots)= \tag{42}$$
$$=\beta_1^{\theta(i)-1}\beta_i+q_{\theta(i),\theta(i)+i-1}$$

where

$$q_{\theta(i),\theta(i)+i-1}\in\mathbb{C}[\beta_1,\ldots,\beta_k][t].$$

It follows that the elements $\frac{d}{dt}A^\sigma(t(e_1+e_i)1)\in \mathrm{Lie}(G^\sigma)$ are independent, where $t(e_1+e_i)=(t,0,\ldots,0,t,0,\ldots,0)$ with the t's are in the 1st and ith position if $i>1$ but interpreted as $(2t,0,\ldots,0)$ if $i=1$. This completes the proof of Proposition 6.22. □

6.4. **Two extra dimensions in the stabilizer of the limit point.** In order to prove Theorem 6.5, it is now enough to prove its statements for the maximal orbits, i.e. to prove

Proposition 6.24.

$$\dim G_{\mathbf{z}_{\lambda^\sigma}}\geq k+n(n-k)+2 \textit{ if } 2\leq\sigma\leq k.$$

According to Proposition 6.22 and our strategy described at the end of §6.1, this follows from

Proposition 6.25. *There exists a 2-dimensional subgroup $B^\sigma\subset G_{\mathbf{z}_{\lambda^\sigma}}$ with $G^\sigma\cap B^\sigma=0$.*

Proof. First, note that the maximal torus of the principal $k\times k$ minor in G^σ is 1-dimensional, and has the form $\mathrm{diag}(\chi,\chi^2,\ldots,\chi^k)$. However, $G_{\mathbf{z}_{\lambda^\sigma}}$ contains a 2-dimensional torus

$$\mathrm{diag}(p_1(\chi,\delta),\ldots,p_k(\chi,\delta)) \tag{43}$$

where

$$p_i(\chi,\delta)=a\chi+b\delta \text{ if } i=b\sigma+a. \tag{44}$$

Indeed, by Proposition 6.20

$$\mathbf{z}_{\lambda^\sigma}=\mathbf{z}_{\lambda^\sigma}[1]\wedge\ldots\wedge\mathbf{z}_{\lambda^\sigma}[k]$$

where

$$\mathbf{z}_{\lambda^\sigma}[i]=\sum_{r_{\varepsilon_i}=r_i^\sigma} e_{\varepsilon_i}. \tag{45}$$

This means that if we give the weight r_i^σ to e_i, then $\mathbf{z}_{\lambda^\sigma}[i]$ is homogeneous of degree

$$r_i^\sigma=i-\lfloor\frac{i}{\sigma}\rfloor\varepsilon=a+b(\sigma-\varepsilon),$$

where $i=b\sigma+a$. Therefore, the torus $\mathrm{diag}(r_1^\sigma,\ldots,r_k^\sigma)$ is in the stabilizer. If we weight r_1 with χ and r_σ with δ then by the same argument, $\mathbf{z}_{\lambda^\sigma}[i]$ is homogeneous of weight $a\chi+b\delta$, and we get the torus in (43).

It remains to find an extra one-dimensional unipotent subgroup of the stabilizer which is not in G^σ. It turns out that we have to distinguish three cases here.

Lemma 6.26. *There exists a one-dimensional unipotent subgroup in $G_{\mathbf{z}_{\lambda^\sigma}} \setminus G^\sigma$ when $\sigma = k$.*

Proof. Let $T \in GL(k)$ denote the transformation

$$T(e_i) = e_i \text{ for } i \neq k-1\ ;\ T(e_{k-1}) = e_{k-1} + \zeta e_k$$

Since e_{k-1} does not occur just in $\mathbf{z}_{\lambda^\sigma}[k-1]$ in (45), we get $T \in G_{\mathbf{z}_{\lambda^\sigma}}$. But $T \notin G^\sigma$, because T is not upper triangular. □

Lemma 6.27. *There exists a one-dimensional unipotent subgroup in $G_{\mathbf{z}_{\lambda^\sigma}} \setminus A^\sigma$ when $\sigma < k$ and $k \neq -1$ mod σ.*

Proof. Let T be the transformation

$$T(e_i) = e_i \text{ for } i \neq k\ ;\ T(e_k) = e_k + \zeta e_\sigma. \tag{46}$$

Since e_k occurs only in $\mathbf{z}_{\lambda^\sigma}[k]$, and $\mathbf{z}_{\lambda^\sigma}[\sigma] = \sigma$ (see (45)), we have

$$\begin{aligned} T \cdot \mathbf{z}_{\lambda^\sigma} &= \mathbf{z}_{\lambda^\sigma}(e_1, \ldots, e_{k-1}, e_k + \zeta e_\sigma) = \\ &= \mathbf{z}_{\lambda^\sigma}[1] \wedge \ldots \wedge \mathbf{z}_{\lambda^\sigma}[\sigma-1] \wedge e_\sigma \wedge \mathbf{z}_{\lambda^\sigma}[\sigma+1] \wedge \ldots \wedge \mathbf{z}_{\lambda^\sigma}[k])+ \\ &+ \zeta \cdot \mathbf{z}_{\lambda^\sigma}[1] \wedge \ldots \wedge \mathbf{z}_{\lambda^\sigma}[\sigma-1] \wedge e_\sigma \wedge \mathbf{z}_{\lambda^\sigma}[\sigma+1] \wedge \ldots \wedge \mathbf{z}_{\lambda^\sigma}[k-1] \wedge e_\sigma = \mathbf{z}_{\lambda^\sigma}, \end{aligned} \tag{47}$$

so $T \in G_{\mathbf{z}_{\lambda^\sigma}}$.

It is slightly harder task to show that $T \notin G^\sigma = \lim_{\theta\to 0} G_{\lambda^\sigma(t)\mathbf{z}}$. First, we compute n_i for $i = k - \sigma$. We claim that for $n \neq -1$ mod σ

$$n_{k-\sigma+1} = \lambda_k^\sigma - \lambda_\sigma^\sigma = \lambda_{k-\sigma+1}^\sigma - \lambda_1^\sigma. \tag{48}$$

Indeed,

$$\lambda_{j+k-\sigma-1} - \lambda_j = \ldots\ldots \leq \lambda_k^\sigma - \lambda_\sigma^\sigma = \lambda_{k-\sigma+1}^\sigma - \lambda_1^\sigma$$

This means that we can choose $\theta(k - \sigma + 1) = \sigma$ in (40) and substitute into (42)

$$(G^\sigma)_{\sigma,k} = \beta_1^{\sigma-1}\beta_{k-\sigma+1} + q_{\sigma,k}(\beta_1, \ldots, \beta_k), \tag{49}$$

where $q_{\sigma,k}(\beta_1, \ldots, \beta_k)$ is a polynomial, whose monomials $\beta_{i_1}^{b_1} \ldots \beta_{i_\sigma}^{b_\sigma}$ satisfy

$$i_1 b_1 + \ldots + i_\sigma b_\sigma = k. \tag{50}$$

Moreover, we can also choose $\theta(k - \sigma + 1) = 1$, by (48), and then (42) gives us

$$(G^\sigma)_{1,k-\sigma+1} = \beta_{k-\sigma+1}. \tag{51}$$

Suppose now that $T \in G^\sigma$, that is

$$T = G^\sigma(\beta_1, \ldots, \beta_k) \text{ for some } \beta_1 \in \mathbb{C}^*, \beta_2, \ldots, b_k \in \mathbb{C}. \tag{52}$$

Let $(T)_{i,j}$ denote the (i, j) entry of T. Then

$$(T)_{\sigma,k} = \zeta\ ,\ (T)_{i,j} = 0 \text{ for } i \neq j\ ,\ (T)_{i,i} = 1.$$

Comparing the $(1,1)$ and $(1, k-\sigma+1)$ entries of T and G^σ we get

$$\beta_1 = 1, \beta_{\delta-\sigma+1} = 0. \tag{53}$$

Choose $\theta(i)$ for $i = 2, \ldots, k$ as in (40) and let $\theta(k-\sigma+1) = \sigma$. Since all off-diagonal entries of T but the (σ, k) are zero, (52) forces the following equations

$$\beta_i + q_{\theta(i),\theta(i)+i-1} = 0 \text{ for } i \neq k-\sigma+1, \tag{54}$$

$$\beta_{k-\sigma+1} + q_{\sigma,k} = \zeta. \tag{55}$$

By (53), these are $k-1$ polynomial equations in $k-2$ variables, and the Jacobian at 0 is the origin, so we have finitely many solutions near the origin. Therefore, for some ζ, it follows that T is not in G^σ. □

Lemma 6.28. *There exists a one-dimensional unipotent subgroup in $G_{\mathbf{z}_{\lambda^\sigma}} \setminus G^\sigma$ when $\sigma < k$ and $d = -1$ mod σ.*

Proof. This case works very similarly to the previous one. Suppose $k-1 > \sigma$, that is, if $k = a\sigma - 1$ where $a \geq 2$ (this holds because $k \geq \sigma$), the condition is that $a\sigma - 2 > \sigma$. This is true for all $k > 3$. For $k = 3$, $\sigma = 2$ Theorem 6.5 is not true.

Let T be the transformation

$$T(e_i) = e_i \text{ for } i \neq k, k-1 \;;\; T(e_{k-1}) = e_{k-1} + \zeta e_\sigma \;;\; T(e_k) = e_k + \zeta e_\sigma \tag{56}$$

First we check again that $T \in G_{\mathbf{z}_{\lambda^\sigma}}$. By (45)

$$\mathbf{z}_{\lambda^\sigma}[\sigma] = e_\sigma \;;$$
$$\mathbf{z}_{\lambda^\sigma}[\sigma+1] = e_{\sigma+1} + e_1 e_\sigma \;;$$
$$\mathbf{z}_{\lambda^\sigma}[k] = e_k + \sum_{i=1}^{k-1} e_i e_{k-i} \;.$$

An easy computation shows that

$$\begin{aligned} T \cdot \mathbf{z}_{\lambda^\sigma} &= \mathbf{z}_{\lambda^\sigma}(e_1, \ldots, e_{k-2}, e_{k-1} + \zeta e_\sigma, e_k + \zeta e_{\sigma+1}) = \\ &= \mathbf{z}_{\lambda^\sigma}[1] \wedge \ldots \wedge \mathbf{z}_{\lambda^\sigma}[k-2] \wedge (\mathbf{z}_{\lambda^\sigma}[k-1] + \zeta \mathbf{z}_{\lambda^\sigma}[\sigma]) \wedge (\mathbf{z}_{\lambda^\sigma}[k] + \zeta \mathbf{z}_{\lambda^\sigma}[\sigma+1] = \\ &= \mathbf{z}_{\lambda^\sigma}[1] \wedge \ldots \wedge \mathbf{z}_{\lambda^\sigma}[k] = \mathbf{z}_{\lambda^\sigma}. \end{aligned} \tag{57}$$

Now we prove that $T \notin G^\sigma$ in a similar way to the second case covered by Lemma 6.27. Since $k-1 \neq -1$ mod σ we can substitute $k-1$ instead of k in (48):

$$n_{k-\sigma} = \lambda^\sigma_{k-1} - \lambda^\sigma_\sigma = \lambda^\sigma_{k-\sigma} - \lambda^\sigma_1. \tag{58}$$

Moreover, we also get the extra equation

$$n_{k-\sigma} = \lambda^\sigma_k - \lambda^\sigma_{\sigma+1}, \tag{59}$$

and similarly to (49) and (51) it follows that

$$(G^{\sigma})_{\sigma,k-1} = \beta_1^{\sigma-1}\beta_{k-\sigma} + q_{\sigma,k-1}(\beta_1,\dots,\beta_k); \tag{60}$$

$$(G^{\sigma})_{\sigma+1,k} = \beta_1^{\sigma}\beta_{k-\sigma} + q_{\sigma+1,k}(\beta_1,\dots,\beta_k); \tag{61}$$

$$(G^{\sigma})_{1,k-\sigma} = \beta_{k-\sigma}. \tag{62}$$

Since T differs from the identity matrix only by the entries

$$(T)_{\sigma,k-1} = (T)_{\sigma+1,k} = \zeta,$$

the equality

$$T = G^{\sigma}(\beta_1,\dots,\beta_k)$$

forces $\beta_{k-\sigma} = 0, \beta_1 = 1$ and the analogue of (54) ,(55):

$$\beta_i + q_{\theta(i),\theta(i)+i-1} = 0 \text{ for } i \neq k-\sigma \tag{63}$$

$$\beta_{k-\sigma} + q_{\sigma,k-1} = \zeta \tag{64}$$

$$\beta_{k-\sigma} + q_{\sigma+1,k} = \zeta \tag{65}$$

which are, again, $k+1$ nondegenerate polynomial equations in $k-1$ variables, and there is no solution for some ζ. □

We have now proved Proposition 6.25, which together with Proposition 6.22 completes the proof of Proposition 6.24 and thus of Theorem 6.5 when $k \leq n$.

