ISOTOPES OF HURWITZ ALGEBRAS

ERIK DARPÖ

Abstract. We study algebras that are isotopic to Hurwitz algebras. Isomorphism classes of such algebras are shown to correspond to orbits of a certain group action. An explicit, geometrically intutive description of the category of isotopes of Hamilton's quaternions is given. As an application, some known results concerning the classification of finite-dimensional composition algebras are deduced.

1. Introduction

Let k be a field, and V a vector space over k. We shall say that a quadratic form $q : V \to k$ is *non-degenerate* if the associated bilinear form $\langle x, y\rangle = q(x + y) - q(x) - q(y)$ is non-degenerate (i.e., $\langle x, V\rangle = 0$ only if $x = 0$). A *composition algebra* is a non-zero (not necessarily associative) algebra A over a field k, equipped with a non-degenerate quadratic form $n : A \to k$ such that $n(ab) = n(a)n(b)$ for all $a, b \in A$. The form n is usually called the *norm* of A. If A possesses an identity element, it is called a *Hurwitz algebra*. Every Hurwitz algebra has dimension one, two, four or eight (thus, in particular, it is finite dimensional), and can be constructed via an iterative method known as the *Cayley-Dickson process*.[1] The facts about Hurwitz algebras referred in this section are described in detail in [12], Chapter VIII.

Two algebras A and B over a field k are said to be *isotopic* if there exist invertible linear maps $\alpha, \beta, \gamma : A \to B$ such that $\gamma(ab) = \alpha(x)\beta(y)$. Clearly, isotopy is an equivalence relation among k-algebras. If A and B are isotopic then there exist $\alpha, \beta \in \mathrm{GL}(A)$ such that the algebra $(A, \circ)$, with multiplication $x \circ y = \alpha(x)\beta(y)$, is isomorphic to B. The algebra $(A, \circ)$ is called the *principal isotope* of A determined by α and β, and is denoted by $A_{\alpha,\beta}$.

Several important classes of non-associative algebras can be constructed by isotopy from the Hurwitz algebras. Examples include:

(1) All finite-dimensional composition algebras [11, p. 957]. This includes in particular all finite-dimensional absolute valued algebras, which are precisely the finite-dimensional composition algebras over $\mathbb{R}$ whose norm is anisotropic (i.e., $n(x) = 0$ only if $x = 0$). However, there exist infinite-dimensional composition algebras that are not isotopic to any Hurwitz algebra; see e.g. [2, 5].
(2) All division algebras of dimension two over a field of characteristic different from two [14].
(3) All eight-dimensional division algebras A with the following property: for all non-zero $a \in A$ there exists a $b \in A$ such that $b(ax) = x$ for all $x \in A$.

The purpose of the present article is to give a uniform description of isotopes of Hurwitz algebras. Generalising ideas that have earlier been used in more specialised situations (for example in [14, 1]), we give a general description of all algebras isotopic to a Hurwitz algebra, encompassing also the case of characteristic two. As a consequence of our study we get a comprehensive picture of all isotopes of Hamilton's quaternion algebra $\mathbb{H}$ – a class of real division algebras that has not been studied before.

In Section 2, a general description is given of the category of isotopes of a Hurwitz algebra A. A more elaborate study of the case where (A, n) is a Euclidean space is given in Section 3, bringing about the promised description of isotopes of $\mathbb{H}$. Finally, Section 4 treats composition algebras, showing how a description of these can be deduced from the results in Section 2.

[1]Some authors use a weaker notion of non-degeneracy for the norm n, requiring that $n(x + y) = n(y)$ for all y implies $x = 0$. This definition gives rise to additional unital composition algebras over fields k of characteristic two, in form of purely inseparable field extensions of k [11]. If $\mathrm{char}\, k \neq 2$, the two definitions are equivalent.

From here on, let k denote a field. All algebras are, unless otherwise stated, assumed to be finite dimensional over k. Every element a of an algebra A determines linear endomorphisms L_a and R_a of A, defined by $\mathrm{L}_a(x) = ax$ and $\mathrm{R}_a(x) = xa$. An algebra A is said to be a *division algebra* if $\dim A > 0$ and L_a and R_a are bijective for all non-zero $a \in A$. Moreover, A is *alternative* if the identities $x^2y = x(xy)$ and $xy^2 = (xy)y$ hold for all $x, y \in A$.

Any element x in a Hurwitz algebra $A = (A, n)$ satisfies $x^2 = \langle x, 1\rangle x - n(x)$. Hence, the norm n is uniquely determined by the algebra structure of A, and every algebra morphism of Hurwitz algebras that respects the identity element also preserves the norm. Every Hurwitz algebra has unique non-trivial involution[2] $\kappa : A \to A$, $x \mapsto \bar{x}$ satisfying $x + \bar{x} \in k1$ and $x\bar{x} = \bar{x}x = n(x)1$ for all $x \in A$. Moreover, two Hurwitz algebras are isomorphic if and only if their respective norms are equivalent (this was first proved in [9] in characteristic different from two). Quadratic forms occurring as norms of Hurwitz algebras are precisely the m-fold Pfister forms over k, $m \in \{0, 1, 2, 3\}$. If A is a Hurwitz algebra and $a \in A$, then L_a and R_a are invertible if and only if $n(a) \neq 0$. This is also equivalent to the existence of an inverse a^{-1} of a in A: since $a\bar{a} = \bar{a}a = n(a)1$, we have $a^{-1} = n(a)^{-1}\bar{a}$ if $n(a) \neq 0$. Moreover, $\mathrm{L}_a^{-1} = \mathrm{L}_{a^{-1}}$ and $\mathrm{R}_a^{-1} = \mathrm{R}_{a^{-1}}$ in this case. The invertible elements of any alternative algebra A form a *Moufang loop* under multiplication (the concept was introduced by Moufang in [13] under the name *quasi-group*), denoted by A^*. In case A is associative, A^* is a group.

Two-dimensional Hurwitz algebras are quadratic étale algebras, i.e., either separable field extensions of k or isomorphic to $k \times k$. The Hurwitz algebras of dimension four are all *quaternion algebras*, that is all four-dimensional central simple associative algebras. Eight-dimensional Hurwitz algebras are precisely the central simple alternative algebras that are not associative [18] (these are called *octonion algebras*). A Hurwitz algebra A is commutative if and only if $\dim A \leqslant 2$, and associative if and only if $\dim A \leqslant 4$.

For any algebra A, the *nucleus* is defined as $N(A) = \{a \in A \mid (xy)z = x(yz) \text{ for } a \in \{x, y, z\}\}$. The nucleus is an associative subalgebra of A. If A is a Hurwitz algebra, then $a \in N(A)$ if and only if $(xa)y = x(ay)$ for all $x, y \in A$. If $\dim A \leqslant 4$ then A is associative and thus $N(A) = A$; the nucleus of an eight-dimensional Hurwitz algebra is $k1$.

A *similitude* of a non-zero quadratic space $V = (V, q)$ is an invertible linear map $\varphi : V \to V$ such that $q(\varphi(x)) = \mu(\varphi)q(x)$ for all $x \in V$, where $\mu(\varphi) \in k$ is a scalar independent of x. The element $\mu(\varphi)$ is called the *multiplier* of φ. If $l = \dim V$ is even, then $\det(\varphi) = \pm\mu(\varphi)^{l/2}$ [12, 12A]. If $\mathrm{char}\, k \neq 2$, a similitude φ satisfying $\det(\varphi) = \mu(\varphi)^{l/2}$ are said to be *proper*. In the characteristic two case, a similitude $\varphi : V \to V$ is proper if its Dickson invariant (see [12, 12.12]) is zero. The group of all similitudes of V is denoted by $\mathrm{GO}(V, q)$, or $\mathrm{GO}(V)$ for short. The proper similitudes form a normal subgroup $\mathrm{GO}^+(V) \subset \mathrm{GO}(V)$ of index two. Elements in $\mathrm{GO}(V) \setminus \mathrm{GO}^+(V)$ are called *improper* similitudes. The map $\mu : \mathrm{GO}(V) \to k^*, \varphi \mapsto \mu(\varphi)$ is a group homomorphism, the kernel of which is the orthogonal group $\mathrm{O}(V)$. We write $\mathrm{O}^+(V) = \mathrm{O}(V) \cap \mathrm{GO}^+(V)$, or $\mathrm{SO}(V) = \mathrm{O}^+(V)$ in case (V, q) is a Euclidean space.

