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CATEGORIES OF PARTIAL ALGEBRAS FOR CRITICAL POINTS BETWEEN VARIETIES OF ALGEBRAS

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ABSTRACT. We denote by $\mathrm{Con_c}\, A$ the $(\vee, 0)$-semilattice of all finitely generated congruences of an algebra A. A *lifting* of a $(\vee, 0)$-semilattice S is an algebra A such that $S \cong \mathrm{Con_c}\, A$.

The aim of this work is to give a categorical theory of partial algebras endowed with a partial subalgebra together with a semilattice-valued distance, that we call *gamps*. This part of the theory is formulated in any variety of (universal) algebras.

Let $\mathcal{V}$ and $\mathcal{W}$ be varieties of algebras (on a finite similarity type). Let P be a finite lattice of order-dimension $d > 0$. Let $\vec{A}$ be a P-indexed diagram of finite algebras in $\mathcal{V}$. If $\mathrm{Con_c} \circ \vec{A}$ has no *partial lifting* in the category of gamps of $\mathcal{W}$, then there is an algebra $A \in \mathcal{V}$ of cardinality $\aleph_{d-1}$ such that $\mathrm{Con_c}\, A$ is not isomorphic to $\mathrm{Con_c}\, B$ for any $B \in \mathcal{W}$.

We already knew a similar result for diagrams $\vec{A}$ such that $\mathrm{Con_c} \circ \vec{A}$ has no lifting in $\mathcal{W}$, however the algebra A constructed here has cardinality $\aleph_d$.

Gamps are also used to study congruence-preserving extensions. Denote by $\mathcal{M}_3$ the variety generated by the lattice of length two, with three atoms. We construct a lattice $A \in \mathcal{M}_3$ of cardinality $\aleph_1$ with no congruence n-permutable, congruence-preserving extension, for each $n \geq 2$.

1. INTRODUCTION

For an algebra A we denote by $\mathrm{Con}\, A$ the lattice of all congruences of A under inclusion. Given $x, y \in A$, we denote by $\Theta_A(x, y)$ the smallest congruence of A that identifies x and y, such a congruence is called *principal*. A congruence is *finitely generated* if it is a finite join of principal congruences. The lattice $\mathrm{Con}\, A$ is algebraic and the compact element of $\mathrm{Con}\, A$ are the finitely generated congruences.

The lattice $\mathrm{Con}\, A$ is determined by the $(\vee, 0)$-semilattice $\mathrm{Con_c}\, A$ of compact congruences of A. In this paper we mostly refer to $\mathrm{Con_c}\, A$ instead of $\mathrm{Con}\, A$. If $\mathrm{Con_c}\, A$ is isomorphic to a $(\vee, 0)$-semilattice S, we call A a *lifting* of S.

Given a class of algebras $\mathcal{K}$ we denote by $\mathrm{Con_c}\, \mathcal{K}$ the class of all $(\vee, 0)$-semilattices with a lifting in $\mathcal{K}$. In general, even if $\mathcal{K}$ is a variety of algebras, there is no good description of $\mathrm{Con_c}\, \mathcal{K}$. The negative solution to the congruence lattice problem (CLP) in [16] is a good example of the difficulty to find such a description.

The study of CLP led to the following related questions. Fix two classes of algebras $\mathcal{V}$ and $\mathcal{W}$.

(**Q1**) Given $A \in \mathcal{V}$, does there exist $B \in \mathcal{W}$ such that $\mathrm{Con_c}\, A \cong \mathrm{Con_c}\, B$?

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(**Q2**) Given $A \in \mathcal{V}$, does there exist a congruence-preserving extension $B \in \mathcal{W}$ of A?

A positive answer to (**Q1**) is equivalent to $\operatorname{Con_c} \mathcal{V} \subseteq \operatorname{Con_c} \mathcal{W}$. The "containment defect" of $\operatorname{Con_c} \mathcal{V}$ into $\operatorname{Con_c} \mathcal{W}$ is measured (cf. [15, 3]) by the *critical point* between $\mathcal{V}$ and $\mathcal{W}$, defined as

$$\operatorname{crit}(\mathcal{V}; \mathcal{W}) = \begin{cases} \min\{\operatorname{card} S \mid S \in (\operatorname{Con_c} \mathcal{V}) - (\operatorname{Con_c} \mathcal{W})\}, & \text{if } \operatorname{Con_c} \mathcal{V} \nsubseteq \operatorname{Con_c} \mathcal{W}, \\ \infty, & \text{if } \operatorname{Con_c} \mathcal{V} \subseteq \operatorname{Con_c} \mathcal{W}. \end{cases}$$

This critical point has been already studied, for different families of varieties of lattices, in [10, 11, 3].

We now give an illustration of (**Q2**). Every countable locally finite lattice has a relatively complemented, congruence-preserving extension (cf. [7]). In particular every countable locally finite lattice has a congruence-permutable, congruence-preserving extension. However, in every non-distributive variety of lattices, the free lattice on $\aleph_1$ generators has no congruence-permutable, congruence-preserving extension (cf. [5, Chapter 5]). A precise answer to (**Q2**) also depends on the cardinality of A.

In order to study a similar problem, Pudlák in [13] uses an approach based on liftings of diagrams. The assignment $A \mapsto \operatorname{Con_c} A$ can be extended to a functor. This leads to the following questions:

(**Q1′**) Given a diagram $\vec{A}$ in $\mathcal{V}$, does there exist a diagram $\vec{B}$ in $\mathcal{W}$ such that $\operatorname{Con_c} \circ \vec{A} \cong \operatorname{Con_c} \circ \vec{B}$?

(**Q2′**) Given a diagram $\vec{A}$ in $\mathcal{V}$, does there exist a diagram $\vec{B}$ in $\mathcal{W}$ which is a congruence-preserving extension of $\vec{A}$?

The functor $\operatorname{Con_c}$ preserves directed colimits, thus, in many cases, a positive answer for the finite case of (**Q1′**) implies a positive answer to (**Q1**).

Proposition 1.1. *Assume that $\mathcal{V}$ and $\mathcal{W}$ are varieties of algebras. If $\mathcal{V}$ is locally finite and for every lattice-indexed diagram $\vec{A}$ of finite algebras in $\mathcal{V}$ there exists a diagram $\vec{B}$ in $\mathcal{W}$ such that $\operatorname{Con_c} \circ \vec{A} \cong \operatorname{Con_c} \circ \vec{B}$, then* (**Q1**) *has a positive answer.*

In this proposition, we consider infinite diagrams of finite algebras. However if $\mathcal{W}$ is finitely generated and congruence-distributive, a compactness argument makes it possible to restrict the assumptions to finite diagrams of finite algebras.

In order to study the converse of Proposition 1.1, we shall use the construction of *condensate*, introduced in [3]. This construction was introduced in order to turn diagram counterexamples to object counterexamples. We use it here to turn a diagram counterexample of (**Q1′**) to a counterexample of (**Q1**).

Theorem 1.2. *Assume that $\mathcal{V}$ and $\mathcal{W}$ are varieties of algebras. Let P be a finite lattice. Let $\vec{A}$ be a P-indexed diagram in $\mathcal{V}$. If $\operatorname{Con_c} \circ \vec{A}$ is not liftable in $\mathcal{W}$ then there is a condensate $A \in \mathcal{V}$ of $\vec{A}$, such that $\operatorname{Con_c} A$ is not liftable in $\mathcal{W}$.*

Moreover, if $\mathcal{W}$ has a countable similarity type and all algebras of $\vec{A}$ are countable, then the condensate A can be chosen of cardinal $\aleph_d$, where d is the order-dimension of P. In particular, $\operatorname{crit}(\mathcal{V}; \mathcal{W}) \leq \aleph_d$.

If every algebra of $\vec{A}$ is finite and $\mathcal{W}$ is finitely generated and congruence-distributive, then A can be chosen of cardinal $\aleph_{d-1}$, so $\operatorname{crit}(\mathcal{V}; \mathcal{W}) \leq \aleph_{d-1}$.

The cardinality bound, in case $\mathcal{W}$ is finitely generated and congruence-distributive, is optimal, in the following sense: There are finitely generated varieties $\mathcal{V}$ and $\mathcal{W}$

of lattices such that every countable $(\vee, 0)$-semilattice liftable in $\mathcal{V}$ is liftable in $\mathcal{W}$ and there is a square-indexed diagram of $(\vee, 0)$-semilattices that has a lifting in $\mathcal{V}$ but no lifting in $\mathcal{W}$. In particular there is a $(\vee, 0)$-semilattice of cardinal $\aleph_1$ that is liftable in $\mathcal{V}$ but not liftable in $\mathcal{W}$, thus $\operatorname{crit}(\mathcal{V}; \mathcal{W}) = \aleph_1$. This example appears in [3, Section 8].

Later, we generalized the condensate construction in [5], to a larger categorical context. The best bound of Theorem 1.2 is obtained in a more general case (cf. [5, Theorem 4-9.2]), namely if $\mathcal{W}$ is both *congruence-proper* (cf. [5, Definition 4-8.1]) and locally finite, for example $\mathcal{W}$ is a finitely generated congruence-modular variety. Using the tools introduced in this paper, we give a new version (cf. Theorem 9.6), we assume that $\mathcal{W}$ is congruence-proper and has finite similarity type.

This categorical version of condensate can also apply to turn a counterexample of (**Q2′**) to a counterexample of (**Q2**). For example in [5, Chapter 5] we give a square $\vec{A}$ of finite lattices, that has no congruence-permutable, congruence-preserving extension. A condensate of this square has cardinality $\aleph_1$ and it has no congruence-permutable, congruence-preserving extension.

The largest part of this paper is the introduction of *pregamps* and *gamps*, it is a generalization of semilattice-metric spaces and semilattice-metric covers given in [5, Chapter 5]. The category of gamps of a variety $\mathcal{V}$ has properties similar to a finitely generated congruence-distributive variety.

A *pregamp* is a triplet $\boldsymbol{A} = (A, \delta, S)$, where A is a partial algebra, S is a $(\vee, 0)$-semilattice and $\delta\colon A^2 \to S$ is a *distance*, *compatible with the operations*. A typical example of pregamp is $(A, \Theta_A, \operatorname{Con_c} A)$, for an algebra A. This generalizes to partial algebras the notion of a congruence.

A *gamp* $\boldsymbol{A}$ is a pregamp (A, δ, S) with a partial subalgebra A^* of A. There are many natural properties that a gamp can satisfy (cf. Section 7), for example $\boldsymbol{A}$ is *full* if all operations with parameters in A^* can be evaluated in A. A *morphism* of gamps is a morphism of partial algebras with a morphism of $(\vee, 0)$-semilattices satisfying a compatibility condition with the distances (cf. Definition 6.1).

The class of all gamps (on a given type), with morphisms of gamps, forms a category. Denote by $\boldsymbol{C}$ the forgetful functor from the category of gamps to the category of $(\vee, 0)$-semilattices. A *partial lifting* of a diagram $\vec{S}$ of $(\vee, 0)$-semilattice is a diagram $\vec{\boldsymbol{B}}$ of gamps, with some additional properties, such that $\boldsymbol{C} \circ \vec{\boldsymbol{B}} \cong \vec{S}$.

The category of gamps has properties similar to locally finite, congruence-proper varieties. Let S be a finite $(\vee, 0)$-semilattice, let $\boldsymbol{B}$ be a gamp such that $\boldsymbol{C}(\boldsymbol{B}) \cong S$, there are (arbitrary large) finite subgamps $\boldsymbol{B}'$ of $\boldsymbol{B}$ such that $\boldsymbol{C}(\boldsymbol{B}') \cong S$. There is no equivalent result for algebras: for example, the three-element chain is the congruence lattice of a modular lattice, but not the congruence lattice of any finite modular lattice.

Assume that $\mathcal{V}$ and $\mathcal{W}$ are varieties of algebras. Let P be a finite lattice of order-dimension d. Suppose that we find $\vec{A}$ a P-indexed diagram of finite algebras in $\mathcal{V}$, such that $\operatorname{Con_c} \circ \vec{A}$ has no partial lifting (with maybe some additional *"locally finite properties"*, cf. Section 7) in the category of gamps of $\mathcal{W}$, then $\operatorname{crit}(\mathcal{V}; \mathcal{W}) \leq \aleph_{d-1}$. Hence we obtain the optimal bound, with no assumption on $\mathcal{W}$. However there is no (known) proof that a diagram with no lifting has no partial lifting, but no counterexample has been found.

The *dual* of a lattice L is the lattice L^{d} with reverse order. The *dual* of a variety of lattices $\mathcal{V}$ is $\mathcal{V}^{\mathrm{d}}$ the variety of all duals of lattices in $\mathcal{V}$. Let $\mathcal{V}$ and $\mathcal{W}$ be varieties of lattices, if $\mathcal{V} \subseteq \mathcal{W}$ or $\mathcal{V} \subseteq \mathcal{W}^{\mathrm{d}}$, then $\mathrm{Con_c}\,\mathcal{V} \subseteq \mathrm{Con_c}\,\mathcal{W}$. In a sequel to the present paper (cf. [4]), we shall prove the following result.

Theorem. *Let $\mathcal{V}$ and $\mathcal{W}$ be varieties of lattices. If every simple lattice in $\mathcal{W}$ contains a prime interval, then one of the following statements holds:*

1. $\mathrm{crit}(\mathcal{V};\mathcal{W}) \leq \aleph_2$.
2. $\mathcal{V} \subseteq \mathcal{W}$.
3. $\mathcal{V} \subseteq \mathcal{W}^{\mathrm{d}}$.

The $\aleph_2$ bound is optimal, as there are varieties $\mathcal{V}$ and $\mathcal{W}$ of lattices such that $\mathrm{crit}(\mathcal{V};\mathcal{W}) = \aleph_2$. Without the use of gamps, we would have obtained an upper bound $\aleph_3$ instead of $\aleph_2$.

The gamps can also be used to study congruence-preserving extensions. Denote by $\boldsymbol{P}_{\mathrm{gl}}$ the functor that maps a gamp $(A^*, A, \delta, \widetilde{A})$ to the pregamp $(A^*, \delta, \widetilde{A})$; we also denote by $\boldsymbol{P}_{\mathrm{ga}}$ the functor that maps an algebra A to the pregamp $(A, \Theta_A, \mathrm{Con_c}\,A)$. Let B be a congruence-preserving extension of an algebra A, then $(A, B, \Theta_B, \mathrm{Con_c}\,B)$ is a gamp. Similarly, let $\vec{B} = (B_p, g_{p,q} \mid p \leq q \text{ in } P)$ be a congruence-preserving extension of a diagram $\vec{A} = (A_p, f_{p,q} \mid p \leq q \text{ in } P)$, denote by $\boldsymbol{B}_p = (A, B, \Theta_B, \mathrm{Con_c}\,B)$ and $\boldsymbol{g}_{p,q} = (g_{p,q}, \mathrm{Con_c}\,g_{p,q})$, for all $p \leq q$ in P, then $\vec{\boldsymbol{B}} = (\boldsymbol{B}_p, \boldsymbol{g}_{p,q} \mid p \leq q \text{ in } P)$ is a diagram of gamps. Moreover, $\boldsymbol{P}_{\mathrm{gl}} \circ \vec{\boldsymbol{B}} = \boldsymbol{P}_{\mathrm{ga}} \circ \vec{A}$, up to the identification of $\mathrm{Con_c}\,B$ and $\mathrm{Con_c}\,A$.

In Section 10, given $n \geq 2$, we construct a square $\vec{A}$ of finite lattices in $\mathcal{M}_3$, such that the diagram $\vec{A}$ has no congruence n-permutable, congruence-preserving extension. Another condensate construction gives a result proved in [12], namely the existence of a lattice $A \in \mathcal{M}_3$ of cardinality $\aleph_2$ with no congruence n-permutable, congruence-preserving extension.

Hopefully, once again, the diagram $\vec{A}$ satisfies a stronger statement, there is no *operational* (cf. Definition 10.1) diagram $\vec{\boldsymbol{B}}$ of *congruence n-permutable* gamps of lattices such that $\boldsymbol{P}_{\mathrm{gl}} \circ \vec{\boldsymbol{B}} \cong \boldsymbol{P}_{\mathrm{ga}} \circ \vec{A}$. Using a condensate, we obtain a lattice $\mathcal{M}_3$ of cardinality $\aleph_1$ with no congruence n-permutable, congruence-preserving extension.

2. Basic Concepts

We denote by 0 (resp., 1) the least (resp. largest) element of a poset if it exists. We denote by $\mathbf{2} = \{0, 1\}$ the two-element lattice, or poset (i.e., partially ordered set), or $(\vee, 0)$-semilattice, depending of the context. Given an algebra A, we denote by $\mathbf{0}_A$ the identity congruence of A.

Given subsets P and Q of a poset R, we set

$$P \downarrow Q = \{p \in P \mid (\exists q \in Q)(p \leq q)\}.$$

If $Q = \{q\}$, we simply write $P \downarrow q$ instead of $P \downarrow \{q\}$.

Let $\mathcal{V}$ be a variety of algebras, let κ be a cardinal, we denote by $F_{\mathcal{V}}(\kappa)$ the free algebras in $\mathcal{V}$ with κ generators. Given an algebra A we denote by $\mathbf{Var}\,A$ the variety of algebras generated by A. If A is a lattice we also denote by $\mathbf{Var}^{0,1}\,A$ the variety of bounded lattices generated by A. We denote by $\mathcal{L}$ the variety of lattices.

We denote the *range* of a function $f\colon X \to Y$ by $\mathrm{rng}\, f = \{f(x) \mid x \in X\}$. We use basic set-theoretical notation, for example ω is the first infinite ordinal, and also the set of all nonnegative integers; furthermore, $n = \{0, 1, \ldots, n-1\}$ for every nonnegative integer n. By "countable" we will always mean "at most countable".

