The complex of partial bases for F_n and finite generation of the Torelli subgroup of $\mathrm{Aut}(F_n)$

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Abstract

We study the complex of partial bases of a free group, which is an analogue for $\mathrm{Aut}(F_n)$ of the curve complex for the mapping class group. We prove that it is connected and simply connected, and we also prove that its quotient by the Torelli subgroup of $\mathrm{Aut}(F_n)$ is highly connected. Using these results, we give a new, topological proof of a theorem of Magnus that asserts that the Torelli subgroup of $\mathrm{Aut}(F_n)$ is finitely generated.

1 Introduction

Let Σ_g be a compact orientable genus g surface and let Mod_g be its mapping class group. One of the most important and ubiquitous objects associated to Mod_g is the *curve complex* $\mathcal{C}_g$. By definition, this is the simplicial complex whose simplices are sets of homotopy classes of non-nullhomotopic simple closed curves on Σ_g that can be realized disjointly. It can be viewed as an analogue for Mod_g of the Tits building of an algebraic group. The space $\mathcal{C}_g$ has many remarkable properties; for instance, Harer [10] showed that $\mathcal{C}(\Sigma_g)$ is homotopy equivalent to a bouquet of spheres and Masur-Minsky [21] showed that $\mathcal{C}(\Sigma_g)$ is δ-hyperbolic.

There is a useful analogy between the mapping class group of a surface and the automorphism group of a free group F_n on n letters. Because of this, there have been many proposals for analogues of the curve complex for $\mathrm{Aut}(F_n)$ (for instance, see [3, 11, 12, 13, 16]). The purpose of this paper is to prove some topological results about one of these proposed complexes (the *complex of partial bases* $\mathcal{B}_n$; see below). We apply these results to give a quick proof of a classical theorem of Magnus which provides generators for the *Torelli subgroup* $\mathrm{IA}_n < \mathrm{Aut}(F_n)$, which is the kernel of the natural homomorphism from $\mathrm{Aut}(F_n)$ to $\mathrm{Aut}(F_n^{\mathrm{ab}}) \cong \mathrm{Aut}(\mathbb{Z}^n) \cong \mathrm{GL}_n(\mathbb{Z})$.

Our complex is inspired by a subcomplex of $\mathcal{C}_g$, the *nonseparating curve complex*. This is the subcomplex $\mathcal{C}_g^{\mathrm{nosep}}$ of $\mathcal{C}_g$ consisting of simplices $\{\gamma_1, \ldots, \gamma_k\}$ such that the γ_i can be realized by simple closed curves whose union does not separate Σ_g. The complex $\mathcal{C}_g^{\mathrm{nosep}}$ was introduced by Harer [10] and plays an important role in both homological stability results for Mod_g and its subgroups (see [10, 28]) and the second author's approach to the Torelli subgroup of Mod_g (see [25, 27, 29, 30]).

To motivate the definition of our complex, we start by giving an algebraic characterization of $\mathcal{C}_g^{\mathrm{nosep}}$. There is a bijection between free homotopy classes of oriented closed curves on Σ_g and

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conjugacy classes in $\pi_1(\Sigma)$. We will call a generating set $\{\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g\}$ for $\pi_1(\Sigma_g)$ a *standard basis* if it satisfies the surface relation $[\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g] = 1$. We then have the following folklore result, whose proof is an exercise in using the fact that Mod_g is the outer automorphism group of $\pi_1(\Sigma_g)$. If G is a group and $g \in G$, then denote by $[\![g]\!]$ the conjugacy class of g.

Lemma. *A set $\{c_1, \ldots, c_n\}$ of conjugacy classes in $\pi_1(\Sigma_g)$ corresponds to a simplex of $\mathcal{C}_g^{nosep}$ (with some orientation on each curve in the simplex) if and only if there exists a standard basis $\{\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g\}$ for $\pi_1(\Sigma_g)$ such that $c_i = [\![\alpha_i]\!]$ for $1 \leq i \leq k$.*

This suggests the following definition.

Definition. Fix $n \geq 1$. A *partial basis* for F_n consists of elements $\{v_1, \ldots, v_k\} \subset F_n$ such that there exists $v_{k+1}, \ldots, v_n \in F_n$ with $\{v_1, \ldots, v_n\}$ a free basis for F_n. The *complex of partial bases* of F_n, denoted $\mathcal{B}_n$, is the simplicial complex whose $(k-1)$-simplices are unordered sets $\{[\![v_1]\!], \ldots, [\![v_k]\!]\}$ such that $\{v_1, \ldots, v_k\} \subset F_n$ is a partial basis for F_n.

Remark. This definition first appeared in [16], where it is proven that $\mathcal{B}_n$ has infinite diameter for $n \geq 2$.

Our two main results about $\mathcal{B}_n$ are as follows.

Theorem A. *The space $\mathcal{B}_n$ is connected for $n \geq 2$ and 1-connected for $n \geq 3$.*

Theorem B. *The space $\mathcal{B}_n / \mathrm{IA}_n$ is $(n-2)$-connected.*

The analogues of Theorems A and B for $\mathcal{C}_g^{\text{nosep}}$ are due to Harer [10] and the second author [27], respectively. In fact, Harer proved $\mathcal{C}_g^{\text{nosep}}$ is $(g-2)$-connected, which leads us to make the following conjecture.

Conjecture 1.1. *The space $\mathcal{B}_n$ is $(n-2)$-connected.*

Remark. In their papers [12, 13], Hatcher-Vogtmann prove that various simplicial complexes built out of free factors of F_n are highly connected. Their proofs are quite different from our proof of Theorem A and do not seem to apply (at least directly) to $\mathcal{B}_n$. Also, their complexes are not 1-connected for $n = 3$, which renders them unsuitable for our application below to IA_n (the induction needs the case $n = 3$ to get started).

Next, we need another definition.

Definition. Let $\{v_1, \ldots, v_n\}$ be a fixed free basis for F_n. Choose $1 \leq i \leq n$, and let $w \in F_n$ be an element of the subgroup of F_n spanned by $\{v_j \mid j \neq i\}$.

- For $e \in \{1, -1\}$, let $\mathrm{M}_{v_i^e, w} \in \mathrm{Aut}(F_n)$ denote the automorphism that takes v_i^e to $w v_i^e$ and fixes v_j for $j \neq i$.
- Let $\mathrm{C}_{v_i, w} \in \mathrm{Aut}(F_n)$ denote the automorphism that takes v_i to $w v_i w^{-1}$ and fixes v_j for $j \neq i$.

Remark. Observe that $\mathrm{C}_{v_i, w} = \mathrm{M}_{v_i, w} \mathrm{M}_{v_i^{-1}, w}$.

Remark. In the literature, it is common to instead work with the elements $\mathrm{M}'_{v_i^e, w} \in \mathrm{Aut}(F_n)$ that take v_i^e to $v_i^e w$ and fix v_j for $j \neq i$. If one used this convention, then it would make sense to define $\mathrm{C}_{v_i, w}$ to take v_i to $w^{-1} v_i w$; however, this conjugation convention would make conjugation a right action rather than a left action. We prefer to work with left actions.

Using the complex $\mathcal{B}_n$ and Theorems A and B, we give a new proof of the following theorem of Magnus.

Theorem C (Magnus, [20])**.** *Fix a free basis* $\{v_1, \ldots, v_n\}$ *for* F_n*. The group* IA_n *is then generated by the finite set*

$$\{M_{v_i,[v_j,v_k]} \mid 1 \le i,j,k \le n,\ i \ne j,\ i \ne k,\ j \ne k\} \cup \{C_{v_i,v_j} \mid 1 \le i,j \le n,\ i \ne j\}.$$

Remark. The usual statement of Theorem C contains the elements $\mathrm{M}'_{v_i,[v_j,v_k]}$ instead of $\mathrm{M}_{v_i,[v_j,v_k]}$. The two generating sets are equivalent due to the formula

$$\mathrm{M}'_{v_i,[v_j,v_k]} = \mathrm{C}^{-1}_{v_i,v_j}\mathrm{C}^{-1}_{v_i,v_k}\mathrm{C}_{v_i,v_j}\mathrm{C}_{v_i,v_k}\mathrm{M}_{v_i,[v_j,v_k]}.$$

Remark. It was proven independently by Cohen-Pakianathan [6], Farb [8], and Kawazumi [17] that the size of the generating set in Theorem C is as small as possible.

Magnus's original proof of Theorem C had two steps. The first and most difficult is the following result.