Let $A = (A, n)$ be a Hurwitz algebra. The set of $\varphi \in \mathrm{GO}(A)$ for which there exist $\varphi_1, \varphi_2 \in \mathrm{GO}(A)$ such that $\varphi(xy) = \varphi_1(x)\varphi_2(x)$ for all $x, y \in A$ is $\mathrm{GO}^+(A)$ if $\dim A \geqslant 4$ and $\mathrm{GO}(A)$ if $\dim A \leqslant 2$. This is known as the principle of *triality* for Hurwitz algebras [16, 3.2]. We call φ_1 and φ_2 *triality components* of φ. It is not difficult to see that any other pair of triality components of φ is of the form $(\mathrm{R}_w^{-1}\varphi_1, \mathrm{L}_w\varphi_2)$ for some $w \in N(A)^*$, and that $\varphi_1 = \mathrm{R}_{\varphi_2(1)}^{-1}\varphi$ and $\varphi_2 = \mathrm{L}_{\varphi_1(1)}^{-1}\varphi$. If A is associative, then $\varphi(xy) = \varphi_1(x)\varphi_2(y) = (\varphi(x)\varphi_2(1)^{-1})(\varphi_1(1)^{-1}\varphi(y)) = \varphi(x)\varphi(1)^{-1}\varphi(y)$. Moreover, $\varphi_1, \varphi_2 \in \mathrm{GO}^+(A)$ if $\varphi \in \mathrm{GO}^+(A)$, and $(\varphi^{-1})_i = (\varphi_i)^{-1}$, $i = 1, 2$.

A *groupoid* is a category in which every morphism is an isomorphism. Every G-set X (G being some group), defines a groupoid with object class X, and morphisms $x \to y$ being the set of group elements $g \in G$ satisfying $g \cdot x = y$. We call this the *groupoid of the G-action on X*. Given a vector space V over k, and a quadratic form $q : V \to k$, set $\mathrm{PGL}(V) = \mathrm{GL}(V)/(k^*\mathbb{I})$, $\mathrm{PGO}(V, q) = \mathrm{GO}(V, q)/(k^*\mathbb{I})$ and $\mathrm{PGO}^+(V, q) = \mathrm{GO}^+(V, q)/(k^*\mathbb{I})$. Generally, no notational distinction shall be made between elements in a group/set and cosets or orbits (of some subgroup respecively group action) represented by such elements; for example, any $\alpha \in \mathrm{GL}(A)$ may also be viewed as an

[2]By an involution is meant a self-inverse isomorphism $A \to A^{op}$.

element in $\mathrm{PGL}(A)$, depending on the context. If V is a Euclidean space then $\mathrm{Pds}(V)$ denotes the set of positive definite symmetric endomorphisms of V. The set of isomorphisms from an object A to an object B in a category $\mathscr{C}$ is denoted by $\mathrm{Iso}(A,B)=\mathrm{Iso}_{\mathscr{C}}(A,B)$. Throughout, C_2 denotes the cyclic group of order two, generated by the canonical involution in a Hurwitz algebra: $\mathrm{C}_2=\langle\kappa\rangle=\{\mathbb{I},\kappa\}$.

2. General description

Lemma 1. *Let A be any algebra, and $\alpha,\beta\in\mathrm{GL}(A)$. The isotope $A_{\alpha,\beta}$ is unital if and only if $\alpha=\mathrm{R}_a^{-1}$, $\beta=\mathrm{L}_b^{-1}$ for some $a,b\in A^*$. The identity element in $A_{\mathrm{R}_a^{-1},\mathrm{L}_b^{-1}}$ is ba.*

Proof. The isotope $A_{\alpha,\beta}=(A,\circ)$ is unital if and only if there exists an element $e\in A$ such that $\mathrm{L}_e^\circ=\mathrm{R}_e^\circ=\mathbb{I}_A$. Now $e\circ x=\alpha(e)\beta(x)$, that is, $\mathrm{L}_e^\circ=\mathrm{L}_{\alpha(e)}\,\beta$, so $\mathrm{L}_e^\circ=\mathbb{I}_A$ if and only if $\beta=\mathrm{L}_{\alpha(e)}^{-1}$. Similarly, $\mathrm{R}_e^\circ=\mathrm{R}_{\beta(e)}\,\alpha$ equals the identity map if and only if $\alpha=\mathrm{R}_{\beta(e)}^{-1}$.

It readily verified that $(ba)\circ x=x=x\circ(ba)$ in $A_{\mathrm{R}_a^{-1},\mathrm{L}_b^{-1}}$. □

In particular, if A is alternative then $A_{\alpha,\beta}$ is unital if and only $\alpha=\mathrm{R}_c$, $\beta=\mathrm{L}_d$ for some $c,d\in A^*$, in which case the identity element in $A_{\alpha,\beta}$ is $(cd)^{-1}$.

Proposition 2. *Let A be a Hurwitz algebra. Any isotope B of A that has unity is again a Hurwitz algebra, isomorphic to A, and $n_B=n_A(1_B)^{-1}n_A$.*

Proof. By Lemma 1, any principal isotope $B=(A,\circ)$ of A that is unital has the form $B=A_{\mathrm{R}_c,\mathrm{L}_d}$. Defining $n_B(x)=n_A(cd)n_A(x)$ for all $x\in B$, we have

$$\begin{aligned}n_B(x\circ y)&=n_B((xc)(dy))=n_A(cd)n_A((xc)(dy))\\&=n_A(cd)n_A(x)n_A(cd)n_A(y)=n_B(x)n_B(y)\end{aligned}$$

so B is a Hurwitz algebra. Moreover, $\mathrm{L}_{cd}:(B,n_B)\to(A,n_A)$ is an isometry. Being isometric as quadratic spaces, A and B are isomorphic algebras. Since $1_B=(cd)^{-1}$, it is clear that $n_B=n_A(cd)n_A=n_A(1_B)^{-1}n_A$. □

Corollary 3. *Let A be a Hurwitz algebra, and $\alpha,\beta,\gamma,\delta\in\mathrm{GL}(A)$. Any isomorphism $\varphi:A_{\alpha,\beta}\to A_{\gamma,\delta}$ is a similitude of (A,n_A) with multiplier $n_A(\varphi(1))$.*

Proof. Let $\varphi:A_{\alpha,\beta}\to A_{\gamma,\delta}$ be an isomorphism. It is straightforward to verify that φ is also an isomorphism $A\to B$, where $B=A_{\gamma\varphi\alpha^{-1}\varphi^{-1},\delta\varphi\beta^{-1}\varphi^{-1}}$. Thus, in particular, φ is an orthogonal map $(A,n_A)\to(B,n_B)$. By Proposition 2, $n_B=n_A(1_B)^{-1}n_A$, so $n_A(\varphi(x))=n_A(1_B)n_B(\varphi(x))=n_A(1_B)n_A(x)$. □

As mentioned in the introduction, the triality principle holds for all similitudes if A is a Hurwitz algebra of dimension two, but only for elements in $\mathrm{GO}^+(A)$ if $\dim A\geqslant 4$. This difference (which comes from the fact that if $\dim A\leqslant 2$ then A is commutative and thus $\kappa\in\mathrm{Aut}(A)$), has implications for the theory of isotopes of A. Set

$$\mathrm{G}(A)=\begin{cases}\mathrm{GO}(A) & \text{if }\dim A\leqslant 2\,,\\ \mathrm{GO}^+(A) & \text{if }\dim A\geqslant 4\,,\end{cases}$$

and $\mathrm{PG}(A)=\mathrm{G}(A)/(k^*\mathbb{I})$.

Proposition 4. *Let $A=(A,n)$ be a Hurwitz algebra. If $\varphi\in\mathrm{G}(A)$ and $\alpha,\beta\in\mathrm{GL}(A)$, then $\varphi\in\mathrm{Iso}(A_{\alpha,\beta},A_{\gamma,\delta})$ where*

$$\begin{cases}\gamma=\varphi_1\alpha\varphi^{-1}=\mathrm{R}_{\varphi_2(1)}^{-1}\,\varphi\alpha\varphi^{-1}\\ \delta=\varphi_2\beta\varphi^{-1}=\mathrm{L}_{\varphi_1(1)}^{-1}\,\varphi\beta\varphi^{-1}\end{cases}$$

and $\varphi_1,\varphi_2\in G$ are triality components of φ. Moreover, $\mathrm{Iso}(A_{\alpha,\beta},A_{\gamma,\delta})\subset\mathrm{G}(A)$ for all $\alpha,\beta,\gamma,\delta\in\mathrm{GL}(A)$.