Let X, I be sets, we often denote $\vec{x} = (x_i)_{i \in I}$ an element of X^I. In particular, for $n < \omega$, we denote by $\vec{x} = (x_0, \dots, x_{n-1})$ an n-tuple of X. If $f \colon Y \to Z$ is a function, where $Y \subseteq X$, we denote $f(\vec{x}) = (f(x_0), \dots, f(x_{n-1}))$ whenever it is defined. Similarly, if $f \colon Y \to Z$ is a function, where $Y \subseteq X^n$, we denote $f(\vec{x}) = f(x_0, \dots, x_{n-1})$ whenever it is defined. We also write $f(\vec{x}, \vec{y}) = f(x_0, \dots, x_{m-1}, y_0, \dots, y_{n-1})$ in case $\vec{x} = (x_0, \dots, x_{m-1})$ and $\vec{y} = (y_0, \dots, y_{n-1})$, and so on.

For example, let A and B be algebras of the same similarity type. Let ℓ be an n-ary operation. Let $f \colon A \to B$ a map. The map f is *compatible* with ℓ if $f(\ell(\vec{x})) = \ell(f(\vec{x}))$ for every n-tuple $\vec{x}$ of X. Let $m \leq n \leq \omega$. Let $\vec{X}$ be an n-tuple of X, we denote by $\vec{x} \restriction m$ the m-tuple $(x_k)_{k<m}$.

If X is a set and θ is an equivalence relation on X, we denote by X/θ the set of all equivalence classes of θ. Given $x \in X$ we denote by x/θ the equivalence class of θ containing x. Given an n-tuple $\vec{x}$ of X, we denote $\vec{x}/\theta = (x_0/\theta, \dots, x_{n-1}/\theta)$. Given $Y \subseteq X$, we set $Y/\theta = \{x/\theta \mid x \in Y\}$.

Let $n \geq 2$ an integer. An algebra A is *congruence n-permutable* if the following equality holds:

$$\underbrace{\alpha \circ \beta \circ \alpha \circ \dots}_{n \text{ times}} = \underbrace{\beta \circ \alpha \circ \beta \circ \dots}_{n \text{ times}}, \text{ for all } \alpha, \beta \in \operatorname{Con} A.$$

If $n = 2$ we say that A is *congruence-permutable* instead of *congruence 2-permutable*.

The following statement is folklore.

Proposition 2.1. *Let A be an algebra, let $n \geq 2$ an integer. The following conditions are equivalent:*

(1) *The algebra A is congruence n-permutable.*
(2) *For all $x_0, x_1, \dots, x_n \in A$, there are $x_0 = y_0, y_1, \dots, y_n = x_n \in A$ such that the following containments hold:*

$$\Theta_A(y_k, y_{k+1}) \subseteq \bigvee (\Theta_A(x_i, x_{i+1}) \mid i < n \text{ even}), \qquad \text{for all } k < n \text{ odd},$$

$$\Theta_A(y_k, y_{k+1}) \subseteq \bigvee (\Theta_A(x_i, x_{i+1}) \mid i < n \text{ odd}), \qquad \text{for all } k < n \text{ even}.$$

Let $n \geq 2$. The class of all congruence n-permutable algebras of a given similarity type is closed under directed colimits and quotients. Moreover the class of congruence n-permutable algebras of a congruence-distributive variety is also closed under finite products (the latter statement is known not to extend to arbitrary algebras).

3. Semilattices

In this section we give some well-known facts about $(\vee, 0)$-semilattices. Most notions and results will have later a generalization involving pregamps and gamps.

Proposition 3.1. *Let S, T be $(\vee, 0)$-semilattices, let X be a set, and let $f \colon X \to S$ and $g \colon X \to T$ be maps. Assume that for every $x \in X$, for every positive integer n, and for every n-tuple $\vec{y}$ of X the following implication holds:*

$$f(x) \leq \bigvee_{k<n} f(y_k) \quad \Longrightarrow \quad g(x) \leq \bigvee_{k<n} g(y_k). \tag{3.1}$$

If S is join-generated by $f(X)$, then there exists a unique $(\vee, 0)$-homomorphism $\phi \colon S \to T$ such that $\phi(f(x)) = g(x)$ for each $x \in X$.

If the converse of (3.1) *also holds and* $g(X)$ *also join-generates* T*, then* ϕ *is an isomorphism.*

Definition 3.2. An *ideal* of a $(\vee, 0)$-semilattice S is a lower subset I of S such that $0 \in I$ and $u \vee v \in I$ for all $u, v \in I$. We denote by $\operatorname{Id} S$ the lattice of ideals of S.

Let $\phi\colon S \to T$ be a $(\vee, 0)$-homomorphism. The 0-*kernel* of ϕ is $\ker_0 \phi = \{a \in S \mid \phi(a) = 0\}$; it is an ideal of S. We say that ϕ *separates zero* if $\ker_0 \phi = \{0\}$.

Let P be a poset, let $\vec{S} = (S_p, \phi_{p,q} \mid p \leq q \text{ in } P)$ be a diagram of $(\vee, 0)$-semilattices. An *ideal* of $\vec{S}$ is a family $(I_p)_{p\in P}$ such that I_p is an ideal of S_p and $\phi_{p,q}(I_p) \subseteq I_q$ for all $p \leq q$ in P.

Let $\vec{\phi} = (\phi_p)_{p\in P}\colon \vec{S} \to \vec{T}$ be a natural transformation of P-indexed diagrams of $(\vee, 0)$-semilattices. The 0-*kernel* of $\vec{\phi}$ is $\ker_0 \vec{\phi} = (\ker_0 \phi_p)_{p\in P}$, it is an ideal of $\vec{S}$.

Lemma 3.3. *Let* S *be a* $(\vee, 0)$*-semilattice, let* $I \in \operatorname{Id} S$*. Put:*

$$\theta_I = \{(x, y) \in S^2 \mid (\exists u \in I)(x \vee u = y \vee u)\}.$$

The relation θ_I *is a congruence of* S*.*

Notation 3.4. We denote by S/I the $(\vee, 0)$-semilattice S/θ_I, where θ_I is the congruence defined in Lemma 3.3. Given $a \in S$, we denote by a/I the equivalent class of a for θ_I. The $(\vee, 0)$-homomorphism $\phi\colon S \to S/I$, $a \mapsto a/I$ is *the canonical projection*. Notice that $\ker_0 \phi = I$.

If $I = \{0\}$, we identify S/I and S.

Lemma 3.5. *Let* $\phi\colon S \to T$ *be a* $(\vee, 0)$*-homomorphism, and let* $I \in \operatorname{Id} S$ *and* $J \in \operatorname{Id} T$ *such that* $\phi(I) \subseteq J$*. There exists a unique map* $\psi\colon S/I \to T/J$ *such that* $\psi(a/I) = \phi(a)/J$ *for each* $a \in S$*. Moreover,* ψ *is a* $(\vee, 0)$*-homomorphism.*

Notation 3.6. We say that ϕ *induces* the $(\vee, 0)$-homomorphism $\psi\colon S/I \to T/J$ in Lemma 3.5.

Lemma 3.7. *Let* P *be a poset, let* $\vec{S} = (S_p, \phi_{p,q} \mid p \leq q \text{ in } P)$ *be a diagram of* $(\vee, 0)$*-semilattices, and let* $\vec{I}$ *be an ideal of* $\vec{S}$*. Denote by* $\psi_{p,q}\colon S_p/I_p \to S_q/I_q$ *the* $(\vee, 0)$*-homomorphism induced by* $\phi_{p,q}$*, then* $(S_p/I_p, \psi_{p,q} \mid p \leq q \text{ in } P)$ *is a diagram of* $(\vee, 0)$*-semilattices.*

Notation 3.8. We denote by $\vec{S}/\vec{I}$ the diagram $(S_p/I_p, \psi_{p,q} \mid p \leq q \text{ in } P)$ introduced in Lemma 3.7.

Lemma 3.9. *Let* P *be a poset, let* $\vec{\phi}\colon \vec{S} \to \vec{T}$ *be a natural transformation of* P*-indexed diagrams of* $(\vee, 0)$*-semilattices, let* $\vec{I} \in \operatorname{Id} \vec{S}$ *and* $\vec{J} \in \operatorname{Id} \vec{T}$ *such that* $\phi_p(I_p) \subseteq J_p$ *for all* $p \in P$*. Denote by* $\psi_p\colon S_p/I_p \to T_p/J_p$ *the* $(\vee, 0)$*-homomorphism induced by* ϕ_p*. Then* $\vec{\psi}$ *is a natural transformation from* $\vec{S}/\vec{I}$ *to* $\vec{T}/\vec{J}$*.*

Notation 3.10. We say that $\vec{\phi}$ *induces* $\vec{\psi}\colon \vec{S}/\vec{I} \to \vec{T}/\vec{J}$, the natural transformation defined in Lemma 3.9.

Definition 3.11. A $(\vee, 0)$-homomorphism $\phi\colon S \to T$ is *ideal-induced* if ϕ is surjective and for all $x, y \in S$ with $\phi(x) = \phi(y)$ there exists $z \in S$ such that $x \vee z = y \vee z$ and $\phi(z) = 0$.

Let P be a poset, let $\vec{S} = (S_p, \phi_{p,q} \mid p \leq q \text{ in } P)$ and $\vec{T} = (T_p, \psi_{p,q} \mid p \leq q \text{ in } P)$ be P-indexed diagrams of $(\vee, 0)$-semilattices. A natural transformation $\vec{\pi} = (\pi_p)_{p\in P}\colon \vec{S} \to \vec{T}$ is *ideal-induced* if π_p is ideal-induced for each $p \in P$.

Remark 3.12. Let I be an ideal of a $(\vee,0)$-semilattice A, denote by $\pi\colon A\to A/I$ the canonical projection, then π is ideal-induced.

The next lemmas give a characterization of ideal-induced $(\vee,0)$-homomorphisms.

Lemma 3.13. *Let $\phi\colon S\to T$ be a $(\vee,0)$-homomorphism. The following statements are equivalent*

(1) *ϕ is ideal-induced.*
(2) *The $(\vee,0)$-homomorphism $\psi\colon S/\ker_0\phi\to T$ induced by ϕ is an isomorphism.*

The following lemma expresses that, given a diagram $\vec{S}$ of $(\vee,0)$-semilattices, the colimits of quotients of $\vec{S}$ are the quotients of the colimits of $\vec{S}$.

Lemma 3.14. *Let P be a directed poset, let $\vec{S}=(S_p,\phi_{p,q}\mid p\le q \text{ in } P)$ be a P-indexed diagram in $\mathbf{Sem}_{\vee,0}$, and let $(S,\phi_p\mid p\in P)=\varinjlim\vec{S}$ be a directed colimit cocone in $\mathbf{Sem}_{\vee,0}$. The following statements hold:*

(1) *Let $\vec{I}$ be an ideal of $\vec{S}$. Then $I=\bigcup_{p\in P}\phi_p(I_p)$ is an ideal of S. Moreover, denote by $\psi_p\colon S_p/I_p\to S/I$ the $(\vee,0)$-homomorphism induced by ϕ_p, for each $p\in P$. The following is a directed colimit cocone:*
$$(S/I,\psi_p\mid p\in P)=\varinjlim\vec{S}/\vec{I}\quad \text{in } \mathbf{Sem}_{\vee,0}.$$
(2) *Let $I\in\operatorname{Id}S$. Put $I_p=\phi_p^{-1}(I)$ for each $p\in P$. Then $\vec{I}=(I_p)_{p\in P}$ is an ideal of $\vec{S}$, moreover $I=\bigcup_{p\in P}\phi_p(I_p)$.*

Lemma 3.15. *Let $\pi\colon A\twoheadrightarrow B$ be a surjective morphism of algebras. The $(\vee,0)$-homomorphism $\operatorname{Con_c}\pi$ is ideal-induced. Moreover, $\ker_0(\operatorname{Con_c}\pi)=(\operatorname{Con_c}A)\mathbin{\downarrow}\ker\pi$.*

Proposition 3.16. *Let S and T be $(\vee,0)$-semilattices with T finite, let $\phi\colon S\to T$ be an ideal-induced $(\vee,0)$-homomorphism, and let $X\subseteq S$ finite. There exists a finite $(\vee,0)$-subsemilattice S' of S such that $X\subseteq S'$ and $\phi\restriction S'\colon S'\to T$ is ideal-induced.*

Proof. As ϕ is surjective and X is finite, there exists a finite $(\vee,0)$-subsemilattice Y of S such that $X\subseteq Y$ and $\phi(Y)=T$. Given $x,y\in Y$ with $\phi(x)=\phi(y)$ we fix $u_{x,y}\in S$ such that $\phi(u_{x,y})=0$ and $x\vee u_{x,y}=y\vee u_{x,y}$. Let U be the $(\vee,0)$-subsemilattice of S generated by $\{u_{x,y}\mid x,y\in Y \text{ and } \phi(x)=\phi(y)\}$. As $\phi(u)=0$ for all generators, $\phi(u)=0$ for each $u\in U$.

Let S' be the $(\vee,0)$-subsemilattice of S generated by $Y\cup U$. As S' is finitely generated, it is finite. As $Y\subseteq S'$, $\phi(S')=T$. Let $a,b\in S'$ such that $\phi(a)=\phi(b)$. There exist $x,y\in Y$ and $u,v\in U$ such that $a=x\vee u$ and $b=y\vee v$, thus $\phi(a)=\phi(x\vee u)=\phi(x)\vee\phi(u)=\phi(x)$. Similarly, $\phi(b)=\phi(y)$, hence $\phi(x)=\phi(y)$, moreover $x,y\in Y$, so $u_{x,y}\in U$. The element $w=u\vee v\vee u_{x,y}$ belongs to U, hence $\phi(w)=0$, moreover $w\in S'$. From $x\vee u_{x,y}=y\vee u_{x,y}$ it follows that $a\vee w=b\vee w$. Therefore, $\phi\restriction S'$ is ideal-induced. □

4. Partial algebras

In this section we introduce a few basic properties of partial algebras. We fix a similarity type $\mathscr{L}$. Given $\ell\in\mathscr{L}$ we denote by $\operatorname{ar}(\ell)$ the arity of ℓ.

Definition 4.1. A *partial algebra* A is a set (the *universe* of the partial algebra), given with a set $D_\ell=\operatorname{Def}_\ell(A)\subseteq A^{\operatorname{ar}(\ell)}$ and a map $\ell^A\colon D_\ell\to A$ called a *partial operation*, for each $\ell\in\mathscr{L}$.

Let $\ell \in \mathscr{L}$ be an n-ary operation. If $\vec{x} \in \operatorname{Def}_\ell(A)$ we say that $\ell^A(\vec{x})$ is *defined in* A. We generalize this notion to terms in the usual way. For example, given binary operations ℓ_1 and ℓ_2 of a partial algebra A and $x, y, z \in A$, $\ell_1^A(\ell_2^A(x,y), \ell_1^A(y,z))$ is defined in A if and only if $(x,y) \in \operatorname{Def}_{\ell_2}(A)$, $(y,z) \in \operatorname{Def}_{\ell_1}(A)$, and $(\ell_2^A(x,y), \ell_1^A(y,z)) \in \operatorname{Def}_{\ell_1}(A)$.

Given a term t, we denote by $\operatorname{Def}_t(A)$ the set of all tuples $\vec{x}$ of A such that $t(\vec{x})$ is defined in A.

We denote $\ell(\vec{x})$ instead of $\ell^A(\vec{x})$ when there is no ambiguity. Any algebra A has a natural structure of partial algebra with $\operatorname{Def}_\ell(A) = A^{\operatorname{ar}(\ell)}$ for each $\ell \in \mathscr{L}$.

Definition 4.2. Let A, B be partial algebras. A *morphism of partial algebras* is a map $f\colon A \to B$ such that $f(\vec{x}) \in \operatorname{Def}_\ell(B)$ and $\ell(f(\vec{x})) = f(\ell(\vec{x}))$, for all $\ell \in \mathscr{L}$ and all $\vec{x} \in \operatorname{Def}_\ell(A)$.

The *category of partial algebras*, denoted by $\mathbf{PAlg}_{\mathscr{L}}$, is the category in which the objects are the partial algebras and the arrows are the above-mentioned morphisms of partial algebras.

A morphism $f\colon A \to B$ of partial algebras is *strong* if $(f(A))^{\operatorname{ar}(\ell)} \subseteq \operatorname{Def}_\ell(B)$ for each $\ell \in \mathscr{L}$.

A partial algebra A is *finite* if its universe is finite.

Remark 4.3. A morphism $f\colon A \to B$ of partial algebras is an isomorphism if and only if the following conditions are both satisfied

(1) The map f is bijective.
(2) If $\ell(f(\vec{x}))$ is defined in B then $\ell(\vec{x})$ is defined in A, for each operation $\ell \in \mathscr{L}$ and each tuple $\vec{x}$ of A.

We remind the reader that the converse of (2) is always true.

Definition 4.4. Given a partial algebra A, a *partial subalgebra* B of A is a subset B of A endowed with a structure of partial algebra such that $\operatorname{Def}_\ell(B) \subseteq \operatorname{Def}_\ell(A)$ and $\ell^A(\vec{x}) = \ell^B(\vec{x})$ for all $\ell \in \mathscr{L}$ and all $\vec{x} \in \operatorname{Def}_\ell(B)$. The inclusion map from A into B is a morphism of partial algebras called *the inclusion morphism*.

A partial subalgebra B of A is *full* if whenever $\ell \in \mathscr{L}$ and $\vec{x} \in B^{\operatorname{ar}(\ell)}$ are such that $\ell^A(\vec{x})$ is defined and belongs to B, then $\ell(\vec{x})$ is defined in B. It is equivalent to the following equality:

$$\operatorname{Def}_\ell(B) = \{\vec{x} \in \operatorname{Def}_\ell(A) \cap B^{\operatorname{ar}(\ell)} \mid \ell(\vec{x}) \in B\}, \quad \text{for each } \ell \in \mathscr{L}.$$

A partial subalgebra B of A is *strong* if the inclusion map is a strong morphism, that is, $B^{\operatorname{ar}(\ell)} \subseteq \operatorname{Def}_\ell(A)$ for each $\ell \in \mathscr{L}$.

An *embedding* of partial algebras is a one-to-one morphism of partial algebras.