Theorem D (Magnus, [20])**.** *Fix a free basis* $\{v_1, \ldots, v_n\}$ *for* F_n*. The group* IA_n *is then normally generated as a subgroup of* $\mathrm{Aut}(F_n)$ *by the single element* C_{v_1,v_2}*.*

In §3 below, we give a topological proof of Theorem D using the complex of partial bases. Following Magnus, one can then derive Theorem C from Theorem D by showing that the subgroup of $\mathrm{Aut}(F_n)$ generated by the generating set in Theorem C is normal. For completeness, we give the details of this argument in Appendix A.

Remark. There is another (quite different) topological proof of Theorem D in a recent paper of Bestvina-Bux-Margalit [2]. That paper also contains in an appendix a sketch of Magnus's derivation of Theorem C from Theorem D.

Remark. An analogue of Theorem D for Mod_g was proven by Powell [24], following work of Birman [5]. Their proof resembled Magnus's proof of Theorem D, though the details were far more complicated. Later, an analogue of Theorem C for Mod_g was proven by Johnson [15], again following Magnus's derivation of Theorem C from Theorem D (though again the details are much more complicated). Recently, the second author [25] gave a topological proof of Birman-Powell's theorem using the curve complex. The machinery of the current paper is set up so that the proof of Theorem D follows the topological argument in [25] closely.

Outline and conventions. One of the key tools in this paper is an analogue of the Birman exact sequence for $\mathrm{Aut}(F_n)$ which the authors proved in [7]. This is discussed in §2. Next, Theorem D is proven in §3 (assuming the truth of Theorems A and B). Finally, Theorems A and B are proven in §4 and §5. The last three sections are largely independent of each other and can be read in any order.

Automorphisms act on the left and compose from right to left like functions. Also, if G is a group and $g,h \in G$, then we define $[g,h] = ghg^{-1}h^{-1}$.

2 The Birman exact sequence for $\mathrm{Aut}(F_n)$

A key tool in this paper is a type of Birman exact sequence for $\mathrm{Aut}(F_n)$ which the authors developed in [7]. This is an analogue of the classical Birman exact sequence for mapping class group (see [4]).

We begin with some notation. For $v_1, \ldots, v_k \in F_n$, let

$$\mathrm{Aut}(F_n, [\![v_1]\!], \ldots, [\![v_k]\!]) = \{\phi \in \mathrm{Aut}(F_n) \mid [\![\phi(v_i)]\!] = [\![v_i]\!] \text{ for } 1 \le i \le k\}.$$

Also, if G is a group and $S \subset G$, then let $\langle S \rangle \subset G$ denote the subgroup of G generated by S and $\langle\!\langle S \rangle\!\rangle \subset G$ denote the normal subgroup of G generated by S.

Let $\{v_1, \ldots, v_n\}$ be a basis for F_n. For some $1 \le k \le n$, let $V = \langle\!\langle v_1, \ldots, v_k \rangle\!\rangle$. Since the group $\mathrm{Aut}(F_n, [\![v_1]\!], \ldots, [\![v_k]\!])$ preserves V, we get an induced map

$$\mathrm{Aut}(F_n, [\![v_1]\!], \ldots, [\![v_k]\!]) \to \mathrm{Aut}(F_n/V)$$

which is clearly a split surjection. Define $\mathcal{K}_n([\![v_1]\!], \ldots, [\![v_k]\!])$ to be its kernel, so we have a split short exact sequence

$$1 \longrightarrow \mathcal{K}_n([\![v_1]\!], \ldots, [\![v_k]\!]) \longrightarrow \mathrm{Aut}(F_n, [\![v_1]\!], \ldots, [\![v_k]\!]) \longrightarrow \mathrm{Aut}(F_n/V) \longrightarrow 1. \tag{1}$$

The paper [7] contains numerous results about $\mathcal{K}_n([\![v_1]\!], \ldots, [\![v_k]\!])$; for instance, it is proven that it is finitely generated but not finitely presentable, its abelianization is calculated, and an explicit infinite presentation for it is constructed. We will only need the following result.

Theorem 2.1 ([7])**.** *If $\{v_1, \ldots, v_n\}$ is a basis for F_n and $1 \le k \le n$, then $\mathcal{K}_n([\![v_1]\!], \ldots, [\![v_k]\!])$ is generated by*

$$\{M_{v_i, v_j} \mid k+1 \le i \le n,\ 1 \le j \le k\} \cup \{C_{v_i, v_j} \mid 1 \le i \le k,\ 1 \le j \le n,\ i \ne j\}.$$

The exact sequence (1) is related to the following exact sequence for $\mathrm{Aut}(\mathbb{Z}^n)$. Let $\overline{V} \subset \mathbb{Z}^n$ be a direct summand and define

$$\mathrm{Aut}(\mathbb{Z}^n, \overline{V}) = \{\phi \in \mathrm{Aut}(\mathbb{Z}^n) \mid \phi|_{\overline{V}} = 1\}.$$

For an appropriate choice of basis, $\mathrm{Aut}(\mathbb{Z}^n, \overline{V})$ consists of matrices in $\mathrm{GL}_n(\mathbb{Z})$ with an identity block in the upper left hand corner and a block of zeros in the lower left hand corner. We then have a split short exact sequence

$$1 \longrightarrow \mathrm{Hom}(\mathbb{Z}^n/\overline{V}, \overline{V}) \longrightarrow \mathrm{Aut}(\mathbb{Z}^n, \overline{V}) \longrightarrow \mathrm{Aut}(\mathbb{Z}^n/\overline{V}) \longrightarrow 1. \tag{2}$$

Letting $\pi\colon \mathbb{Z}^n \to \mathbb{Z}^n/V$ be the projection, an element $\varphi \in \mathrm{Hom}(\mathbb{Z}^n/\overline{V}, \overline{V})$ corresponds to the element of $\mathrm{Aut}(\mathbb{Z}^n, \overline{V})$ that takes $x \in \mathbb{Z}^n$ to $x + \varphi(\pi(x))$.

The exact sequences (1) and (2) are related by the following lemma.

Lemma 2.2 (Images of stabilizers)**.** *Fix bases $\{v_1, \ldots, v_n\}$ and $\{\overline{v}_1, \ldots, \overline{v}_n\}$ for F_n and $\mathbb{Z}^n$, respectively, such that $\pi(v_i) = \overline{v}_i$ for $1 \le i \le n$. Choose some $1 \le k \le n$. Set $\overline{V} = \langle \overline{v}_1, \ldots, \overline{v}_k \rangle \subset \mathbb{Z}^n$ and $V = \langle\!\langle v_1, \ldots, v_k \rangle\!\rangle \subset F_n$. There is then a commutative diagram of split short exact sequences*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathcal{K}_n([\![v_1]\!], \ldots, [\![v_k]\!]) & \longrightarrow & \mathrm{Aut}(F_n, [\![v_1]\!], \ldots, [\![v_k]\!]) & \longrightarrow & \mathrm{Aut}(F_n/V) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathrm{Hom}(\mathbb{Z}^n/\overline{V}, \overline{V}) & \longrightarrow & \mathrm{Aut}(\mathbb{Z}^n, \overline{V}) & \longrightarrow & \mathrm{Aut}(\mathbb{Z}^n/\overline{V}) & \longrightarrow & 1 \end{array}$$

whose vertical maps are all surjective.

Remark. By a commutative diagram of split short exact sequences, we mean not only that the diagram commutes and that two sequences are split, but also that the splitting is compatible with the commutative diagram in the obvious way.

Proof of Lemma 2.2. The only nonobvious claims are the surjectivity of the vertical maps. The fact that the map $\mathrm{Aut}(F_n/V) \to \mathrm{Aut}(\mathbb{Z}^n/\overline{V})$ is surjective is classical, while the fact that the map $\mathcal{K}_n([\![v_1]\!], \ldots, [\![v_k]\!]) \to \mathrm{Hom}(\mathbb{Z}^n/\overline{V}, \overline{V})$ is surjective follows from the fact that the elements

$$\{\mathrm{M}_{v_i,v_j} \mid k+1 \le i \le n,\ 1 \le j \le k\} \subset \mathcal{K}_n([\![v_1]\!], \ldots, [\![v_k]\!])$$

project to a basis for $\mathrm{Hom}(\mathbb{Z}^n/\overline{V}, \overline{V})$. The surjectivity of the map $\mathrm{Aut}(F_n, [\![v_1]\!], \ldots, [\![v_k]\!]) \to \mathrm{Aut}(\mathbb{Z}^n, \overline{V})$ now follows from the five lemma. □

Theorem 2.1 and Lemma 2.2 have the following corollary.