Proof. It is straightforward to verify that $\varphi \in \mathrm{G}(A)$ is an isomorphism $A_{\alpha,\beta} \to A_{\gamma,\delta}$ if $\gamma = \varphi_1 \alpha \varphi^{-1}$ and $\delta = \varphi_2 \beta \varphi^{-1}$. Inserting $\varphi_1 = \mathrm{R}_{\varphi_2(1)}^{-1} \varphi$ and $\varphi_2 = lt_{\varphi_1(1)}^{-1} \varphi$ gives $\gamma = \mathrm{R}_{\varphi_2(1)}^{-1} \varphi \alpha \varphi^{-1}$ and $\delta = \mathrm{L}_{\varphi_1(1)}^{-1} \varphi \beta \varphi^{-1}$, respectively.

Suppose $\varphi : A_{\alpha,\beta} \to A_{\gamma,\delta}$ is an isomorphism. By Corollary 3, φ is a similitude with multiplier $n(\varphi(1))$. Assume $\dim A \geqslant 4$. If φ is an improper similitude then $\varphi = \psi\kappa$, where $\psi \in \mathrm{GO}^+(A, n)$. By the first statement of the proposition, $\psi^{-1} \in \mathrm{GO}^+(A, n)$ is an isomorphism $A_{\gamma,\delta} \to A_{\psi_1^{-1}\gamma\psi, \psi_2^{-1}\delta\psi}$, so $\kappa = \psi^{-1}\varphi \in \mathrm{Iso}(A_{\alpha,\beta}, A_{\psi_1^{-1}\gamma\psi, \psi_2^{-1}\delta\psi})$. This implies $\kappa \in \mathrm{Iso}(A, A_{\gamma',\delta'})$, where $(\gamma', \delta') = (\psi_1^{-1}\gamma\psi\kappa\alpha^{-1}\kappa, \psi_2^{-1}\delta\psi\kappa\beta^{-1}\kappa)$. Lemma 1 gives $(\gamma', \delta') = (\mathrm{R}_c, \mathrm{L}_d)$ for some $c, d \in A \setminus \{0\}$.

Now $\kappa \in \mathrm{Iso}(A, A_{\mathrm{R}_c,\mathrm{L}_d})$ means that

$$\bar{y}\bar{x} = \overline{xy} = (\bar{x}c)(d\bar{y}) \qquad \text{for all} \quad x, y \in A. \tag{1}$$

Inserting $y = 1$ yields $\bar{x} = (\bar{x}c)d$ for all $x \in A$, that is, $\mathrm{R}_d \mathrm{R}_c = \mathbb{I}_A$ and hence $c = d^{-1}$. Next, setting $x = \bar{d}$ in (1) gives $\bar{y}d = d\bar{y}$ for all $y \in A$, implying $d \in k1$. But then (1) becomes $\bar{y}\bar{x} = \bar{x}\bar{y}$, contradicting the non-commutativity of A. □

We record the following observations for future use. The first statement is a consequence of the fact, referred in the introduction, that $x \in N(A)$ if and only if $(ax)b = a(xb)$ for all $a, b \in A$; the second follows from Proposition 4.

Lemma 5. *Let A be a Hurwitz algebra, $\alpha, \beta, \gamma, \delta \in \mathrm{GL}(A)$ and $\rho \in k^*$.*

1. *$A_{\alpha,\beta} = A_{\gamma,\delta}$ if and only if $\alpha = \mathrm{R}_w^{-1} \gamma$, $\beta = \mathrm{L}_w \delta$ for some $w \in N(A)^*$.*
2. *The homothety $h_\rho(x) = \rho x$ on A defines an isomorphism $A_{\rho\mathbb{I},\mathbb{I}} \to A$.*

By Lemma 5, isotopes $A_{\alpha,\beta}$ of A are parametrised by orbits of the group action

$$N(A)^* \times \mathrm{GL}(A)^2 \to \mathrm{GL}(A)^2\,,\ (w, (\alpha, \beta)) \mapsto w \cdot (\alpha, \beta) = (\mathrm{R}_w^{-1} \alpha, \mathrm{L}_w \beta)\,. \tag{2}$$

Moreover, $A_{\alpha,\beta} \simeq A_{\gamma,\delta}$ if (α, β) and (γ, δ) represent the same element in $\mathrm{PGL}(A)^2$.

Denote by $X_A = \mathrm{PGL}(A)^2/N(A)^*$ the orbit set of the action

$$w \cdot (\alpha, \beta) = (\mathrm{R}_w^{-1} \alpha, \mathrm{L}_w \beta) \tag{3}$$

of $N(A)^*$ on $\mathrm{PGL}(A)^2$. The group $\mathrm{PG}(A)$ acts on X_A as follows:

$$\varphi \cdot (\alpha, \beta) = (\varphi_1 \alpha \varphi^{-1}, \varphi_2 \beta \varphi^{-1}) \tag{4}$$

where φ_1 and φ_2 are triality components of φ. Note that for any other choice $(\varphi_1', \varphi_2') = (\mathrm{R}_w^{-1} \varphi_1, \mathrm{L}_w \varphi_2)$, $w \in N(A)^*$ of triality components of φ,

$$(\varphi_1' \alpha \varphi^{-1}, \varphi_2' \beta \varphi^{-1}) = (\mathrm{R}_w^{-1} \varphi_1 \alpha \varphi^{-1}, \mathrm{L}_w \varphi_2 \beta \varphi^{-1}) = w \cdot (\varphi_1 \alpha \varphi^{-1}, \varphi_2 \beta \varphi^{-1})$$

with respect to (3), hence the action (4) on $X_A = \mathrm{PGL}(A)^2/N(A)^*$ is well defined.

Let $\mathscr{X}(A) = {}_{\mathrm{PG}(A)} X_A$ be the groupoid of the action (4). For any Hurwitz algebra A, let $\mathscr{I}(A)$ denote the category of principal isotopes of A, and $\check{\mathscr{I}}(A)$ the category obtained from $\mathscr{I}(A)$ by removing all non-isomorphisms between the objects. Note that if A is a division algebra then so are all its isotopes, and thus any non-zero morphism in $\mathscr{I}(A)$ is an isomorphism.

The essence of our findings so far is summarised in the following theorem.

Theorem 6. *For any Hurwitz algebra A, the categories $\check{\mathscr{I}}(A)$ and $\mathscr{X}(A)$ are equivalent. An equivalence $\mathscr{F}_A : \check{\mathscr{I}}(A) \to \mathscr{X}(A)$ is given by $\mathscr{F}_A(A_{\alpha,\beta}) = (\alpha, \beta)$ and $\mathscr{F}_A(\varphi) = \varphi$.*

Proof. Let $(\alpha, \beta), (\gamma, \delta) \in \mathrm{GL}(A)^2$. If $A_{\alpha,\beta} = A_{\gamma,\delta}$ then $(\alpha, \beta), (\gamma, \delta)$ are in the same orbit of the action (2), and hence represent the same object in X_A. Proposition 4 guarantees that $\varphi \cdot (\alpha, \beta) = (\gamma, \delta)$ whenever $\varphi \in \mathrm{Iso}(A_{\alpha,\beta}, A_{\gamma,\delta})$. This shows that $\mathscr{F}_A$ is well defined. Clearly, $\mathscr{F}_A$ is surjective on objects, hence dense as a functor.

If $\varphi, \psi \in \mathrm{Iso}(A_{\alpha,\beta}, A_{\gamma,\delta})$ and $\mathscr{F}_A(\varphi) = \mathscr{F}_A(\psi)$, then $\varphi = \rho\psi$ for some $\rho \in k^*$, so $\rho\mathbb{I}_A = \varphi\psi^{-1} \in \mathrm{Aut}(A_{\alpha,\beta})$. This implies $\rho = 1$ and $\varphi = \psi$; hence $\mathscr{F}_A$ is faithful. Fullness is clear from the construction. □

Remark 7. (1) If k is a Euclidean field[3] (e.g. $k = \mathbb{R}$), then the group $\mathrm{PGO}(A, n)$ is canonically isomorphic to $\mathrm{O}(A, n)/(\pm\mathbb{I})$, via composition of inclusion and quotient projection: $\mathrm{O}(A, n)/(\pm\mathbb{I}) \subset \mathrm{GO}(A, n)/(\pm\mathbb{I}) \twoheadrightarrow \mathrm{GO}(A, n)/k^* = \mathrm{PGO}(A, n)$. This induces an isomorphism $\mathrm{O}^+(A, n)/(\pm\mathbb{I}) \to \mathrm{PGO}^+(A, n)$.
(2) If $\dim A = 8$, since $N(A) = k1$, we have $X_A = \mathrm{PGL}(A)^2$.

If A is associative, then its similitudes have a particularly nice form. Let $\mathrm{L}_{A^*} = \{\mathrm{L}_a \mid a \in A^*\}$.