Notation 4.5. Let $f\colon A \to B$ be a morphism of partial algebras, let X be a partial subalgebra of A. The set $f(X)$ can be endowed with a natural structure of partial algebra, by setting $\operatorname{Def}_\ell(f(X)) = f(\operatorname{Def}_\ell(X)) = \{f(\vec{x}) \mid \vec{x} \in \operatorname{Def}_\ell(X)\}$, for each $\ell \in \mathscr{L}$. Similarly, let Y be a partial subalgebra of B. The set $f^{-1}(Y)$ can be endowed with a natural structure of partial algebra, by setting $\operatorname{Def}_\ell(f^{-1}(Y)) = f^{-1}(\operatorname{Def}_\ell(Y)) = \{\vec{x} \in A \mid f(\vec{x}) \in \operatorname{Def}_\ell(Y)\}$, for each $\ell \in \mathscr{L}$.

Remark. Let $f\colon A \to B$ and $g\colon B \to C$ be morphisms of partial algebras, let X a sub-partial algebra of A, then $(g \circ f)(X) = g(f(X))$ as partial algebras. Let Z be a partial subalgebra of X, then $(g \circ f)^{-1}(Z) = f^{-1}(g^{-1}(Z))$ as partial algebras.

Let $f\colon A \to B$ be a morphism of partial algebras, let X be a partial subalgebra of A. Then X is a partial subalgebra of $f^{-1}(f(X))$. In particular $f^{-1}(f(A)) = A$ as partial algebras. Let Y be a partial subalgebra of B, then $f(f^{-1}(Y))$ is a partial subalgebra of Y.

If $\mathscr{L}$ is infinite, then there are a finite partial algebra A (even with one element) and an infinite chain of partial subalgebras of A with union A. In particular, A is not finitely presented in the category $\mathbf{PAlg}_{\mathscr{L}}$.

A morphism $f\colon A \to B$ of partial algebras is strong if and only if $f(A)$ is a strong partial subalgebra of B.

Lemma 4.6. *An embedding $f\colon A \to B$ of partial algebras is an isomorphism if and only if $f(A) = B$ as partial algebras.*

Proof. Assume that f is an isomorphism, let g be its inverse. Notice that $f(A)$ is a partial subalgebra of B and $B = f(g(B))$ is a partial subalgebra of $f(A)$, therefore $B = f(A)$ as partial algebras.

Conversely, assume that $B = f(A)$ as partial algebras. Then f is surjective, moreover f is an embedding, so f is a bijection. Let $g = f^{-1}$ in **Set**. Let $\vec{y} \in \operatorname{Def}_\ell(B)$. As $B = f(A)$, there exists $\vec{x} \in \operatorname{Def}_\ell(A)$ such that $f(\vec{x}) = \vec{y}$, thus $g(\vec{y}) = \vec{x} \in \operatorname{Def}_\ell(A)$. Moreover $g(\ell(\vec{y})) = g(\ell(f(\vec{x}))) = g(f(\ell(\vec{x}))) = \ell(\vec{x}) = \ell(g(\vec{y}))$. □

Notation 4.7. Let A be a partial algebra, let X be a subset of A. We define inductively, for each $n < \omega$,

$$\langle X\rangle_A^0 = X \cup \{c \mid c \text{ is a constant of } \mathscr{L}\}$$

$$\langle X\rangle_A^{n+1} = \langle X\rangle_A^n \cup \left\{\ell(\vec{x}) \mid \ell \in \mathscr{L},\ \vec{x} \in \operatorname{Def}_\ell(A),\ \vec{x} \text{ is an } \operatorname{ar}(\ell)\text{-tuple of} \langle X\rangle_A^n\right\}$$

We endow $\langle X\rangle_A^n$ with the induced structure of full partial subalgebra of A. If $\mathscr{L}$ and X are both finite, then $\langle X\rangle_A^n$ is finite for each $n < \omega$. If A is understood, we shall simply denote this partial algebra by $\langle X\rangle^n$.

Definition 4.8. A partial algebra A *satisfies an identity* $t_1 = t_2$ if $t_1(\vec{x}) = t_2(\vec{x})$ for each tuple $\vec{x}$ of A such that both $t_1(\vec{x})$ and $t_2(\vec{x})$ are defined in A. Otherwise we say that A *fails* $t_1 = t_2$.

Let $\mathcal{V}$ be a variety of algebras, a partial algebra A is *a partial algebra of* $\mathcal{V}$ if A satisfies all identities of $\mathcal{V}$.

Remark. Let A be a partial algebra, let $\ell \in \mathscr{L}$. If $\operatorname{Def}_\ell(A) = \emptyset$ then A satisfies $\ell(\vec{x}) = y$, vacuously.

If A fails $t_1 = t_2$, then there exists a tuple $\vec{x}$ of A such that $t_1(\vec{x})$ and $t_2(\vec{x})$ are both defined and $t_1(\vec{x}) \neq t_2(\vec{x})$.

Lemma 4.9. *The category $\mathbf{PAlg}_{\mathscr{L}}$ has all directed colimits. Moreover, given a directed poset P, a P-indexed diagram $\vec{A} = (A_p, f_{p,q} \mid p \leq q \text{ in } P)$ in $\mathbf{PAlg}_{\mathscr{L}}$, and a directed colimit cocone:*

$$(A, f_p \mid p \in P) = \varinjlim (A_p, f_{p,q} \mid p \leq q \text{ in } P), \quad \text{in } \mathbf{Set}, \tag{4.1}$$

the set A can be uniquely endowed with a structure of partial algebra such that:

- $\operatorname{Def}_\ell(A) = \{f_p(\vec{x}) \mid p \in P \text{ and } \vec{x} \in \operatorname{Def}_\ell(A_p)\}$*, for each* $\ell \in \mathscr{L}$*;*
- $\ell(f_p(\vec{x})) = f_p(\ell(\vec{x}))$ *for each* $p \in P$*, all* $\ell \in \mathscr{L}$*, and all* $\vec{x} \in \operatorname{Def}_\ell(A_p)$*.*

Moreover, if A is endowed with this structure of partial algebra, the following statements hold:

(1) $(A, f_p \mid p \in P) = \varinjlim(A_p, f_{p,q} \mid p \leq q \text{ in } P)$ *in* $\mathbf{PAlg}_{\mathscr{L}}$.
(2) *Assume that for each* $\ell \in \mathscr{L}$, *each* $p \in P$, *and each* $\operatorname{ar}(\ell)$*-tuple* $\vec{x}$ *of* A_p *there exists* $q \geq p$ *such that* $f_{p,q}(\vec{x}) \in \operatorname{Def}_\ell(A_q)$. *Then* A *is an algebra, that is,* $\operatorname{Def}_\ell(A) = A^{\operatorname{ar}(\ell)}$ *for each* $\ell \in \mathscr{L}$.
(3) *If* P *has no maximal element and* $f_{p,q}$ *is strong for all* $p < q$ *in* P, *then* A *is an algebra.*
(4) $\operatorname{Def}_t(A) = \{f_p(\vec{x}) \mid p \in P \text{ and } \vec{x} \in \operatorname{Def}_t(A_p)\}$ *for each term* t *of* $\mathscr{L}$.
(5) *Let* $t_1 = t_2$ *be an identity. If* A_p *satisfies* $t_1 = t_2$ *for all* $p \in P$, *then* A *satisfies* $t_1 = t_2$.

Proof. Put $\operatorname{Def}_\ell(A) = \{f_p(\vec{x}) \mid p \in P \text{ and } \vec{x} \in \operatorname{Def}_\ell(A_p)\}$, for each $\ell \in \mathscr{L}$.

Let $\ell \in \mathscr{L}$, let $\vec{x} \in \operatorname{Def}_\ell(A)$. There exist $p \in P$ and $\vec{y} \in \operatorname{Def}_\ell(A_p)$ such that $\vec{x} = f_p(\vec{y})$. We first show that $f_p(\ell(\vec{y}))$ does not depend on the choice of p and $\vec{y}$. Let $q \in P$ and $\vec{z} \in \operatorname{Def}_\ell(A_q)$ such that $\vec{x} = f_q(\vec{z})$. As $f_p(\vec{y}) = \vec{x} = f_q(\vec{z})$, it follows from (4.1) that there exists $r \geq p, q$ such that $f_{p,r}(\vec{y}) = f_{q,r}(\vec{z})$. Therefore the following equalities hold:

$$f_p(\ell(\vec{y})) = f_r(f_{p,r}(\ell(\vec{y}))) = f_r(\ell(f_{p,r}(\vec{y}))) = f_r(\ell(f_{q,r}(\vec{z}))) = f_r(f_{q,r}(\ell(\vec{z}))) = f_q(\ell(\vec{z})).$$

Hence $\ell(f_p(\vec{y})) = f_p(\ell(\vec{y}))$ for all $p \in P$ and all $\vec{y} \in \operatorname{Def}_\ell(A_p)$ uniquely define a partial operation $\ell\colon \operatorname{Def}_\ell(A) \to A$. Moreover f_p is a morphism of partial algebras for each $p \in P$.

Let $(B, g_p \mid p \in P)$ be a cocone over $(A_p, f_{p,q} \mid p \leq q \text{ in } P)$ in $\mathbf{PAlg}_{\mathscr{L}}$. In particular, it is a cocone in **Set**, so there exists a unique map $h\colon A \to B$ such that $h \circ f_p = g_p$ for each $p \in P$. Let $\ell \in \mathscr{L}$, let $\vec{x} \in \operatorname{Def}_\ell(A)$. There exist $p \in P$ and $\vec{y} \in \operatorname{Def}_\ell(A_p)$ such that $\vec{x} = f_p(\vec{y})$, thus $h(\vec{x}) = h(f_p(\vec{y})) = g_p(\vec{y})$. As g_p is a morphism of partial algebras and $\vec{y} \in \operatorname{Def}_\ell(A_p)$, we obtain that $h(\vec{x}) \in \operatorname{Def}_\ell(B)$. Moreover the following equalities hold:

$$\ell(h(\vec{x})) = \ell(g_p(\vec{y})) = g_p(\ell(\vec{y})) = h(f_p(\ell(\vec{y}))) = h(\ell(f_p(\vec{y}))) = h(\ell(\vec{x})).$$

Hence h is a morphism of partial algebras. Therefore:

$$(A, f_p \mid p \in P) = \varinjlim(A_p, f_{p,q} \mid p \leq q \text{ in } P) \quad \text{in } \mathbf{PAlg}_{\mathscr{L}}.$$

Assume that for each $\ell \in \mathscr{L}$, for all $p \in P$, and for all $\operatorname{ar}(\ell)$-tuples $\vec{x}$ of A_p, there exists $q \geq p$ such that $f_{p,q}(\vec{x}) \in \operatorname{Def}_\ell(A_q)$.

Let $\ell \in \mathscr{L}$, let $\vec{x}$ be an $\operatorname{ar}(\ell)$-tuple of A. There exist $p \in P$ and a tuple $\vec{y}$ of A_p such that $\vec{x} = f_p(\vec{y})$. Let $q \geq p$ such that $f_{p,q}(\vec{y}) \in \operatorname{Def}_\ell(A_q)$. It follows that $\vec{x} = f_p(\vec{y}) = f_q(f_{p,q}(\vec{y}))$ belongs to $\operatorname{Def}_\ell(A)$. Therefore A is an algebra.

The statement (3) follows directly from (2). The statement (4) is proved by a straightforward induction on terms, and (5) is an easy consequence of (4). □

5. Pregamps

A *pregamp* is a partial algebra endowed with a semilattice-valued "distance" (cf. (1)-(3)) compatible with all operations of A (cf. (4)). It is a generalization of the notion of *semilattice-metric space* defined in [5, Section 5-1].

Definition 5.1. Let A be a partial algebra, let S be a $(\vee, 0)$-semilattice. A S*-valued partial algebra distance* on A is a map $\delta\colon A^2 \to S$ such that:

(1) $\delta(x, y) = 0$ if and only if $x = y$, for all $x, y \in A$.
(2) $\delta(x, y) = \delta(y, x)$, for all $x, y \in A$.
(3) $\delta(x, y) \leq \delta(x, z) \vee \delta(z, y)$, for all $x, y, z \in A$.

(4) $\delta(\ell(\vec{x}),\ell(\vec{y})) \leq \bigvee_{k<\operatorname{ar}(\ell)} \delta(x_k,y_k)$, for all $\ell \in \mathscr{L}$ and all $\vec{x},\vec{y} \in \operatorname{Def}_\ell(A)$.

Then we say that $\boldsymbol{A} = (A,\delta,S)$ is a *pregamp*. We shall generally write $\delta_{\boldsymbol{A}} = \delta$ and $\widetilde{A} = S$.

The pregamp is *distance-generated* if it satisfies the following additional property:

(5) S is join-generated by $\delta_{\boldsymbol{A}}(A^2)$. That is, for all $\alpha \in S$ there are $n \geq 0$ and n-tuples $\vec{x},\vec{y}$ of A such that $\alpha = \bigvee_{k<n} \delta_{\boldsymbol{A}}(x_k,y_k)$.

Example 5.2. Let A be an algebra. We remind the reader that $\Theta_A(x,y)$ denotes the smallest congruence that identifies x and y, for all $x,y \in A$. This defines a distance $\Theta_A\colon A^2 \to \operatorname{Con_c} A$. Moreover, $(A,\Theta_A,\operatorname{Con_c} A)$ is a distance-generated pregamp.

A straightforward induction argument on the length of the term t yields the following lemma.

Lemma 5.3. *Let $\boldsymbol{A}$ be a pregamp, let t be an n-ary term, and let $\vec{x},\vec{y} \in \operatorname{Def}_t(A)$. The following inequality holds:*

$$\delta_{\boldsymbol{A}}(t(\vec{x}),t(\vec{y})) \leq \bigvee_{k<n} \delta_{\boldsymbol{A}}(x_k,y_k).$$

We say that $\delta_{\boldsymbol{A}}$ and t are compatible.

Definition 5.4. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be pregamps. A *morphism* from $\boldsymbol{A}$ to $\boldsymbol{B}$ is an ordered pair $\boldsymbol{f} = (f,\widetilde{f})$ such that $f\colon A \to B$ is a morphism of partial algebras, $\widetilde{f}\colon \widetilde{A} \to \widetilde{B}$ is a $(\vee,0)$-homomorphism, and $\delta_{\boldsymbol{B}}(f(x),f(y)) = \widetilde{f}(\delta_{\boldsymbol{A}}(x,y))$ for all $x,y \in A$.

Given morphisms $\boldsymbol{f}\colon \boldsymbol{A} \to \boldsymbol{B}$ and $\boldsymbol{g}\colon \boldsymbol{B} \to \boldsymbol{C}$ of pregamps, the pair $\boldsymbol{g} \circ \boldsymbol{f} = (g \circ f, \widetilde{g} \circ \widetilde{f})$ is a morphism from $\boldsymbol{A}$ to $\boldsymbol{C}$.

We denote by $\mathbf{PGamp}_{\mathscr{L}}$ the category of pregamps with the morphisms defined above.

We denote by $\boldsymbol{P}_{\mathrm{ga}}$ the functor from the category of $\mathscr{L}$-algebras to $\mathbf{PGamp}_{\mathscr{L}}$ that maps an algebra A to $(A,\Theta_A,\operatorname{Con_c} A)$, and a morphism of algebras f to $(f,\operatorname{Con_c} f)$. We denote by $\boldsymbol{C}_{\mathrm{pg}}$ the functor from $\mathbf{PGamp}_{\mathscr{L}}$ to $\mathbf{Sem}_{\vee,0}$ that maps a pregamp $\boldsymbol{A}$ to $\widetilde{A}$, and maps a morphism of pregamps $\boldsymbol{f}\colon \boldsymbol{A} \to \boldsymbol{B}$ to the $(\vee,0)$-homomorphism $\widetilde{f}$.

Remark 5.5. A morphism $\boldsymbol{f}\colon \boldsymbol{A} \to \boldsymbol{B}$ of pregamps is an isomorphism if and only if f is an isomorphism of partial algebras and $\widetilde{f}$ is an isomorphism of $(\vee,0)$-semilattices.

Notice that $\boldsymbol{C}_{\mathrm{pg}} \circ \boldsymbol{P}_{\mathrm{ga}} = \operatorname{Con_c}$.

We leave to the reader the straightforward proof of the following lemma.

Lemma 5.6. *The category $\mathbf{PGamp}_{\mathscr{L}}$ has all directed colimits. Moreover, given a directed poset P, a P-indexed diagram $\vec{\boldsymbol{A}} = (\boldsymbol{A}_p,\boldsymbol{f}_{p,q} \mid p \leq q \text{ in } P)$ in $\mathbf{PGamp}_{\mathscr{L}}$, a directed colimit cocone $(A,f_p \mid p \in P) = \varinjlim(A_p,f_{p,q} \mid p \leq q \text{ in } P)$ in $\mathbf{PAlg}_{\mathscr{L}}$, and a directed colimit cocone $(\widetilde{A},\widetilde{f}_p \mid p \in P) = \varinjlim(\widetilde{A}_p,\widetilde{f}_{p,q} \mid p \leq q \text{ in } P)$ in $\mathbf{Sem}_{\vee,0}$, there exists a unique $\widetilde{A}$-valued partial algebra distance $\delta_{\boldsymbol{A}}$ on A such that $\delta_{\boldsymbol{A}}(f_p(x),f_p(y)) = \widetilde{f}_p(\delta_{\boldsymbol{A}_p}(x,y))$ for all $p \in P$ and all $x,y \in A_p$.*

Furthermore $\boldsymbol{A} = (A,\delta_{\boldsymbol{A}},\widetilde{A})$ is a pregamp, $\boldsymbol{f}_p\colon \boldsymbol{A}_p \to \boldsymbol{A}$ is a morphism of pregamps for each $p \in P$, and the following is a directed colimit cocone:

$$(\boldsymbol{A},\boldsymbol{f}_p \mid p \in P) = \varinjlim(\boldsymbol{A}_p,\boldsymbol{f}_{p,q} \mid p \leq q \text{ in } P), \qquad \text{in } \mathbf{PGamp}_{\mathscr{L}}.$$

Moreover if $\boldsymbol{A}_p$ is distance-generated for each $p \in P$, then $\boldsymbol{A}$ is distance-generated.