Corollary 2.3. *Let $\{v_1, \ldots, v_n\}$ be a basis for F_n. Then for $1 \le k \le n$, the group $\mathcal{K}_n([\![v_1]\!], \ldots, [\![v_k]\!]) \cap \mathrm{IA}_n$ is normally generated as a subgroup of $\mathcal{K}_n([\![v_1]\!], \ldots, [\![v_k]\!])$ by*

$$\{C_{v_i,v_j} \mid 1 \le i \le k,\ 1 \le j \le n,\ i \ne j\}.$$

Proof. Let $\pi: F_n \to \mathbb{Z}^n$ be the abelianization map and let $\overline{V} = \langle \pi(v_1), \ldots, \pi(v_k)\rangle$. Using Lemma 2.2, we have that $\mathcal{K}_n([\![v_1]\!], \ldots, [\![v_k]\!]) \cap \mathrm{IA}_n$ is the kernel of a surjective map

$$\Phi: \mathcal{K}_n([\![v_1]\!], \ldots, [\![v_k]\!]) \longrightarrow \mathrm{Hom}(\mathbb{Z}^n/\overline{V}, \overline{V}).$$

Let $S_{\mathcal{K}}$ be the generating set for $\mathcal{K}_n([\![v_1]\!], \ldots, [\![v_k]\!])$ given by Theorem 2.1 and let $S_{\mathcal{K},\mathrm{IA}} \subset S_{\mathcal{K}}$ be the set we are trying to prove normally generates $\mathcal{K}_n([\![v_1]\!], \ldots, [\![v_k]\!]) \cap \mathrm{IA}_n$. Also, let

$$T = S_{\mathcal{K}} \setminus S_{\mathcal{K},\mathrm{IA}} = \{\mathrm{M}_{v_i,v_j} \mid k+1 \le i \le n,\ 1 \le j \le k\}.$$

Observe the following two facts.

- $S_{\mathcal{K},\mathrm{IA}} \subset \ker(\Phi)$.
- $\Phi|_T$ is injective and $\Phi(T)$ is a basis for the free abelian group $\mathrm{Hom}(\mathbb{Z}^n/\overline{V}, \overline{V})$.

From these two facts, it follows immediately that $\ker(\Phi)$ is normally generated by the set $S_{\mathcal{K},\mathrm{IA}} \cup \{[t_1, t_2] \mid t_1, t_2 \in T\}$. We must show that the set $\{[t_1, t_2] \mid t_1, t_2 \in T\}$ is in the normal closure of $S_{\mathcal{K},\mathrm{IA}}$.

Consider $\mathrm{M}_{v_i,v_j}, \mathrm{M}_{v_{i'},v_{j'}} \in T$. We want to express $[\mathrm{M}_{v_i,v_j}, \mathrm{M}_{v_{i'},v_{j'}}]$ as a product of conjugates of elements of $S_{\mathcal{K},\mathrm{IA}}$. If either $i \ne i'$ or $j = j'$, then $[\mathrm{M}_{v_i,v_j}, \mathrm{M}_{v_{i'},v_{j'}}] = 1$ and the claim is trivial (this uses the fact that $\{v_i, v_{i'}\} \cap \{v_j, v_{j'}\} = \varnothing$). Assume, therefore, that $i = i'$ and $j \ne j'$. We then have the easily verified identity

$$[\mathrm{M}_{v_i,v_j}, \mathrm{M}_{v_{i'},v_{j'}}] = \mathrm{M}_{v_i,[v_{j'}^{-1}, v_j^{-1}]} = (\mathrm{M}_{v_i,v_j} \mathrm{C}_{v_i,v_{j'}} \mathrm{M}_{v_i,v_j}^{-1}) \mathrm{C}_{v_i,v_{j'}}^{-1},$$

and the corollary follows. □

3 Generators for IA_n

We will now assume the truth of Theorems A and B and prove Theorem D, which gives generators for IA_n. The proof will closely follow the proof in [25] of the analogous fact for the mapping class group. We start with a definition.

Definition. A group G acts on a simplicial complex X *without rotations* if for all simplices s of X, the stabilizer G_s stabilizes s pointwise.

For example, IA_n acts without rotations on $\mathcal{B}_n$, but $\mathrm{Aut}(F_n)$ does not act without rotations on $\mathcal{B}_n$ if $n \geq 2$.

The key to our proof will be the following theorem of Armstrong [1]. Our formulation is a little different from Armstrong's; see [25, Theorem 2.1] for a description of how to extract it from [1].

Theorem 3.1 (Armstrong, [1])**.** *Let G act without rotations on a 1-connected simplicial complex X. Then G is generated by the set*

$$\bigcup_{v \in X^{(0)}} G_v$$

if and only if X/G is 1-connected.

We now proceed to the proof.

Proof of Theorem D. Let $\{v_1, \ldots, v_n\}$ be a basis for F_n. Our goal is to show that IA_n is normally generated as a subgroup of $\mathrm{Aut}(F_n)$ by C_{v_1,v_2}. We first observe that the conjugacy class of C_{v_1,v_2} is independent of the initial choice of basis. It is thus enough to show that IA_n is generated by the automorphisms that (for some choice of basis) conjugate one basis element by another while fixing the rest of the basis. We will call such an automorphism a *basis-conjugating automorphism*.

The proof will be by induction on n. The case $n = 1$ is trivial, while the case $n = 2$ follows from a classical theorem of Nielsen [23] that asserts that $\mathrm{Out}(F_2) \cong \mathrm{GL}_2(\mathbb{Z})$, and thus that IA_2 consists entirely of inner automorphisms (see [18, Proposition 4.5] for a modern account of Nielsen's theorem). Assume, therefore, that $n \geq 3$ and that the result is true for all smaller n. For $v \in F_n$, define

$$\mathrm{IA}_n([\![v]\!]) = \{\phi \in \mathrm{IA}_n \mid [\![\phi(v)]\!] = [\![v]\!]\}.$$

Using Theorems A and B, we can apply Theorem 3.1 and deduce that IA_n is generated by the set

$$\bigcup_{[\![v]\!] \in (\mathcal{B}_n)^{(0)}} \mathrm{IA}_n([\![v]\!]).$$

Consider some $[\![v]\!] \in (\mathcal{B}_n)^{(0)}$. Applying Lemma 2.2, we obtain a split short exact sequence

$$1 \longrightarrow \mathcal{K}_n([\![v]\!]) \cap \mathrm{IA}_n \longrightarrow \mathrm{IA}_n([\![v]\!]) \longrightarrow \mathrm{IA}_{n-1} \longrightarrow 1.$$

The inductive hypothesis says that IA_{n-1} is generated by basis-conjugating automorphisms, and Corollary 2.3 says that $\mathcal{K}_n([\![v]\!]) \cap \mathrm{IA}_n$ is generated by basis-conjugating automorphisms. We conclude that $\mathrm{IA}_n([\![v]\!])$, and hence IA_n, is generated by basis-conjugating automorphisms, as desired. □

4 The connectivity of $\mathcal{B}_n$

In this section, we prove Theorem A. The proof uses a trick that was introduced by the second author in [26]. We will need two results for the proof. For the first, let $\mathrm{SAut}(F_n)$ be the *special automorphism group* of F_n, i.e. the subgroup of $\mathrm{Aut}(F_n)$ consisting of automorphisms whose images in $\mathrm{Aut}(\mathbb{Z}^n)$ have determinant 1. Also, if S is a fixed basis for F_n and $a, b \in S^{\pm 1}$ satisfy $a \neq b^{-1}$, then denote by $w_{a,b}$ the automorphism of F_n that takes a to b^{-1}, that takes b to a, and fixes all the elements of $S^{\pm 1} \smallsetminus \{a^{\pm 1}, b^{\pm 1}\}$.