Proposition 8. *If A is an associative Hurwitz algebra then L_{A^*} is a normal subgroup of $\mathrm{G}(A)$, and $\mathrm{G}(A) = \mathrm{L}_{A^*} \rtimes \mathrm{Aut}(A)$.*

Proof. Clearly, L_{A^*} is a subgroup of $\mathrm{G}(A)$. It is normal, since for all $a \in A^*$ and $\varphi \in \mathrm{G}(A)$,

$$\varphi \,\mathrm{L}_a\, \varphi^{-1} = \mathrm{L}_{\varphi_1(a)}\, \varphi_2 \varphi^{-1} = \mathrm{L}_{\varphi_1(a)}\, \mathrm{L}^{-1}_{\varphi_1(1)} = \mathrm{L}_{\varphi_1(a)\varphi_1(1)^{-1}}$$

where $\varphi_1, \varphi_2 \in \mathrm{G}(A)$ are triality components of φ. Moreover, $\mathrm{L}_a \in \mathrm{Aut}(A)$ if and only if $a = 1$, so $\mathrm{L}_{A^*} \cap \mathrm{Aut}(A) = \{\mathbb{I}\}$. The inclusion $\mathrm{Aut}(A) \subset \mathrm{G}(A)$ is obvious.

If $\varphi \in G(A)$ then $\varphi(xy) = \varphi(x)\varphi(1)^{-1}\varphi(y)$. Hence $\mathrm{L}^{-1}_{\varphi(1)}\, \varphi(xy) = \mathrm{L}^{-1}_{\varphi(1)}\, \varphi(x)\, \mathrm{L}^{-1}_{\varphi(1)}\, \varphi(y)$, so $\mathrm{L}^{-1}_{\varphi(1)}\, \varphi \in \mathrm{Aut}(A)$. This implies $\mathrm{G}(A) = \mathrm{L}_{A^*}\, \mathrm{Aut}(A)$, which concludes the proof of the proposition. □

Note that if $\dim A = 2$ then $\mathrm{Aut}(A) = \mathrm{C}_2$. If A is a quaterion algebra then it is central simple, and the Skolem-Noether Theorem gives [10, p. 222], $\mathrm{Aut}(A) = \{\mathrm{L}_a\, \mathrm{R}^{-1}_a \mid a \in A^*\}$. Hence every $\varphi \in \mathrm{GO}^+(A)$ can be written as $\varphi = \mathrm{L}_a\, \mathrm{L}_b\, \mathrm{R}^{-1}_b = \mathrm{L}_{ab}\, \mathrm{R}_{b^{-1}}$. It is also easy to see that $\mathrm{L}_a\, \mathrm{R}_b = \mathbb{I}$ if and only if $b^{-1} = a \in k1$. Hence we have the following result. It has been proved by Stampfli-Rollier [17, 3.5. Hilfsatz] for ortogonal maps under the assumption $\operatorname{char} k \neq 2$.

Corollary 9. *Every proper similitude of a quaterion algebra A has the form $\mathrm{L}_a\, \mathrm{R}_b$ for some $a, b \in A^*$. The kernel of the group epimorphism $A^* \times (A^*)^{op} \to \mathrm{GO}^+(A)$, $(a, b) \mapsto \mathrm{L}_a\, \mathrm{R}_b$ is $\{(\rho, \rho^{-1}) \mid \rho \in k^*\} \subset A^* \times (A^*)^{op}$.*

Whenever A is an associative Hurwitz algebra, the action (4) of an element $\mathrm{L}_a\, \psi \in \mathrm{L}_{A^*} \rtimes \mathrm{Aut}(A)$ $= \mathrm{G}(A)$ on X_A can be written as

$$\mathrm{L}_a\, \psi \cdot (\alpha, \beta) = (\mathrm{L}_a\, \psi\alpha\psi^{-1}\, \mathrm{L}^{-1}_a, \psi\beta\psi^{-1}\, \mathrm{L}^{-1}_a)\,. \tag{5}$$

For $\dim A = 2$, this description of the groupoid $\mathscr{X}(A)$ is equivalent to the isomorphism criterion 1.12 in [14], applied to isotopes of quadratic étale algebras.

We conclude this section by introducing a numerical, easily computed isomorphism invariant. Let $k^{*l} = \{\rho^l \in k^* \mid \rho \in k^*\} \subset k^*$.

Proposition 10. *Let A be a Hurwitz algebra of dimension $l \geqslant 2$, not isomorphic to $k \times k$, and $\alpha, \beta \in \mathrm{GL}(A)$. Then the pair $(\det(\alpha), \det(\beta)) \in \left(k^*/k^{*l}\right)^2$ is an isomorphism invariant for $A_{\alpha,\beta}$.*

Proof. If $\varphi : A_{\alpha,\beta} \to A_{\gamma,\delta}$ is an isomorphism then $\varphi\, \mathrm{L}_{\alpha(x)}\, \beta\varphi^{-1} = \mathrm{L}_{\gamma\varphi(x)}\, \delta$, hence $\det(\mathrm{L}_{\alpha(x)}) \det(\beta)$ $= \det(\mathrm{L}_{\gamma\varphi(x)}) \det(\delta)$. It is easy to show that L_a and R_a are proper similitudes with multiplier $n_A(a)^2$ for any $a \in A^*$ (this, however, is not true for $A \simeq k \times k$). Consequently, $n_A(\alpha(x))^l \det(\beta) = n_A(\gamma\varphi(x))^l \det(\delta)$, so $\det(\beta) \det(\delta)^{-1} \in k^l$. Similarly, the identity $\varphi\, \mathrm{R}_{\beta(y)}\, \alpha\varphi^{-1} = \mathrm{R}_{\delta\varphi(y)}\, \gamma$ implies $\det(\alpha) \det(\gamma)^{-1} \in k^{*l}$. □

Let A and l ba as in Proposition 10. For $i, j \in k^*/k^{*l}$, setting

$$\mathscr{X}(A)_{i,j} = \left\{(\alpha, \beta) \in \mathscr{X}(A) \mid (\det(\alpha), \det(\beta)) = (i, j) \text{ in } \left(k^*/k^{*l}\right)^2\right\} \subset \mathscr{X}(A)\,,$$

the groupoid $\mathscr{X}(A)$ can be written as a coproduct

$$\mathscr{X}(A) = \coprod_{i,j \in k^*/k^{*l}} \mathscr{X}(A)_{i,j}\,. \tag{6}$$

Hence, any subcategory $\mathscr{A} \subset \mathscr{X}(A)$ can be classified by classifying each of the subcategories $\mathscr{A}_{i,j} = \mathscr{A} \cap \mathscr{X}(A)_{i,j} \subset \mathscr{X}(A)_{i,j}$.

[3]A field k is called Euclidean if k^{*2} forms and an ordering of k.

As for the real ground field, $[\mathbb{R}^* : \mathbb{R}^{*l}] = 2$ for any even number l, the two cosets being represented by 1 and -1, and the quotient projection $\mathbb{R}^* \to \mathbb{R}^*/\mathbb{R}^{*l}$ is given by $\rho \mapsto \operatorname{sign}(\rho)$. If A is either $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$, $\alpha, \beta \in \mathrm{GL}(A)$ and $A_{\alpha,\beta} = (A, \circ)$, then $\det(\mathrm{L}_x^\circ) = \det(\mathrm{L}_{\alpha(x)}) \det(\beta)$. Since $\det(\mathrm{L}_{\alpha(x)}) = n(\alpha(x))^{\dim A} > 0$ for any $x \neq 0$, it follows that $\operatorname{sign}(\det(\mathrm{L}_x^\circ)) = \operatorname{sign}(\det(\beta))$, and similarly $\operatorname{sign}(\det(\mathrm{R}_x)) = \operatorname{sign}(\det(\alpha))$. This means that the decomposition $\mathscr{X}(A) = \coprod_{i,j\in\{-1,1\}} \mathscr{X}(A)_{i,j}$ here coincides with the "double sign" decomposition for real division algebras, introduced in [3].

3. The Euclidean case

The aim of this section is to give a more detailed account for isotopes of real Hurwitz algebras whose underlying quadratic space is Euclidean. Such a Hurwitz algebra is isomorphic to either $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$, and the isotopes are precisely the real division algebras that are isotopic to a Hurwitz algebra. The isotopes of $\mathbb{C}$ are all the two-dimensional real division algebras, and their classification has been described in [8, 4]. We therefore focus on the higher-dimensional cases, and let A be either $\mathbb{H}$ or $\mathbb{O}$. Our main tool will be polar decomposition of linear maps.