Remark 5.7. As an immediate application of Lemma 5.6, and the fact that $\mathrm{Con_c}$ preserves directed colimits, we obtain that both $\boldsymbol{C}_{\mathrm{pg}}$ and $\boldsymbol{P}_{\mathrm{ga}}$ preserve directed colimits.

Definition 5.8. An *embedding* $\boldsymbol{f} \colon \boldsymbol{B} \to \boldsymbol{A}$ of pregamps is a morphism of pregamps such that f and $\widetilde{f}$ are both one-to-one.

A *sub-pregamp* of a pregamp $\boldsymbol{A}$ is a pregamp $\boldsymbol{B}$ such that B is a partial subalgebra of A, $\widetilde{B}$ is a $(\vee, 0)$-subsemilattice of $\widetilde{A}$, and $\delta_{\boldsymbol{B}} = \delta_{\boldsymbol{A}} \restriction B^2$.

If $f \colon B \to A$ and $\widetilde{f} \colon \widetilde{B} \to \widetilde{A}$ denote the inclusion maps, the morphism of pregamps $\boldsymbol{f} = (f, \widetilde{f})$ is called *the canonical embedding*.

Notation 5.9. Let $\boldsymbol{f} \colon \boldsymbol{B} \to \boldsymbol{A}$ be a morphism of pregamps. Given a sub-pregamp $\boldsymbol{C}$ of $\boldsymbol{B}$, the triple $\boldsymbol{f}(\boldsymbol{C}) = (f(C), \delta_{\boldsymbol{A}} \restriction (f(C))^2, \widetilde{f}(\widetilde{C}))$ (see Notation 4.5) is a sub-pregamp of $\boldsymbol{A}$.

For a sub-pregamp $\boldsymbol{C}$ of $\boldsymbol{A}$, the triple $\boldsymbol{f}^{-1}(\boldsymbol{C}) = (f^{-1}(C), \delta_{\boldsymbol{B}} \restriction (f^{-1}(C))^2, \widetilde{f}^{-1}(\widetilde{C}))$ is a sub-pregamp of $\boldsymbol{B}$.

We leave to the reader the straightforward proof of the following description of sub-pregamps and embeddings.

Proposition 5.10. *The following statements hold.*

1. *Let $\boldsymbol{A}$ be a pregamp, let B be a partial subalgebra of A, let $\widetilde{B}$ be a $(\vee, 0)$-subsemilattice of $\widetilde{A}$ that contains $\delta_{\boldsymbol{A}}(B^2)$. Put $\delta_{\boldsymbol{B}} = \delta_{\boldsymbol{A}} \restriction B^2$. Then $(B, \delta_{\boldsymbol{B}}, \widetilde{B})$ is a sub-pregamp of $\boldsymbol{A}$. Moreover, all sub-pregamps of $\boldsymbol{A}$ are of this form.*
2. *Let $\boldsymbol{f} \colon \boldsymbol{B} \to \boldsymbol{A}$ be a morphism of pregamps. Then f is an embedding of partial algebras if and only if $\widetilde{f}$ separates 0. Moreover $\boldsymbol{f}$ is an embedding if and only if $\widetilde{f}$ is an embedding.*
3. *Let $\boldsymbol{f} \colon \boldsymbol{B} \to \boldsymbol{A}$ be an embedding of pregamps. The restriction $\boldsymbol{f} \colon \boldsymbol{B} \to \boldsymbol{f}(\boldsymbol{B})$ is an isomorphism of pregamps.*

The following result appears in [9, Theorem 10.4]. It gives a description of finitely generated congruences of a general algebra.

Lemma 5.11. *Let B be an algebra, let m be a positive integer, let $x, y \in B$, and let $\vec{x}, \vec{y}$ be m-tuples of B. Then $\Theta_B(x, y) \leq \bigvee_{i<m} \Theta_B(x_i, y_i)$ if and only if there are a positive integer n, a list $\vec{z}$ of parameters from B, and terms t_0, …, t_n such that*

$$\begin{aligned} x &= t_0(\vec{x}, \vec{y}, \vec{z}), \\ y &= t_n(\vec{x}, \vec{y}, \vec{z}), \\ t_j(\vec{y}, \vec{x}, \vec{z}) &= t_{j+1}(\vec{x}, \vec{y}, \vec{z}) \quad (\text{for all } j < n). \end{aligned}$$

The following lemma shows that the obvious direction of Lemma 5.11 holds for pregamps.

Lemma 5.12. *Let $\boldsymbol{B}$ be a pregamp, let m be a positive integer, let $x, y \in B$, and let $\vec{x}, \vec{y}$ be m-tuples of B. Assume that there are a positive integer n, a list $\vec{z}$ of parameters from B, and terms t_0, …, t_n such that the following equalities hold and* *all evaluations are defined*

$$x = t_0(\vec{x}, \vec{y}, \vec{z}),$$
$$y = t_n(\vec{x}, \vec{y}, \vec{z}),$$
$$t_j(\vec{y}, \vec{x}, \vec{z}) = t_{j+1}(\vec{x}, \vec{y}, \vec{z}), \quad (\textit{for all } j < n).$$

Then $\delta_{\boldsymbol{B}}(x, y) \leq \bigvee_{i<m} \delta_{\boldsymbol{B}}(x_i, y_i)$.

Proof. As $\delta_{\boldsymbol{B}}$ is compatible with terms (cf. Lemma 5.3), and $\delta_{\boldsymbol{B}}(u, u) = 0$ for each $u \in B$, the following inequality holds:

$$\delta_{\boldsymbol{B}}(t_j(\vec{x}, \vec{y}, \vec{z}), t_j(\vec{y}, \vec{x}, \vec{z})) \leq \bigvee_{k<n} \delta_{\boldsymbol{B}}(x_k, y_k), \quad \text{for all } j < n.$$

Hence:

$$\delta_{\boldsymbol{B}}(x, y) \leq \bigvee_{j<n} \delta_{\boldsymbol{B}}(t_j(\vec{x}, \vec{y}, \vec{z}), t_j(\vec{y}, \vec{x}, \vec{z})) \leq \bigvee_{k<n} \delta_{\boldsymbol{B}}(x_k, y_k). \qquad \square$$

The following definition expresses that whenever two elements of A are identified by a "congruence" of A, then there is a "good reason" for this in B (cf. Lemma 5.11).

Definition 5.13. A morphism $\boldsymbol{f}\colon \boldsymbol{A} \to \boldsymbol{B}$ of pregamps is *congruence-tractable* if for all $m < \omega$ and for all $x, y, x_0, y_0, \dots, x_{m-1}, y_{m-1}$ in A such that:

$$\delta_{\boldsymbol{B}}(x, y) \leq \bigvee_{k<m} \delta_{\boldsymbol{B}}(x_k, y_k),$$

there are a positive integer n, a list $\vec{z}$ of parameters from B, and terms $t_0, \dots, t_n$ such that the following equations are satisfied in B (in particular, all the corresponding terms are defined).

$$f(x) = t_0(f(\vec{x}), f(\vec{y}), \vec{z}),$$
$$f(y) = t_n(f(\vec{x}), f(\vec{y}), \vec{z}),$$
$$t_j(f(\vec{y}), f(\vec{x}), \vec{z}) = t_{j+1}(f(\vec{x}), f(\vec{y}), \vec{z}) \quad (\text{for all } j < n).$$

Lemma 5.14. *Let P be directed poset and let $\vec{\boldsymbol{A}} = (\boldsymbol{A}_p, \boldsymbol{f}_{p,q} \mid p \leq q \text{ in } P)$ be a direct system of pregamps. Assume that for each $p \in P$ there exists $q \geq p$ in P such that $\boldsymbol{f}_{p,q}$ is congruence-tractable. Let:*

$$(\boldsymbol{A}, \boldsymbol{f}_p \mid p \in P) = \varinjlim(\boldsymbol{A}_p, \boldsymbol{f}_{p,q} \mid p \leq q \text{ in } P), \quad \textit{in } \mathbf{PGamp}_{\mathscr{L}}.$$

If A is an algebra then the following statements hold:

(1) *Let $x, y \in A$, let $m < \omega$, let $x_0, y_0, \dots, x_{m-1}, y_{m-1}$ in A. The following two inequalities are equivalent:*

$$\delta_{\boldsymbol{A}}(x, y) \leq \bigvee_{k<m} \delta_{\boldsymbol{A}}(x_k, y_k), \tag{5.1}$$

$$\Theta_A(x, y) \leq \bigvee_{k<m} \Theta_A(x_k, y_k). \tag{5.2}$$

(2) *There exists a unique $(\vee, 0)$-homomorphism $\phi\colon \operatorname{Con_c} A \to \widetilde{A}$ such that:*

$$\phi(\Theta_A(x, y)) = \delta_{\boldsymbol{A}}(x, y), \quad \textit{for all } x, y \in A.$$

Moreover ϕ is an embedding.

(3) *If $\boldsymbol{A}$ is distance-generated, then the $(\vee,0)$-homomorphism ϕ above is an isomorphism.*

Proof. Lemma 5.6 and Lemma 4.9 imply that the following are directed colimits cocones

$$(A, f_p \mid p \in P) = \varinjlim(A_p, f_{p,q} \mid p \leq q \text{ in } P), \quad \text{in } \mathbf{PAlg}_{\mathscr{L}}. \tag{5.3}$$

$$(A, f_p \mid p \in P) = \varinjlim(A_p, f_{p,q} \mid p \leq q \text{ in } P), \quad \text{in } \mathbf{Set}. \tag{5.4}$$

$$(\widetilde{A}, \widetilde{f}_p \mid p \in P) = \varinjlim(\widetilde{A}_p, \widetilde{f}_{p,q} \mid p \leq q \text{ in } P), \quad \text{in } \mathbf{Sem}_{\vee,0}. \tag{5.5}$$

(1) Let $x, y \in A$, let $m < \omega$, and let $\vec{x}, \vec{y}$ be m-tuples of A.

Assume that (5.1) holds. It follows from (5.4) that there are $p \in P$, $x', y' \in A$, and m-tuples $\vec{x}', \vec{y}'$ of A_p, such that $x = f_p(x')$, $y = f_p(y')$, $\vec{x} = f_p(\vec{x}')$, and $\vec{y} = f_p(\vec{y}')$. The inequality (5.1) can be written

$$\delta_{\boldsymbol{A}}(f_p(x'), f_p(y')) \leq \bigvee_{k<m} \delta_{\boldsymbol{A}}(f_p(x'_k), f_p(y'_k)).$$

This implies:

$$\widetilde{f}_p\left(\delta_{\boldsymbol{A}_p}(x', y')\right) \leq \widetilde{f}_p\left(\bigvee_{k<m} \delta_{\boldsymbol{A}_p}(x'_k, y'_k)\right).$$

Hence, it follows from (5.5) that there exists $q \geq p$ with:

$$\widetilde{f}_{p,q}\left(\delta_{\boldsymbol{A}_p}(x', y')\right) \leq \widetilde{f}_{p,q}\left(\bigvee_{k<m} \delta_{\boldsymbol{A}_p}(x'_k, y'_k)\right),$$

so, changing p to q, x' to $f_{p,q}(x')$, y' to $f_{p,q}(y')$, $\vec{x}'$ to $f_{p,q}(\vec{x}')$, and $\vec{y}'$ to $f_{p,q}(\vec{y}')$, we can assume that:

$$\delta_{\boldsymbol{A}_p}(x', y') \leq \bigvee_{k<m} \delta_{\boldsymbol{A}_p}(x'_k, y'_k).$$

Let $q \geq p$ in P such that $f_{p,q}$ is congruence-tractable. There are a positive integer n, a list $\vec{z}$ of parameters from A_q, and terms $t_0, \ldots, t_n$ such that the following equations are satisfied in A_q:

$$\begin{aligned} f_{p,q}(x') &= t_0(f_{p,q}(\vec{x}'), f_{p,q}(\vec{y}'), \vec{z}), \\ f_{p,q}(y') &= t_n(f_{p,q}(\vec{x}'), f_{p,q}(\vec{y}'), \vec{z}), \\ t_k(f_{p,q}(\vec{y}'), f_{p,q}(\vec{x}'), \vec{z}) &= t_{k+1}(f_{p,q}(\vec{x}'), f_{p,q}(\vec{y}'), \vec{z}), \quad \text{(for all } k < n). \end{aligned}$$

Hence, applying f_q, we obtain

$$\begin{aligned} x &= t_0(\vec{x}, \vec{y}, f_q(\vec{z})), \\ y &= t_n(\vec{x}, \vec{y}, f_q(\vec{z})), \\ t_k(\vec{y}, \vec{x}, f_q(\vec{z})) &= t_{k+1}(\vec{x}, \vec{y}, f_q(\vec{z})), \quad \text{(for all } k < n). \end{aligned}$$

Therefore, it follows from Lemma 5.11 that (5.2) holds.

Conversely, assume that (5.2) holds. It follows from Lemma 5.11 that there are a positive integer n, a list $\vec{z}$ of parameters from A, and terms $t_0, \ldots, t_n$ such that

$$\begin{aligned} x &= t_0(\vec{x}, \vec{y}, \vec{z}), \\ y &= t_n(\vec{x}, \vec{y}, \vec{z}), \\ t_j(\vec{y}, \vec{x}, \vec{z}) &= t_{j+1}(\vec{x}, \vec{y}, \vec{z}), \quad \text{(for all } j < n). \end{aligned}$$

We conclude, using Lemma 5.12, that (5.1) holds.

As $\mathrm{Con_c}\, A$ is generated by $\{\Theta_A(x,y) \mid x,y \in A\}$, the statement (2) follows from Proposition 3.1. Moreover if we assume that $\boldsymbol{A}$ is distance-generated, that is $\widetilde{A}$ is join-generated by $\{\delta_{\boldsymbol{A}}(x,y) \mid x,y \in A\}$, then ϕ is an isomorphism. □

As an immediate application, we obtain that a "true" directed colimit of "good" pregamps is an algebra together with its congruences.

Corollary 5.15. *Let P be directed poset with no maximal element and let $\vec{\boldsymbol{A}} = (\boldsymbol{A}_p, \boldsymbol{f}_{p,q} \mid p \leq q \text{ in } P)$ be a P-indexed diagram of distance-generated pregamps. If $\boldsymbol{f}_{p,q}$ is congruence-tractable and $f_{p,q}$ is strong for all $p < q$ in P, then there exists a unique $(\vee, 0)$-homomorphism $\phi\colon \mathrm{Con_c}\, A \to \widetilde{A}$ such that:*

$$\phi(\Theta_A(x,y)) = \delta_{\boldsymbol{A}}(x,y) \quad \text{for all } x,y \in A.$$

Moreover, ϕ is an isomorphism.

Definition 5.16. An *ideal* of a pregamp $\boldsymbol{A}$ is an ideal of $\widetilde{A}$. Denote by $\operatorname{Id} \boldsymbol{A} = \operatorname{Id} \widetilde{A}$ the set of all ideals of $\boldsymbol{A}$.

Let P be a poset, let $\vec{\boldsymbol{A}} = (\boldsymbol{A}_p, \boldsymbol{f}_{p,q} \mid p \leq q \text{ in } P)$ be a P-indexed diagram in $\mathbf{PGamp}_{\mathscr{L}}$. An *ideal* of $\vec{\boldsymbol{A}}$ is an ideal of $(\widetilde{A}_p, \widetilde{f}_{p,q} \mid p \leq q \text{ in } P)$ (cf. Definition 3.2).

Definition 5.17. Let $\boldsymbol{\pi}\colon \boldsymbol{A} \to \boldsymbol{B}$ be a morphism of pregamps. The 0-*kernel of* $\boldsymbol{\pi}$, denoted by $\ker_0 \boldsymbol{\pi}$, is the 0-kernel of $\widetilde{\pi}$ (cf. Definition 3.2).

Let P be a poset and let $\vec{\boldsymbol{\pi}} = (\boldsymbol{\pi}_p)_{p \in P}\colon \vec{\boldsymbol{A}} \to \vec{\boldsymbol{B}}$ be a natural transformation of P-indexed diagrams of pregamps. The 0-*kernel of* $\vec{\boldsymbol{\pi}}$ is $\vec{I} = (\ker_0 \boldsymbol{\pi}_p)_{p \in P}$.

Remark 5.18. The 0-kernel of $\boldsymbol{\pi}$ is an ideal of $\boldsymbol{A}$. Similarly the 0-kernel of $\vec{\boldsymbol{\pi}}$ is an ideal of $\vec{\boldsymbol{A}}$.

If $\pi\colon A \to B$ is a morphism of algebras, then $\ker_0 \boldsymbol{P}_{\mathrm{ga}}(\pi)$ is the set of all compact congruences of A below $\ker \pi$, that is, $\ker_0 \boldsymbol{P}_{\mathrm{ga}}(\pi) = (\mathrm{Con_c}\, A) \downarrow \ker \pi$.

Definition 5.19. A morphism $\boldsymbol{f}\colon \boldsymbol{A} \to \boldsymbol{B}$ of pregamps, is *ideal-induced* if $f(A) = B$ as partial algebras and $\widetilde{f}$ is ideal-induced. In that case we say that $\boldsymbol{B}$ is an *ideal-induced image of* $\boldsymbol{A}$.

Let P be a poset, let $\vec{\boldsymbol{A}}$ and $\vec{\boldsymbol{B}}$ be P-indexed diagrams of pregamps. A natural transformation $\vec{\boldsymbol{f}} = (\boldsymbol{f}_p)_{p \in P}\colon \vec{\boldsymbol{A}} \to \vec{\boldsymbol{B}}$ is *ideal-induced* if $\boldsymbol{f}_p$ is ideal-induced for each $p \in P$.

Remark 5.20. A morphism $\boldsymbol{f}\colon \boldsymbol{A} \to \boldsymbol{B}$ of pregamps is ideal-induced if $\widetilde{f}$ is ideal-induced, f is surjective, and for each $\ell \in \mathscr{L}$ and each tuple $\vec{b}$ of B, $\ell(\vec{b})$ is defined in B if and only if there exists a tuple $\vec{a}$ in A such that $\vec{b} = f(\vec{a})$ and $\ell(\vec{a})$ is defined in A.