□

6.5. **Boundary components for $k > n$.** When $k > n$ the argument used in §§6.1-6.4 to prove Theorem 6.5 in the case when $k \leq n$ breaks down since the image $\phi^{\mathrm{Proj}}(J_k^{\mathrm{nondeg}}(1,n))$ is not a $GL(n)$ orbit, and therefore we cannot localize the boundary points in the same way using 1-parameter subgroups. The embedding ϕ^{Proj} is still $GL(n)$-invariant, but the image is the union of infinitely many $GL(n)$-orbits. In fact, however, as we will see below, Theorem 6.5 for $k > n$ follows from Theorem 6.5 for $k \leq n$ which we have already proved.

Let $k > n$ and $\mathbf{i} = (i_1 < i_2 < \dots < i_n)$ be an n-element subset of $\{1,\dots,k\}$. Fix a basis $e_1,\dots,e_k$ of $\mathbb{C}^k$, and denote

$$\mathbb{C}^n_{\mathbf{i}} = \mathbb{C}e_{i_1} \oplus \dots \oplus \mathbb{C}e_{i_n} \subset \mathbb{C}^k$$

the coordinate subspace spanned by bases elements from $\mathbf{i}$. Define the corresponding subspace

$$\wedge^k \operatorname{Sym}^{\leq k}\mathbb{C}^n_{\mathbf{i}} \subset \wedge^k \operatorname{Sym}^{\leq k}\mathbb{C}^k,$$

and let $\pi_{\mathbf{i}} : \wedge^k \operatorname{Sym}^{\leq k}\mathbb{C}^k \to \wedge^k \operatorname{Sym}^{\leq k}\mathbb{C}^n_{\mathbf{i}}$ denote the projection. Their direct sum

$$\pi = \oplus_{\mathbf{i}\in\binom{k}{n}}\pi_{\mathbf{i}} : \wedge^k \operatorname{Sym}^{\leq k}\mathbb{C}^k \to \bigoplus_{\mathbf{i}\in\binom{k}{n}} \wedge^k \operatorname{Sym}^{\leq k}\mathbb{C}^n_{\mathbf{i}} \tag{66}$$

descends to a rational map

$$\bar{\pi} : \mathbb{P}(\wedge^k \operatorname{Sym}^{\leq k}\mathbb{C}^k) \dashrightarrow \mathbb{P}\left(\bigoplus_{\mathbf{i}\in\binom{k}{n}} \wedge^k \operatorname{Sym}^{\leq k}\mathbb{C}^n_{\mathbf{i}}\right),$$

which is well-defined on $\overline{Y_{k,k}} \subset \mathbb{P}(\wedge^k \operatorname{Sym}^{\leq k}\mathbb{C}^k)$. In fact, it is well-defined on $\mathbb{C}^k \wedge \operatorname{Sym}^2\mathbb{C}^k \wedge \ldots \wedge \operatorname{Sym}^k\mathbb{C}^k \subset \wedge^k \operatorname{Sym}^{\leq k}\mathbb{C}^k$, and $\overline{Y_{k,k}}$ sits in this subspace.

Now we have a well-defined map

$$\bar{\pi} : \overline{Y_{k,k}} \to \mathbb{P}\left(\bigoplus_{\mathbf{i}\in\binom{k}{n}} \wedge^k \operatorname{Sym}^{\leq k}\mathbb{C}^n_{\mathbf{i}}\right),$$

and

$$\iota : \overline{Y_{n,k}} \hookrightarrow \mathbb{P}(\wedge^k \operatorname{Sym}^{\leq k}\mathbb{C}^n_{1,\ldots,n}) \hookrightarrow \mathbb{P}\left(\bigoplus_{\mathbf{i}\in\binom{k}{n}} \wedge^k \operatorname{Sym}^{\leq k}\mathbb{C}^n_{\mathbf{i}}\right)$$

defines an embedding. Moreover, $J_k^{\mathrm{reg}}(1,n) \subset J_k^{\mathrm{reg}}(1,k)$ simply by adding the $(k-n)\times k$ zero matrix to get a $k\times k$ matrix from an $n\times k$ matrix. The diagram

$$\begin{tikzcd}
J_k^{\mathrm{reg}}(1,k) & J_k^{\mathrm{reg}}(1,n) \arrow[l, hook'] \arrow[r, "\phi_k^{\mathrm{proj}}"] \arrow[d, "="] & Y_{k,k} \arrow[r, hook] \arrow[d, "\bar{\pi}"] & \mathbb{P}(\wedge^k \operatorname{Sym}^{\leq k}\mathbb{C}^k) \\
& J_k^{\mathrm{reg}}(1,n) \arrow[r, "\phi^{\mathrm{proj}}"] & Y_{n,k} \arrow[r, hook, "\iota"] & \mathbb{P}\left(\bigoplus_{\mathbf{i}\in\binom{k}{n}} \wedge^k \operatorname{Sym}^{\leq k}\mathbb{C}^n_{\mathbf{i}}\right)
\end{tikzcd} \tag{67}$$

commutes by definition, so $Y_{n,k} \subset \bar{\pi}(Y_{k,k})$, and therefore it extends to a surjective morphism

$$\bar{\pi} : \overline{Y}_{k,k} \to \overline{Y}_{n,k}.$$

Since $\dim(Y_{k,k}) = k(k-1)$ and $\dim(Y_{n,k}) = k(n-1)$, the generic fiber has dimension $k(k-n)$. Furthermore,

$$\overline{Y}_{n,k} \setminus Y_{n,k} \subset \bar{\pi}(\overline{Y}_{k,k} \setminus Y_{k,k}),$$

and by Proposition 6.21

$$\overline{Y}_{n,k} \setminus Y_{n,k} \subset \bigcup_{\mathbf{z}_\lambda \in A_{k,k}} \bar{\pi}(\overline{O_\lambda}).$$

Now, $\overline{O_\lambda}$ is irreducible, and therefore $\bar{\pi}(\overline{O_\lambda})$, too. We want to prove that for a generic point $z \in \bar{\pi}(\overline{O_\lambda}) \cap \overline{Y}_{n,k}$

$$\dim(\bar{\pi}^{-1}(z)) \cap \overline{O_\lambda} \geq k(k-n)$$

holds. The generic z sits in $\overline{\pi}(O_\lambda)$, and therefore has the form

$$z = \overline{\pi}(\mathbf{z}_\lambda[1](\mathbf{v}) \wedge \ldots \wedge \mathbf{z}_\lambda[k](\mathbf{v}))$$

for some $\mathbf{v} \in J_k^{\mathrm{reg}}(1,k)$. Let $\kappa : \mathbb{C}^k \to \mathbb{C}^n_{1,\ldots,n}$ denote the projection to the subspace spanned by the first n basis vectors of $\mathbb{C}^k$. Now $\kappa(\mathbf{v}) \in J_k(1,n) \subset (\mathbb{C}^n_{1,\ldots,n})^k$, and by the diagram (67)

$$\mathbf{z}_\lambda[1](\kappa^{-1}(\kappa(\mathbf{v}))) \wedge \ldots \wedge \mathbf{z}_\lambda[k](\kappa^{-1}(\kappa(\mathbf{v}))) \subset \overline{\pi}^{-1}(z) \cap \overline{O_\lambda}.$$

So we have a commutative diagram

$$\begin{tikzcd} J_k(1,k) \arrow[r, hook, "{(\mathbf{z}_\lambda[1],\ldots,\mathbf{z}_\lambda[k])}"] \arrow[d, "\kappa"'] & \mathrm{Hom}\,(\mathbb{C}^k,\ \mathrm{Sym}^{\leq k}\mathbb{C}^k) \arrow[d, "\tilde{\kappa}"'] \\ J_k(1,n) \arrow[r, hook, "{(\mathbf{z}_\lambda[1],\ldots,\mathbf{z}_\lambda[k])}"] & \mathrm{Hom}\,(\mathbb{C}^k,\ \mathrm{Sym}^{\leq k}\mathbb{C}^n_{1,\ldots,n}) \end{tikzcd} \tag{68}$$

where the horizontal maps are injective, because $\mathbf{z}_\lambda \in A_{k,k}$ implies that $v_i \in \mathbf{z}_\lambda[i]$ for $1 \leq i \leq k$. So we can apply the following observation with $\mathrm{Sym}^{\leq k}\mathbb{C}^n_{1,\ldots,n} \subset \mathrm{Sym}^{\leq k}\mathbb{C}^k$.

Observation: Let $V \subset W$ be complex vector spaces. Then

$$\tilde{\kappa} : \mathrm{Hom}\,(\mathbb{C}^k, W) \to \mathrm{Hom}\,(\mathbb{C}^k, V)$$

is a $GL(k)$-equivariant projection, and the stabilizer of a point $p \in \mathrm{Hom}\,(\mathbb{C}^k, W)$ is a subroup of the stabilizer of $\tilde{\kappa}(p) \in \mathrm{Hom}\,(\mathbb{C}^k, V)$.

This implies that the dimension of $\overline{\pi}^{-1}(z) \cap \overline{O_\lambda} \subset \mathrm{Hom}\,(\mathbb{C}^k, \mathrm{Sym}^{\leq k}\mathbb{C}^k)/GL(k)$ is greater or equal to the dimension of the fibre of κ, which is $k(k-n)$. Combining this with the codimension two property for O_λ which has already been proved, we find that

$$\dim(\pi(\overline{O_\lambda})) \leq \dim(\overline{O_\lambda}) - k(k-n) \leq \dim Y_{k,k} - 2 - k(k-n) = kn - k - 2 = \dim Y_{n,k} - 2,$$

proving Theorem 6.5 for $k > n$.

7. Geometric description of Demailly-Semple invariants

By using Theorem 6.5 in the case when $n = k$, we can now prove Theorem 3.3 in the case when $p = 1$.

Theorem 7.1. *If $k \geq 4$ then $\mathbb{G}_k$ is a Grosshans subgroup of the general linear group $GL(k)$, so that every linear action of $\mathbb{G}_k$ which extends to a linear action of $GL(k)$ has finitely generated invariants.*

Proof. We take $n = k$ in Theorem 6.5 to obtain an affine variety $\overline{X}_{n,n}$ containing $GL(k)/\mathbb{G}_k$ as a dense open subset with complement of codimension at least two. □

In particular we have the special case of Theorem 3.2 when $p = 1$.