Any $\alpha \in \mathrm{GL}(A)$ can be written as $\alpha = \alpha'\lambda$, where $\det(\alpha') = |\det(\alpha)| > 0$ and $\lambda \in \mathrm{C}_2 = \{\mathbb{I}, \kappa\}$. Polar decomposition now yields $\alpha' = \zeta\delta$, with $\zeta \in \mathrm{SO}(A)$ and $\delta \in \mathrm{Pds}(A)$. Hence $\alpha = \zeta\delta\lambda$, and this decomposition is unique [7, §14].

Passing to the projective setting, the above implies that every $\alpha \in \mathrm{PGL}(A)$ factorises uniquely as $\alpha = \zeta\delta\lambda$ with $\zeta \in \mathrm{SO}(A)/(\pm\mathbb{I})$, $\delta \in \mathrm{SPds}(A) = \mathrm{Pds}(A) \cap \mathrm{SL}(A)$, $\lambda \in \mathrm{C}_2$. As noted in Remark 7, $\mathrm{SO}(A)/(\pm\mathbb{I}) \simeq \mathrm{PGO}^+(A)$, and ζ can indeed be viewed as an element in $\mathrm{PGO}^+(A)$ instead of $\mathrm{SO}(A)/(\pm\mathbb{I})$. The cyclic group C_2 acts on $\mathrm{PGO}^+(A)$ by $\varphi^\lambda = \lambda\varphi\lambda$ ($\lambda \in \mathrm{C}_2$, $\varphi \in \mathrm{PGO}^+(A)$). Note that $\mathrm{L}_a^\kappa = \mathrm{R}_{\bar{a}} = \mathrm{R}_a^{-1}$ and $\mathrm{R}_a^\kappa = \mathrm{L}_{\bar{a}} = \mathrm{L}_a^{-1}$ in $\mathrm{PGO}^+(A)$. We write $\varphi^{-\lambda}$ for $\left(\varphi^{-1}\right)^\lambda = \left(\varphi^\lambda\right)^{-1}$.

Let $\alpha, \beta \in \mathrm{PGL}(A)$, $\alpha = \zeta\delta\lambda$ and $\beta = \eta\epsilon\mu$, where $\zeta, \eta \in \mathrm{PGO}^+(A)$, $\delta, \epsilon \in \mathrm{SPds}(A)$ and $\lambda, \mu \in \mathrm{C}_2$. The action (4) of $\varphi \in \mathrm{PGO}^+(A)$ on $(\alpha, \beta) \in X_A$ is given by

$$\varphi \cdot (\alpha, \beta) = (\varphi_1\zeta\delta\lambda\varphi^{-1}\,,\ \varphi_2\eta\epsilon\mu\varphi^{-1}) = (\varphi_1\zeta\varphi^{-\lambda}(\varphi^\lambda\delta\varphi^{-\lambda})\lambda\,,\ \varphi_2\eta\varphi^{-\mu}(\varphi^\mu\epsilon\varphi^{-\mu})\mu)\,,$$

and $\varphi_1\zeta\varphi^{-\lambda}$, $\varphi_2\eta\varphi^{-\mu} \in \mathrm{PGO}^+(A)$, $\varphi^\lambda\delta\varphi^{-\lambda}$, $\varphi^\mu\epsilon\varphi^{-\mu} \in \mathrm{SPds}(A)$.

The group $N(A)^*$ acts on $\mathrm{PGO}^+(A)^2$ by $w \cdot (\zeta, \eta) = (\mathrm{R}_w^{-1}\,\zeta\,,\ \mathrm{L}_w\,\eta)$, and we denote the orbit set of this action by $\mathrm{PGO}^+(A)^2/N(A)^*$. The argument in the preceding paragraph shows that the $\mathrm{PGO}^+(A)$-action on $\mathrm{PGO}^+(A)^2/N(A)^* \times \mathrm{SPds}(A)^2 \times \mathrm{C}_2^2$ by

$$\varphi \cdot ((\zeta, \eta), (\delta, \epsilon), (\lambda, \mu)) = \big((\varphi_1\zeta\varphi^{-\lambda}, \varphi_2\eta\varphi^{-\mu}), (\varphi^\lambda\delta\varphi^{-\lambda}, \varphi^\mu\epsilon\varphi^{-\mu}), (\lambda, \mu)\big) \tag{7}$$

is equivalent to the $\mathrm{PGO}^+(A)$-action (4) on X_A via the map

$$m : \mathrm{PGO}^+(A)^2/N(A)^* \times \mathrm{SPds}(A)^2 \times \mathrm{C}_2^2 \to X_A\,,\ ((\zeta, \eta), (\delta, \epsilon), (\lambda, \mu)) \mapsto (\zeta\delta\lambda, \eta\epsilon\mu)$$

(i.e., $\varphi m = m\varphi$ for all $\varphi \in \mathrm{PGO}^+(A)$). This means, in particular, that the groupoid $\mathscr{Y}(A) = {}_{\mathrm{PGO}^+(A)}\left(\mathrm{PGO}^+(A)^2/N(A)^* \times \mathrm{SPds}(A)^2 \times \mathrm{C}_2^2\right)$ of the action (7) is isomorphic to $\mathscr{X}(A)$:

Proposition 11. *The functor $\mathscr{G}_A : \mathscr{Y}(A) \to \mathscr{X}(A)$ defined by $\mathscr{G}_A(y) = m(y)$ for $y \in \mathscr{Y}(A)$, and $\mathscr{G}_A(\varphi) = \varphi$ for morphisms $\varphi \in \mathrm{PGO}^+(A)$, is an isomorphism of categories.*

From here on we focus exclusively on the four-dimensional case, in which $A \simeq \mathbb{H}$. Corollary 9 implies that every element $\zeta \in \mathrm{PGO}^+(\mathbb{H})$ has the form $\zeta = \mathrm{L}_a\,\mathrm{R}_b$ for some $a, b \in \mathbb{H}^*$. Since $\mathbb{H}$ is associative, it follows that if $\varphi = \mathrm{L}_a\,\mathrm{R}_b$ then $\varphi_1 = \mathrm{L}_a$, $\varphi_2 = \mathrm{R}_b$ are triality components of φ. Moreover, since $[\mathrm{L}_a, \mathrm{R}_b] = 0$ for all $a, b \in \mathbb{H}$, we have

$$(\mathrm{L}_a\,\mathrm{R}_b, \mathrm{L}_c\,\mathrm{R}_d) = c^{-1} \cdot (\mathrm{L}_a\,\mathrm{R}_b, \mathrm{L}_c\,\mathrm{R}_d) = (\mathrm{R}_c\,\mathrm{L}_a\,\mathrm{R}_b, \mathrm{R}_d) = (\mathrm{L}_a\,\mathrm{R}_{bc}, \mathrm{R}_d)$$

in $\mathrm{PGO}^+(\mathbb{H})^2/N(\mathbb{H})^* = \mathrm{PGO}^+(\mathbb{H})^2/\mathbb{H}^*$. Hence every element $(\zeta, \eta) \in \mathrm{PGO}^+(\mathbb{H})^2/\mathbb{H}^*$ can be written on the form $(\zeta, \eta) = (\mathrm{L}_a\,\mathrm{R}_b, \mathrm{R}_c)$, with $a, b, c \in \mathbb{H}^*$. Now the action of $\varphi = \mathrm{L}_s\,\mathrm{R}_t \in \mathrm{PGO}^+(\mathbb{H})$ ($s, t \in \mathbb{H}^*$) on $((\zeta, \eta), (\delta, \epsilon), (\lambda, \mu)) \in \mathscr{Y}(\mathbb{H})$ is given by

$$\varphi \cdot ((\zeta, \eta), (\delta, \epsilon), (\lambda, \mu)) = \Big(\Big(\mathrm{L}_s\,\mathrm{L}_a\,\mathrm{R}_b\,\mathrm{L}_s^{-\lambda}\,\mathrm{R}_t^{-\lambda}, \mathrm{R}_t\,\mathrm{R}_c\,\mathrm{L}_s^{-\mu}\,\mathrm{R}_t^{-\mu}\Big)\,,\ \big(\varphi^\lambda\delta\varphi^{-\lambda}, \varphi^\mu\epsilon\varphi^{-\mu}\big)\,,\ (\lambda, \mu)\Big)\,. \tag{8}$$

For $(i,j) \in \{-1,1\}^2$, set $\mathscr{Y}(\mathbb{H})_{i,j} = \mathscr{G}_A^{-1}(\mathscr{X}(\mathbb{H})_{i,j}) \subset \mathscr{Y}(\mathbb{H})$. Note that $((\zeta,\eta),(\delta,\epsilon),(\kappa^i,\kappa^j)) \in \mathscr{Y}(\mathbb{H})_{(-1)^i,(-1)^j}$. This gives the decomposition

$$\mathscr{Y}(\mathbb{H}) = \coprod_{i,j\in\{-1,1\}} \mathscr{Y}(\mathbb{H})_{i,j}\,,$$

and the action of $\varphi = \mathrm{L}_s\,\mathrm{R}_t$ on each of the cofactors $\mathscr{Y}(\mathbb{H})_{i,j}$ can be studied separately.