If $f\colon A \twoheadrightarrow B$ is a surjective morphism of algebras, then $\boldsymbol{P}_{\mathrm{ga}}(f)$ is ideal-induced.

If $\boldsymbol{f}\colon \boldsymbol{A} \to \boldsymbol{B}$ and $\boldsymbol{g}\colon \boldsymbol{B} \to \boldsymbol{C}$ are ideal-induced morphisms of pregamps, then $\boldsymbol{g} \circ \boldsymbol{f}$ is ideal-induced.

The following proposition gives a description of *quotients* of pregamps.

Proposition 5.21. *Let $\boldsymbol{A}$ be a pregamp and let $I \in \operatorname{Id} \widetilde{A}$. The binary relation $\theta_I = \{(x,y) \in A^2 \mid \delta_{\boldsymbol{A}}(x,y) \in I\}$ is an equivalence relation on A. Given $a \in A$ denote by a/I the θ_I-equivalence class containing a, and set $A/I = A/\theta_I$. We can* *define a structure of partial algebra on A/I in the following way. Given $\ell \in \mathscr{L}$, we put:*

$$\operatorname{Def}_\ell(A/I) = \{\vec{x}/I \mid \vec{x} \in \operatorname{Def}_\ell(A)\},$$
$$\ell^{A/I}(\vec{x}/I) = \ell^A(\vec{x})/I, \quad \text{for all } \vec{x} \in \operatorname{Def}_\ell(A).$$

Moreover $\delta_{\boldsymbol{A}/I} \colon (A/I)^2 \to \widetilde{A}/I$, $(x/I, y/I) \mapsto \delta_{\boldsymbol{A}}(x,y)/I$ defines an $\widetilde{A}/I$-valued partial algebra distance, and the following statements hold:

1. *$\boldsymbol{A}/I = (A/I, \delta_{\boldsymbol{A}/I}, \widetilde{A}/I)$ is a pregamp.*
2. *Put $\pi \colon A \to A/I$, $x \mapsto x/I$, and denote by $\widetilde{\pi} \colon \widetilde{A} \to \widetilde{A}/I$, $d \mapsto d/I$ the canonical projection. Then $\boldsymbol{\pi} = (\pi, \widetilde{\pi})$ is an ideal-induced morphism of pregamps from $\boldsymbol{A}$ to $\boldsymbol{A}/I$.*
3. *The 0-kernel of $\boldsymbol{\pi}$ is I.*
4. *If $\boldsymbol{A}$ is distance-generated, then $\boldsymbol{A}/I$ is distance-generated.*

Proof. The relation θ_I is reflexive (it follows from Definition 5.1(1)), symmetric (see Definition 5.1(2)) and transitive (see Definition 5.1(3)), thus it is an equivalence relation.

Let $\ell \in \mathscr{L}$, let $\vec{x}, \vec{y} \in \operatorname{Def}_\ell(A)$ such that $x_k/I = y_k/I$ for each $k < \operatorname{ar}(\ell)$. It follows from Definition 5.1(4) that $\delta_{\boldsymbol{A}}(\ell^A(\vec{x}), \ell^A(\vec{y})) \leq \bigvee_{k<\operatorname{ar}(\ell)} \delta_{\boldsymbol{A}}(x_k, y_k) \in I$, so $\ell^A(\vec{x})/I = \ell^A(\vec{y})/I$. Therefore the partial operation $\ell^{A/I} \colon \operatorname{Def}_\ell(A/I) \to A/I$ is well-defined.

Let $x, x', y, y' \in A$, assume that $x/I = x'/I$ and $y/I = y'/I$. The following inequality holds:

$$\delta_{\boldsymbol{A}}(x,y) \leq \delta_{\boldsymbol{A}}(x,x') \vee \delta_{\boldsymbol{A}}(x',y') \vee \delta_{\boldsymbol{A}}(y',y).$$

However, $\delta_{\boldsymbol{A}}(x,x')$ and $\delta_{\boldsymbol{A}}(y,y')$ both belong to I, hence $\delta_{\boldsymbol{A}}(x,y)/I \leq \delta_{\boldsymbol{A}}(x',y')/I$. Similarly $\delta_{\boldsymbol{A}}(x',y')/I \leq \delta_{\boldsymbol{A}}(x,y)/I$. So the map $\delta_{\boldsymbol{A}/I} \colon (A/I)^2 \to \widetilde{A}/I$ is well-defined.

Let $x, y \in A$, the following equivalences hold:

$$\delta_{\boldsymbol{A}/I}(x/I, y/I) = 0/I \iff \delta_{\boldsymbol{A}}(x,y) \in I \iff x/I = y/I.$$

That is, Definition 5.1(1) holds. Each of the conditions of Definition 5.1(2)-(5) for $\delta_{\boldsymbol{A}}$ implies its analogue for $\delta_{\boldsymbol{A}/I}$.

It is easy to check that $\boldsymbol{\pi}$ is well-defined and that it is a morphism of pregamps. □

Notation 5.22. The notations $\boldsymbol{A}/I$, A/I, and $\delta_{\boldsymbol{A}/I}$ used in Proposition 5.21 will be used throughout the paper. The map $\boldsymbol{\pi}$ is *the canonical projection*.

If $I = \{0\}$, we identify $\boldsymbol{A}/I$ and $\boldsymbol{A}$.

If X is a partial subalgebra of A, then we denote $X/I = \{x/I \mid x \in X\}$, with its natural structure of partial subalgebra of A, inherited from X, with $\operatorname{Def}_\ell(X/I) = \{\vec{x}/I \mid \vec{x} \in \operatorname{Def}_\ell(X)\}$ for each $\ell \in \mathscr{L}$. That is $X/I = \pi(X)$ as partial algebras.

Let $\boldsymbol{B}$ be a sub-pregamp of $\boldsymbol{A}$ and let I be a common ideal of $\boldsymbol{A}$ and $\boldsymbol{B}$. Then we identify the quotient $\boldsymbol{B}/I$ with the corresponding sub-pregamp of $\boldsymbol{A}/I$.

Remark. It is easy to construct a pregamp $\boldsymbol{A}$, a term t, a tuple $\vec{x}$ of A, and an ideal I of $\widetilde{A}$, such that $t(\vec{x})$ is not defined in A, but $t(\vec{x}/I)$ is defined in A/I.

The following proposition gives a description of how morphisms of pregamps factorize through quotients. It is related to Lemma 3.5.

Proposition 5.23. *Let $\boldsymbol{f}\colon \boldsymbol{A}\to\boldsymbol{B}$ be a morphism of pregamps, let $I\in\operatorname{Id}\boldsymbol{A}$, and let $J\in\operatorname{Id}\boldsymbol{B}$. If $\widetilde{f}(I)\subseteq J$, then the following maps are well-defined:*

$$\begin{aligned} g\colon A/I &\to B/J\\ x/I &\mapsto f(x)/J,\end{aligned}$$

$$\begin{aligned} \widetilde{g}\colon \widetilde{A}/I &\to \widetilde{B}/J\\ \alpha/I &\mapsto \widetilde{f}(\alpha)/J.\end{aligned}$$

Moreover, $\boldsymbol{g}=(g,\widetilde{g})$ is a morphism of pregamps from $\boldsymbol{A}/I$ to $\boldsymbol{B}/J$. If $\boldsymbol{\pi}_I\colon\boldsymbol{A}\to\boldsymbol{A}/I$ and $\boldsymbol{\pi}_J\colon\boldsymbol{A}\to\boldsymbol{A}/I$ denote the canonical projections, then the following diagram commutes:

$$\begin{array}{ccc} \boldsymbol{A} & \xrightarrow{\ \boldsymbol{f}\ } & \boldsymbol{B}\\ {\scriptstyle\boldsymbol{\pi}_I}\big\downarrow & & \big\downarrow{\scriptstyle\boldsymbol{\pi}_J}\\ \boldsymbol{A}/I & \xrightarrow[\ \boldsymbol{g}\ ]{} & \boldsymbol{B}/J \end{array}$$

Proof. Observe that $\widetilde{g}\colon\widetilde{A}/I\to\widetilde{B}/J$ is the $(\vee,0)$-homomorphism induced by $\widetilde{f}$. Let $x,y\in A$ such that $x/I=y/I$, that is, $\delta_{\boldsymbol{A}}(x,y)\in I$. It follows that $\delta_{\boldsymbol{B}}(f(x),f(y))=\widetilde{f}(\delta_{\boldsymbol{A}}(x,y))\in J$, so $f(x)/J=f(y)/J$. Therefore the map g is well-defined.

Let $\ell\in\mathscr{L}$, let $\vec{a}\in\operatorname{Def}_\ell(A/I)$, and let $\vec{x}\in\operatorname{Def}_\ell(A)$ such that $\vec{a}=\vec{x}/I$. The following equalities hold:

$$g(\ell(\vec{x}/I))=g(\ell(\vec{x})/I)=f(\ell(\vec{x}))/J=\ell(f(\vec{x}))/J=\ell(f(\vec{x})/J)=\ell(g(\vec{x}/I)).$$

Thus $g(\ell(\vec{a}))=\ell(g(\vec{a}))$. Therefore g is a morphism of partial algebras.

Let $x,y\in A$. It is easy to check $\widetilde{g}(\delta_{\boldsymbol{A}/I}(x/I,y/I))=\delta_{\boldsymbol{B}/J}(g(x/J),g(y/J))$. Therefore $\boldsymbol{g}\colon\boldsymbol{A}/I\to\boldsymbol{B}/J$ is a morphism of pregamps. Moreover $\boldsymbol{\pi}_J\circ\boldsymbol{f}=\boldsymbol{g}\circ\boldsymbol{\pi}_I$ is obvious. □

Notation 5.24. We say that $\boldsymbol{f}$ *induces* $\boldsymbol{g}\colon\boldsymbol{A}/I\to\boldsymbol{B}/J$, the morphism of Proposition 5.23.

Let P be a poset, let $\vec{\boldsymbol{A}}=(\boldsymbol{A}_p,\boldsymbol{f}_{p,q}\mid p\le q\text{ in }P)$ be a P-indexed diagram in $\mathbf{PGamp}_{\mathscr{L}}$, let $\vec{I}=(I_p)_{p\in P}$ be an ideal of $\vec{\boldsymbol{A}}$, and let $\boldsymbol{g}_{p,q}\colon\boldsymbol{A}_p/I_p\to\boldsymbol{A}_q/I_q$ the morphism induced by $\boldsymbol{f}_{p,q}$, for all $p\le q$ in P. We denote by $\vec{\boldsymbol{A}}/\vec{I}=(\boldsymbol{A}_p/I_p,\boldsymbol{g}_{p,q}\mid p\le q\text{ in }P)$.

The diagram $\vec{\boldsymbol{A}}/\vec{I}$ is a *quotient* of $\vec{\boldsymbol{A}}$.

Remark 5.25. It is easy to check that $\vec{\boldsymbol{A}}/\vec{I}$ is indeed a diagram. Given $p\le q\le r$ in P and $x\in A_p$ the following equalities hold:

$$g_{q,r}(g_{p,q}(x/I_p))=g_{q,r}(f_{p,q}(x)/I_q)=f_{q,r}(f_{p,q}(x))/I_r=f_{p,r}(x)/I_r=g_{p,r}(x/I_p).$$

Proposition 5.23 can be easily extended to diagrams in the following way. It is also related to Lemma 3.7.

Proposition 5.26. *Let P be a poset, let $\vec{\boldsymbol{A}}=(\boldsymbol{A}_p,\boldsymbol{f}_{p,q}\mid p\le q$ in $P)$ and $\vec{\boldsymbol{B}}=(\boldsymbol{B}_p,\boldsymbol{g}_{p,q}\mid p\le q$ in $P)$ be P-indexed diagrams in $\mathbf{PGamp}_{\mathscr{L}}$. Let $\vec{I}$ be an ideal of $\vec{\boldsymbol{A}}$, let $\vec{J}$ be an ideal of $\vec{\boldsymbol{B}}$. Let $\vec{\boldsymbol{\xi}}=(\boldsymbol{\xi}_p)_{p\in P}\colon\vec{\boldsymbol{A}}\to\vec{\boldsymbol{B}}$ be a natural transformation such that $\xi_p(I_p)\subseteq J_p$ for each $p\in P$. Denote by $\boldsymbol{\chi}_p\colon\boldsymbol{A}_p/I_p\to\boldsymbol{B}_p/J_p$ the morphism* *induced by $\boldsymbol{\xi}_p$, for each $p \in P$. Then $\vec{\boldsymbol{\chi}} = (\boldsymbol{\chi}_p)_{p \in P}$ is a natural transformation from $\vec{\boldsymbol{A}}/\vec{I}$ to $\vec{\boldsymbol{B}}/\vec{J}$.*

Notation 5.27. With the notation of Proposition 5.26. We say that $\vec{\boldsymbol{\chi}}\colon \vec{\boldsymbol{A}}/\vec{I} \to \vec{\boldsymbol{B}}/\vec{J}$ is *induced by* $\vec{\boldsymbol{\xi}}$.

The following lemma expresses that ideal-induced images of pregamps correspond, up to isomorphism, to quotients of pregamps. It is related to Lemma 3.13.

Lemma 5.28. *Let $\boldsymbol{f}\colon \boldsymbol{A} \to \boldsymbol{B}$ be a morphism of pregamps. The following statements are equivalent:*

(1) *$\boldsymbol{f}$ is ideal-induced.*
(2) *$\boldsymbol{f}$ induces an isomorphism $\boldsymbol{g}\colon \boldsymbol{A}/\ker_0 \boldsymbol{f} \to \boldsymbol{B}$.*

Proof. Denote by $\boldsymbol{\pi}\colon \boldsymbol{A} \to \boldsymbol{A}/\ker_0 \boldsymbol{f}$ the canonical projection, so $\boldsymbol{g} \circ \boldsymbol{\pi} = \boldsymbol{f}$.

Assume that $\boldsymbol{f}$ is ideal-induced. As $\widetilde{f}\colon \widetilde{A} \to \widetilde{B}$ is ideal-induced, Lemma 3.13 implies that $\widetilde{f}$ induces an isomorphism $\widetilde{g}\colon \widetilde{A}/\ker_0 \boldsymbol{f} \to \widetilde{B}$. It follows that $\widetilde{g}$ separates 0, thus (cf. Proposition 5.10(2)) g is an embedding. Moreover $g(A/\ker_0 \boldsymbol{f}) = g(\pi(A)) = f(A) = B$ as partial algebras. Therefore it follows from Lemma 4.6 that g is an isomorphism of partial algebras, thus $\boldsymbol{g}$ is an isomorphism of pregamps (cf. Remark 5.5).

Assume that $\boldsymbol{g}$ is an isomorphism. It follows that $\widetilde{g}$ is an isomorphism, so Lemma 3.13 implies that $\widetilde{f}$ is ideal-induced. Moreover g is an isomorphism, thus $f(A) = g(\pi(A)) = g(A/\ker_0 \boldsymbol{f}) = B$ as partial algebras. Therefore $\boldsymbol{f}$ is ideal-induced. □

The following proposition expresses that a quotient of a quotient is a quotient. It follows from Lemma 5.28, together with the fact that a composition of ideal-induced morphisms of pregamps is ideal-induced.

Proposition 5.29. *Let $\boldsymbol{A}$ be a pregamp, let I be an ideal of $\boldsymbol{A}$, let J be an ideal of $\boldsymbol{A}/I$. Then $(\boldsymbol{A}/I)/J$ is isomorphic to a quotient of $\boldsymbol{A}$.*

The following results expresses that, up to isomorphism, quotients of sub-pregamps are sub-pregamps of quotients.

Proposition 5.30. *Let $\boldsymbol{A}$ be a pregamp, let $\boldsymbol{B}$ be a sub-pregamp of $\boldsymbol{A}$, and let $I \in \operatorname{Id} \boldsymbol{B}$. Then there exist $J \in \operatorname{Id} \boldsymbol{A}$, a sub-pregamp $\boldsymbol{C}$ of $\boldsymbol{A}/J$, and an isomorphism $\boldsymbol{f}\colon \boldsymbol{B}/I \to \boldsymbol{C}$.*

Let $\boldsymbol{A}$ be a pregamp, let $I \in \operatorname{Id} \boldsymbol{A}$, and let $\boldsymbol{B}$ be a sub-pregamp of $\boldsymbol{A}/I$. There exists a sub-pregamp $\boldsymbol{C}$ of $\boldsymbol{A}$ such that $\boldsymbol{B}$ is isomorphic to some quotient of $\boldsymbol{C}$.

Proof. Let $\boldsymbol{A}$ be a pregamp, let $\boldsymbol{B}$ be a sub-pregamp of $\boldsymbol{A}$, let $I \in \operatorname{Id} \boldsymbol{B}$. Put $J = \widetilde{A} \downarrow I$. As I is an ideal of $\widetilde{B}$, it is directed, therefore J is an ideal of $\widetilde{A}$.

Let $\boldsymbol{f}\colon \boldsymbol{B} \to \boldsymbol{A}$ be the canonical embedding. Notice that $\widetilde{f}(I) \subseteq J$; denote by $\boldsymbol{g}\colon \boldsymbol{B}/I \to \boldsymbol{A}/J$ the morphism induced by $\boldsymbol{f}$ (cf. Proposition 5.23).

Let $d, d' \in \widetilde{B}$ such that $\widetilde{g}(d/I) = \widetilde{g}(d'/I)$, that is, $d/J = d'/J$, so there exists $u \in J$ such that $d \vee u = d' \vee u$. As $J = \widetilde{A} \downarrow I$, there exists $v \in I$ such that $u \leq v$, hence $d \vee v = d' \vee v$, that is, $d/I = d'/I$. Therefore $\widetilde{g}$ is an embedding. It follows from Proposition 5.10 that $\boldsymbol{g} = (g, \widetilde{g})$ is an embedding and induces an isomorphism $\boldsymbol{B}/I \to \boldsymbol{g}(\boldsymbol{B}/I)$; the latter is a sub-pregamp of $\boldsymbol{A}/J$.