Theorem 7.2. *If $k \geq 4$ the fibre $\mathcal{O}((J_k)_x)_\ell^{\mathbb{U}_k} \cong \mathcal{O}((J_k)_x \times \mathbb{C})^{\mathbb{G}_k}$ of the bundle E_k^n is a finitely generated graded complex algebra when ℓ is divisible by $k(k+1)/2$.*

Proof. If ℓ is divisible by $k(k+1)/2$ then the action of $\mathbb{G}_k$ on $(J_k)_x \times \mathbb{C}$ extends to a linear action of $GL(k)$ (with $GL(k)$ acting on $\mathbb{C}$ as multiplication by a power of the determinant) and so Theorem 7.1 applies. □

Theorem 6.5 also allows us to describe the subalgebra $\mathcal{O}((J_k)_x)_{k(k+1)/2}^{\mathbb{U}_k}$ of the Demailly-Semple algebra. This is the invariant ring

$$\mathcal{O}(J_k^{\mathrm{reg}}(1,n))^{\mathbb{U}_k \rtimes \mu_{k(k+1)/2}} = \mathcal{O}(J_k^{\mathrm{reg}}(1,n) \times \mathbb{C})^{\mathbb{G}_k},$$

that is the ring of invariant polynomials in the entries of $\mathrm{Hom}^{\mathrm{reg}}(\mathbb{C}^k, \mathbb{C}^n)$ under the linear action on the right of the semi-direct product $\mathbb{U}_k \rtimes \mu_{k(k+1)/2}$ of $\mathbb{U}_k$ with the group of $k(k+1)/2$th roots of 1 in $\mathbb{C}$, or equivalently the ring of $\mathbb{G}_k$-invariant polynomials in the entries of $\mathrm{Hom}^{\mathrm{reg}}(\mathbb{C}^k, \mathbb{C}^n) \times \mathbb{C}$.

In §5 we constructed an embedding

$$\phi^{\mathrm{Proj}} : J_k^{\mathrm{reg}}(1,n)/\mathbb{G}_k \hookrightarrow \mathbb{P}(\wedge^k(\mathrm{Sym}^{\leq k}\mathbb{C}^n))$$

of $J_k^{\mathrm{reg}}(1,n)/\mathbb{G}_k$ in the projective space $\mathbb{P}(\wedge^k(\mathrm{Sym}^{\leq k}\mathbb{C}^n))$ and in Theorem 6.5 we proved that the boundary components of the closure $\overline{Y}_{n,k}$ of its image $Y_{n,k} = \mathrm{Im}(\phi^{\mathrm{Proj}}) \subset \mathbb{P}(\wedge^k(\mathrm{Sym}^{\leq k}\mathbb{C}^n))$ have codimension at least two. Equivalently this construction gives us an embedding of $J_k^{\mathrm{reg}}(1,n)/(\mathbb{U}_k \rtimes \mu_{k(k+1)/2})$ in the affine space $\wedge^k(\mathrm{Sym}^{\leq k}\mathbb{C}^n)$ such that the boundary components of the closure of its image (which is the affine cone over $\overline{Y}_{n,k}$ have codimension at least two. Let $\mathcal{O}_{\mathbb{P}}(1)$ be the tautological line bundle on $\mathbb{P}(\wedge^k(\mathrm{Sym}^{\leq k}\mathbb{C}^n))$. The global sections of $\mathcal{O}_{\mathbb{P}}(1)$ pull back to $\mathbb{U}_k \rtimes \mu_{k(k+1)/2}$-invariant polynomials of weighted degree $1+2+\ldots+k = k(k+1)/2$ on $J_k^{\mathrm{reg}}(1,n)$, and since the complement of $J_k^{\mathrm{reg}}(1,n)$ in $J_{k,n}$ has codimension at least two, these polynomials all extend to $\mathbb{U}_k \rtimes \mu_{k(k+1)/2}$-invariant polynomials of weighted degree $1+2+\ldots+k = k(k+1)/2$ on $J_{k,n}$. Moreover the fact that the boundary components of the closure of $J_k^{\mathrm{reg}}(1,n)/(\mathbb{U}_k \rtimes \mu_{k(k+1)/2})$ in the affine space $\wedge^k(\mathrm{Sym}^{\leq k}\mathbb{C}^n)$ have codimension at least two tells us that every $\mathbb{U}_k \rtimes \mu_{k(k+1)/2}$-invariant polynomial on $J_{k,n}$ is the pullback of a polynomial in the global sections of $\mathcal{O}_{\mathbb{P}}(1)$ (or equivalently in the Plücker coordinates on the Grassmannian $\mathrm{Grass}_{k(k+1)/2}(\mathrm{Sym}^{\leq k}\mathbb{C}^n)$). Thus we obtain the following corollary of Theorem 6.5:

Theorem 7.3. (i) *If $k \geq 4$ the subalgebra*

$$\mathcal{O}((J_k)_x)_{k(k+1)/2}^{\mathbb{U}_k} = \mathcal{O}(J_k^{\mathrm{reg}}(1,n))^{\mathbb{U}_k \rtimes \mu_{k(k+1)/2}} = \mathcal{O}(J_k^{\mathrm{reg}}(1,n) \times \mathbb{C})^{\mathbb{G}_k}$$

of the Demailly-Semple algebra spanned by the $\mathbb{U}_k$-invariant polynomials which are homogeneous of degree divisible by $k(k+1)/2$ is generated by the Plücker coordinates on $\mathbb{P}(\wedge^k(\mathrm{Sym}^{\leq k}\mathbb{C}^n))$. These can be expressed as

$$\{\Delta_{\mathbf{i}_1,\ldots,\mathbf{i}_s} : s \leq n\},$$

where $\mathbf{i}_j$ *denotes a multi-index corresponding to basis elements of* $\mathrm{Sym}^{\leq k}\mathbb{C}^n$*, and* $\Delta_{\mathbf{i}_1,\dots,\mathbf{i}_s}$ *is the corresponding minor of* $\phi(f' \dots, f^{(n)}) \in \mathrm{Hom}(\mathbb{C}^n, \mathrm{Sym}^{\leq k}\mathbb{C}^n)$.

(ii) *A polynomial p in* $\mathcal{O}((J_k)_x)$ *which is homogeneous of degree h with respect to the* $\mathbb{C}^*$*-action is* $\mathbb{U}_k$*-invariant (or equivalently lies in the Demailly-Semple algebra) if and only if* $p^{k(k+1)/2}$ *lies in the subalgebra* $\mathcal{O}((J_k)_x)^{\mathbb{U}_k}_{k(k+1)/2}$ *generated by the Plucker coordinates* $\{\Delta_{\mathbf{i}_1,\dots,\mathbf{i}_s} : s \leq n\}$.

Example 7.4. $n = k = 2$. *Although our codimension-two property has been proved only for* $k \geq 4$*, we get all the invariants as Plücker coordinates in this case too. Now*

$$J_2^{\mathrm{reg}}(1,2) = \{(f_1', f_2', f_1'', f_2'') \in (\mathbb{C}^2)^2; (f_1', f_2') \neq (0,0)\},$$

and fixing a basis $\{e_1, e_2\}$ *of* $\mathbb{C}^2$ *and the induced basis* $\{e_1, e_2, e_1^2, e_1e_2, e_2^2\}$ *of* $\mathbb{C}^2 \oplus \mathrm{Sym}^2\mathbb{C}^2$*, the map* $\phi : J_2(1,2) = \mathrm{Hom}(\mathbb{C}^2, \mathbb{C}^2) \to \mathrm{Hom}(\mathbb{C}^2, \mathrm{Sym}^{\leq 2}\mathbb{C}^2)$ *of* (14) *is given by*

$$(f_1', f_2', f_1'', f_2'') \mapsto \begin{pmatrix} f_1' & f_2' & 0 & 0 & 0 \\ \frac{1}{2!}f_1'' & \frac{1}{2!}f_2'' & (f_1')^2 & f_1'f_2' & (f_2')^2 \end{pmatrix}.$$

The 2×2 *minors of this* 2×5 *matrix are* $(f_1')^3$, $(f_1')^2 f_2'$, $f_1'(f_2')^2, (f_2')^3$ *and*

$$\Delta_{[1,2]} = f_1' f_2'' - f_1'' f_2'.$$

These give generators of the subalgebra $\mathcal{O}((J_k)_x)_3^{\mathbb{U}_2}$ *of the Demailly-Semple algebra* $\mathcal{O}((J_k)_x)^{\mathbb{U}_2}$*. In fact the Demailly-Semple algebra itself is generated by* f_1', f_2' *and* $\Delta_{[1,2]}$.

Example 7.5. $n = k = 3$. *Recall that Theorem 6.5 requires* $k \geq 4$*, and in fact it fails for* $k = 3$ *as indeed happens in this example, though nonetheless the Demailly-Semple algebra* $\mathcal{O}((J_k)_x)^{\mathbb{U}_k}$ *is finitely generated in this case as proved by Rousseau in* [34]*. We have*

$$J_3^{\mathrm{reg}}(1,3) = \{(f_1', f_2', f_3', f_1'', f_2'', f_3'', f_1''', f_2''', f_3''') \in (\mathbb{C}^3)^3; (f_1', f_2', f_3') \neq (0,0,0)\},$$

and if we fix a basis $\{e_1, e_2, e_3\}$ *of* $\mathbb{C}^2$ *and the induced basis*

$$\{e_1, e_2, e_3, e_1^2, e_1e_2, e_2^2, e_1e_3, e_2e_3, e_3^2, e_1^3, e_1^2e_2, \dots, e_3^3\}$$

of $\mathbb{C}^3 \oplus \mathrm{Sym}^2\mathbb{C}^3 \oplus \mathrm{Sym}^3\mathbb{C}^3$*, the map* $\phi : \mathrm{Hom}(\mathbb{C}^3, \mathbb{C}^3) \to \mathrm{Hom}(\mathbb{C}^3, \mathrm{Sym}^{\leq 3}\mathbb{C}^3)$ *in* (14) *sends*

$$(f_1', f_2', f_3', f_1'', f_2'', f_3'', f_1''', f_2''', f_3''')$$

to a 3×19 *matrix, whose first* 9 *columns (corresponding to* $\mathrm{Sym}^{\leq 2}\mathbb{C}^3$*) are*

$$\begin{pmatrix} f_1' & f_2' & f_3' & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2!}f_1'' & \frac{1}{2!}f_2'' & \frac{1}{2!}f_3'' & (f_1')^2 & f_1'f_2' & (f_2')^2 & f_1'f_3' & f_2'f_3' & (f_3')^2 \\ \frac{1}{3!}f_1''' & \frac{1}{3!}f_2''' & \frac{1}{3!}f_3''' & f_1'f_1'' & f_1'f_2'' + f_1''f_2' & f_2'f_2'' & f_1'f_3'' + f_3'f_1'' & f_2'f_3'' + f_2''f_3' & f_3'f_3'' \end{pmatrix},$$

and the remaining 10 *columns (corresponding to* $\mathrm{Sym}^3\mathbb{C}^3$*) are*

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ (f_1')^3 & (f_1')^2f_2' & f_1'(f_2')^2 & (f_2')^3 & f_1'(f_3')^2 & (f_1')^2f_3' & (f_2')^2f_3' & f_2'(f_3')^2 & (f_3')^3 & f_1'f_2'f_3' \end{pmatrix}.$$