For $((\zeta,\eta),(\delta,\epsilon),(\mathbb{I},\mathbb{I})) \in \mathscr{X}(\mathbb{H})_{1,1}$, we have

$$\begin{aligned}\varphi\cdot((\zeta,\eta),(\delta,\epsilon),(\mathbb{I},\mathbb{I})) &= \left(\left(\mathrm{L}_s\,\mathrm{L}_a\,\mathrm{R}_b\,\mathrm{L}_s^{-1}\,\mathrm{R}_t^{-1}, \mathrm{R}_t\,\mathrm{R}_c\,\mathrm{L}_s^{-1}\,\mathrm{R}_t^{-1}\right),\, \left(\varphi\delta\varphi^{-1},\varphi\epsilon\varphi^{-1}\right),\, (\mathbb{I},\mathbb{I})\right)\\ &= \left(\left(\mathrm{R}_s^{-1}\,\mathrm{R}_b\,\mathrm{R}_t^{-1}\,\mathrm{L}_s\,\mathrm{L}_a\,\mathrm{L}_s^{-1}, \mathrm{R}_t\,\mathrm{R}_c\,\mathrm{R}_t^{-1}\right),\, \left(\varphi\delta\varphi^{-1},\varphi\epsilon\varphi^{-1}\right),\, (\mathbb{I},\mathbb{I})\right)\,.\end{aligned}$$

In particular, if $s=1$, $t=b$, so that $\varphi = \mathrm{R}_b$, then

$$\varphi\cdot((\zeta,\eta),(\delta,\epsilon),(\mathbb{I},\mathbb{I})) = \left((\mathrm{L}_a,\mathrm{R}_{bcb^{-1}}),\, \left(\mathrm{R}_b\,\delta\,\mathrm{R}_b^{-1}, \mathrm{R}_b\,\epsilon\,\mathrm{R}_b^{-1}\right),\, (\mathbb{I},\mathbb{I})\right)\,.$$

Hence every orbit in $\mathscr{Y}(\mathbb{H})_{1,1}$ contains an element of the form $((\mathrm{L}_a,\mathrm{R}_b),(\delta,\epsilon),(\mathbb{I},\mathbb{I}))$. Moreover, the action of $\varphi = \mathrm{L}_s\,\mathrm{R}_t$ on such an element is given by

$$\begin{aligned}\varphi\cdot((\mathrm{L}_a,\mathrm{R}_b),(\delta,\epsilon),(\mathbb{I},\mathbb{I})) &= \left(\left(\mathrm{L}_s\,\mathrm{L}_a\,\mathrm{L}_s^{-1}\,\mathrm{R}_t^{-1}, \mathrm{R}_t\,\mathrm{R}_b\,\mathrm{L}_s^{-1}\,\mathrm{R}_t^{-1}\right), (\varphi\delta\varphi^{-1},\varphi\epsilon\varphi^{-1}), (\mathbb{I},\mathbb{I})\right)\\ &= \left(\left(\mathrm{R}_{(st)^{-1}}\,\mathrm{L}_{sas^{-1}}, \mathrm{R}_{t^{-1}bt}\right), (\varphi\delta\varphi^{-1},\varphi\epsilon\varphi^{-1}), (\mathbb{I},\mathbb{I})\right)\end{aligned} \tag{9}$$

which has the form $((\mathrm{L}_c,\mathrm{R}_d),(\delta,\epsilon),(\mathbb{I},\mathbb{I}))$ if and only if $t = s^{-1}$, that is, $\varphi = \mathrm{L}_s\,\mathrm{R}_s^{-1}$.

Next, consider $((\zeta,\eta),(\delta,\epsilon),(\kappa,\mathbb{I})) \in \mathscr{Y}(\mathbb{H})_{-1,1}$. In this case,

$$\begin{aligned}\varphi\cdot((\zeta,\eta),(\delta,\epsilon),(\kappa,\mathbb{I})) &= \left(\left(\mathrm{L}_s\,\mathrm{L}_a\,\mathrm{R}_b\,\mathrm{L}_s^{-\kappa}\,\mathrm{R}_t^{-\kappa}, \mathrm{R}_t\,\mathrm{R}_c\,\mathrm{L}_s^{-1}\,\mathrm{R}_t^{-1}\right),\, \left(\varphi^\kappa\delta\varphi^{-\kappa},\varphi\epsilon\varphi^{-1}\right),\, (\kappa,\mathbb{I})\right)\\ &= \left(\left(\mathrm{L}_s\,\mathrm{L}_a\,\mathrm{L}_t\,\mathrm{R}_s^{-1}\,\mathrm{R}_b\,\mathrm{R}_s, \mathrm{R}_t\,\mathrm{R}_c\,\mathrm{R}_t^{-1}\right),\, \left(\varphi^\kappa\delta\varphi^{-\kappa},\varphi\epsilon\varphi^{-1}\right),\, (\kappa,\mathbb{I})\right)\end{aligned}$$

Again, setting $s=1$ and $t=a^{-1}$ gives $\varphi = \mathrm{R}_a^{-1}$ and

$$\mathrm{R}_a^{-1}\cdot((\zeta,\eta),(\delta,\epsilon),(\kappa,\mathbb{I})) = \left((\mathrm{R}_b,\mathrm{R}_{aca^{-1}}),\, \left(\mathrm{L}_a\,\delta\,\mathrm{L}_a^{-1}, \mathrm{R}_a^{-1}\,\epsilon\,\mathrm{R}_a\right),\, (\kappa,\mathbb{I})\right)$$

so the orbit of $((\zeta,\eta),(\delta,\epsilon),(\kappa,\mathbb{I}))$ contains an element of the form $((\zeta,\eta)=(\mathrm{R}_a,\mathrm{R}_b),(\delta',\epsilon'),(\kappa,\mathbb{I}))$. Similarly to the previous case with $\mathscr{Y}(\mathbb{H})_{1,1}$, the group elements $\varphi \in \mathrm{PGO}^+(\mathbb{H})$ stabilising this form, so that $\varphi\cdot((\mathrm{R}_a,\mathrm{R}_b),(\delta,\epsilon),(\kappa,\mathbb{I})) = ((\mathrm{R}_c,\mathrm{R}_d),(\delta',\epsilon'),(\kappa,\mathbb{I}))$ for some $c,d\in\mathbb{H}$, are precisely those of the form $\varphi = \mathrm{L}_s\,\mathrm{R}_s^{-1}$, $s\in\mathbb{H}^*$. Note that if $\varphi = \mathrm{L}_s\,\mathrm{R}_s^{-1}$ then $\varphi^\kappa = \varphi$, hence

$$\varphi\cdot((\mathrm{R}_a,\mathrm{R}_b),(\delta,\epsilon),(\kappa,\mathbb{I})) = \left((\mathrm{R}_{sas^{-1}},\mathrm{R}_{sbs^{-1}}), (\varphi\delta\varphi^{-1},\varphi\epsilon\varphi^{-1}), (\kappa,\mathbb{I})\right)\,.$$

Similar computations can be made for $\mathscr{Y}(\mathbb{H})_{1,-1}$ and $\mathscr{Y}(\mathbb{H})_{-1,-1}$. The results are summarised in Proposition 12 below.

For $a\in\mathbb{H}^*$, set $c_a = \mathrm{L}_a\,\mathrm{R}_a^{-1} \in \mathrm{PGO}^+(\mathbb{H})$. Note that the kernel of the morphism $\mathbb{H}^*\to\mathrm{PGO}^+(\mathbb{H})$, $a\mapsto c_a$ is $\mathbb{R}^*1$.