Theorem 4.1 (Gersten, [9])**.** *For $n \geq 3$, the group $\mathrm{SAut}(F_n)$ has the following presentation. Let S_F be a fixed basis for F_n.*

- *The generating set consists of*

$$\{M_{a,b} \mid a, b \in S_F^{\pm 1},\ a \neq b^{\pm 1}\} \cup \{C_{a,b} \mid a, b \in S_F,\ a \neq b\} \cup \{w_{a,b} \mid a, b \in S_F^{\pm 1}, a \neq b^{\pm 1}\}.$$

- *The relations consist of the following.*
  1. $M_{a,b} M_{a,b^{-1}} = 1$ *for* $a, b \in S_F^{\pm 1}$ *with* $a \neq b^{\pm 1}$.
  2. $[M_{a,b}, M_{c,d}] = 1$ *for* $a, b, c, d \in S_F^{\pm 1}$ *with* $b \neq a^{\pm 1}, c^{\pm 1}$ *and* $d \neq c^{\pm 1}, a^{\pm 1}$ *and* $a \neq c$.
  3. $[M_{b,a^{-1}}, M_{c,b^{-1}}] = M_{c,a}$ *for* $a, b, c \in S_F^{\pm 1}$ *with* $a \neq b^{\pm 1}, c^{\pm 1}$ *and* $b \neq c^{\pm 1}$.
  4. $w_{a,b} = w_{a^{-1},b^{-1}}$ *for* $a, b \in S_F^{\pm 1}$ *with* $a \neq b^{\pm 1}$.
  5. $w_{a,b} = M_{b^{-1},a^{-1}} M_{a^{-1},b} M_{b,a}$ *for* $a, b \in S_F^{\pm 1}$ *with* $a \neq b^{\pm 1}$.
  6. $w_{a,b}^4 = 1$ *for* $a, b \in S_F^{\pm 1}$ *with* $a \neq b^{\pm 1}$.
  7. $C_{a,b} = M_{a,b} M_{a^{-1},b}$ *for* $a, b \in S_F$ *with* $a \neq b$.

Remark. The presentation above differs from the presentation in [9] in three ways. First, our convention is to write functions on the left, while in [9] functions are written on the right. Second, our convention is that $\mathrm{M}_{a,b}$ multiplies a on the left by b, while in [9] the analogous elements multiply a on the right by b. Third, we have added the additional generators $w_{a,b}$ and $\mathrm{C}_{a,b}$ together with relations 5 and 7, which express $w_{a,b}$ and $\mathrm{C}_{b,a}$ in terms of the generators $\mathrm{M}_{a,b}$ of [9]. This will simplify our proof later on.

The second result we will need is as follows. For a partial basis $\{v_1, \ldots, v_k\}$ of F_n, define

$$\mathrm{SAut}(F_n, [\![v_1]\!], \ldots, [\![v_k]\!]) = \{\phi \in \mathrm{SAut}(F_n) \mid [\![\phi(v_i)]\!] = [\![v_i]\!] \text{ for } 1 \leq i \leq k\}.$$

Lemma 4.2. *Let $S = \{v_1, \ldots, v_n\}$ be a fixed free basis of F_n. Then for $1 \leq k \leq n$, the group $\mathrm{SAut}(F_n, [\![v_1]\!], \ldots, [\![v_k]\!])$ is generated by the set*

$$\{M_{a,b} \mid a, b \in S^{\pm 1},\ a \neq b^{\pm 1},\ \text{and } a \notin \{v_1^{\pm 1}, \ldots, v_k^{\pm 1}\}\} \cup \{C_{a,b} \mid a, b \in S^{\pm 1} \text{ and } a \neq b^{\pm 1}\}$$

Proof. We will assume that $k < n$; the case of $k = n$ is similar but easier. Let T be the indicated generating set. Also, let $I_n \in \mathrm{Aut}(F_n, [\![v_1]\!], \ldots, [\![v_k]\!])$ be the automorphism that takes v_n to v_n^{-1} and fixes v_i for $1 \leq i < n$. The short exact sequence

$$1 \longrightarrow \mathrm{SAut}(F_n, [\![v_1]\!], \ldots, [\![v_k]\!]) \longrightarrow \mathrm{Aut}(F_n, [\![v_1]\!], \ldots, [\![v_k]\!]) \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 1$$

splits via a splitting taking the generator of $\mathbb{Z}/2\mathbb{Z}$ to I_n. In other words,

$$\mathrm{Aut}(F_n, [\![v_1]\!], \dots, [\![v_k]\!]) = \mathrm{SAut}(F_n, [\![v_1]\!], \dots, [\![v_k]\!]) \rtimes \langle I_n \rangle.$$

Observe that $I_n^{-1} s I_n \in T$ for all $s \in T$. To prove the lemma, it therefore suffices to show that $T \cup \{I_n\}$ generates $\mathrm{Aut}(F_n, [\![v_1]\!], \dots, [\![v_k]\!])$

Let $G \subset \mathrm{Aut}(F_n, [\![v_1]\!], \dots, [\![v_k]\!])$ be the subgroup consisting of automorphisms ϕ with the following properties.

- $\phi(v_i) = v_i$ for $1 \le i \le k$.
- $\phi(v_i) \in \{v_{k+1}^{\pm 1}, \dots, v_n^{\pm 1}\}$ for $k+1 \le i \le n$.

Making use of a deep theorem of McCool [22], Jensen-Wahl [14] proved that $T \cup G$ generates $\mathrm{Aut}(F_n, [\![v_1]\!], \dots, [\![v_k]\!])$. It is an easy exercise to show that $G \cap \mathrm{SAut}(F_n, [\![v_1]\!], \dots, [\![v_k]\!])$ is contained in the subgroup generated by T. Since G is generated by the union of $\{I_n\}$ and $G \cap \mathrm{SAut}(F_n, [\![v_1]\!], \dots, [\![v_k]\!])$, we conclude that $T \cup \{I_n\}$ generates $\mathrm{Aut}(F_n, [\![v_1]\!], \dots, [\![v_k]\!])$, as desired. □

Finally, we recall the following definition.

Definition. Let G be a group with a generating set S. The *Cayley graph* of G, denoted $\mathrm{Cay}(G, S)$, is the graph with vertex set G and with $g \in G$ connected by an edge to $g' \in G$ precisely when $g' = gs$ for some $s \in S^{\pm 1}$.

We can now prove Theorem A.

Proof of Theorem A. Let $S_F = \{v_1, \dots, v_n\}$ be a free basis for F_n and let S be the corresponding generating set for $\mathrm{SAut}(F_n)$ from Theorem 4.1. Observe that for $s \in S$, either $[\![s(v_1)]\!] = [\![v_1]\!]$ or $\{v_1, s(v_1)\}$ forms a partial basis for F_n. This implies that the map

$$\begin{aligned} \mathrm{SAut}(F_n) &\to \mathcal{B}_n \\ f &\mapsto [\![f(v_1)]\!] \end{aligned}$$

extends to a $\mathrm{SAut}(F_n)$-equivariant simplicial map $\Phi\colon \mathrm{Cay}(\mathrm{SAut}(F_n), S) \to \mathcal{B}_n$. When $n \ge 2$, the group $\mathrm{SAut}(F_n)$ acts transitively on the vertices of $\mathcal{B}_n$, so the image of Φ contains every vertex of $\mathcal{B}_n$. Since $\mathrm{Cay}(\mathrm{SAut}(F_n), S)$ is connected, this implies that $\mathcal{B}_n$ is connected for $n \ge 2$.

Assume now that $n \ge 3$. Our goal is to prove that $\mathcal{B}_n$ is 1-connected. We will prove below that the induced map

$$\Phi_*\colon \pi_1(\mathrm{Cay}(\mathrm{SAut}(F_n), S), 1) \longrightarrow \pi_1(\mathcal{B}_n, [\![v_1]\!])$$

is surjective and has image 1. The desired result will then follow.

Throughout the proof, we will use the following notation. A *simplicial path* in a simplicial complex C is a sequence of vertices $x_1, \dots, x_k$ in C such that for $1 \le i < k$, the vertex x_i is either equal to x_{i+1} or is joined by an edge to x_{i+1}. We will denote such a path by $x_1 - x_2 - \cdots - x_k$. If $x_k = x_1$, then we will call this a *simplicial loop*.

Claim 1. *The image of the map* $\Phi_*\colon \pi_1(\mathrm{Cay}(\mathrm{SAut}(F_n), S), 1) \longrightarrow \pi_1(\mathcal{B}_n, [\![v_1]\!])$ *is* 1.

It is well known that one can construct a 1-connected space from the Cayley graph of a finitely-presented group by attaching discs to the orbits of the loops associated to any complete set of relations. Let X be the space obtained from $\mathrm{Cay}(\mathrm{SAut}(F_n), S), 1)$ in this way using the presentation from Theorem 4.1. We will show that the images in $\mathcal{B}_n$ of the loops associated to these relations are contractible. This will imply that we can extend Φ to X. Since X is 1-connected, we will be able to conclude that Φ_* is the zero map, as desired.