The 3×3 *minors of this matrix in the ring of polynomials in* $f_1', f_2', f_3', f_1'', f_2'', f_3'', f_1''', f_2''', f_3'''$ *localized with respect to* f_1' *generate the invariants with weight (for the* $\mathbb{C}^*$*-action) divisible by 6. The necessity to localize with respect to* f_1' *corresponds to the fact that* $Y_{n,k}$ *has a boundary component of codimension* 1 *in this case (so that Theorem 6.5 fails here); this boundary component is described as the closure of the subset*

$$\{v_1 \wedge v_2 \wedge (v_3 \oplus v_1 v_2) : v_1, v_2, v_3 \in \mathbb{C}^3, (v_1, v_2, v_3) \in GL_3\} \subset \mathbb{P}(\wedge^3 \operatorname{Sym}^{\leq 3}\mathbb{C}^3).$$

Example 7.6. $n = 2, k = 4$. *In this case*

$$J_4^{\mathrm{reg}}(1,2) = \{(f_1', f_2', f_1'', f_2'', f_1''', f_2''', f_1'''', f_2'''') \in (\mathbb{C}^2)^4; (f_1', f_2') \neq (0,0)\},$$

and fixing a basis $\{e_1, e_2\}$ *of* $\mathbb{C}^2$ *and*

$$\{e_1, e_2, e_1^2, e_1e_2, e_2^2, e_1^3, \ldots, e_1e_2^4, e_2^4\}$$

of $\operatorname{Sym}^{\leq 4}\mathbb{C}^2$ *the map* $\phi : J_4(1,2) \to \operatorname{Hom}(\mathbb{C}^4, \operatorname{Sym}^{\leq 4}\mathbb{C}^2)$ *in* (14) *sends*

$$(f_1', f_2', f_1'', f_2'', f_1''', f_2''', f_1'''', f_2''''$$

to a 4×15 *matrix, whose first* 5 *columns (corresponding to* $\operatorname{Sym}^{\leq 2}\mathbb{C}^2$*) are*

$$\begin{pmatrix} f_1' & f_2' & 0 & 0 & 0 \\ \frac{1}{2!}f_1'' & \frac{1}{2!}f_2'' & (f_1')^2 & f_1'f_2' & (f_2')^2 \\ \frac{1}{3!}f_1''' & \frac{1}{3!}f_2''' & f_1'f_1'' & (f_1'f_2'' + f_1''f_2') & f_2'f_2'' \\ \frac{1}{4!}f_1'''' & \frac{1}{4!}f_2'''' & \frac{2}{3!}f_1'f_1''' + \frac{1}{2!2!}(f_1'')^2 & \frac{2}{3!}(f_1'f_2''' + f_1'''f_2') + \frac{1}{2!}f_1''f_2'' & \frac{2}{3!}f_2'f_2''' + \frac{1}{2!2!}(f_2'')^2 \end{pmatrix},$$

and next four columns (corresponding to $\operatorname{Sym}^3\mathbb{C}^2$*) are*

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ (f_1')^3 & (f_1')^2 f_2' & f_1'(f_2')^2 & (f_2')^3 \\ \frac{3}{2!}((f_1')^2 f_1'') & \frac{3}{2!}((f_1')^2 f_2'' + 2f_1'f_2'f_1'') & \frac{3}{2!}((f_2')^2 f_1'' + 2f_2'f_1'f_2'') & \frac{3}{2!}((f_2')^2 f_2'') \end{pmatrix},$$

and the remaining five columns (corresponding to $\operatorname{Sym}^3\mathbb{C}^3$*) are*

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ (f_1')^4 & (f_1')^3 f_2' & (f_1')^2(f_2')^2 & f_1'(f_2')^3 & (f_2')^4 \end{pmatrix}.$$

Then the weight $1+2+3+4 = 10$ *piece* $\mathcal{O}((J_4)_x)_{10}^{\mathbb{U}_4}$ *of the invariant algebra* $\mathcal{O}((J_4)_x)^{\mathbb{U}_4}$ *is generated by the* 4×4 *minors of this* 4×15 *matrix.*

7.1. **Construction of the bi-invariants.** In this section we deal with the problem of finding a decomposition of the Demailly-Semple bundle $E_{k,m}$ (whose fibre at $x \in X$ consists of the $\mathbb{U}_k$-invariant jet differentials of order k and weight m) into a direct sum of irreducible Schur bundles

$$\Gamma^{(l_1, l_2, \ldots, l_n)} T_X^*$$

with $l_1 \geq l_2 \geq \ldots \geq l_n$, where $n = \dim X$. According to the strategy of Merker [24] a suitable description of these basic bricks of the Demailly-Semple bundle should lead to cohomology computations, as follows. Knowing the (asymptotic) Euler characteristic of the Schur bundles $\Gamma^{(l_1,l_2,\ldots,l_n)}T^*_X$, and upper bounds for the higher Betti numbers gives us a lower bound for $h^0(\Gamma^{(l_1,l_2,\ldots,l_n)}T^*_X)$ and therefore $h^0(X, E_{k,m})$. The existence of global sections of $E_{k,m}$ forces differential equations to be satisfied by all entire holomorphic curves in X, which is the basis of Demailly's strategy for solving the Kobayashi conjecture. In [24] Merker carries out this strategy for small values of k and n.

On the fibre $\oplus_{m\geq 0}\mathcal{E}^n_{k,m} = \mathcal{O}((K_k)_x)^{\mathbb{U}_k}$ of $\oplus_{m\geq 0}E_{k,m}$ at x there is a $GL(n)$ action, where $n = \dim X$. To describe this action recall that the fibre is identified with $\mathcal{O}(J_k(1,n))^{\mathbb{G}'_k}$ consisting of polynomials $Q(f', f'', \ldots f^{(k)})$ invariant under the unipotent reparametrization group $\mathbb{G}'_k = \mathbb{U}_k$. Here $f \in J_k(1,n)$ can be identified with an $n \times k$ matrix M_f; that is, an element of $\mathrm{Hom}\,(\mathbb{C}^k, \mathbb{C}^n)$, as in §5, by putting the components of $f^{(i)}$ in the ith column. The matrix $w = (w_{ij}) \in GL(n)$ acts on $J_k(1,n)$ by multiplication on the right

$$w \cdot f = M_f w.$$

In more detail, the action on f^λ is given by

$$w \cdot f^\lambda = (\sum_{j=1}^n w_{1j} f^\lambda_j, \ldots, \sum_{j=1}^n w_{nj} f^\lambda_j).$$

Moreover, this action commutes with the action of $\mathbb{G}_k$.

According to elementary representation theory, the fibre $\mathcal{E}^n_{k,m}$ of the Demailly-Semple bundle $E_{k,m}$ then decomposes into a direct sum of irreducible $GL(n)$-representations. General reasons ensure that this decomposition of the fibre extends to a global decomposition of $E_{k,m}$, which is the Schur decomposition. So the task is to find the highest weights of the $GL(n)$-representation $\oplus_{m\geq 0}\mathcal{E}^n_{k,m} = \mathcal{O}((J_k)_x)^{\mathbb{U}_k}$.

$GL(n)$ has a natural action on $\mathrm{Sym}^{\leq k}\mathbb{C}^n$, and therefore on $\mathrm{Hom}^{\mathrm{reg}}(\mathbb{C}^k, \mathrm{Sym}^{\leq k}\mathbb{C}^n)$. This induces an action on $\mathrm{Grass}(k, \mathrm{Sym}^{\leq k}\mathbb{C}^n)$, and ϕ^{Proj} is $GL(n)$-equivariant.

The description of the highest weight minors in $\mathrm{Hom}^{\mathrm{reg}}(\mathbb{C}^k, \mathrm{Sym}^{\leq k}\mathbb{C}^n)$ goes as follows. Recall that $\mathrm{Sym}^{\leq k}\mathbb{C}^n$ has a basis $e_{\mathbf{i}}$ indexed by series $\mathbf{i} = (i_1, \ldots, i_s)$ with $1 \leq i_1 \leq \ldots \leq i_s \leq n$ for some $s \leq k$. Let $<_{\mathrm{lex}}$ be the lexicographic partial order on the set of indices, that is, $(i_1, \ldots, i_s) <_{\mathrm{lex}} (j_1, \ldots, j_t)$ if and only if $s = t$ and $i_l < j_l$ for the first index l with $i_l \neq j_l$. We call a set of indices Λ *descendent* if

$$\mathbf{i} \in \Lambda \Rightarrow \mathbf{i}' \in \Lambda \text{ for all } \mathbf{i}' < \mathbf{i}.$$

Theorem 7.7. (i) $\Delta_{\mathbf{i}_1,\ldots,\mathbf{i}_s}$ *is a highest weight if and only if* $\{\mathbf{i}_1, \ldots, \mathbf{i}_s\}$ *is descendent.*

(ii) *The $k(k+1)/2$th graded piece of the ring of bi-invariants* ${}^{GL(n)}\mathcal{O}((J_k)_x)^{\mathbb{U}_k}_{k(k+1)/2}$ *is generated by*

$$\{\Delta_{\mathbf{i}_1,\ldots,\mathbf{i}_s} : \{\mathbf{i}_1, \ldots, \mathbf{i}_s\} \text{ descendent}\}.$$

Example 7.8. *$n = k = 2$. We have three descendent sets of indices $\mathbf{i}_1, \mathbf{i}_2$, namely $(\mathbf{i}_1, \mathbf{i}_2) = ((1), (2))$, $(\mathbf{i}_1, \mathbf{i}_2) = ((1), (1, 1))$ and $(\mathbf{i}_1, \mathbf{i}_2) = ((1, 1), (1, 2))$. The corresponding minors are*

$$\Delta_{((1),(2))} = \Delta_{[1,2]}, \quad \Delta_{((1),(1,1))} = (f_1')^3, \quad \Delta_{((1,1),(1,2))} = 0,$$

so the weight-3 piece ${}^{GL(2)}\mathcal{O}((J_2)_x)_3^{\mathbb{U}_2}$ of the bi-invariant algebra ${}^{GL(2)}\mathcal{O}((J_2)_x)^{\mathbb{U}_2}$ is generated by $\Delta_{[1,2]}$ and $(f_1')^3$, and the remaining minors $(f_1')^2 f_2'$, $f_1'(f_2')^2$, and $(f_2')^3$ are not bi-invariants.

Example 7.9. *$n = 2, k = 4$. We list some of the descendent sets of indices $(\mathbf{i}_1 < \mathbf{i}_2 < \mathbf{i}_3 < \mathbf{i}_4)$ in the following table:*

$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_4$
(1)	(2)	(1, 1)	(1, 2)
(1)	(2)	(1, 1, 1)	(1, 1, 2)
(1)	(2)	(1, 1, 1, 1)	(1, 1, 1, 2)
(1, 1)	(1, 2)	(1, 1, 1)	(1, 1, 2)
(1, 1)	(1, 2)	(1, 1, 1, 1)	(1, 1, 1, 2)
(1, 1, 1)	(1, 1, 2)	(1, 1, 1, 1)	(1, 1, 1, 2)
(1)	(1, 1)	(1, 2)	(2, 2)
(1)	(1, 1, 1)	(1, 1, 2)	(1, 2, 2)
(1, 1, 1)	(1, 1, 2)	(1, 2, 2)	(2, 2, 2)
...	...	...	...

According to [6, 34, 24]*, the bi-invariant algebra is generated by* 5 *independent invariants:*

$$I^1 = f_1'; \;\; I^3 = \Delta_{[1,2]} = f_1' f_2'' - f_1'' f_2'; \;\; I^5 = [I^3, f_1']; \;\; I^7 = [I^5, f_1']; \;\; I^8 = \frac{[I^5, I^3]}{f_1'}.$$

It can be checked that the weight-10 piece $\mathcal{O}((J_4)_x)_{10}^{\mathbb{U}_4}$ of the bi-invariant algebra is indeed generated by all the descendent minors of the 4×15 matrix in Example 7.6.