Proposition 12. *Let $(i,j)\in\{-1,1\}^2$. Each of the subcategories $\mathscr{Y}(\mathbb{H})_{i,j}$ of $\mathscr{Y}(\mathbb{H})$ is equivalent to the groupoid $\mathscr{Z} = {}_{\mathbb{H}^*/\mathbb{R}^*}\left((\mathbb{H}^*/\mathbb{R}^*)^2\times\mathrm{SPds}(\mathbb{H})^2\right)$ of the action*

$$s\cdot((a,b),(\delta,\epsilon)) = ((sas^{-1},sbs^{-1}),(c_s\delta c_s^{-1},c_s\epsilon c_s^{-1}))\,.$$

An equivalence $\mathscr{H}_{i,j}:\mathscr{Z}\to\mathscr{Y}(\mathbb{H})_{i,j}$ is given by $\mathscr{H}_{i,j}(s) = c_s$ for morphisms $s\in\mathbb{H}^*/\mathbb{R}^*$ and

$$\mathscr{H}_{1,1}((a,b),(\delta,\epsilon)) = ((\mathrm{L}_a,\mathrm{R}_b),(\delta,\epsilon),(\mathbb{I},\mathbb{I}))\,,\qquad \mathscr{H}_{-1,1}((a,b),(\delta,\epsilon)) = ((\mathrm{R}_a,\mathrm{R}_b),(\delta,\epsilon),(\kappa,\mathbb{I}))\,,$$

$$\mathscr{H}_{1,-1}((a,b),(\delta,\epsilon)) = ((\mathrm{L}_a,\mathrm{L}_b),(\delta,\epsilon),(\mathbb{I},\kappa))\,,\qquad \mathscr{H}_{-1,-1}((a,b),(\delta,\epsilon)) = ((\mathrm{L}_a,\mathrm{R}_b),(\delta,\epsilon),(\kappa,\kappa))\,.$$

Remark 13. Proposition 12 shows, in particular, that $\mathscr{X}(\mathbb{H})_{i,j}\simeq\mathscr{X}(\mathbb{H})_{i',j'}$ for all $(i,j),(i',j')\in\{-1,1\}^2$.

The image of the group monomorphism $\mathbb{H}^*/\mathbb{R}^*\to\mathrm{PGO}^+(\mathbb{H})$, $s\mapsto c_s$ is $\{\varphi\in\mathrm{PGO}^+(\mathbb{H}) \mid \varphi(\mathbb{R}^*1)=\mathbb{R}^*1\}$, which can be identified with $\mathrm{SO}_1(\mathbb{H}) = \{\varphi\in\mathrm{SO}(\mathbb{H}) \mid \varphi(1)=1\}\simeq\mathrm{SO}(1^\perp)$. Thus the map $f:\mathbb{H}^*/\mathbb{R}^*\to\mathrm{SO}_1(\mathbb{H})$, $s\mapsto c_s$ is an isomorphism, inducing an action of $\mathrm{SO}_1(\mathbb{H})$ on the object set $(\mathbb{H}^*/\mathbb{R}^*)^2\times\mathrm{SPds}(\mathbb{H})^2$ of $\mathscr{Z}$, given by $\varphi\cdot((a,b),(\delta,\epsilon)) = f^{-1}(\varphi)\cdot((a,b),(\delta,\epsilon)) = ((\varphi(a),\varphi(b)),(\varphi\delta\varphi^{-1},\varphi\epsilon\varphi^{-1}))$.

This allows for the following geometric interpretation of the category $\mathscr{Z}$. Elements in $\mathbb{H}^*/\mathbb{R}^*$ are viewed as lines through the origin in $\mathbb{H}$ (that is, elements in the real projective space $\mathbb{P}(\mathbb{H})$), and $\delta \in \mathrm{SPds}(A)$ is identified with the three-dimensional hyper-ellipsoid $E_\delta = \{x \in \mathbb{H} \mid \langle x, \delta(x)\rangle = 1\} \subset \mathbb{H}$. The set $\mathcal{E} = \{E_\delta \mid \delta \in \mathrm{SPds}(\mathbb{H})\}$ consists of all hyper-ellipsoids centered in the origin and satisfying $\lambda_1\lambda_2\lambda_3\lambda_4 = 1$, where $\lambda_i \in \mathbb{R}_{>0}$, $i = 1, 2, 3, 4$ are the lengths of the principal axes. Moreover, $\mathbb{H}$ is identified with $\mathbb{R}^4$ in the natural way, and $1_{\mathbb{H}}^{\perp}$ with $V = \mathrm{span}\{e_2, e_3, e_4\} \subset \mathbb{R}^4$. Now objects in $\mathscr{X}$ can be viewed as configurations in $\mathbb{R}^4$ consisting of two lines $a, b \in \mathbb{P}(\mathbb{H})$, and two hyper-ellipsoids $E_\delta, E_\epsilon \in \mathcal{E}$. A morphism $(a, b, E_\delta, E_\epsilon) \to (a', b', E_{\delta'}, E_{\epsilon'})$ between two such configurations is an element $\varphi \in \mathrm{SO}(V) \subset \mathrm{SO}(\mathbb{R}^4)$ transforming one configuration to the other: $(\varphi(a), \varphi(b), \varphi(E_\delta), \varphi(E_\epsilon)) = (a', b', E_{\delta'}, E_{\epsilon'})$.

4. Composition algebras

Lemma 14. *Let $A = (A, n_A)$ be a Hurwitz algebra, and $\alpha, \beta \in \mathrm{GL}(A)$. The isotope $A_{\alpha,\beta}$ is a composition algebra if and only if $\alpha, \beta \in \mathrm{GO}(A, n_A)$.*

Proof. If α, β are similitudes, then $A_{\alpha,\beta}$ is a composition algebra with respect to the norm $n = \mu(\alpha)\mu(\beta)n_A$. Conversely, suppose $A_{\alpha,\beta}$ is a composition algebra with norm n. Then, by standard arguments (see e.g. [11, p. 957]), there exists a Hurwitz algebra B isotopic to $A_{\alpha,\beta}$, with norm $n_B = n$. Since B is an isotope of $A_{\alpha,\beta}$, it is also isotopic to A, and thus, by Proposition 2, $n = \rho n_A$ for some $\rho \in k^*$. Hence

$$n_A(\alpha(x))n_A(\beta(y)) = n_A(\alpha(x)\beta(y)) = n_A(x \circ y) = \rho^{-1}n(x \circ y) = \rho^{-1}n(x)n(y) = \rho n_A(x)n_A(y)$$

for all $x, y \in A$, from which follows that α and β are similitudes with respect to n_A. □

Given a Hurwitz algebra A, let $\mathscr{X}^c(A) \subset \mathscr{X}(A)$ be the full subcategory formed by all (α, β) such that $\mathscr{F}_A(\alpha, \beta)$ is a composition algebra. Lemma 14 implies that $\mathscr{X}^c(A) = \mathrm{PGO}(A, n_A)^2/N(A)^* \subset \mathrm{PGL}(A, n_A)^2/N(A)^* = \mathscr{X}(A)$.

For $\dim A \geqslant 2$, the property of being a proper respectively improper similitude is retained by the factors α, β of $(\alpha, \beta) \in \mathscr{X}^c(A)$ under the action of $\mathrm{PGO}^+(A)$ (respectively $\mathrm{PGO}(A)$ in the two-dimensional case). This means that for each $(i, j) \in \{-1, 1\}^2$, the subset

$$\mathscr{X}^{c(i,j)}(A) = \left(\mathrm{PGO}^i(A) \times \mathrm{PGO}^j(A)\right)/N(A)^* \subset \mathscr{X}^c(A)$$

(where we use the notational convention $\mathrm{PGO}^{\pm 1}(A) = \mathrm{PGO}^{\pm}(A)$) is invariant under this action, and the category $\mathscr{X}^c(A)$ decomposes as

$$\mathscr{X}^c(A) = \coprod_{i,j \in \{-1,1\}} \mathscr{X}^{c(i,j)}(A)\,.$$

This refines the decomposition (6) for $A \not\simeq k \times k$: $\mathscr{X}^{c(i,j)}(A) \subset \mathscr{X}^c(A)_{i,j}$ for all $i, j \in \{-1, 1\}$. If $-1 \notin k^l$, $l = \dim A$, then $\mathscr{X}^{c(i,j)}(A) = \mathscr{X}^c(A)_{i,j}$, but whereas the four sets $\mathscr{X}^{c(i,j)}(A)$ are always distinct, $\mathscr{X}^c(A)_{i,j} = \mathscr{X}^c(A)$ for all $i, j \in \{-1, 1\}$ in case $-1 \in k^l$.

If (A, n_A) is a Euclidean space, the groups $\mathrm{PGO}(A)$ and $\mathrm{PGO}^+(A)$ are canonically identified with $\mathrm{O}(A)/(\pm\mathbb{I}_A)$ and $\mathrm{SO}(A)/(\pm\mathbb{I}_A)$ respectively. This, together with Theorem 6, gives a description of all finite-dimensional absolute valued algebras equivalent to Theorem 4.3 in [1].