Consider one of the relations $s_1 \cdots s_k = 1$ with $s_i \in S^{\pm 1}$ from Theorem 4.1. The associated simplicial loop in $\mathrm{Cay}(\mathrm{SAut}(F_n), S)$ and its image in $\mathcal{B}_n$ are

$$1 - s_1 - s_1 s_2 - \cdots - s_1 s_2 \cdots s_k = 1$$

and

$$[\![v_1]\!] - [\![s_1(v_1)]\!] - [\![s_1 s_2(v_1)]\!] - \cdots - [\![s_1 s_2 \cdots s_k(v_1)]\!] = [\![v_1]\!], \tag{3}$$

respectively. Examining the relations in Theorem 4.1, we see that there exists some $x \in S_F$ such that $s_i(x) = x$ for all $1 \le i \le k$ (this uses the fact that $n \ge 3$). Since $[\![v_1]\!]$ and $[\![x]\!]$ are either equal or joined by an edge, the vertices $[\![s_1 \cdots s_i(v_1)]\!]$ and $[\![s_1 \cdots s_i(x)]\!] = [\![x]\!]$ are either equal or joined by an edge for all $0 \le i \le k$. We deduce that if we can show that the loops $[\![x]\!] - [\![s_1 \cdots s_i(v_1)]\!] - [\![s_1 \cdots s_{i+1}(v_1)]\!] - [\![x]\!]$ are contractible for all $0 \le i < k$, then we can contract the loop in (3) to $[\![x]\!]$.

Using the $\mathrm{SAut}(F_n)$-action, we see that it is enough to show that the loops $[\![x]\!] - [\![v_1]\!] - [\![s_i(v_1)]\!] - [\![x]\!]$ are contractible for all $1 \le i \le k$. There are three cases. In the first, $[\![v_1]\!] = [\![s_i(v_1)]\!]$ and the result is trivial. In the second, the set $\{v_1, s_i(v_1), x\}$ is a partial basis for F_n, and again the result is trivial. In the third, the set $\{v_1, s_i(v_1), x\}$ is not a partial basis for F_n. Looking at the generators in S, we see that this can hold only if $s_i = \mathrm{M}^{e''}_{v_1^{e'}, x^e}$ for some $e, e', e'' \in \{1, -1\}$. Letting $y \in S_F$ be a generator distinct from x and v_1, the loop $[\![x]\!] - [\![v_1]\!] - [\![s_i(v_1)]\!] - [\![x]\!]$ can be contracted to $[\![y]\!]$, and we are done.

Claim 2. *The map* $\Phi_* \colon \pi_1(\mathrm{Cay}(\mathrm{SAut}(F_n), S), 1) \longrightarrow \pi_1(\mathcal{B}_n, [\![v_1]\!])$ *is surjective.*

Consider a loop $[\![w_0]\!] - [\![w_1]\!] - \cdots - [\![w_k]\!]$ in $\mathcal{B}_n$ with $[\![w_0]\!] = [\![w_k]\!] = [\![v_1]\!]$. We will show that this loop is in the image of Φ_*. By choosing an appropriate representative w_i for each conjugacy class $[\![w_i]\!]$, we can assume that $w_0 = v_1$ and that $\{w_i, w_{i+1}\}$ is a partial basis of F_n for $0 \le i < k$. Since $\{v_1, w_1\}$ is a partial basis for F_n and $\mathrm{SAut}(F_n)$ acts transitively on two-element partial bases (this uses the fact that $n \ge 3$), there exists some $\phi_1 \in \mathrm{SAut}(F_n, [\![v_1]\!])$ such that $\phi_1(v_2) = w_1$ (in fact, we could assume that ϕ_1 fixes v_1 and not just its conjugacy class, but that is not needed). It follows that $w_1 = \phi_1 w_{v_2, v_1}(v_1)$. Next, since $\{w_1, w_2\}$ is a partial basis, so is

$$\{(\phi_1 w_{v_2, v_1})^{-1}(w_1), (\phi_1 w_{v_2, v_1})^{-1}(w_2)\} = \{v_1, (\phi_1 w_{v_2, v_1})^{-1}(w_2)\}.$$

As before, there exists some $\phi_2 \in \mathrm{SAut}(F_n, [\![v_1]\!])$ such that $\phi_2(v_2) = (\phi_1 w_{v_2, v_1})^{-1}(w_2)$. We conclude that $w_2 = (\phi_1 w_{v_2, v_1})(\phi_2 w_{v_2, v_1})(v_1)$. Repeating this argument, we obtain elements $\phi_1, \ldots, \phi_k \in \mathrm{SAut}(F_n, [\![v_1]\!])$ such that $w_i = (\phi_1 w_{v_2, v_1}) \cdots (\phi_i w_{v_2, v_1})(v_1)$ for all $1 \le i \le k$.

Set $\phi_{k+1} = ((\phi_1 w_{v_2, v_1}) \cdots (\phi_i w_{v_2, v_1}))^{-1}$. Since

$$[\![(\phi_1 w_{v_2, v_1}) \cdots (\phi_k w_{v_2, v_1})(v_1)]\!] = [\![w_k]\!] = [\![v_1]\!],$$

we have that $\phi_{k+1} \in \mathrm{SAut}(F_n, [\![v_1]\!])$.

By Lemma 4.2, for $1 \le i \le k+1$ there exists $s_1^i, \ldots, s_{m_i}^i \in S^{\pm 1}$ such that $s_1^i \cdots s_{m_i}^i = \phi_i$ and such that $s_j^i \in \mathrm{SAut}(F_n, [\![v_1]\!])$ for $1 \le j \le m_i$. Observe now that we have a relation

$$(s_1^1 \cdots s_{m_1}^1 w_{v_2,v_1}) \cdots (s_1^k \cdots s_{m_k}^k w_{v_2,v_1})(s_1^{k+1} \cdots s_{m_{k+1}}^{k+1}) = 1$$

in $\mathrm{SAut}(F_n)$. The image under Φ_* of the corresponding loop in $\mathrm{Cay}(\mathrm{SAut}(F_n), S)$ is

$$[\![v_1]\!] - [\![s_1^1(v_1)]\!] - [\![s_1^1 s_2^1(v_1)]\!] - \cdots - [\![s_1^1 s_2^1 \cdots s_{m_1}^1 w_{v_2,v_1}(v_1)]\!] - \cdots.$$

Since $[\![s_j^i(v_1)]\!] = [\![v_1]\!]$ for all $1 \le i \le k+1$ and $1 \le j \le m_i$, after deleting repeated vertices this path equals

$$[\![v_1]\!] - [\![s_1^1 s_2^1 \cdots s_{m_1}^1 w_{v_2,v_1}(v_1)]\!] - [\![(s_1^1 s_2^1 \cdots s_{m_1}^1 w_{v_2,v_1})(s_1^2 s_2^2 \cdots s_{m_2}^2 w_{v_2,v_1})(v_1)]\!] - \\ \cdots - [\![(s_1^1 s_2^1 \cdots s_{m_1}^1 w_{v_2,v_1}) \cdots (s_1^k s_2^k \cdots s_{m_k}^k w_{v_2,v_1})(v_1)]\!].$$

By construction, this equals $[\![w_0]\!] - [\![w_1]\!] - \cdots - [\![w_k]\!]$, as desired. ☐

5 The connectivity of $\mathcal{B}_n / \mathrm{IA}_n$

The goal of this section is to prove Theorem B. The proof is contained in §5.2, which is prefaced with some background information on simplicial complexes in §5.1.

5.1 Simplicial complexes

Our basic reference for simplicial complexes is [32, Chapter 3]. Let us recall the definition of a simplicial complex given there.

Definition. A *simplicial complex* X is a set of nonempty finite sets (called *simplices*) such that if $\Delta \in X$ and $\varnothing \ne \Delta' \subset \Delta$, then $\Delta' \in X$. The *dimension* of a simplex $\Delta \in X$ is $|\Delta| - 1$ and is denoted $\dim(\Delta)$. For $k \ge 0$, the subcomplex of X consisting of all simplices of dimension at most k (known as the *k-skeleton of X*) will be denoted $X^{(k)}$. If X and Y are simplicial complexes, then a *simplicial map* from X to Y is a function $f\colon X^{(0)} \to Y^{(0)}$ such that if $\Delta \in X$, then $f(\Delta) \in Y$.

If X is a simplicial complex, then we will define the geometric realization $|X|$ of X in the standard way (see [32, Chapter 3]). When we say that X has some topological property (e.g. simple-connectivity), we will mean that $|X|$ possesses that property.

Next, we will need the following definitions.

Definition. Consider a simplex Δ of a simplicial complex X.

- The *star* of Δ (denoted $\mathrm{star}_X(\Delta)$) is the subcomplex of X consisting of all $\Delta' \in X$ such that there is some $\Delta'' \in X$ with $\Delta, \Delta' \subset \Delta''$. By convention, we will also define $\mathrm{star}_X(\varnothing) = X$.
- The *link* of Δ (denoted $\mathrm{link}_X(\Delta)$) is the subcomplex of $\mathrm{star}_X(\Delta)$ consisting of all simplices that do not intersect Δ. By convention, we will also define $\mathrm{link}_X(\varnothing) = X$.