8. Generalized Demailly-Semple jet bundles

The aim of this section is to extend the earlier constructions for $p = 1$ to generalized Demailly-Semple invariant jet differentials when $p > 1$.

Let X be a compact, complex manifold of dimension n. We fix a parameter $1 \le p \le n$, and study the maps $\mathbb{C}^p \to X$. Recall that as before we fix the degree k of the map, and introduce the bundle $J_{k,p} \to X$ of k-jets of maps $\mathbb{C}^p \to X$, so that the fibre over $x \in X$ is the set of equivalence classes of germs of holomorphic maps $f : (\mathbb{C}^p, 0) \to (X, x)$, with the equivalence relation $f \sim g$ if and only if all derivatives $f^{(j)}(0) = g^{(j)}(0)$ are equal for $0 \le j \le k$. Recall also that $\mathbb{G}_{k,p}$ is the group of k-jets of germs of biholomorphisms of $(\mathbb{C}^p, 0)$, which has a natural fibrewise right action on $J_{k,p}$ with the matrix representation given by

$$G_{k,p} = \begin{pmatrix} \Phi_1 & \Phi_2 & \Phi_3 & \dots & \Phi_k \\ 0 & \Phi_1^2 & \Phi_1\Phi_2 & \dots & \\ 0 & 0 & \Phi_1^3 & \dots & \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & & \Phi_1^k \end{pmatrix}, \tag{69}$$

where $\Phi_i \in \mathrm{Hom}(\mathrm{Sym}^i\mathbb{C}^p, \mathbb{C}^p)$ and $\det\Phi_1 \neq 0$, and that $\mathbb{G}_{k,p}$ is generated along its first p rows, in the sense that the parameters in the first p row are independent, and all the remaining entries are polynomials in these parameters. The parameters in the $(1,m)$ block are indexed by a basis of $\mathrm{Sym}^m(\mathbb{C}^p) \times \mathbb{C}^p$, so they are of the form α_ν^l where $\nu \in \binom{p+m-1}{m-1}$ is an m-tuple and $1 \leq l \leq p$, and the polynomial in the (l,m) block and entry indexed by $\tau = (\tau[1], \dots, \tau[l]) \in \binom{p+l-1}{l-1}$ and $\nu \in \binom{p+m-1}{m-1}$ is given by

$$(G_{k,p})_{\tau,\nu} = \sum_{\nu_1+\ldots+\nu_l=\nu} \alpha_{\nu_1}^{\tau[1]}\alpha_{\nu_2}^{\tau[2]}\dots\alpha_{\nu_l}^{\tau[l]}. \tag{70}$$

Recall also that $\mathbb{G}_{k,p} = \mathbb{U}_{k,p} \rtimes GL_p$ is an extension of its unipotent radical $\mathbb{U}_{k,p}$ by GL_p, and that the generalized Demailly-Semple jet bundle $E_{k,p,m} \to X$ of invariant jet differentials of order k and weighted degree $(m, \dots, m)$ consists of the jet differentials which transform under any reparametrization $\phi \in \mathbb{G}_{k,p}$ of $(\mathbb{C}^p, 0)$ as

$$Q(f \circ \phi) = (J_\phi)^m Q(f) \circ \phi,$$

where $J_\phi = \det\Phi_1$ denotes the Jacobian of ϕ, so that $E_{k,p} = \oplus_{m\geq 0}E_{k,p,m}$ is the graded algebra of $\mathbb{U}_{k,p}$-invariants.

8.1. **Geometric description for $p > 1$.** As in the case when $p = 1$ our goal is to give a geometric description of the invariants by finding a suitable projective embedding of the quotient $J_{k,p}^{reg}/G_{k,p}$.

Remark 8.1. In [32] Pacienza and Rousseau generalize the inductive process given in [5] of constructing a smooth compactification of the Demailly-Semple jet bundles. Using the concept of a directed manifold, they define a bundle $X_{k,p} \to X$ with smooth fibres, and the effective locus $Z_{k,p} \subset X_{k,p}$, and a holomorphic embedding $J_{k,p}^{reg}/G_{k,p} \hookrightarrow Z_{k,p}$ which identifies $J_{k,p}^{reg}/G_{k,p}$ with $Z_{k,p}^{reg} = X_{k,p}^{reg} \cap Z_{k,p}$, and therefore $Z_{k,p}$ is a relative compactification of $J_{k,p}/G_{k,p}$. We choose a different approach, generalizing the test curve model, resulting in a holomorphic embedding $J_{k,p}/G_{k,p}$ into a partial flag manifold and a different compactification, which is a singular subvariety of the patial flag manifold, such that the invariant jet differentials of degree divisible by $\mathrm{sym}^{\leq k}p$ are given by polynomial expressions in the Plücker coordinates.

Fix $x \in X$ and an identification of T_xX with $\mathbb{C}^n$; then let $J_k(p,n) = J_{k,p,x}$ as defined in §2. Let

$$J_k^{\mathrm{reg}}(p,n) = \{\gamma \in J_k(p,n) : \Gamma_1 \text{ is non-degenerate}\}$$

where γ is represented by

$$\mathbf{u} \mapsto \gamma(\mathbf{u}) = \Gamma_1\mathbf{u} + \Gamma_2\mathbf{u}^2 + \ldots + \Gamma_k\mathbf{u}^k$$

with $\Gamma_i \in \operatorname{Hom}(\operatorname{Sym}^i\mathbb{C}^p, \mathbb{C}^p)$. Let $N \geq n$ be any integer and define

$$\Upsilon_{k,p} = \left\{\Psi \in J_k(n,N) : \exists\gamma \in J_k^{\mathrm{reg}}(p,n) : \Psi\circ\gamma = 0\right\}.$$

Remark 8.2. The global singularity theory description of $\Upsilon_{k,p}$ is

$$\Upsilon_{k,p} \doteq \left\{p = (p_1,\ldots,p_N) \in J_k(n,N) : \mathbb{C}[z_1,\ldots,z_n]/\langle p_1,\ldots,p_N\rangle \cong \mathbb{C}[x,y]/\langle z_1,\ldots,z_n\rangle^{k+1}\right\}.$$

Note, again, as in the $p = 1$ case, that if $\gamma \in J_k^{\mathrm{reg}}(p,n)$ is a test surface of $\Psi \in \Upsilon_{k,p}$, and $\varphi \in \mathbb{G}_k$ is a holomorphic reparametrization of $\mathbb{C}^p$, then $\gamma\circ\varphi$ is, again, a test surface of Ψ:

$$\mathbb{C}^p \xrightarrow{\varphi} \mathbb{C}^p \xrightarrow{\gamma} \mathbb{C}^n \xrightarrow{\Psi} \mathbb{C}^N \tag{71}$$

$$\Psi\circ\gamma = 0 \quad\Rightarrow\quad \Psi\circ(\gamma\circ\varphi) = 0$$

Example 8.3. $k = 2, p = 2.$
Let $\Psi(\mathbf{z}) = \Psi'\mathbf{z} + \Psi''\mathbf{z}^2$ *for* $\mathbf{z} \in \mathbb{C}^n$, *and*

$$\gamma(u_1,u_2) = \gamma_{10}u_1 + \gamma_{01}u_2 + \gamma_{20}u_1^2 + \gamma_{11}u_1u_2 + \gamma_{02}u_2^2,\ \gamma_{ij} \in \mathbb{C}^n.$$

Then $\Psi\circ\gamma = 0$ *has the form*

$$\Psi'(\gamma_{10}) = 0\ ;\ \Psi'(\gamma_{01}) = 0 \tag{72}$$

$$\Psi'(\gamma_{20}) + \Psi''(\gamma_{10},\gamma_{10}) = 0,\ ;\ \Psi'(\gamma_{11}) + 2\Psi''(\gamma_{10},\gamma_{01}) = 0,\ ;\ \Psi'(\gamma_{01}) + \Psi''(\gamma_{01},\gamma_{01}) = 0,$$

We introduce

$$\mathcal{S}_\gamma = \{\Psi \in J_k(n,N) : \Psi\circ\gamma = 0\}$$

and the following analogue of $J_k^o(1,n)$:

$$J_k^o(n,N) = \{\Psi \in J_k(n,N) : \dim\ker\Psi = p\}.$$

The proof of the following proposition is analogous to that of Proposition 4.7 in [3], and we omit the details. We use the notation

$$\operatorname{sym}^i(p) = \dim(\operatorname{Sym}^i\mathbb{C}^p);\ \operatorname{sym}^{\leq k}(p) = \dim(\mathbb{C}^p \oplus \operatorname{Sym}^2\mathbb{C}^p \oplus \ldots \oplus \operatorname{Sym}^k\mathbb{C}^p) = \sum_{i=1}^k \operatorname{sym}^i p.$$

Proposition 8.4. (i) *If* $\gamma \in J_k^{\mathrm{reg}}(p,n)$ *then* $\mathcal{S}_\gamma \subset J_k(n,N)$ *is a linear subspace of codimension* $N\operatorname{sym}^{\leq k}(p)$.

(ii) *For any* $\gamma \in J_k^{\mathrm{reg}}(p,n)$, *the subset* $\mathcal{S}_\gamma \cap J_k^o(n,N)$ *of* $\mathcal{S}_\gamma$ *is dense.*

(iii) *If* $\Psi \in J_k^o(n,N)$, *then* Ψ *belongs to at most one of the spaces* $\mathcal{S}_\gamma$. *More precisely, if*

$$\gamma_1,\gamma_2 \in J_k^{\mathrm{reg}}(p,n),\quad \Psi \in J_k^o(n,N) \text{ and } \Psi\circ\gamma_1 = \Psi\circ\gamma_2 = 0,$$

then there exists $\varphi \in J_k^{\mathrm{reg}}(p,p)$ *such that* $\gamma_1 = \gamma_2\circ\varphi$.

(iv) *Given* $\gamma_1, \gamma_2 \in J_k^{\mathrm{reg}}(1,n)$*, we have* $\mathcal{S}_{\gamma_1} = \mathcal{S}_{\gamma_2}$ *if and only if there is some* $\varphi \in J_k^{\mathrm{reg}}(1,1)$ *such that* $\gamma_1 = \gamma_2 \circ \varphi$*.*

With the notation

$$\Upsilon_{k,p} = \Upsilon_{k,p} \cap J_k^o(n,N),$$

we deduce from Proposition 8.4 the following

Corollary 8.5. $\Upsilon^0_{k,p}$ *is a dense subset of* $\Upsilon_{k,p}$*, and* $\Upsilon^0_{k,p}$ *has a fibration over the orbit space* $J_k^{\mathrm{reg}}(p,n)/J_k^{\mathrm{reg}}(p,p) = J_k^{\mathrm{reg}}(p,n)/G_{k,p}$ *with linear fibres.*

Remark 8.6. In fact, Proposition 8.4 says a bit more, namely that $\Upsilon^0_{k,p}$ is fibrewise dense in $\Upsilon_{k,p}$ over $J_k^{\mathrm{reg}}(p,n)/G_{k,p}$, but we will not use this stronger statement.