Now let $I \in \operatorname{Id} \boldsymbol{A}$ and let $\boldsymbol{B}$ be a sub-pregamp of $\boldsymbol{A}/I$. Denote by $\boldsymbol{\pi}\colon \boldsymbol{A} \to \boldsymbol{A}/I$ the canonical projection, put $\boldsymbol{C} = \boldsymbol{\pi}^{-1}(\boldsymbol{B})$ (cf. Notation 5.9). As $\boldsymbol{\pi}$ is ideal-induced, it is easy to check that $\boldsymbol{\pi}(\boldsymbol{C}) = \boldsymbol{B}$, and the restriction $\boldsymbol{\pi} \restriction \boldsymbol{C} \to \boldsymbol{B}$ is ideal-induced. □

The following lemma, in conjunction with Lemma 3.14, proves that, given a direct system $\vec{\boldsymbol{A}}$ of pregamps, every quotient of the colimit of $\vec{\boldsymbol{A}}$ is the colimit of a quotient of $\vec{\boldsymbol{A}}$.

Lemma 5.31. *Let P be directed poset and let $\vec{\boldsymbol{A}} = (\boldsymbol{A}_p, \boldsymbol{f}_{p,q} \mid p \leq q \text{ in } P)$ be a P-indexed diagram in $\mathbf{PGamp}_{\mathscr{L}}$. Let $(\boldsymbol{A}, \boldsymbol{f}_p \mid p \in P) = \varinjlim(\boldsymbol{A}_p, \boldsymbol{f}_{p,q} \mid p \leq q \text{ in } P)$ be a directed colimit cocone in $\mathbf{PGamp}_{\mathscr{L}}$. Let $\vec{I} = (I_p)_{p\in P}$ be an ideal of $\vec{\boldsymbol{A}}$. Then $I = \bigcup_{p\in P} \widetilde{f}_p(I_p)$ is an ideal of $\boldsymbol{A}$.*

Let $\boldsymbol{g}_p\colon \boldsymbol{A}_p/I_p \to \boldsymbol{A}/I$ be the morphism induced by $\boldsymbol{f}_p$, let $\boldsymbol{g}_{p,q}\colon \boldsymbol{A}_p/I_p \to \boldsymbol{A}_q/I_q$ be the morphism induced by $\boldsymbol{f}_{p,q}$, for all $p \leq q$ in P. The following is a directed colimit cocone:

$$(\boldsymbol{A}/I, \boldsymbol{g}_p \mid p \in P) = \varinjlim(\boldsymbol{A}_p/I_p, \boldsymbol{f}_{p,q} \mid p \leq q \text{ in } P) \text{ in } \mathbf{PGamp}_{\mathscr{L}}.$$

Proof. Lemma 4.9 and Lemma 5.6 imply that the following are colimits cocones:

$$(A, f_p \mid p \in P) = \varinjlim(A_p, f_{p,q} \mid p \leq q \text{ in } P) \text{ in } \mathbf{Set}, \tag{5.6}$$

$$(\widetilde{A}, \widetilde{f}_p \mid p \in P) = \varinjlim(\widetilde{A}_p, \widetilde{f}_{p,q} \mid p \leq q \text{ in } P) \text{ in } \mathbf{Sem}_{\vee,0} \tag{5.7}$$

$$(A, f_p \mid p \in P) = \varinjlim(A_p, f_{p,q} \mid p \leq q \text{ in } P) \text{ in } \mathbf{PAlg}_{\mathscr{L}} \tag{5.8}$$

Moreover, Lemma 3.14 implies that I is an ideal of $\widetilde{A}$ and that the following is a directed colimit cocone:

$$(\widetilde{A}/I, \widetilde{g}_p \mid p \in P) = \varinjlim(\widetilde{A}_p/I_p, \widetilde{g}_{p,q} \mid p \leq q \text{ in } P) \text{ in } \mathbf{Sem}_{\vee,0} \tag{5.9}$$

Let $p \in P$, let $x, y \in A_p$ such that $g_p(x/I_p) = g_p(y/I_p)$. It follows that $f_p(x)/I = f_p(y)/I$, that is, $\delta_{\boldsymbol{A}}(f_p(x), f_p(y)) \in I$. So there exist $q \in P$ and $\alpha \in \widetilde{A}_q$ such that $\widetilde{f}_p(\delta_{\boldsymbol{A}_p}(x, y)) = \delta_{\boldsymbol{A}}(f_p(x), f_p(y)) = \widetilde{f}_q(\alpha)$. It follows from (5.7) that there exists $r \geq p, q$ such that $\delta_{\boldsymbol{A}_r}(f_{p,r}(x), f_{p,r}(y)) = \widetilde{f}_{p,r}(\delta_{\boldsymbol{A}_p}(x, y)) = \widetilde{f}_{q,r}(\alpha)$. However, $\widetilde{f}_{q,r}(\alpha) \in \widetilde{f}_{q,r}(I_q) \subseteq I_r$, so $f_{p,r}(x)/I_r = f_{p,r}(y)/I_r$, and so $g_{p,r}(x/I_p) = g_{p,r}(y/I_p)$. Moreover $A/I = \bigcup_{p\in P} f_p(A_p)/I = \bigcup_{p\in P} g_p(A_p/I_p)$. Hence the following is a directed colimit cocone:

$$(A/I, g_p \mid p \in P) = \varinjlim(A_p/I_p, g_{p,q} \mid p \leq q \text{ in } P) \text{ in } \mathbf{Set}.$$

Let $\ell \in \mathscr{L}$. The following equalities hold:

$$\operatorname{Def}_\ell(A/I) = \operatorname{Def}_\ell(A)/I = \bigcup_{p\in P} f_p(\operatorname{Def}_\ell(A_p))/I = \bigcup_{p\in P} g_p(\operatorname{Def}_\ell(A_p/I_p)).$$

Let $p \in P$, let $\vec{a} \in \operatorname{Def}_\ell(A_p/I)$. Then, as g_p is a morphism of partial algebras, $g_p(\ell(\vec{a})) = \ell(g_p(\vec{a}))$. So, by Lemma 4.9, the following is a directed colimit cocone:

$$(A/I, g_p \mid p \in P) = \varinjlim(A_p/I_p, g_{p,q} \mid p \leq q \text{ in } P) \text{ in } \mathbf{PAlg}_{\mathscr{L}}.$$

As $\boldsymbol{f}_p$ is a morphism of pregamps, $\delta_{\boldsymbol{A}/I}(g_p(x), g_p(y)) = \widetilde{g}_p(\delta_{\boldsymbol{A}_p/I_p}(x, y))$ for all $p \in P$ and all $x, y \in A_p/I_p$, thus Lemma 5.6 implies that the following is a directed colimit cocone:

$$(\boldsymbol{A}/I, \boldsymbol{g}_p \mid p \in P) = \varinjlim(\boldsymbol{A}_p/I_p, \boldsymbol{g}_{p,q} \mid p \leq q \text{ in } P) \text{ in } \mathbf{PGamp}_{\mathscr{L}}.$$ □

Definition 5.32. A pregamp $\boldsymbol{A}$ *satisfies* an identity $t_1 = t_2$ if A/I satisfies $t_1 = t_2$ for each $I \in \operatorname{Id} \boldsymbol{A}$.

Let $\mathcal{V}$ be a variety of algebras. A pregamp $\boldsymbol{A}$ is a *pregamp of* $\mathcal{V}$ if it satisfies all identities of $\mathcal{V}$.

Remark. It is not hard to construct a pregamp $\boldsymbol{A}$, an identity $t_1 = t_2$, and an ideal I of $\widetilde{A}$ such that A satisfies $t_1 = t_2$, but A/I fails $t_1 = t_2$.

A pregamp $\boldsymbol{A}$ satisfies an identity $t_1 = t_2$ if and only if for each ideal-induced morphism $\boldsymbol{f} \colon \boldsymbol{A} \to \boldsymbol{B}$ of pregamps, the partial algebra B satisfies $t_1 = t_2$.

Definition 5.33. Let $\mathcal{V}$ be a variety of algebras. *The category of pregamps of* $\mathcal{V}$, denoted by $\mathbf{PGamp}(\mathcal{V})$, is the full subcategory of $\mathbf{PGamp}_{\mathscr{L}}$ in which the objects are all the pregamps of $\mathcal{V}$.

As an immediate application of Lemma 5.31 and Lemma 3.14, we obtain that the class of all pregamps that satisfy a given identity is closed under directed colimits.

Corollary 5.34. *Let $\mathcal{V}$ be a variety of algebras and let P be a directed poset. Let $\vec{\boldsymbol{A}} = (\boldsymbol{A}_p, \boldsymbol{f}_{p,q} \mid p \leq q \text{ in } P)$ be a P-indexed diagram in $\mathbf{PGamp}(\mathcal{V})$. Let $(\boldsymbol{A}, \boldsymbol{f}_p \mid p \in P) = \varinjlim(\boldsymbol{A}_p, \boldsymbol{f}_{p,q} \mid p \leq q \text{ in } P)$ be a directed colimit cocone in $\mathbf{PGamp}_{\mathscr{L}}$. Then $\boldsymbol{A}$ is a pregamp of $\mathcal{V}$.*

Similarly, it follows from Proposition 5.29 and Proposition 5.30 that the class of all pregamps that satisfy a given identity is closed under ideal-induced images and sub-pregamps.

Corollary 5.35. *Let $\boldsymbol{A}$ be a pregamp of a variety $\mathcal{V}$, let $\boldsymbol{B}$ be a pregamp, let $\boldsymbol{f} \colon \boldsymbol{A} \to \boldsymbol{B}$ be an ideal-induced morphism of pregamps, then $\boldsymbol{B}$ is a pregamp of $\mathcal{V}$. Furthermore, every sub-pregamp of $\boldsymbol{A}$ is a pregamp of $\mathcal{V}$.*

6. Gamps

A *gamp* of a variety $\mathcal{V}$ is a pregamp that "belongs" to $\mathcal{V}$ (cf. (1)), together with a partial subalgebra (cf. (2)). The main interest of this new notion is to express later some additional properties that reflect properties of algebras (cf. Definition 6.3). It is a generalization of the notion of a *semilattice-metric cover as defined in* [5, Section 5-1].

Definition 6.1. Let $\mathcal{V}$ be a variety of $\mathscr{L}$-algebras. A *gamp* (resp., a *gamp of* $\mathcal{V}$) is a quadruple $\boldsymbol{A} = (A^*, A, \delta_{\boldsymbol{A}}, \widetilde{A})$ such that

1. $(A, \delta_{\boldsymbol{A}}, \widetilde{A})$ is a pregamp (resp., a pregamp of $\mathcal{V}$) (cf. Definitions 5.1 and 5.32).
2. A^* is a partial subalgebra of A.

A *realization* of $\boldsymbol{A}$ is an ordered pair (A', χ) such that $A' \in \mathcal{V}$, A is a partial subalgebra of A', $\chi \colon \widetilde{A} \to \operatorname{Con_c} A'$ is a $(\vee, 0)$-embedding, and $\chi(\delta_{\boldsymbol{A}}(x, y)) = \Theta_{A'}(x, y)$ for all $x, y \in A$. A realization is *isomorphic* if χ is an isomorphism.

A gamp $\boldsymbol{A}$ is *finite* if both A and $\widetilde{A}$ are finite.

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be gamps. A morphism $\boldsymbol{f} \colon (A, \delta_{\boldsymbol{A}}, \widetilde{A}) \to (B, \delta_{\boldsymbol{B}}, \widetilde{B})$ of pregamps is a *morphism of gamps* from $\boldsymbol{A}$ to $\boldsymbol{B}$ if $f(A^*)$ is a partial subalgebra of B^*.

The *category of gamps of* $\mathcal{V}$, denoted by $\mathbf{Gamp}(\mathcal{V})$, is the category in which the objects are the gamps of $\mathcal{V}$ and the arrows are the morphisms of gamps.

A *subgamp* of a gamp $\boldsymbol{A}$ is a gamp $\boldsymbol{B} = (B^*, B, \delta_{\boldsymbol{B}}, \widetilde{B})$ such that B^* is a partial subalgebra of A^*, B is a partial subalgebra of A, $\delta_{\boldsymbol{B}} = \delta_{\boldsymbol{A}} \restriction B^2$, and $\widetilde{B}$ is a $(\vee, 0)$-subsemilattice of $\widetilde{A}$. Let $f \colon B \to A$ and $\widetilde{f} \colon \widetilde{B} \to \widetilde{A}$ be the inclusion maps. The ordered pair $(f, \widetilde{f})$ is a morphism of gamps from $\boldsymbol{B}$ to $\boldsymbol{A}$, called *the canonical embedding*.

Remark. A gamp might have no realization. A realization of a finite gamp does not need to be finite.

Let $\boldsymbol{f} \colon \boldsymbol{A} \to \boldsymbol{B}$ be a morphism of gamps, let (A', χ) be a realization of $\boldsymbol{A}$, and let (B', ξ) be a realization of $\boldsymbol{B}$. There might not exist any morphism $g \colon A' \to B'$.

Definition 6.2. A *gamp of lattices* is a gamp of the variety of all lattices.

Let $\boldsymbol{B}$ be a gamp of lattices. A *chain* of $\boldsymbol{B}$ is a sequence $x_0, x_1, \dots, x_{n-1}$ of B^* such that $x_i \wedge x_j = x_i$ in B for all $i \leq j < n$. We sometime denote such a chain as $x_0 \leq x_1 \leq \cdots \leq x_{n-1}$. If $x_i \neq x_j$ for all $i < j < n$, we denote the chain as $x_0 < x_1 < \cdots < x_{n-1}$.

Let $u < v$ be a chain of $\boldsymbol{B}$, we say that v is a *cover* of u, and then we write $u \prec v$, if there is no chain $u < x < v$ in $\boldsymbol{B}$.

The following properties for a gamp come from algebra. It follows from Definition 6.3(1) that there are many operations defined in A. With (2) or (3) all "congruences" have a set of "generators". Condition (4) expresses that whenever two elements are identified by a "congruence" of A^*, then there is a "good reason" for this in A (cf. Lemma 5.11). Conditions (6) and (7) are related to the transitive closure of relations. Condition (8) is related to congruence n-permutability (cf. Proposition 2.1).

Definition 6.3. A gamp $\boldsymbol{A}$ is *strong* if the following holds:

(1) A^* is a strong partial subalgebra of A (cf. Definition 4.4).

A gamp $\boldsymbol{A}$ is *distance-generated* if it satisfies the following condition:

(2) Every element of $\widetilde{A}$ is a finite join of elements of the form $\delta_{\boldsymbol{A}}(x, y)$ where $x, y \in A^*$.

A gamp $\boldsymbol{A}$ of lattices is *distance-generated with chains* if

(3) For all $\alpha \in \widetilde{A}$ there are a positive integer n, and chains $x_0 < y_0, x_1 < y_1, \dots, x_{n-1} < y_{n-1}$ of $\boldsymbol{A}$ such that $\alpha = \bigvee_{k<n} \delta_{\boldsymbol{A}}(x_k, y_k)$.

A gamp $\boldsymbol{A}$ is *congruence-tractable* (cf. Lemma 5.11) if

(4) For all $x, y, x_0, y_0, \dots, x_{m-1}, y_{m-1}$ in A^*, if $\delta_{\boldsymbol{A}}(x, y) \leq \bigvee_{k<m} \delta_{\boldsymbol{A}}(x_k, y_k)$ then there are a positive integer n, a list $\vec{z}$ of parameters from A, and terms $t_0, \dots, t_n$ such that the following equations are satisfied in A.

$$x = t_0(\vec{x}, \vec{y}, \vec{z}),$$
$$y = t_n(\vec{x}, \vec{y}, \vec{z}),$$
$$t_k(\vec{y}, \vec{x}, \vec{z}) = t_{k+1}(\vec{x}, \vec{y}, \vec{z}) \quad \text{(for all } k < n).$$

A morphism $\boldsymbol{f} \colon \boldsymbol{A} \to \boldsymbol{B}$ of gamps is *strong* if

(5) $f(A)$ is a strong partial subalgebra of B^* (cf. Definition 4.4).

A morphism $\boldsymbol{f} \colon \boldsymbol{A} \to \boldsymbol{B}$ of gamps is *congruence-cuttable* if

(6) $f(A)$ is a partial sublattice of B^* and given a finite subset X of $\widetilde{B}$ and $x, y \in A$ with $\delta_{\boldsymbol{A}}(f(x), f(y)) \leq \bigvee X$, there are $n < \omega$ and $f(x) = x_0, \dots, x_n = f(y)$ in B^* such that $\delta_{\boldsymbol{A}}(x_k, x_{k+1}) \in \widetilde{B} \downarrow X$ for all $k < n$.

A morphism $\boldsymbol{f} \colon \boldsymbol{A} \to \boldsymbol{B}$ of gamps of the variety of all lattices is *congruence-cuttable with chains* if

(7) $f(A)$ is a partial sublattice of B^* and given a finite subset X of $\widetilde{B}$ and $x, y \in A$ with $\delta_{\boldsymbol{A}}(f(x), f(y)) \leq \bigvee X$, there is a chain $x_0 < \cdots < x_n$ of $\boldsymbol{B}$ such that $x_0 = f(x) \wedge f(y)$, $x_n = f(x) \vee f(y)$, and $\delta_{\boldsymbol{A}}(x_k, x_{k+1}) \in \widetilde{B} \downarrow X$ for all $k < n$.

Let $n \geq 1$ be an integer. A gamp $\boldsymbol{A}$ is *congruence n-permutable* if the following statement holds:

(8) For all $x_0, x_1, \dots, x_n \in A^*$, there are $x_0 = y_0, y_1, \dots, y_n = x_n$ in A such that:

$$\delta_{\boldsymbol{A}}(y_k, y_{k+1}) \leq \bigvee(\delta_{\boldsymbol{A}}(x_i, x_{i+1}) \mid i < n \text{ even}), \qquad \text{for all } k < n \text{ odd},$$

$$\delta_{\boldsymbol{A}}(y_k, y_{k+1}) \leq \bigvee(\delta_{\boldsymbol{A}}(x_i, x_{i+1}) \mid i < n \text{ odd}), \qquad \text{for all } k < n \text{ even}.$$

The following lemma shows that chains in strong gamps of lattices behave the same way as chains in lattices.