For $n \le -1$, we will say that the empty set is both an n-sphere and a closed n-ball. Also, if X is a space then we will say that $\pi_{-1}(X) = 0$ if X is nonempty and that $\pi_k(X) = 0$ for all $k \le -2$. With these conventions, it is true for all $n \in \mathbb{Z}$ that a space X satisfies $\pi_n(X) = 0$ if and only if every map of an n-sphere into X can be extended to a map of a closed $(n+1)$-ball into X.

Finally, we will need the following definition. A basic reference is [31].

Definition. For $n \geq 0$, a *combinatorial n-manifold M* is a nonempty simplicial complex that satisfies the following inductive property. If $\Delta \in M$, then $\dim(\Delta) \leq n$. Additionally, if $n - \dim(\Delta) - 1 \geq 0$, then $\text{link}_M(\Delta)$ is a combinatorial $(n - \dim(\Delta) - 1)$-manifold homeomorphic to either an $(n - \dim(\Delta) - 1)$-sphere or a closed $(n - \dim(\Delta) - 1)$-ball. We will denote by ∂M the subcomplex of M consisting of all simplices Δ such that $\dim(\Delta) < n$ and such that $\text{link}_M(\Delta)$ is homeomorphic to a closed $(n - \dim(\Delta) - 1)$-ball. If $\partial M = \varnothing$ then M is said to be *closed*. A combinatorial n-manifold homeomorphic to an n-sphere (resp. a closed n-ball) will be called a *combinatorial n-sphere* (resp. a *combinatorial n-ball*).

It is well-known that if $\partial M \neq \varnothing$, then ∂M is a closed combinatorial $(n-1)$-manifold and that if B is a combinatorial n-ball, then ∂B is a combinatorial $(n-1)$-sphere.

Warning. There exist simplicial complexes that are homeomorphic to manifolds but are *not* combinatorial manifolds.

The following is an immediate consequence of the Zeeman's extension [33] of the simplicial approximation theorem.

Lemma 5.1. *Let X be a simplicial complex and $n \geq 0$. The following hold.*

1. *Every element of $\pi_n(X)$ is represented by a simplicial map $S \to X$, where S is a combinatorial n-sphere.*
2. *If S is a combinatorial n-sphere and $f\colon S \to X$ is a nullhomotopic simplicial map, then there is a combinatorial $(n+1)$-ball B with $\partial B = S$ and a simplicial map $g\colon B \to X$ such that $g|_S = f$.*

5.2 The proof of Theorem B

We will need two lemmas. First, some notation. For $\{v_1, \ldots, v_k\} \subset F_n$, define

$$\text{Aut}(F_n, v_1, \ldots, v_k) = \{\phi \in \text{Aut}(F_n) \mid \phi(v_i) = v_i \text{ for } 1 \leq i \leq k\}.$$

We then have the following lemma, whose proof is identical to the proof of Lemma 2.2.

Lemma 5.2 (Images of stabilizers II)**.** *Let $\{v_1, \ldots, v_k\}$ and $\{\overline{v}_1, \ldots, \overline{v}_k\}$ be partial bases for F_n and $\mathbb{Z}^n$, respectively, such that $\pi(v_i) = \overline{v}_i$ for $1 \leq i \leq k$. Set $\overline{V} = \langle \overline{v}_1, \ldots, \overline{v}_k \rangle \subset \mathbb{Z}^n$. Then the map $\text{Aut}(F_n, v_1, \ldots, v_k) \to \text{Aut}(\mathbb{Z}^n, \overline{V})$ is surjective.*

Next, we prove the following.

Lemma 5.3 (Basis completion lemma)**.** *Let $\{\overline{v}_1, \ldots, \overline{v}_n\}$ be a basis for $\mathbb{Z}^n$ and let $\pi\colon F_n \to \mathbb{Z}^n$ be the abelianization map. For some $0 \leq k \leq n$, let $\{v_1, \ldots, v_k\} \subset F_n$ be a partial basis for F_n such that $\pi(v_i) = \overline{v}_i$ for $1 \leq i \leq k$. There then exists $\{v_{k+1}, \ldots, v_n\} \subset F_n$ such that $\{v_1, \ldots, v_n\}$ is a basis for F_n and such that $\pi(v_i) = \overline{v}_i$ for $1 \leq i \leq n$.*

Proof. Complete the partial basis $\{v_1, \ldots, v_k\}$ to a basis $\{v_1, \ldots, v_k, v'_{k+1}, \ldots, v'_n\}$ for F_n and set $\overline{v}'_i = \pi(v'_i)$ for $k+1 \leq i \leq n$. Define $\overline{V} = \langle \overline{v}_1, \ldots, \overline{v}_k \rangle$. There then exists some $\overline{\phi} \in \text{Aut}(\mathbb{Z}^n, \overline{V})$ such that $\overline{\phi}(\overline{v}'_i) = \overline{v}_i$ for $k+1 \leq i \leq n$. By Lemma 5.2, there exists some $\phi \in \text{Aut}(F_n, v_1, \ldots, v_k)$ that induces $\overline{\phi}$ on $\text{H}_1(F_n; \mathbb{Z}) = \mathbb{Z}^n$. The desired basis for F_n is then $\{v_1, \ldots, v_k, \phi(v'_{k+1}), \ldots, \phi(v'_n)\}$. □

We now proceed to the proof of Theorem B.

Proof of Theorem B. Recall that this theorem asserts that $\mathcal{B}_n/\mathrm{IA}_n$ is $(n-2)$-connected. Our proof will have two steps. In the first, we will show that $\mathcal{B}_n/\mathrm{IA}_n$ is isomorphic to a more concrete space $\mathcal{B}_n(\mathbb{Z})$, and in the second, we will prove that $\mathcal{B}_n(\mathbb{Z})$ is $(n-2)$-connected. We begin by defining $\mathcal{B}_n(\mathbb{Z})$.

Definition. Let $\mathcal{B}_n(\mathbb{Z})$ denote the simplicial complex whose $(k-1)$-simplices are sets $\{x_1,\ldots,x_k\} \subset \mathbb{Z}^n$ whose span is a k-dimensional direct summand of $\mathbb{Z}^n$.

Now on to the proof.

Step 1. *We have* $\mathcal{B}_n/\mathrm{IA}_n \cong \mathcal{B}_n(\mathbb{Z})$.

Let $\pi: F_n \to \mathbb{Z}^n$ be the projection. There is a map $\psi: \mathcal{B}_n \to \mathcal{B}_n(\mathbb{Z})$ that takes a simplex $\{[\![w_1]\!],\ldots,[\![w_m]\!]\}$ of $\mathcal{B}_n$ to $\{\pi(w_1),\ldots,\pi(w_m)\}$. By Lemma 5.3, the map ψ is surjective. Also, ψ is invariant under the action of IA_n. It is thus enough to prove that the induced map $\mathcal{B}_n/\mathrm{IA}_n \to \mathcal{B}_n(\mathbb{Z})$ is injective. To do this, it is enough to show that if $s = \{[\![v_1]\!],\ldots,[\![v_k]\!]\}$ and $s' = \{[\![v'_1]\!],\ldots,[\![v'_k]\!]\}$ are two simplices of $\mathcal{B}_n$ such that $\psi(s) = \psi(s')$, then there exists some $f \in \mathrm{IA}_n$ such that $f(s) = s'$.

Reordering our simplices if necessary, we can assume that $\pi(v_i) = \pi(v'_i)$ for $1 \le i \le k$. Define $\overline{v}_i = \pi(v_i) \in \mathbb{Z}^n$. We can complete the partial basis $\{\overline{v}_1,\ldots,\overline{v}_k\}$ for $\mathbb{Z}^n$ to a basis $\{\overline{v}_1,\ldots,\overline{v}_n\}$. Applying Lemma 5.3 twice, we obtain bases $\{v_1,\ldots,v_n\}$ and $\{v'_1,\ldots,v'_n\}$ for F_n such that $\pi(v_i) = \pi(v'_i) = \overline{v}_i$ for $1 \le i \le n$. Define $f \in \mathrm{Aut}(F_n)$ by $f(v_i) = v'_i$. Clearly $f(s) = s'$, and by construction $f \in \mathrm{IA}_n$, as desired.