By the first part of Proposition 8.4 the assignment $\gamma \to \mathcal{S}_\gamma$ defines a map

$$\nu : J_k^{\mathrm{reg}}(p,n) \to \mathrm{Grass}(kN, J_k(n,N))$$

which, by the fourth part, descends to the quotient

$$\bar{\nu} : J_k^{\mathrm{reg}}(p,n)/G_{k,p} \hookrightarrow \mathrm{Grass}(kN, J_k(n,N)) \tag{73}$$

(cf. Proposition 4.4). Next, we want to rewrite this embedding in terms of the identifications introduced in §5. So we

- identify $J_k(p,n)$ with $\mathrm{Hom}\,(\mathbb{C}^{\mathrm{sym}^1 p} \oplus \ldots \oplus \mathbb{C}^{\mathrm{sym}^k p}, \mathbb{C}^n) = \mathrm{Hom}\,(\mathbb{C}^{\mathrm{sym}^{\le k}(p)}, \mathbb{C}^n)$ where $\mathrm{sym}^j p = \dim \mathrm{Sym}^j \mathbb{C}^p$ and $\mathrm{sym}^{\le k}(p) = \sum_{j=1}^k \mathrm{sym}^j p$;
- identify $J_k(n,1)^*$ with $\mathrm{Sym}^{\le k}\mathbb{C}^n = \oplus_{l=1}^k \mathrm{Sym}^{\,l}\mathbb{C}^n$.

We think of an element v of $\mathrm{Hom}\,(\mathbb{C}^{\mathrm{sym}^{\le k}(p)}, \mathbb{C}^n)$ as an $n \times \mathrm{sym}^{\le k}(p)$ matrix, with column vectors in $\mathbb{C}^n$. These columns correspond to basis elements of $\mathbb{C}^{\mathrm{sym}^1 p} \oplus \ldots \oplus \mathbb{C}^{\mathrm{sym}^k p}$, and the columns in the ith component are indexed by i-tuples $1 \le t_1 \le t_2 \le \ldots \le t_i \le p$, or equivalently by

$$(e_{t_1} + e_{t_2} + \ldots + e_{t_i}) \in \mathbb{Z}^p_{\ge 0}$$

where $e_j = (0, \ldots, 1, \ldots, 0)$ with 1 in the jth place, giving us

$$v = (v_{10,\ldots 0}, v_{01\ldots 0}, \ldots, v_{0\ldots 0k}) \in \mathrm{Hom}\,(\mathbb{C}^{\mathrm{sym}^{\le k}(p)}, \mathbb{C}^n).$$

The elements of $J_k^{\mathrm{reg}}(p,n)$ correspond to matrices whose first p columns are linearly independent. When $n \ge \mathrm{sym}^{\le k}(p)$ there is a smaller dense open subset $J_k^{\mathrm{nondeg}}(p,n) \subset J_k^{\mathrm{reg}}(p,n)$ consisting of the $n \times \mathrm{sym}^{\le k}(p)$ matrices of rank $\mathrm{sym}^{\le k}(p)$.

Define the following map, whose components correspond to the equations in (72):

$$\begin{aligned} \phi : \mathrm{Hom}\,(\mathbb{C}^{\mathrm{sym}^{\le k}(p)}, \mathbb{C}^n) &\to \mathrm{Hom}\,(\mathbb{C}^{\mathrm{sym}^{\le k}(p)}, \mathrm{Sym}^{\le k}\mathbb{C}^n) \\ (v_{10,\ldots 0}, v_{01\ldots 0}, \ldots, v_{0\ldots 0k}) &\mapsto (\ldots, \textstyle\sum_{\mathbf{s}_1 + \mathbf{s}_2 + \ldots + \mathbf{s}_j = \mathbf{s}} v_{\mathbf{s}_1} v_{\mathbf{s}_2} \ldots v_{\mathbf{s}_j}, \ldots), \end{aligned} \tag{74}$$

where on the right hand side $\mathbf{s} \in \mathbb{Z}^p_{\ge 0}$.

Example 8.7. *If $k = p = 2$ then ϕ is given by*

$$\phi(v_{10}, v_{01}, v_{20}, v_{11}, v_{02}) = (v_{10}, v_{01}, v_{20} + v_{10}^2, v_{11} + 2v_{10}v_{01}, v_{02} + v_{01}^2).$$

Let $P_{k,p} \subset GL_{\mathrm{sym}^{\leq k}(p)}$ denote the parabolic subgroup with Levi subgroup

$$GL(\mathrm{sym}^1 p) \times \ldots \times GL(\mathrm{sym}^k p),$$

where $\mathrm{sym}^j p = \dim \mathrm{Sym}^j \mathbb{C}^p$ and $\mathrm{sym}^{\leq k}(p) = \sum_{j=1}^k \mathrm{sym}^j p$. Then (73) has the following reformulation, analogous to Proposition 5.1.

Proposition 8.8. *The map ϕ in* (74) *is a $\mathbb{G}_{k,p}$-invariant algebraic morphism*

$$\phi : J_k^{\mathrm{reg}}(p, n) \to \mathrm{Hom}\,(\mathbb{C}^{\mathrm{sym}(p)}, \mathrm{Sym}^{\leq k}\mathbb{C}^n)$$

which induces an injective map ϕ^{Grass} on the $\mathbb{G}_{k,p}$-orbits:

$$\phi^{\mathrm{Grass}} : J_k^{\mathrm{reg}}(p, n)/\mathbb{G}_{k,p} \hookrightarrow \mathrm{Grass}_{\mathrm{sym}^{\leq k}(p)}(\mathrm{Sym}^{\leq k}\mathbb{C}^n)$$

and

$$\phi^{\mathrm{Flag}} : J_k^{\mathrm{reg}}(p, n)/\mathbb{G}_{k,p} \hookrightarrow \mathrm{Flag}_{\mathrm{sym}^1(p),\ldots,\mathrm{sym}^k(p)}(\mathrm{Sym}^{\leq k}\mathbb{C}^n) \hookrightarrow \mathrm{Hom}\,(\mathbb{C}^{\mathrm{sym}(p)}, \mathrm{Sym}^{\leq k}\mathbb{C}^n)/P_{k,p}.$$

Composition with the Plücker embedding gives

$$\phi^{\mathrm{Proj}} = \mathrm{Pluck} \circ \phi^{\mathrm{Grass}} : J_k^{\mathrm{reg}}(p, n)/\mathbb{G}_{k,p} \hookrightarrow \mathbb{P}(\wedge^{\mathrm{sym}^{\leq k}(p)}\mathrm{Sym}^{\leq k}\mathbb{C}^n).$$

As in the case when $p = 1$, we introduce the following notation

$$X_{n,k,p} = \phi^{\mathrm{Proj}}(J_k^{\mathrm{reg}}(p, n)), \quad Y_{n,k,p} = \phi^{\mathrm{Proj}}(J_k^{\mathrm{nondeg}}(p, n)) \subset \mathbb{P}(\wedge^{\mathrm{sym}^{\leq k}}(\mathrm{Sym}^{\leq k}\mathbb{C}^n)).$$

9. Boundary components for $p > 1$

In this section we study the boundary components of $X_{n,k,p}$ and $Y_{n,k,p}$.

Definition 9.1. *Let $n \geq \mathrm{sym}^{\leq k}(p) = \mathrm{sym}^1(p) + \ldots + \mathrm{sym}^k(p)$. Then the open subset of $\mathbb{P}(\wedge^{\mathrm{sym}^{\leq k}(p)}(\mathrm{Sym}^{\leq k}\mathbb{C}^n))$ where the projection to $\wedge^{\mathrm{sym}^{\leq k}(p)}\mathbb{C}^n$ is nonzero is denoted by $A_{n,k,p}$.*

Since ϕ^{Grass} and ϕ^{Proj} are $GL(n)$-equivariant, and for $n \geq \mathrm{sym}^{\leq k}(p)$ the action of $GL(n)$ is transitive on $\mathrm{Hom}^{\mathrm{nondeg}}(\mathbb{C}^{\mathrm{sym}^{\leq k}(p)}, \mathbb{C}^n)$, we have

Lemma 9.2. (i) *If $n \geq \mathrm{sym}^{\leq k}(p)$ then $X_{n,k,p}$ is the $GL(n)$ orbit of*

$$\mathbf{z} = \phi^{\mathrm{Proj}}(e_1, \ldots, e_{\mathrm{Sym}^{\leq k}(p)}) = [\wedge_{j_1+\ldots+j_p \leq k} \sum_{\mathbf{i}_1+\ldots+\mathbf{i}_s=(j_1,\ldots,j_p)} e_{\mathbf{i}_1} \ldots e_{\mathbf{i}_s}] \tag{75}$$

in $\mathbb{P}(\wedge^{\mathrm{sym}^{\leq k}(p)}(\mathrm{Sym}^{\leq k}\mathbb{C}^n))$.

(ii) *If $n \geq \mathrm{sym}^{\leq k}(p)$ then $X_{n,k,p}$ and $Y_{n,k,p}$ are finite unions of $GL(n)$ orbits.*

(iii) *For $k > n$ the images $X_{n,k,p}$ and $Y_{n,k,p}$ are $GL(n)$-invariant quasi-projective varieties, though they have no dense $GL(n)$ orbit.*

Similar statements hold for the closure of the image in the Grassmannian $Grass_{\mathrm{sym}^{\leq k}(p)}(\mathrm{Sym}^{\leq k}\mathbb{C}^n)$ (or equivalently in the projective space $\mathbb{P}(\wedge^{\mathrm{sym}^{\leq k(p)}}(\mathrm{Sym}^{\leq k}\mathbb{C}^n))$).

Lemma 9.3. *Let $n \ge \mathrm{sym}^{\le k}(\mathbb{C}^n)$; then*

(i) *$A_{n,k,p}$ is invariant under the $GL(n)$ action on $\mathbb{P}(\wedge^{\mathrm{sym}^{\le k}(p)}(\mathrm{Sym}^{\le k}\mathbb{C}^n))$;*
(ii) *$X_{n,k,p} \subset A_{n,k,p}$, although $Y_{n,k,p} \not\subseteq A_{n,k,p}$;*
(iii) *$\overline{X}_{n,k,p}$ is the union of finitely many $GL(n)$-orbits.*

The image of $J_k^{\mathrm{reg}}(p,n)/\mathbb{G}_{k,p}$ is contained in $A_{n,k,p}$, and the goal is to prove the following generalization of Theorem 6.5:

Theorem 9.4. (i) *Assume that $p > 1$ and $n \ge \mathrm{sym}^{\le k}(p)$ where $\mathrm{sym}^{\le k}(p) = \dim \mathrm{Sym}^{\le k}\mathbb{C}^p$. Then the intersection with $A_{n,k,p}$ of the boundary components of $X_{n,k,p}$ have codimension at least two.*
(ii) *The boundary components of $Y_{n,k,p} \subset \mathbb{P}(\wedge^{\mathrm{sym}^{\le k}(p)}(\mathrm{Sym}^{\le k}\mathbb{C}^n))$ have codimension at least two.*

Note that for $p > 1$ the condition that $k \ge 4$ is not necessary. The proof follows the ideas of the case when $p = 1$ and therefore we do not give all the details. The strategy of the proof is the same: first we notice that the dimension of the stabilizer of any point in $X_{n,k,p}$ is $p \cdot \mathrm{sym}^{\le k}(p)$, and then we prove that the dimension of the stabilizer of any point in $\overline{Y}_{n,k,p} \setminus Y_{n,k,p}$ and $(\overline{X}_{n,k,p} \setminus X_{n,k,p}) \cap A_{n,k,p}$ is at least $p \cdot \mathrm{sym}^{\le k}(p) + 2$. The first part is clear from the observation that the stabilizer of any point in $X_{n,k,p}$ is conjugate to (in shorthand)

$$G_{\mathbf{z}} = \{ \begin{pmatrix} \mathbb{G}_{k,p} & * \\ 0 & GL(n - \mathrm{sym}^{\le k}(p)) \end{pmatrix} \}.$$

9.1. **Orbit structure and maximal boundary orbits.** Let

$$Z_{n,k,p} = T \cdot \mathbf{z} \subset \mathbb{P}(\wedge^{\mathrm{sym}^{\le k}(p)}(\mathrm{Sym}^{\le k}(\mathbb{C}^n))$$

be the torus orbit. Proposition 6.8, Corollary 6.11 and Lemma 6.12 remain valid for $p > 1$, allowing us to identify the boundary components using 1-parameter subgroups of T. Note that the stabilizer of $\mathbf{z}$ contains a $p + (n - \mathrm{sym}^{\le k}(p)$-dimensional torus inside T, which is the maximal torus in $GL(p) \times GL(n - \mathrm{sym}^{\le k}(p)) \subset G_{\mathbf{z}}$.