Lemma 6.4. *Let $x_0 < \cdots < x_n$ a chain of a strong gamp of lattices $\boldsymbol{B}$. The equalities $x_i \wedge x_j = x_j \wedge x_i = x_i$ and $x_i \vee x_j = x_j \vee x_i = x_j$ hold in B for all $i \leq j \leq n$.*

Moreover the following statements hold:

$$\delta_{\boldsymbol{B}}(x_k, x_{k'}) \leq \delta_{\boldsymbol{B}}(x_i, x_j), \quad \textit{for all } i \leq k \leq k' \leq j \leq n. \tag{6.1}$$

$$\delta_{\boldsymbol{B}}(x_i, x_j) = \bigvee_{i \leq k < j} \delta_{\boldsymbol{B}}(x_k, x_{k+1}), \quad \textit{for all } i \leq j \leq n. \tag{6.2}$$

Proof. Let $i \leq j \leq n$. As $x_i, x_j \in B^*$, all the elements $x_i \wedge x_j$, $x_j \wedge x_i$, $x_i \vee x_j$, and $x_j \vee x_i$ are defined in B.

As $u \wedge v = v \wedge u$ is an identity of lattices, it follows that $x_j \wedge x_i = x_i \wedge x_j = x_i$.

As $x_j \wedge x_i = x_i$, $(x_i \wedge x_j) \vee x_j = x_i \vee x_j$ in B, and $(u \wedge v) \vee v = v$ is an identity of lattices, $x_i \vee x_j = (x_i \wedge x_j) \vee x_j = x_j$. Similarly $x_j \vee x_i = x_j$.

Let $i \leq k \leq k' \leq j \leq n$. As $x_k \wedge x_{k'} = x_k$ and $x_j \wedge x_{k'} = x_{k'}$, we obtain from Definition 5.1(4) the inequality:

$$\delta_{\boldsymbol{B}}(x_k, x_{k'}) = \delta_{\boldsymbol{B}}(x_k \wedge x_{k'}, x_j \wedge x_{k'}) \leq \delta_{\boldsymbol{B}}(x_k, x_j).$$

Similarly, as $x_k = x_k \vee x_i$ and $x_j = x_j \vee x_i$, the inequality $\delta_{\boldsymbol{B}}(x_k, x_j) \leq \delta_{\boldsymbol{B}}(x_i, x_j)$ holds. Therefore (6.1) holds.

Let $i < j \leq n$. Definition 5.1(3) implies the inequality:

$$\delta_{\boldsymbol{B}}(x_i, x_j) \leq \bigvee_{i \leq k < j} \delta_{\boldsymbol{B}}(x_k, x_{k+1}).$$

Moreover (6.1) implies $\delta_{\boldsymbol{B}}(x_k, x_{k+1}) \leq \delta_{\boldsymbol{B}}(x_i, x_j)$ for all $i \leq k < j$, it follows that (6.2) is true. □

The following proposition gives a description of *quotient* of gamps.

Proposition 6.5. *Let $\boldsymbol{A}$ be a gamp of $\mathcal{V}$ and let I be an ideal of $\widetilde{A}$. Let $(A, \delta_{\boldsymbol{A}}, \widetilde{A})/I = (A/I, \delta_{\boldsymbol{A}/I}, \widetilde{A}/I)$ be the quotient pregamp and set $A^*/I = \{a/I \mid a \in A^*\}$ (cf. Notation 5.22). The following statements hold:*

(1) *$\boldsymbol{A}/I = (A^*/I, A/I, \delta_{\boldsymbol{A}/I}, \widetilde{A}/I)$ is a gamp of $\mathcal{V}$.*

(2) *The canonical projection $\boldsymbol{\pi}\colon (A, \delta_{\boldsymbol{A}}, \widetilde{A}) \to (A/I, \delta_{\boldsymbol{A}/I}, \widetilde{A}/I)$ of pregamps is a morphism of gamps from $\boldsymbol{A}$ to $\boldsymbol{A}/I$.*

(3) *If* (A',χ) *is a realization of* $\boldsymbol{A}$ *in* $\mathcal{V}$*, then* $(A'/\bigvee\chi(I),\chi')$ *is a realization of* $\boldsymbol{A}/I$ *in* $\mathcal{V}$*, where*

$$\chi'\colon \widetilde{A}/I \to \mathrm{Con_c}(A'/\bigvee\chi(I))$$
$$d/I \mapsto \chi(d)/\bigvee\chi(I).$$

Moreover, if (A',χ) *is an isomorphic realization of* $\boldsymbol{A}$*, then* $(A'/\bigvee\chi(I),\chi')$ *is an isomorphic realization of* $\boldsymbol{A}/I$*.*

(4) *If* $\boldsymbol{A}$ *is strong, then* $\boldsymbol{A}/I$ *is strong.*

(5) *If* $\boldsymbol{A}$ *is distance-generated (resp., distance-generated with chains) then* $\boldsymbol{A}/I$ *is distance-generated (resp., distance-generated with chains).*

(6) *If* $\boldsymbol{A}$ *is distance-generated and congruence-tractable, then* $\boldsymbol{A}/I$ *is congruence-tractable.*

(7) *Let* $n\geq 2$ *an integer. If* $\boldsymbol{A}$ *is congruence* n*-permutable then* $\boldsymbol{A}/I$ *is congruence* n*-permutable.*

(8) *If* $\boldsymbol{A}$ *is a gamp of lattices and* $x_0\leq x_1\leq\cdots\leq x_n$ *is a chain of* $\boldsymbol{A}$*, then* $x_0/I\leq x_1/I\leq\cdots\leq x_n/I$ *is a chain of* $\boldsymbol{A}/I$*.*

Proof. The statement (1) follows from Corollary 5.35. Denote by $\boldsymbol{\pi}\colon (A,\delta_{\boldsymbol{A}},\widetilde{A})\to (A/I,\delta_{\boldsymbol{A}/I},\widetilde{A}/I)$ the canonical projection of pregamps. The fact that $\pi(A^*)=A^*/I$ as partial algebras follows from the definition of A^*/I. Thus (3) holds.

Let (A',χ) be a realization of $\boldsymbol{A}$ in $\mathcal{V}$, let $d,d'\in\widetilde{A}$ such that $d/I=d'/I$. Hence there exists $u\in I$ such that $d\vee u=d'\vee u$, it follows that $\chi(d)/\chi(u)=\chi(d')/\chi(u)$, hence $\chi(d)/\bigvee\chi(I)=\chi(d')/\bigvee\chi(I)$. Therefore the map χ' is well-defined. It is easy to check that χ' is a $(\vee,0)$-homomorphism. Assume that $\chi'(d/I)\leq\chi'(d'/I)$ for some $d,d'\in\widetilde{A}$. Hence $\chi(d)/\bigvee\chi(I)\leq\chi(d')/\bigvee\chi(I)$, so $\chi(d)\leq\chi(d')\vee\bigvee\chi(I)$. However, $\chi(d)$ is a compact congruence of A', so there exist $u\in I$ such that $\chi(d)\leq\chi(d')\vee\chi(u)=\chi(d'\vee u)$, as χ is an embedding, it follows that $d\leq d'\vee u$, so $d/I\leq d'/I$. Therefore χ' is an embedding.

Let $x,y\in A$. The following equivalences hold:

$$\begin{aligned}x/I=y/I &\Longleftrightarrow \delta_{\boldsymbol{A}}(x,y)\in I\\ &\Longleftrightarrow \chi(\delta_{\boldsymbol{A}}(x,y))\leq\bigvee\chi(I)\\ &\Longleftrightarrow \Theta_{A'}(x,y)\leq\bigvee\chi(I)\\ &\Longleftrightarrow x/\bigvee\chi(I)=y/\bigvee\chi(I).\end{aligned}$$

So we can identify A/I with the corresponding subset of $A'/\bigvee\chi(I)$. Moreover, given $\vec{a}\in\mathrm{Def}_\ell(A/I)$, there exists $\vec{x}\in\mathrm{Def}_\ell(A)$ such that $\vec{a}=\vec{x}/I$, hence $\ell(\vec{a})=\ell(\vec{x})/I$ is identified with $\ell(\vec{x})/\bigvee\chi(I)=\ell(\vec{x}/\bigvee\chi(I))$. So this identification preserves the operations.

Now assume that the realization is isomorphic. Then χ is surjective, thus χ' is surjective, and thus bijective, hence the realization $(A'/\bigvee\chi(I),\chi')$ is isomorphic. Therefore (3) holds.

The proofs of the statements (4), (5), (7), and (8) are straightforward.

Assume that $\boldsymbol{A}$ is distance-generated and congruence-tractable. Let $x,y\in A^*$, let $m<\omega$, let $\vec{x},\vec{y}$ be m-tuples of A^*. Assume that:

$$\delta_{\boldsymbol{A}/I}(x/I,y/I)\leq\bigvee_{k<m}\delta_{\boldsymbol{A}/I}(x_k/I,y_k/I).$$

It follows that there exists $u \in I$ with:

$$\delta_{\boldsymbol{A}}(x,y) \leq u \vee \bigvee_{k<m} \delta_{\boldsymbol{A}}(x_k, y_k).$$

However, as $\boldsymbol{A}$ is distance-generated, there exist $x'_0, \dots, x'_{p-1}, y'_0, \dots, y'_{p-1}$ in A^* such that $u = \bigvee_{k<p} \delta_{\boldsymbol{A}}(x'_k, y'_k)$. As $\delta_{\boldsymbol{A}}(x'_k, y'_k) \leq u \in I$, $x'_k/I = y'_k/I$ for all $k < p$. Moreover the following inequality holds:

$$\delta_{\boldsymbol{A}}(x,y) \leq \bigvee_{k<m} \delta_{\boldsymbol{A}}(x_k, y_k) \vee \bigvee_{k<p} \delta_{\boldsymbol{A}}(x'_k, y'_k).$$

As $\boldsymbol{A}$ is congruence-tractable, there are a positive integer n, a list $\vec{z}$ of parameters from A, and terms $t_0, \dots, t_n$ such that, the following equations are satisfied in A:

$$x = t_0(\vec{x}, \vec{x}', \vec{y}, \vec{y}', \vec{z}),$$
$$y = t_n(\vec{x}, \vec{x}', \vec{y}, \vec{y}', \vec{z}),$$
$$t_k(\vec{y}, \vec{y}', \vec{x}, \vec{x}', \vec{z}) = t_{k+1}(\vec{x}, \vec{x}', \vec{y}, \vec{y}', \vec{z}) \quad \text{(for all } k < n).$$

Put $t'_k(\vec{a}, \vec{b}, \vec{c}, \vec{d}) = t_k(\vec{a}, \vec{d}, \vec{b}, \vec{d}, \vec{c})$, for all tuples $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ and all $k \leq n$. As $\vec{x}'/I = \vec{y}'/I$ the following equations are satisfied in A/I:

$$x/I = t'_0(\vec{x}/I, \vec{y}/I, \vec{z}/I, \vec{x}'/I),$$
$$y/I = t'_n(\vec{x}/I, \vec{y}/I, \vec{z}/I, \vec{x}'/I),$$
$$t'_k(\vec{y}/I, \vec{x}/I, \vec{z}/I, \vec{x}'/I) = t'_{k+1}(\vec{x}/I, \vec{y}/I, \vec{z}/I, \vec{x}'/I) \quad \text{(for all } k < n).$$

Therefore $\boldsymbol{A}/I$ is congruence-tractable. □

Definition 6.6. The gamp $\boldsymbol{A}/I$ described in Proposition 6.5 is a *quotient of* $\boldsymbol{A}$, the morphism $\boldsymbol{\pi}$ is *the canonical projection.*

The following proposition describes how morphisms factorize through quotients of gamps.

Proposition 6.7. *Let* $\boldsymbol{f}\colon \boldsymbol{A} \to \boldsymbol{B}$ *be a morphism of gamps, let* I *be an ideal of* $\widetilde{A}$, *and let* J *be an ideal of* $\widetilde{B}$. *Assume that* $\widetilde{f}(I) \subseteq J$ *and denote by*

$$\boldsymbol{g}\colon (A/I, \delta_{\boldsymbol{A}/I}, \widetilde{A}/I) \to (B/J, \delta_{\boldsymbol{B}/J}, \widetilde{B}/J)$$

the morphism of pregamps induced by $\boldsymbol{f}$. *The following statements hold.*

(1) $\boldsymbol{g}\colon \boldsymbol{A}/I \to \boldsymbol{B}/J$ *is a morphism of gamps.*
(2) *If* $\boldsymbol{f}$ *is strong, then* $\boldsymbol{g}$ *is strong.*
(3) *If* $\boldsymbol{f}$ *is congruence-cuttable then* $\boldsymbol{g}$ *is congruence-cuttable.*
(4) *If* $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *are gamps of lattices and* $\boldsymbol{f}$ *is congruence-cuttable with chains, then* $\boldsymbol{g}$ *is congruence-cuttable with chains.*

Proof. The equality $g(A^*/I) = f(A^*)/J$ of partial algebras holds. Moreover, as $f(A^*)$ is a partial subalgebra of B, $g(A^*/I)$ is a partial subalgebra of B/J. Therefore (1) holds.

The statement (2) follows from the definitions of a quotient gamp (cf. Proposition 6.5) and of a strong morphism (Definition 6.3).

Assume that $\boldsymbol{f}$ is congruence-cuttable. As $f(A)$ is a partial subalgebra of B^* it follows that $g(A/I)$ is a partial subalgebra of B^*/J. Let X be a finite subset of $\widetilde{B}$, let $x, y \in A$ such that $\delta_{\boldsymbol{B}/J}(g(x/I), g(y/I)) \leq \bigvee X/I$. If $X = \emptyset$, then $x/I = y/I$, hence the case is immediate.

If $X \neq \emptyset$, let $u \in J$ such that $\delta_{\boldsymbol{B}}(g(x), g(y)) \leq u \vee \bigvee X$. Put $X' = X \cup \{u\}$. There are $n < \omega$ and $f(x) = x_0, \dots, x_n = f(y)$ in B^* such that $\delta_{\boldsymbol{B}}(x_k, x_{k+1}) \in \widetilde{B} \downarrow X'$ for each $k < n$. If $\delta_{\boldsymbol{B}}(x_k, y_k) \leq u$, then $\delta_{\boldsymbol{B}/J}(x_k/J, y_k/J) = \delta_{\boldsymbol{B}}(x_k, y_k)/J = 0/J \in (\widetilde{B}/J) \downarrow X/J$. Otherwise $\delta_{\boldsymbol{B}}(x_k, y_k) \in \widetilde{B} \downarrow X$, thus $\delta_{\boldsymbol{B}/J}(x_k/J, y_k/J) = \delta_{\boldsymbol{B}}(x_k, y_k)/J \in (\widetilde{B}/J) \downarrow X/J$. Therefore (3) holds.

The proof of (4) is similar to the proof of (3). $\square$

We introduce in the following definitions a functor $\boldsymbol{G}\colon \mathcal{V} \to \mathbf{Gamp}(\mathcal{V})$, a functor $\boldsymbol{C}\colon \mathbf{Gamp}(\mathcal{V}) \to \mathbf{Sem}_{\vee,0}$ and functors $\boldsymbol{P}_{\mathrm{gl}}, \boldsymbol{P}_{\mathrm{gr}}\colon \mathbf{Gamp}(\mathcal{V}) \to \mathbf{PGamp}(\mathcal{V})$.

Definition 6.8. Let A be a member of a variety $\mathcal{V}$ of algebras. Then the quadruple $\boldsymbol{G}(A) = (A, A, \Theta_A, \operatorname{Con_c} A)$ is a gamp of $\mathcal{V}$ (we recall that $\Theta_A(x, y)$ denotes the smallest congruence that identifies x and y). If $f\colon A \to B$ is a morphism of algebras, then $\boldsymbol{G}(f) = (f, \operatorname{Con_c} f)$ is a morphism of gamps from $\boldsymbol{G}(A)$ to $\boldsymbol{G}(B)$. It defines a functor from the category $\mathcal{V}$ to the category $\mathbf{Gamp}(\mathcal{V})$.

A gamp $\boldsymbol{A}$ is an *algebra* if $\boldsymbol{A}$ is isomorphic to $\boldsymbol{G}(B)$ for some B. A gamp $\boldsymbol{A}$ is an *algebra* of a variety $\mathcal{V}$ if $\boldsymbol{A}$ is isomorphic to $\boldsymbol{G}(B)$ for some $B \in \mathcal{V}$.

Let $\boldsymbol{A}$ be a gamp of $\mathcal{V}$, we set $\boldsymbol{C}(\boldsymbol{A}) = \widetilde{A}$. Let $\boldsymbol{f}\colon \boldsymbol{A} \to \boldsymbol{B}$ be a morphism of gamps of $\mathcal{V}$, we set $\boldsymbol{C}(\boldsymbol{f}) = \widetilde{f}$. This defines a functor $\boldsymbol{C}\colon \mathbf{Gamp}(\mathcal{V}) \to \mathbf{Sem}_{\vee,0}$.

Let $\boldsymbol{A}$ be a gamp of $\mathcal{V}$, we set $\boldsymbol{P}_{\mathrm{gr}}(\boldsymbol{A}) = (A, \delta_{\boldsymbol{A}}, \widetilde{A})$. Let $\boldsymbol{f}\colon \boldsymbol{A} \to \boldsymbol{B}$ be a morphism of gamps of $\mathcal{V}$, we put $\boldsymbol{P}_{\mathrm{gr}}(\boldsymbol{f}) = \boldsymbol{f}$ as a morphism of pregamps. This defines a functor $\boldsymbol{P}_{\mathrm{gr}}\colon \mathbf{Gamp}(\mathcal{V}) \to \mathbf{PGamp}(\mathcal{V})$.