Step 2. *The space* $\mathcal{B}_n(\mathbb{Z})$ *is* $(n-2)$*-connected.*

This result is contained in the 1979 PhD thesis of Maazen [19]. Since this thesis was never published, we include a proof. Our proof is somewhat different from Maazen's proof and is modeled after proofs of related results due to the second author (see [27, Proposition 6.13] and [28, Proposition 6.8]). Fix a basis $\{e_1,\ldots,e_n\}$ for $\mathbb{Z}^n$, and for $0 \le k < n$ define $\mathcal{B}_n^k(\mathbb{Z}) = \mathrm{link}_{\mathcal{B}_n(\mathbb{Z})}(\{e_1,\ldots,e_k\})$. We will prove a more general statement, namely, that $\pi_\ell(\mathcal{B}_n^k(\mathbb{Z})) = 0$ for $0 \le k < n$ and $-1 \le \ell \le n-k-2$.

The proof will be by induction on ℓ. The base case $\ell = -1$ is equivalent to the observation that if $k < n$, then $\mathcal{B}_n^k(\mathbb{Z})$ is nonempty. Assume now that $0 \le \ell \le n-k-2$ and that $\pi_{\ell'}(\mathcal{B}_{n'}^{k'}(\mathbb{Z})) = 0$ for all $0 \le k' < n'$ and $-1 \le \ell' \le n'-k'-2$ such that $\ell' < \ell$. Let S be a combinatorial ℓ-sphere and let $\phi: S \to \mathcal{B}_n^k(\mathbb{Z})$ be a simplicial map. By Lemma 5.1, it is enough to show that ϕ may be homotoped to a constant map.

We begin with some notation. Consider $w \in \mathbb{Z}^n$. Write $w = \sum_{i=1}^n c_i e_i$ with $c_i \in \mathbb{Z}$, and define $\mathrm{Rank}(w) = |c_n|$. Now set

$$R = \max\{\mathrm{Rank}(\phi(x)) \mid x \in S^{(0)}\}.$$

If $R = 0$, then $\phi(S) \subset \mathrm{star}_{\mathcal{B}_n^k(\mathbb{Z})}(\{e_n\})$, and hence the map ϕ can be homotoped to the constant map e_n. Assume, therefore, that $R > 0$. Let Δ' be a simplex of S such that $\mathrm{Rank}(\phi(x)) = R$ for all vertices x of Δ'. Choose Δ' so that $m := \dim(\Delta')$ is maximal, which implies that $\mathrm{Rank}(\phi(x)) < R$ for all vertices x of $\mathrm{link}_S(\Delta')$. Now, $\mathrm{link}_S(\Delta')$ is a combinatorial $(\ell-m-1)$-sphere and $\phi(\mathrm{link}_S(\Delta'))$ is contained in

$$\mathrm{link}_{\mathcal{B}_n^k(\mathbb{Z})}(\phi(\Delta')) \cong \mathcal{B}_n^{k+m'}(\mathbb{Z})$$

for some $m' \le m$ (it may be less than m if $\phi|_{\Delta'}$ is not injective). The inductive hypothesis together with Lemma 5.1 therefore tells us that there a combinatorial $(\ell-m)$-ball B with $\partial B = \mathrm{link}_S(\Delta')$ and a simplicial map $f: B \to \mathrm{link}_{\mathcal{B}_n^k(\mathbb{Z})}(\phi(\Delta'))$ such that $f|_{\partial B} = \phi|_{\mathrm{link}_S(\Delta')}$.

$t \in T$	$s \in S \cup S^{-1}$	sts^{-1}
$\mathrm{M}_{c,[a,b]}$	$\mathrm{M}_{x,c}$	$\mathrm{C}_{x,c}[\mathrm{C}_{x,b}^{-1}, \mathrm{C}_{x,a}^{-1}]\mathrm{M}_{x,[b,a]}\mathrm{M}_{c,[a,b]}\mathrm{C}_{x,c}^{-1}$
	$\mathrm{M}_{x,c}^{-1}$	$\mathrm{M}_{x,[a,b]}\mathrm{M}_{c,[a,b]}$
	$\mathrm{M}_{a,x}$	$\mathrm{M}_{c,[x,b]}\mathrm{C}_{c,x}^{-1}\mathrm{M}_{c,[a,b]}\mathrm{C}_{c,x}$
	$\mathrm{M}_{a,x}^{-1}$	$\mathrm{C}_{a,x}^{-1}\mathrm{M}_{c,[a,b]}\mathrm{C}_{c,a}^{-1}\mathrm{C}_{c,x}\mathrm{M}_{c,[b,x]}\mathrm{C}_{c,x}^{-1}\mathrm{C}_{c,a}\mathrm{C}_{a,x}$
	$\mathrm{M}_{a^{-1},x}$	$\mathrm{M}_{c,[a,b]}\mathrm{C}_{c,a}^{-1}\mathrm{C}_{c,x}\mathrm{M}_{c,[b,x]}\mathrm{C}_{c,x}^{-1}\mathrm{C}_{c,a}$
	$\mathrm{M}_{a^{-1},x}^{-1}$	$\mathrm{C}_{a,x}^{-1}\mathrm{M}_{c,[x,b]}\mathrm{C}_{c,x}^{-1}\mathrm{M}_{c,[a,b]}\mathrm{C}_{c,x}\mathrm{C}_{a,x}$
	$\mathrm{M}_{b,x}$	$\mathrm{C}_{c,x}^{-1}\mathrm{M}_{c,[a,b]}\mathrm{C}_{c,x}\mathrm{M}_{c,[a,x]}$
	$\mathrm{M}_{b,x}^{-1}$	$\mathrm{C}_{b,x}^{-1}\mathrm{C}_{c,b}^{-1}\mathrm{C}_{c,x}\mathrm{M}_{c,[x,a]}\mathrm{C}_{c,x}^{-1}\mathrm{C}_{c,b}\mathrm{M}_{c,[a,b]}\mathrm{C}_{b,x}$
	$\mathrm{M}_{b^{-1},x}$	$\mathrm{C}_{c,b}^{-1}\mathrm{C}_{c,x}\mathrm{M}_{c,[x,a]}\mathrm{C}_{c,x}^{-1}\mathrm{C}_{c,b}\mathrm{M}_{c,[a,b]}$
	$\mathrm{M}_{b^{-1},x}^{-1}$	$\mathrm{C}_{b,x}^{-1}\mathrm{C}_{c,x}^{-1}\mathrm{M}_{c,[a,b]}\mathrm{C}_{c,x}\mathrm{M}_{c,[a,x]}\mathrm{C}_{b,x}$
	$\mathrm{M}_{c,x}^{\epsilon}$	$\mathrm{C}_{c,x}^{\epsilon}\mathrm{M}_{c,[a,b]}\mathrm{C}_{c,x}^{-\epsilon}$
	$\mathrm{M}_{a,b}^{\epsilon}$	$\mathrm{C}_{c,b}^{-\epsilon}\mathrm{M}_{c,[a,b]}\mathrm{C}_{c,b}^{\epsilon}$
	$\mathrm{M}_{a,c}$	$\mathrm{C}_{a,c}\mathrm{C}_{a,b}\mathrm{M}_{a,[b,c]}\mathrm{C}_{a,b}^{-1}\mathrm{C}_{a,c}^{-1}\mathrm{C}_{a,b}[\mathrm{C}_{c,a}^{-1}, \mathrm{C}_{c,b}^{-1}]\mathrm{M}_{c,[a,b]}\mathrm{C}_{c,b}^{-1}$
	$\mathrm{M}_{a,c}^{-1}$	$\mathrm{C}_{c,b}\mathrm{M}_{c,[a,b]}\mathrm{C}_{c,a}^{-1}\mathrm{C}_{a,c}\mathrm{M}_{a,[b,c]}\mathrm{C}_{a,b}^{-1}\mathrm{C}_{a,c}^{-1}\mathrm{C}_{c,a}$
	$\mathrm{M}_{a^{-1},c}$	$\mathrm{C}_{a,c}\mathrm{C}_{c,b}\mathrm{M}_{c,[a,b]}\mathrm{C}_{c,a}^{-1}\mathrm{C}_{a,c}\mathrm{M}_{a,[b,c]}\mathrm{C}_{a,b}^{-1}\mathrm{C}_{a,c}^{-1}\mathrm{C}_{c,a}\mathrm{C}_{a,c}^{-1}$
	$\mathrm{M}_{a^{-1},c}^{-1}$	$\mathrm{C}_{a,b}\mathrm{M}_{a,[b,c]}\mathrm{C}_{a,b}^{-1}\mathrm{C}_{a,c}^{-1}\mathrm{C}_{a,b}[\mathrm{C}_{c,a}^{-1}, \mathrm{C}_{c,b}^{-1}]\mathrm{M}_{c,[a,b]}\mathrm{C}_{c,b}^{-1}\mathrm{C}_{a,c}$
	$\mathrm{M}_{b,a}^{\epsilon}$	$\mathrm{C}_{c,a}^{-\epsilon}\mathrm{M}_{c,[a,b]}\mathrm{C}_{c,a}^{\epsilon}$
	$\mathrm{M}_{b,c}$	$\mathrm{C}_{c,a}\mathrm{M}_{c,[a,b]}[\mathrm{C}_{c,a}^{-1}, \mathrm{C}_{c,b}^{-1}]\mathrm{C}_{b,a}^{-1}\mathrm{C}_{b,c}\mathrm{C}_{b,a}\mathrm{M}_{b,[c,a]}\mathrm{C}_{b,a}^{-1}\mathrm{C}_{b,c}^{-1}$
	$\mathrm{M}_{b,c}^{-1}$	$\mathrm{C}_{c,b}^{-1}\mathrm{C}_{b,c}\mathrm{C}_{b,a}\mathrm{M}_{b,[c,a]}\mathrm{C}_{b,c}^{-1}\mathrm{C}_{c,b}\mathrm{M}_{c,[a,b]}\mathrm{C}_{c,a}^{-1}$
	$\mathrm{M}_{b^{-1},c}$	$\mathrm{C}_{b,c}\mathrm{C}_{c,b}^{-1}\mathrm{C}_{b,c}\mathrm{C}_{b,a}\mathrm{M}_{b,[c,a]}\mathrm{C}_{b,c}^{-1}\mathrm{C}_{c,b}\mathrm{M}_{c,[a,b]}\mathrm{C}_{c,a}^{-1}\mathrm{C}_{b,c}^{-1}$
	$\mathrm{M}_{b^{-1},c}^{-1}$	$\mathrm{C}_{b,c}^{-1}\mathrm{C}_{c,a}\mathrm{M}_{c,[a,b]}[\mathrm{C}_{c,a}^{-1}, \mathrm{C}_{c,b}^{-1}]\mathrm{C}_{b,a}^{-1}\mathrm{C}_{b,c}\mathrm{C}_{b,a}\mathrm{M}_{b,[c,a]}\mathrm{C}_{b,a}^{-1}$
	$\mathrm{M}_{c,a}^{\epsilon}$	$\mathrm{C}_{c,a}^{\epsilon}\mathrm{M}_{c,[a,b]}\mathrm{C}_{c,a}^{-\epsilon}$
	$\mathrm{M}_{c,b}^{\epsilon}$	$\mathrm{C}_{c,b}^{\epsilon}\mathrm{M}_{c,[a,b]}\mathrm{C}_{c,b}^{-\epsilon}$
$\mathrm{C}_{c,a}$	$\mathrm{M}_{x,c}$	$\mathrm{C}_{c,a}\mathrm{C}_{x,c}\mathrm{C}_{x,a}\mathrm{M}_{x,[c,a]}\mathrm{C}_{x,a}^{-1}\mathrm{C}_{x,c}^{-1}$
	$\mathrm{M}_{x,c}^{-1}$	$\mathrm{C}_{c,a}\mathrm{C}_{x,a}\mathrm{M}_{x,[a,c]}\mathrm{C}_{x,a}^{-1}$
	$\mathrm{M}_{x^{-1},c}$	$\mathrm{C}_{x,c}\mathrm{C}_{c,a}\mathrm{C}_{x,a}\mathrm{M}_{x,[a,c]}\mathrm{C}_{x,a}^{-1}\mathrm{C}_{x,c}^{-1}$
	$\mathrm{M}_{x^{-1},c}^{-1}$	$\mathrm{C}_{x,c}^{-1}\mathrm{C}_{c,a}\mathrm{C}_{x,c}\mathrm{C}_{x,a}\mathrm{M}_{x,[c,a]}\mathrm{C}_{x,a}^{-1}$
	$\mathrm{M}_{a^{\epsilon},x}^{\delta}$	$(\mathrm{C}_{c,a}^{\epsilon}\mathrm{C}_{c,x}^{\delta})^{\epsilon}$
	$\mathrm{M}_{c,x}$	$\mathrm{C}_{c,a}\mathrm{C}_{c,x}\mathrm{M}_{c,[a,x]}\mathrm{C}_{c,x}^{-1}$
	$\mathrm{M}_{c,x}^{-1}$	$\mathrm{C}_{c,a}\mathrm{M}_{c,[x,a]}$
	$\mathrm{M}_{c^{-1},x}$	$\mathrm{C}_{c,x}\mathrm{C}_{c,a}\mathrm{M}_{c,[x,a]}\mathrm{C}_{c,x}^{-1}$
	$\mathrm{M}_{c^{-1},x}^{-1}$	$\mathrm{C}_{c,x}^{-1}\mathrm{C}_{c,a}\mathrm{C}_{c,x}\mathrm{M}_{c,[a,x]}$
	$\mathrm{M}_{a^{\epsilon},c}^{\delta}$	$(\mathrm{C}_{a,c}^{\delta}\mathrm{C}_{c,a}^{\epsilon})^{\epsilon}$