Take a one-parameter subgroup λ of the maximal torus in $GL(n)$ given by

$$\lambda(t) = \begin{pmatrix} t^{\lambda_1} & & & \\ & \cdot & & \\ & & \cdot & \\ & & & t^{\lambda_n} \end{pmatrix}$$

and note that the action of $\lambda(t)$ on $\mathbf{z}$ does not depend on the last $n - \mathrm{sym}^{\le k}(p)$ coordinates of λ.

Definition 9.5. We will often index the first $\mathrm{sym}^{\le k}(p)$ coordinates of λ by ordered p-tuples $(i_1, \ldots, i_p)$ with $i_1 + \ldots + i_p \le k$, and

$$\underline{\mathrm{sym}}^{\le k}(p) = \underline{\mathrm{sym}}^1(p) \cup \underline{\mathrm{sym}}^2(p) \cup \cdots \cup \underline{\mathrm{sym}}^k(p)$$

will denote the set of these p-tuples. We will use the following notation:

- for $\mathbf{i} = (i_1, \ldots, i_p) \in \underline{\mathrm{sym}}^{\leq k}(p)$ let $m_\lambda[\mathbf{i}] = \min_{\varepsilon_1+\ldots+\varepsilon_s=\mathbf{i}} \lambda_\varepsilon$ for $1 \leq i \leq k$, where $\varepsilon_j \in \underline{\mathrm{sym}}^{\leq k}(p)$.
- $z_\lambda[\mathbf{i}] = \sum_{\Sigma\varepsilon=\mathbf{i},\lambda_\varepsilon=m_\lambda[\mathbf{i}]} e_\varepsilon$ where $e_\varepsilon = e_{\varepsilon_1} \ldots e_{\varepsilon_s}$.

Then

(i) the boundary components of $Z_{n,k,p}$ (respectively $X_{n,k,p}$) are $\overline{T \cdot \mathbf{z}_\lambda}$ (respectively $\overline{G \cdot \mathbf{z}_\lambda}$) for some one-parameter subgroups λ, where $\mathbf{z}_\lambda = \wedge_{i_1}^{\mathrm{sym}^{\leq k}(p)} z_\lambda[i]$;

(1) $\mathbf{z}_\lambda \in A_{n,k,p}$ if and only if $\lambda_{\mathbf{i}} = m_\lambda[\mathbf{i}]$ for all $\mathbf{i} \in \mathrm{sym}^{\leq k}(p)$.

Let O_λ denote the $GL(n)$-orbit of $\mathbf{z}_\lambda$, and recall that $\mathrm{sym}^j(p) = \dim \mathrm{Sym}^j \mathbb{C}^p$ where $\mathrm{sym}^{\leq k}(p) = \dim \mathrm{Sym}^{\leq k} \mathbb{C}^p$. The stabilizer $G_{\mathbf{z}}$ contains the maximal torus T^p of GL_p, embedded as $\mathrm{diag}(t^{\tilde{\lambda}_\tau} : \tau \in \mathrm{sym}^{\leq k}(p)) \subset GL_n$, where

$$\tilde{\lambda}_{(0,\ldots,1^i,\ldots,0)} = \lambda_i \text{ for } \tau = (0, \ldots, 1^i, \ldots, 0) \in \mathrm{sym}^1(p)$$

and

$$\tilde{\lambda}_{(i_1,\ldots,i_p)} = i_1\lambda_1 + \ldots + i_p\lambda_p \text{ for } \tau = (i_1, \ldots, i_p) \in \mathrm{sym}^{i_1+\ldots+i_p}(p). \tag{76}$$

The last $n - \mathrm{sym}^{\leq k}(p)$ coordinates are irrelevant for $\tilde{\lambda}$, so we can define them to be 0.

Now we define the 1-parameter subgroups which move $\mathbf{z}$ to the maximal boundary components.

Definition 9.6. *Choose a positive $\varepsilon << 1$ and $\sigma \in \underline{\mathrm{sym}}^2(p) \cup \underline{\mathrm{sym}}^3(p) \cup \ldots \cup \underline{\mathrm{sym}}^k(p)$. For $\tau \in \underline{\mathrm{sym}}^{\leq k}(p)$ we denote by $L(\tau, \sigma) = \tau/\sigma$ the quotient of the two p-tuples, i.e. the greatest integer such that*

$$\tau = L(\tau, \sigma)\sigma + \xi$$

for some $\xi \in \mathrm{sym}^{\leq k}(p)$. Let λ^σ and μ^σ be the one-parameter subgroups of the maximal torus T of $GL(p) \times GL(n - \mathrm{sym}^{\leq k}(p))$ such that

$$\lambda^\sigma_\tau = \begin{cases} \tilde{\lambda}_\tau - L(\tau, \sigma)\,\varepsilon \text{ if } \tau \in \underline{\mathrm{sym}}^{\leq k}(p) \\ 0 \text{ if } \tau \notin \underline{\mathrm{sym}}^{\leq k}(p) \end{cases}$$

$$\mu^\sigma_\tau = \begin{cases} \tilde{\lambda}_\tau \text{ if } \tau \neq \sigma \text{ and } \tau \in \underline{\mathrm{sym}}^{\leq k}(p) \\ \tilde{\lambda}_\sigma \text{ if } \tau = \sigma \\ 0 \text{ if } \tau \notin \underline{\mathrm{sym}}^{\leq k}(p). \end{cases}$$

A short computation shows that

$$\mathbf{z}_{\lambda^\sigma}[\sigma] = e_\sigma$$

where $\mathbf{z}_{\lambda^\sigma}[\sigma]$ is defined as in Definition 9.5. For $\tau_1, \tau_2 \in \underline{\mathrm{sym}}^{\leq k}(p)$ let $\tau_1 < \tau_2$ if either $\Sigma(\tau_1) < \Sigma(\tau_2)$ or $\Sigma(\tau_1) = \Sigma(\tau_2)$ and τ_1 is smaller with respect to the lexicographic order. We call $\sigma = \mathrm{Head}(\lambda)$ the head of $\lambda = (\lambda_\tau : \tau \in \mathrm{sym}^{\leq k}(p))$ if $\lambda_{\mathbf{i}} = \tilde{\lambda}_{\mathbf{i}}$ for $\mathbf{i} < \sigma$, but $\lambda_\sigma \neq \tilde{\lambda}_\sigma$. If $\lambda_\sigma < \tilde{\lambda}_\sigma$ then we call λ regular, otherwise degenerate.

Using Definition 6.17, we can see just as for $p = 1$ that the dimension of the maximal torus in the stabilizer of $\mathbf{z}_\lambda$ is equal to the toral dimension of λ, and again, we have to focus on those λ whose toral dimension is 1 or 2. The following description of the maximal boundary components of $X_{n,k,p}$ can be proved similarly to Lemma 6.18:

Lemma 9.7. (i) *The maximal regular 1-parameter subgroups have toral dimension at least* 2*. Those with toral dimension* 2 *are* λ^σ *such that* $\sigma \in \underline{\mathrm{sym}}^2(p) \cup \ldots \cup \underline{\mathrm{sym}}^k(p)$*; in other words for a regular* λ *with Head*$(\lambda) = \sigma$ $O_\lambda \subset \overline{O_{\lambda^\sigma}}$*. The regular boundary components lie in* $A_{n,k,p}$*.*

(ii) *The maximal degenerate 1-parameter subgroups are* μ^σ *such that* $\sigma \in \underline{\mathrm{sym}}^2(p) \cup \ldots \cup \underline{\mathrm{sym}}^k(p)$*; in other words for a regular* μ *with Head*$(\mu) = \sigma$ *we have* $O_\mu \subset \overline{O_{\mu^\sigma}}$*. The degenerate boundary components lie outside* $A_{n,k,p}$*.*

(iii) $\mathbf{z}_{\mu^\sigma} \in Y_{n,k,p}$ *and therefore the degenerate boundary orbits sit in* $Y_{n,k,p}$*, and they are not boundary orbits.*

According to Lemma 9.7, the codimension-at-least-two property has to be proved only for the regular boundary components. The following analogue of Proposition 6.21 (with the same proof) identifies the boundary orbits of $Y_{n,k,p}$.

Proposition 9.8. *Let* $n \geq \mathrm{sym}^{\leq k}(p)$*. The boundary orbits of* $Y_{n,k,p}$ *lie in the closures of boundary orbits in* $A_{n,k,p}$*.*

9.2. **The limit of the stabilizers.** The next step is to prove

Proposition 9.9. $G^\sigma = \lim_{t\to 0} G_{\lambda^\sigma(t)\mathbf{z}} \subset GL(n)$ *is a subgroup of* $G_{\mathbf{z}_{\lambda^\sigma}}$ *with dimension*

$$\dim G_{\mathbf{z}} = p(\mathrm{sym}^{\leq k}(p)) + n(n - \mathrm{sym}^{\leq k}(p)).$$

Proof. The (τ, ν) entry of the stabilizer of $\lambda^s(t)\mathbf{z}$ is

$$(G_{\lambda^\sigma(t)\mathbf{z}})_{\tau,\nu} = t^{\lambda^\sigma_\tau - \lambda^\sigma_\nu} p_{\tau,\nu}(\alpha). \tag{77}$$

To determine the limit as $t \to 0$ we study the Lie algebra $\mathfrak{g}_{\mathbf{z}} = \mathrm{Lie}(G_{\mathbf{z}})$. We are only interested in the upper left $\mathrm{sym}^{\leq k}(p) \times \mathrm{sym}^{\leq k}(p)$ minor, which is $\mathfrak{g}_{k,p} = \mathrm{Lie}(G_{k,p})$. This is generated along the first p rows, and the entries in the other rows are linear forms in these $p \cdot \mathrm{sym}^{\leq k}(p)$ variables. For a parameter a let E_a denote the set of those entries of $\mathfrak{g}_{k,p}$ where a occurs with nonzero coefficient, and define

$$n^\sigma_a = \max_{(\tau,\nu)\in E_a} \lambda^\sigma_\tau - \lambda^\sigma_\nu$$

Note that by definition $n^\sigma_1 = 0$ for all σ. Then the following analogue of Lemma 6.23 holds:

Lemma 9.10. (i) *Under the substitution*

$$\beta^\sigma_a = t^{-n^\sigma_a} \alpha^\sigma_a$$

we have

$$G_{\lambda^\sigma(t)\mathbf{z}}(\beta_{1,1},\dots,\beta_{p,\mathrm{sym}^{\le k}(p)}) \in GL(\mathbb{C}[\beta_{1,1},\dots,\beta_{p,\mathrm{sym}^{\le k}(p)}][t],$$

so the entries are polynomials in t with coefficients in $\mathbb{C}[\beta_{1,1},\dots,\beta_{p,\mathrm{sym}^{\le k}(p)}]$.

(ii) $A^\sigma = \lim_{t\to 0} G_{\lambda^\sigma(t)\mathbf{z}}$ *has dimension* $p\cdot \mathrm{sym}^{\le k}(p) + n(n-\mathrm{sym}^{\le k}(p))$.