We proceed to describe all composition algebras isotopic to a fixed quaternion algebra $A = (A, n)$ over k. Every $(\alpha, \beta) \in \mathscr{X}^c(A)$ can be written as $(\alpha, \beta) = (\zeta\lambda, \eta\mu)$ with $\zeta, \eta \in \mathrm{GO}^+(A)$ and $\lambda, \mu \in \mathrm{C}_2$. Corollary 9 now gives $\zeta = \mathrm{L}_a \mathrm{R}_b$, $\eta = \mathrm{L}_c \mathrm{R}_d$ for some $a, b, c, d \in A^*$, and since $N(A) = A$, $(\alpha, \beta) = (\mathrm{L}_a \mathrm{R}_b \lambda, \mathrm{L}_c \mathrm{R}_d \mu) = (\mathrm{L}_a \mathrm{R}_{bc} \lambda, \mathrm{R}_d)$ in $X_A = \mathrm{PGL}(A)^2/N(A)^*$. Thus, analogously with the Euclidean case treated in Section 3, (α, β) can be written as $(\alpha, \beta) = (\mathrm{L}_a \mathrm{R}_b \lambda, \mathrm{R}_c \mu)$. Moreover, one reads off that $(\alpha, \beta) \in \mathscr{X}^{c(\det(\lambda), \det(\mu))}(A)$.

Let $(\alpha, \beta) = (\mathrm{L}_a \mathrm{R}_b, \mathrm{R}_c) \in \mathscr{X}^{c(1,1)}(A)$. Now $\mathrm{R}_b \cdot (\alpha, \beta) = (\mathrm{L}_a, \mathrm{R}_{b^{-1}cb})$, so every $\mathrm{PGO}^+(A)$-orbit in $\mathscr{X}^{c(1,1)}(A)$ contains an element of the form $(\mathrm{L}_b, \mathrm{R}_b)$. Similarly, every orbit in $\mathscr{X}^{c(-1,1)}(A)$ contains an element of the form $(\mathrm{R}_a \kappa, \mathrm{R}_b)$, every orbit in $\mathscr{X}^{c(1,-1)}(A)$ contains some $(\mathrm{L}_a, \mathrm{L}_b \kappa)$, and every orbit in $\mathscr{X}^{c(-1,-1)}(A)$ contains an element of the form $(\mathrm{L}_a \kappa, \mathrm{R}_b \kappa)$. By computations similar to (9), one proves that $\varphi \in \mathrm{PGO}^+(A)$ stabilises each of these forms if and only if $\varphi = c_s$

for some $s \in A^*$, and that

$$\begin{aligned} c_s \cdot (\mathrm{L}_a, \mathrm{R}_b) &= (\mathrm{L}_s \mathrm{L}_a \mathrm{L}_s^{-1}, \mathrm{R}_s^{-1} \mathrm{R}_b \mathrm{R}_s) = (\mathrm{L}_{sas^{-1}}, \mathrm{R}_{sbs^{-1}}) \,, \\ c_s \cdot (\mathrm{R}_a \kappa, \mathrm{R}_b) &= (\mathrm{R}_s^{-1} \mathrm{R}_a \mathrm{R}_s \kappa, \mathrm{R}_s^{-1} \mathrm{R}_b \mathrm{R}_s) = (\mathrm{R}_{sas^{-1}} \kappa, \mathrm{R}_{sbs^{-1}}) \,, \\ c_s \cdot (\mathrm{L}_a, \mathrm{L}_b \kappa) &= (\mathrm{L}_s \mathrm{L}_a \mathrm{L}_s^{-1}, \mathrm{L}_s \mathrm{L}_b \mathrm{L}_s^{-1} \kappa) = (\mathrm{L}_{sas^{-1}}, \mathrm{L}_{sbs^{-1}} \kappa) \,, \\ c_s \cdot (\mathrm{L}_a \kappa, \mathrm{R}_b \kappa) &= (\mathrm{L}_s \mathrm{L}_a \mathrm{L}_s^{-1} \kappa, \mathrm{R}_s^{-1} \mathrm{R}_b \mathrm{R}_s \kappa) = (\mathrm{L}_{sas^{-1}} \kappa, \mathrm{R}_{sbs^{-1}} \kappa) \,. \end{aligned}$$

This proves the following result, which gives an isomorphism criterion for all composition algebras isotopic to A.

Proposition 15. *Each of the categories $\mathscr{X}^{c(i,j)}(A)$ is equivalent to the groupoid $\mathscr{Z}^c(A) = {}_{A^*/k^*}(A^*/k^*)^2$ of the action $s \cdot (a,b) = (sas^{-1}, sbs^{-1})$. Equivalences $\mathscr{H}^c_{i,j} : \mathscr{Z}^c(A) \to \mathscr{X}^c(A)_{i,j}$ are given by $\mathscr{H}^c_{i,j}(s) = c_s$ for morphisms, and*

$$\begin{aligned} \mathscr{H}^c_{1,1}(a,b) &= (\mathrm{L}_a, \mathrm{R}_b) \,, & \mathscr{H}^c_{-1,1}(a,b) &= (\mathrm{R}_a, \mathrm{R}_b) \,, \\ \mathscr{H}^c_{1,-1}(a,b) &= (\mathrm{L}_a, \mathrm{L}_b) \,, & \mathscr{H}^c_{-1,-1}(a,b) &= (\mathrm{L}_a, \mathrm{R}_b) \,. \end{aligned}$$

A characterisation of all four-dimensional composition algebras over k, similar to Proposition 15, is given by Stampfli-Rollier in Section 3–4 of [17]. Her exposition also contains explicit isomorphism criteria in terms of the parameters a, b. The Euclidean case, comprising all composition algebras isotopic to $\mathbb{H}$, that is, all four-dimensional absolute valued algebras, has also been described by Ramírez Álvarez [15]. Forsberg, in his master thesis [6], refined that description to give an explicit cross-section for the isomorphism classes of these algebras.

References

[1] A. Caderón, A. Kaidi, C. Martín, A. Morales, M. Ramírez, and A. Rochdi. Finite-dimensional absolute valued algebras. To appear in Israel J. Math.

[2] J. A. Cuenca Mira. On one-sided division infinite-dimensional normed real algebras. *Publ. Mat.*, 36(2A):485–488 (1993), 1992.

[3] E. Darpö and E. Dieterich. The double sign of a finite-dimensional real division algebra. In preparation.

[4] E. Dieterich. Classification, automorphism groups and categorical structure of the two-dimensional real division algebras. *J. Algebra Appl.*, 4(5):517–538, 2005.

[5] A. Elduque and H. C. Myung. On flexible composition algebras. *Comm. Algebra*, 21(7):2481–2505, 1993.

[6] L. Forsberg. Four-dimensional absolute valued algebras. Master's thesis, Uppsala University, 2009. U.U.D.M. Project Report 2009:9.

[7] F. R. Gantmacher. *The theory of matrices. Vol. 1.* Translated by K. A. Hirsch. Chelsea Publishing Co., New York, 1959.

[8] M. Hübner and H. P. Petersson. Two-dimensional real division algebras revisited. *Beiträge Algebra Geom.*, 45(1):29–36, 2004.

[9] N. Jacobson. Composition algebras and their automorphisms. *Rend. Circ. Mat. Palermo*, 7(2):55–80, 1958.

[10] N. Jacobson. *Basic Algebra II.* W.H. Freeman and Company, second edition, 1985.

[11] I. Kaplansky. Infinite-dimensional quadratic forms admitting composition. *Proceedings of the American mathematical society*, 4(6):956–960, 1953.

[12] M.-A. Knus, A. Merkurjev, M. Rost, and J.-P. Tignol. *The book of involutions*, volume 44 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 1998.

[13] R. Moufang. Zur Struktur von Alternativkörpern. *Math. Ann.*, 110(1):416–430, 1935.

[14] H. P. Petersson. The classification of two-dimensional nonassociative algebras. *Results Math.*, 37(1-2):120–154, 2000.

[15] M. I. Ramírez Álvarez. On four-dimensional absolute-valued algebras. In *Proceedings of the International Conference on Jordan Structures (Málaga, 1997)*, pages 169–173, Málaga, 1999. Univ. Málaga.

[16] T. A. Springer and F. D. Veldkamp. *Octonions, Jordan algebras and exceptional groups.* Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2000.

[17] C. Stampfli-Rollier. 4-dimensionale Quasikompositionsalgebren. *Arch. Math. (Basel)*, 40(6):516–525, 1983.

[18] M. Zorn. Theorie der alternativen Ringe. *Abh. Math. Sem. Hamburg*, 8:123–147, 1931.

E-mail address: darpo@maths.ox.ac.uk

Mathematical Institute, 24–29 St Giles', Oxford OX1 3LD, United Kingdom