Table 1: Conjugating T by $S \cup S^{-1}$. Here a, b, c and x are distinct elements of the generating set for F_n and $\epsilon, \delta = \pm 1$. If a particular choice of s does not appear in the table, then $sts^{-1} = s^{-1}ts = t$.

Our goal now is to adjust f so that $\mathrm{Rank}(\phi(x)) < R$ for all $x \in B^{(0)}$. Let $v \in \mathbb{Z}^n$ be a vector corresponding to a vertex of $\phi(\Delta')$. Observe that the e_n-coordinate of v is $\pm R$. We define a map $f' \colon B \to \mathrm{link}_{\mathcal{B}_n^k(\mathbb{Z})}(\phi(\Delta'))$ in the following way. Consider $x \in B^{(0)}$, and let $v_x = f(x) \in \mathbb{Z}^n$. By the division algorithm, there exists some $q_x \in \mathbb{Z}$ such that $\mathrm{Rank}(v_x + q_x v) < R$. Moreover, by the maximality of m we can choose q_x such that $q_x = 0$ if $x \in (\partial B)^{(0)}$. Define $f'(x) = v_x + q_x v$. It is clear that the map f' extends to a map $f' \colon B \to \text{link}_{\mathcal{B}_n^k(\mathbb{Z})}(\phi(\Delta'))$. Additionally, $f'|_{\partial B} = f|_{\partial B} = \phi|_{\text{link}_S(\Delta')}$. We conclude that we can homotope ϕ so as to replace $\phi|_{\text{star}_S(\Delta')}$ with f'. Since $\text{Rank}(f'(x)) < R$ for all $x \in B$, we have removed Δ' from S without introducing any vertices whose images have rank greater than or equal to R. Continuing in this manner allows us to simplify ϕ until $R = 0$, and we are done. □

A Appendix : Derivation of Theorem C from Theorem D

Fixing a basis $\{v_1, \ldots, v_n\}$ for F_n, let T be the purported generating set for IA_n from Theorem C and let $\Gamma < \text{Aut}(F_n)$ be the subgroup generated by T. Fix some $f \in \text{Aut}(F_n)$. By Theorem D, to show that $\Gamma = \text{Aut}(F_n)$, it is enough to show that $f \text{C}_{v_1,v_2} f^{-1} \in \Gamma$. Recall that $\text{SAut}(F_n)$ consists of all elements of $\text{Aut}(F_n)$ whose images in $\text{Aut}(\mathbb{Z}^n)$ have determinant 1. Letting $I_1 \in \text{Aut}(F_n)$ be the automorphism that takes v_1 to v_1^{-1} and fixes v_i for $i > 1$, we can write $f = g \cdot I_1^k$ for some $g \in \text{SAut}(F_n)$ and some $k \in \mathbb{Z}$. We then have

$$f \text{C}_{v_1,v_2} f^{-1} = g \cdot I_1^k \text{C}_{v_1,v_2} I_1^{-k} \cdot g^{-1} = g \text{C}_{v_1,v_2} g^{-1}.$$

Since $\text{C}_{v_1,v_2} \in \Gamma$, to prove that $g \text{C}_{v_1,v_2} g^{-1} \in \Gamma$ it is enough to prove that Γ is a normal subgroup of $\text{SAut}(F_n)$. Define

$$S = \{\text{M}_{v_i,v_j} \mid 1 \le i,j \le n, \, i \ne j\}.$$

The group $\text{SAut}(F_n)$ is generated by S (see Theorem 4.1), so it is enough to show that for $s \in S \cup S^{-1}$ and $t \in T$, the automorphism sts^{-1} can be written as a word in $T \cup T^{-1}$. The various cases of this are contained in Table 1.

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