□

9.3. **Two extra dimensions in the stabilizer of the limit.** Finally, we prove the analogue of Proposition 6.25 in this more general situation, and find two extra dimensions in the stabilizer.

Proposition 9.11. *There exists a 2-dimensional subgroup $B^\sigma \subset G_{\mathbf{z}_{\lambda^\sigma}}$ with $G^\sigma \cap B^\sigma = 0$.*

Proof. One can easily check that $\mathbf{z}_{\lambda^\sigma}[\sigma]=e_\sigma$ implies that there is an $p+1$-dimensional torus in the upper left $\mathrm{sym}^{\le k}(p)\times\mathrm{sym}^{\le k}(p)$ minor – which we call temporarily the main minor – of $G_{\mathbf{z}^\sigma_\lambda}$. Indeed, giving the weight λ^σ_τ to e_τ, $\mathbf{z}_{\lambda^\sigma}[\tau]$ is homogeneous of degree

$$\lambda^\sigma_\tau = \lambda_\tau - L(\tau,\sigma)\varepsilon.$$

Therefore, the p-dimensional torus $\mathrm{diag}(\lambda^\sigma_\tau : \tau\in\mathrm{sym}^{\le k}(p))$ is in the stabilizer.

This implies that the weights

$$\lambda_{(0,\dots,1^i,\dots,0)} = \lambda_i,\ i=1,\dots,p,\ t_\sigma = \chi$$

induce a 1-parameter subgroup sitting in the stabilizer $G_{\mathbf{z}_{\lambda^\sigma}}$:

$$\lambda_\tau = \tilde{\lambda}_\tau + L(\tau,\sigma)\chi \text{ for } \tau\in\mathrm{sym}^{\ge 2}(p).$$

This is the $p+1$-dimensional torus in the main minor of $G_{\mathbf{z}_{\lambda^\sigma}}$, and it can be easily shown that there is no higher dimensional torus in the main minor.

It remains to find an extra dimension in the unipotent radical of $G_{\mathbf{z}_{\lambda^\sigma}}$. We will see that here there are only two cases, corresponding to Lemma 6.26 and Lemma 6.27; we do not have to study the situation in Lemma 6.28 separately. For this reason, we do not need the condition $k\ge 4$ which was required when $p=1$.

Lemma 9.12. *There exists a one-dimensional unipotent subgroup in $G_{\mathbf{z}_{\lambda^\sigma}}\setminus\lim_{t\to 0}G_{\lambda^\sigma(t)\mathbf{z}}$ when $\sigma\in\underline{\mathrm{sym}}^k(p)$.*

Proof. Fix $\delta\in\underline{\mathrm{sym}}^{k-1}(p)$. Let $T\in GL(n)$ denote the transformation

$$T(e_\tau)=e_\tau \text{ for } \tau\neq\delta\ ;\ T(e_\delta)=e_\delta+\zeta e_\sigma$$

For the same reason as in the case when $p=1$ this is in $G_{\mathbf{z}_{\lambda^\sigma}}\setminus\lim_{t\to 0}G_{\lambda^\sigma(t)\mathbf{z}}$. □

Lemma 9.13. *There exists a one-dimensional unipotent subgroup in $G_{\mathbf{z}_{\lambda^\sigma}}\setminus\lim_{t\to 0}G_{\lambda^\sigma(t)\mathbf{z}}$ when $\sigma\in\underline{\mathrm{sym}}^i(p)$ with $i<k$, $p>1$.*

Proof. For $\delta \in \underline{\mathrm{sym}}^k(p)$ let T_δ be the transformation

$$T(e_\tau) = e_\tau \text{ for } \tau \neq \delta\,;\ T(e_\delta) = e_\delta + \zeta e_\sigma. \tag{78}$$

Since $\mathbf{z}_{\lambda^\sigma}[\sigma] = e_\sigma$, it is clear that $T_\delta \in G_{\mathbf{z}_{\lambda^\sigma}}$ for all $\delta \in \underline{\mathrm{sym}}^k(p)$. We show that there is some $\delta \in \underline{\mathrm{sym}}^k(p)$ such that $T_\delta \notin \lim_{t\to 0} G_{\lambda^\sigma(t)\mathbf{z}}$.

Case 1. If $\overline{\sigma} = (0, i_2 \ldots, i_p)$ with $i_2, \ldots, i_p > 0$ then $\delta = (k, 0 \ldots, 0)$ is a good choice. Indeed, in this case $(\mathfrak{g}_\mathbf{z})_{\sigma,\delta} = 0$, and therefore $(\lim_{t\to 0} G_{\lambda^\sigma(t)\mathbf{z}})_{\sigma,\delta} = 0$, so $T_\delta \notin \lim_{t\to 0} G_{\lambda^\sigma(t)\mathbf{z}}$. The same reasoning works for any σ with at least one 0 coordinate.

Case 2. If $\sigma = (i_1, \ldots, i_p)$ with positive entries, let δ be a p-tuple in $\overline{\mathrm{sym}}^k(p)$ such that $L(\delta, \sigma) = \delta/\sigma$ is maximal. We will prove that for this choice $T_\delta \notin \lim_{t\to 0} G_{\lambda^\sigma(t)\mathbf{z}}$.

We have $\delta = L(\delta,\sigma)\sigma + \xi$ for some $\xi \in \underline{\mathrm{sym}}^{k-L(\delta,\sigma)i}(p)$, and therefore

$$\lambda^\sigma_\delta - \lambda^\sigma_\sigma = \lambda_\delta - \lambda_\sigma - (L(\delta,\sigma) - 1)\varepsilon.$$

The (δ,σ)-entry of $\mathfrak{g}_\mathbf{z}$ is a linear form in the parameters $\alpha_{i,\tau}$ such that $1 \le i \le p$ and $\tau \in \underline{\mathrm{sym}}^{\le k}(p)$, namely for $\delta = (i_1, \ldots, i_p)$ according to Proposition 2.2 $(\mathfrak{g}_\mathbf{z})_{\sigma,\delta}$ contains a monomial term of the form $C \cdot \alpha^i_\nu$ for some $C \neq 0$. The key observation is that all the parameters appear in one of the entries

$$\{(\mathfrak{g}_\mathbf{z})_{(0,\ldots,i^s,\ldots,0),\tau} : 1 \le s \le p, \tau \in \mathrm{sym}^k(p)\}$$

and if a parameter $\alpha_{m,\nu}$ appears with nonzero coefficient in $(\mathfrak{g}_\mathbf{z})_{(0,\ldots,i^s,\ldots,0),\tau}$ then by the definition of δ and λ^s

$$\lambda^\sigma_\delta - \lambda^\sigma_\sigma \le \lambda^\sigma_{(0,\ldots,k^s,\ldots,0)} - \lambda^\sigma_\tau,$$

so

$$(\lim_{t\to 0} \mathfrak{g}_{\lambda^\sigma(t)\mathbf{z}})_{\sigma,\delta}$$

is either 0 or a linear form in the parameters which appear also in other entries of $\lim_{t\to 0} \mathfrak{g}_{\lambda^\sigma(t)\mathbf{z}}$. But the (σ,δ) entry of $\mathrm{Lie}T_\delta$ is independent of the remaining entries, so

$$\mathrm{Lie}T_\delta \notin \lim_{t\to 0} \mathfrak{g}_{\lambda^\sigma(t)\mathbf{z}}.$$

$\square$

Proposition 9.11 and Theorem 9.4 are now proved for $n \ge \mathrm{sym}^{\le k}(p)$.

Theorem 9.4 for the case when $n < \mathrm{sym}^{\le k}(p)$ can now be proved in exactly the same way as for $p = 1$. Namely, the projection $\mathbb{C}^{\mathrm{sym}^{\le k}(p)} \to \mathbb{C}^n$ induces $\pi : \wedge^{\mathrm{sym}^{\le k}(p)} \mathrm{Sym}^{\,k}\mathbb{C}^{\mathrm{sym}^{\le k}(p)} \to \wedge^{\mathrm{sym}^{\le k}(p)} \mathrm{Sym}^{\,\le k}\mathbb{C}^n$, and a rational map

$$\pi : \mathbb{P}(\wedge^{\mathrm{sym}^{\le k}(p)} \mathrm{Sym}^{\,\le k}\mathbb{C}^{\mathrm{sym}^{\le k}(p)}) \dashrightarrow \mathbb{P}\Big(\bigoplus_{\underline{i} \in \binom{\mathrm{sym}^{\le k}(p)}{n}} \wedge^{\mathrm{sym}^{\le k}(p)} \mathrm{Sym}^{\,\le k}\mathbb{C}^n_{\underline{i}}\Big)$$

which restricts to a morphism

$$\overline{\pi} : \overline{Y}_{\mathrm{sym}^{\le k}(p),k,p} \to \overline{Y}_{n,k,p}.$$

If $z \in \pi(O_\lambda)$ is a generic point of the boundary of $Y_{n,k,p}$ then O_λ is a maximal boundary orbit of $Y_{\mathrm{sym}^{\le k}(p),k,p}$ sitting in $A_{\mathrm{sym}^{\le k}(p),k,p}$, and

$$\dim(\pi^{-1}(z) \cap \overline{O_\lambda}) \ge \dim Y_{\mathrm{sym}^{\le k}(p),k,p} - \dim Y_{n,k,p}.$$

Since the codimension of O_λ is at least two in $\overline{Y}_{\mathrm{sym}^{\le k}(p),k,p}$, the same holds for the boundary component containing z. □

It now follows just as in §7 for the case when $p = 1$ that

Theorem 9.14. *If $p > 1$, then $\mathbb{G}_{k,p}$ is a Grosshans subgroup of the general linear group $GL(\mathrm{sym}^{\le k} p)$ where*

$$\mathrm{sym}^{\le k} p = \sum_{i=1}^{k} \dim \mathrm{Sym}^i \mathbb{C}^p = \binom{k+p-1}{k-1},$$

so that every linear action of $\mathbb{G}_{k,p}$ which extends to a linear action of $GL(\mathrm{sym}^{\le k} p)$ has finitely generated invariants.

In particular we have

Theorem 9.15. *When $p > 1$ the fibres $O((J_{k,p})_x)_\ell^{\mathbb{G}'_{k,p}} \cong O((J_{k,p})_x \times \mathbb{C})^{\mathbb{G}_{k,p}}$ of the bundle $E^n_{k,p}$ are finitely generated graded complex algebras when ℓ is divisible by*

$$s(k,p) = \sum_{i=1}^{k} i(\dim \mathrm{Sym}^i \mathbb{C}^p).$$

Moreover just as in §7 we have generators of the subalgebra

$$O((J_{k,p})_x)_{s(k,p)}^{\mathbb{G}'_{k,p}}$$

of the generalized Demailly-Semple algebra $O((J_{k,p})_x)^{\mathbb{G}'_{k,p}}$ spanned by homogeneous $\mathbb{G}'_{k,p}$-invariant polynomials of weight divisible by $s(k,p)$ (where the weight is with respect to the central 1-parameter subgroup $\mathbb{C}^*$ of $GL(p) \le \mathbb{U}_{k,p} \rtimes GL(p) = \mathbb{G}_{k,p}$) given by the Plücker coordinates on the Grassmannian

$$\mathrm{Grass}_{\mathrm{sym}^{\le k}(p)}(\mathrm{Sym}^{\le k}\mathbb{C}^n) \subseteq \mathbb{P}(\wedge^{\mathrm{sym}^{\le k}(p)}(\mathrm{Sym}^{\le k}\mathbb{C}^n))$$

as in Theorem 9.4.

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