Let $\boldsymbol{A}$ be a gamp of $\mathcal{V}$, we set $\boldsymbol{P}_{\mathrm{gl}}(\boldsymbol{A}) = (A^*, \delta_{\boldsymbol{A}} \restriction (A^*)^2, \widetilde{A})$. Let $\boldsymbol{f}\colon \boldsymbol{A} \to \boldsymbol{B}$ be a morphism of gamps of $\mathcal{V}$, we denote by $\boldsymbol{P}_{\mathrm{gl}}(\boldsymbol{f})$ the restriction $(f, \widetilde{f})\colon \boldsymbol{P}_{\mathrm{gl}}(\boldsymbol{A}) \to \boldsymbol{P}_{\mathrm{gl}}(\boldsymbol{B})$. This defines a functor $\boldsymbol{P}_{\mathrm{gl}}\colon \mathbf{Gamp}(\mathcal{V}) \to \mathbf{PGamp}(\mathcal{V})$.

Remark 6.9. The following assertions hold.

(1) The following equations, between the functors introduced in Definition 5.4 and Definition 6.8, are satisfied:

$$\begin{aligned} \boldsymbol{C} \circ \boldsymbol{G} &= \operatorname{Con_c} = \boldsymbol{C}_{\mathrm{pg}} \circ \boldsymbol{P}_{\mathrm{ga}}, \\ \boldsymbol{P}_{\mathrm{gr}} \circ \boldsymbol{G} &= \boldsymbol{P}_{\mathrm{gl}} \circ \boldsymbol{G} = \boldsymbol{P}_{\mathrm{ga}}, \\ \boldsymbol{C}_{\mathrm{pg}} \circ \boldsymbol{P}_{\mathrm{gr}} &= \boldsymbol{C}_{\mathrm{pg}} \circ \boldsymbol{P}_{\mathrm{gl}} = \boldsymbol{C}. \end{aligned}$$

(2) If A is a subalgebra of B, then, in general, $\boldsymbol{G}(A)$ is not a subgamp of $\boldsymbol{G}(B)$. The different “congruences” of a subgamp can be extended in a natural way to different “congruences” of the gamp.

(3) Let $\boldsymbol{A}$ be a gamp. If $\boldsymbol{A}$ is an algebra, then there is a unique, up to isomorphism, algebra B such that $\boldsymbol{A} \cong \boldsymbol{G}(B)$. Moreover if $\boldsymbol{A}$ is a gamp in a variety $\mathcal{V}$, then $B \in \mathcal{V}$. Indeed A is an algebra and $\boldsymbol{A} \cong \boldsymbol{G}(A)$.

(4) Let $\boldsymbol{f}\colon \boldsymbol{A} \to \boldsymbol{B}$ be a morphism of gamps. If $\boldsymbol{A}$ and $\boldsymbol{B}$ are algebras, then $f\colon A \to B$ is a morphism of algebras.

(5) Let $\vec{\boldsymbol{A}} = (\boldsymbol{A}_p, \boldsymbol{f}_{p,q} \mid p \leq q \text{ in } P)$ be a diagram of gamps. If $\boldsymbol{A}_p$ is an algebra, for all $p \in P$, then $\vec{A} = (A_p, f_{p,q} \mid p \leq q \text{ in } P)$ is a diagram of algebras, moreover $\vec{\boldsymbol{A}} \cong \boldsymbol{G} \circ \vec{A}$.

(6) Let $B \in \mathcal{V}$ and let I be an ideal of $\operatorname{Con_c} B$. There is an isomorphism $\boldsymbol{f}\colon \boldsymbol{G}(B)/I \cong \boldsymbol{G}(B/\bigvee I)$ satisfying $f(x/I) = x/\bigvee I$ and $\widetilde{f}(\alpha/I) = \alpha/\bigvee I$ for each $x \in B$ and each $\alpha \in \operatorname{Con_c} B$.

(7) Let B be an algebra. Then B is congruence n-permutable if and only if $\boldsymbol{G}(B)$ is congruence n-permutable (this follows immediately from Proposition 2.1).

Lemma 6.10. *Let $\mathcal{V}$ be a variety of algebras. The category $\mathbf{Gamp}(\mathcal{V})$ has all directed colimits. Suppose that we are given a directed poset P, a P-indexed diagram $\vec{\boldsymbol{A}} = (\boldsymbol{A}_p, \boldsymbol{f}_{p,q} \mid p \leq q \text{ in } P)$ in $\mathbf{Gamp}(\mathcal{V})$, and a directed colimit cocone:*

$$((A, \delta_{\boldsymbol{A}}, \widetilde{A}), \boldsymbol{f}_p \mid p \in P) = \varinjlim((A_p, \delta_{\boldsymbol{A}_p}, \widetilde{A}_p), \boldsymbol{f}_{p,q} \mid p \leq q \text{ in } P) \quad \text{in } \mathbf{PGamp}_{\mathscr{L}}.$$

Put $A^* = \bigcup_{p \in P} f(A_p^*)$ with its natural structure of partial algebra (cf. Lemma 4.9), then $\boldsymbol{A} = (A^*, A, \delta_{\boldsymbol{A}}, \widetilde{A})$ is a gamp of $\mathcal{V}$, $\boldsymbol{f}_p \colon \boldsymbol{A}_p \to \boldsymbol{A}$ is a morphism of gamps, and the following is a directed colimit cocone:

$$(\boldsymbol{A}, \boldsymbol{f}_p \mid p \in P) = \varinjlim(\boldsymbol{A}_p, \boldsymbol{f}_{p,q} \mid p \leq q \text{ in } P), \quad \text{in } \mathbf{Gamp}(\mathcal{V}).$$

Moreover the following statements hold:

(1) *If $\boldsymbol{A}_p$ is distance-generated for each $p \in P$, then $\boldsymbol{A}$ is distance-generated.*
(2) *If $\mathcal{V}$ is a variety of lattices and $\boldsymbol{A}_p$ is distance-generated with chains for each $p \in P$, then $\boldsymbol{A}$ is distance-generated with chains.*
(3) *Let n be a positive integer. If $\boldsymbol{A}_p$ is congruence n-permutable for each $p \in P$, then $\boldsymbol{A}$ is congruence n-permutable.*

Proof. It follows from Corollary 5.34 that $(A, \delta_{\boldsymbol{A}}, \widetilde{A})$ is a pregamp of $\mathcal{V}$. Moreover A^* is a partial subalgebra of A. Hence $\boldsymbol{A} = (A^*, A, \delta_{\boldsymbol{A}}, \widetilde{A})$ is a gamp of $\mathcal{V}$. As $f(A_p^*)$ is a partial subalgebra of A^*, $\boldsymbol{f}_p \colon \boldsymbol{A}_p \to \boldsymbol{A}$ is a morphism of gamps. It is easy to check that the following is a directed colimit cocone:

$$(\boldsymbol{A}, \boldsymbol{f}_p \mid p \in P) = \varinjlim(\boldsymbol{A}_p, \boldsymbol{f}_{p,q} \mid p \leq q \text{ in } P), \quad \text{in } \mathbf{Gamp}(\mathcal{V}).$$

Assume that $\boldsymbol{A}_p$ is distance-generated for each $p \in P$. Let $\alpha \in \widetilde{A}$, then there are $p \in P$ and $\beta \in \widetilde{A}_p$ such that $\alpha = \widetilde{f}_p(\beta)$. As $\boldsymbol{A}_p$ is distance-generated, there are an integer $n \geq 0$ and n-tuple $\vec{x}, \vec{y}$ of A_p^* such that $\beta = \bigvee_{k<n} \delta_{\boldsymbol{A}_p}(x_k, y_k)$. Therefore the following equalities hold:

$$\alpha = \widetilde{f}_p(\beta) = \widetilde{f}_p\left(\bigvee_{k<n} \delta_{\boldsymbol{A}_p}(x_k, y_k)\right) = \bigvee_{k<n} \widetilde{f}_p(\delta_{\boldsymbol{A}_p}(x_k, y_k)) = \bigvee_{k<n} \delta_{\boldsymbol{A}}(f_p(x_k), f_p(y_k)).$$

Thus $\boldsymbol{A}$ is distance-generated.

The proofs of (2) and (3) are similar. □

As an immediate application we obtain the following corollary.

Corollary 6.11. *The functors $\boldsymbol{G}$, $\boldsymbol{C}$, $\boldsymbol{P}_{\mathrm{gl}}$, and $\boldsymbol{P}_{\mathrm{gr}}$ preserves directed colimits.*

Proof. It follows from the description of directed colimits of gamps (cf. Lemma 6.10) and pregamps (cf. Lemma 5.6) that $\boldsymbol{C}$, $\boldsymbol{P}_{\mathrm{gl}}$, and $\boldsymbol{P}_{\mathrm{gr}}$ preserve directed colimits. As $\mathrm{Con_c}$ preserves directed colimits, $\boldsymbol{G}$ also preserves directed colimits. □

Definition 6.12. Let $\mathcal{V}$ be a variety of algebras, let P be a poset, and let $\vec{\boldsymbol{A}} = (\boldsymbol{A}_p, \boldsymbol{f}_{p,q} \mid p \leq q \text{ in } P)$ in $\mathbf{Gamp}(\mathcal{V})$. An *ideal* of $\vec{\boldsymbol{A}}$ is an ideal of $\boldsymbol{C} \circ \vec{A}$. It consists of a family $\vec{I} = (I_p)_{p \in P}$ such that I_p is an ideal of $\widetilde{A}_p$ and $\widetilde{f}_{p,q}(I_p) \subseteq I_q$ for all $p \leq q$ in P.

We denote by $\vec{\boldsymbol{A}}/\vec{I} = (\boldsymbol{A}_p/I_p, \boldsymbol{g}_{p,q} \mid p \leq q \text{ in } P)$, where $\boldsymbol{g}_{p,q} \colon \boldsymbol{A}_p/I_p \to \boldsymbol{A}_q/I_q$ is induced by $\boldsymbol{f}_{p,q}$, for all $p \leq q$ in P.

The diagram $\vec{\boldsymbol{A}}/\vec{I}$ is a *quotient* of $\vec{\boldsymbol{A}}$.

Remark 6.13. In the context of Definition 6.12 the following equalities hold:

$$(\boldsymbol{P}_{\mathrm{gr}} \circ \vec{\boldsymbol{A}})/\vec{I} = \boldsymbol{P}_{\mathrm{gr}} \circ (\vec{\boldsymbol{A}}/\vec{I}).$$

$$(\boldsymbol{C} \circ \vec{\boldsymbol{A}})/\vec{I} = \boldsymbol{C} \circ (\vec{\boldsymbol{A}}/\vec{I}).$$

Moreover, up to a natural identification (cf. Notation 5.22)

$$(\boldsymbol{P}_{\mathrm{gl}} \circ \vec{\boldsymbol{A}})/\vec{I} = \boldsymbol{P}_{\mathrm{gl}} \circ (\vec{\boldsymbol{A}}/\vec{I}).$$

Definition 6.14. Let $\mathcal{V}$ be a variety of algebras, let P be a poset. A *partial lifting* in $\mathcal{V}$ is a diagram $\vec{\boldsymbol{A}} = (\boldsymbol{A}_p, \boldsymbol{f}_{p,q} \mid p \leq q \text{ in } P)$ in $\mathbf{Gamp}(\mathcal{V})$ such that the following statements hold:

(1) The gamp $\boldsymbol{A}_p$ is strong, congruence-tractable, distance-generated, and has an isomorphic realization (cf. Definitions 6.3 and 6.1), for each $p \in P$.
(2) The morphisms $\boldsymbol{f}_{p,q}$ is strong and congruence-cuttable (cf. Definition 6.3), for all $p < q$ in P.

The partial lifting is a *lattice partial lifting* if $\mathcal{V}$ is a variety of lattices, $\boldsymbol{B}_p$ is distance-generated with chains for each $p \in P$, and $\boldsymbol{f}_{p,q}$ is congruence-cuttable with chains for all $p < q$ in P.

A partial lifting $\vec{\boldsymbol{A}}$ is *congruence n-permutable* if $\boldsymbol{A}_p$ is congruence n-permutable for each $p \in P$.

Let $\vec{S} = (S_p, \sigma_{p,q})$ be a diagram in $\mathbf{Sem}_{\vee,0}$. A *partial lifting of* $\vec{S}$ is a partial lifting of $\vec{\boldsymbol{A}}$ such that $\boldsymbol{C} \circ \vec{\boldsymbol{A}} \cong \vec{S}$.

Remark 6.15. If $\vec{\boldsymbol{A}}$ is a partial lifting of $\vec{S}$, then there exists a diagram $\vec{\boldsymbol{A}}' \cong \vec{\boldsymbol{A}}$ such that $\boldsymbol{C} \circ \vec{\boldsymbol{A}}' = \vec{S}$. Hence we can assume that $\boldsymbol{C} \circ \vec{\boldsymbol{A}} = \vec{S}$.

The following result expresses the fact that a subdiagram or a quotient of a partial lifting is a partial lifting.

Lemma 6.16. *Let $\mathcal{V}$ be a variety of algebras, let P be a poset, and let $\vec{\boldsymbol{A}}$ be a partial lifting in $\mathcal{V}$ of a diagram $\vec{S}$. The following statements hold:*

(1) *Let $\vec{I}$ be an ideal of $\vec{\boldsymbol{A}}$; then $\vec{\boldsymbol{A}}/\vec{I}$ is a partial lifting of $\vec{S}/I$.*
(2) *Let $Q \subseteq P$; then $\vec{\boldsymbol{A}} \restriction Q$ is a partial lifting of $\vec{S} \restriction Q$.*

Proof. The statement (1) follows from Proposition 6.5 and Proposition 6.7. The statement (2) is immediate. □

Lemma 6.17. *Let $\mathcal{V}$ be a variety of algebras, let P be a directed poset with no maximal element, and let $\vec{\boldsymbol{A}} = (\boldsymbol{A}_p, \boldsymbol{f}_{p,q} \mid p \leq q$ in $P)$ be a partial lifting in $\mathcal{V}$. Consider a colimit cocone:*

$$(\boldsymbol{A}, \boldsymbol{f}_p \mid p \in P) = \varinjlim \vec{\boldsymbol{A}} \quad \text{in } \mathbf{Gamp}(\mathcal{V}).$$

Then $\boldsymbol{A}$ is an algebra in $\mathcal{V}$. Moreover, for any $n \geq 2$, if all $\boldsymbol{A}_p$ are congruence n-permutable, then the algebra corresponding to $\boldsymbol{A}$ is congruence n-permutable.

Proof. The morphism $\boldsymbol{f}_{p,q} \colon (A_p, \delta_{\boldsymbol{A}_p}, \widetilde{A}_p) \to (A_q, \delta_{\boldsymbol{A}_q}, \widetilde{A}_q)$ of pregamps is both strong and congruence-tractable for all $p < q$ in P. It follows from the description of colimits in $\mathbf{Gamp}(\mathcal{V})$ (cf. Lemma 6.10 and Corollary 5.15) that A is an algebra and there is an isomorphism $\phi \colon \operatorname{Con_c} A \to \widetilde{A}$ satisfying:

$$\phi(\Theta_A(x, y)) = \delta_{\boldsymbol{A}}(x, y), \quad \text{for all } x, y \in A.$$

As $A^* = \bigcup_{p\in P} f_p(A^*_p)$ and $f_{p,q}(A_p) \subseteq A^*_q$ for all $p<q$ in P, it follows that $A^* = A$. Therefore $(\mathrm{id}_A, \phi)\colon \boldsymbol{G}(A) \to \boldsymbol{A}$ is an isomorphism of gamps.

Now assume that all $\boldsymbol{A}_p$ are congruence n-permutable. It follows from Lemma 6.10 that $\boldsymbol{A}$ is congruence n-permutable. As $\boldsymbol{A}$ is an algebra, the conclusion follows from Proposition 2.1. □

7. Locally finite properties

The aim of this section is to prove Lemma 7.8, which is a special version of the Buttress Lemma of [5] adapted to gamps for the functor $\boldsymbol{C}$.

We use the following generalizations of (2), (3), (4), (6), and (7) of Definition 6.3.

Definition 7.1. Fix a gamp $\boldsymbol{A}$ in a variety $\mathcal{V}$, a $(\vee,0)$-semilattice S, and a $(\vee,0)$-homomorphism $\phi\colon \widetilde{A} \to S$. The gamp $\boldsymbol{A}$ is *distance-generated through* ϕ if the following statement holds:

(2′) For each $s\in S$ there are $n<\omega$ and n-tuple $\vec{x},\vec{y}$ of A^* such that:

$$s = \bigvee_{k<n} \phi(\delta_{\boldsymbol{A}}(x_k,y_k))$$

If $\boldsymbol{A}$ is a gamp of lattices, we say $\boldsymbol{A}$ is *distance-generated with chains through* ϕ if

(3′) For each $s\in S$ there are $n<\omega$ and chains $x_0<y_0,\dots,x_{n-1}<y_{n-1}$ of $\boldsymbol{A}$ such that:

$$s = \bigvee_{k<n} \phi(\delta_{\boldsymbol{A}}(x_k,y_k))$$

The gamp $\boldsymbol{A}$ is *congruence-tractable through* ϕ if the following statement holds:

(4′) Let $x,y,x_0,y_0,\dots,x_{m-1},y_{m-1}$ in A^*, if $\phi(\delta_{\boldsymbol{A}}(x,y)) \le \bigvee_{k<m}\phi(\delta_{\boldsymbol{A}}(x_k,y_k))$ then there are a positive integer n, a list $\vec{z}$ of parameters from A, and terms t_0, …, t_n such that, the following equations are satisfied in A:

$$\begin{aligned}\phi(\delta_{\boldsymbol{A}}(x,t_0(\vec{x},\vec{y},\vec{z}))) &= 0,\\ \phi(\delta_{\boldsymbol{A}}(y,t_n(\vec{x},\vec{y},\vec{z}))) &= 0,\\ \phi(\delta_{\boldsymbol{A}}(t_j(\vec{y},\vec{x},\vec{z}),t_{j+1}(\vec{x},\vec{y},\vec{z}))) &= 0 \quad \text{(for all } j<n).\end{aligned}$$

A morphism $\boldsymbol{f}\colon \boldsymbol{U}\to\boldsymbol{A}$ of gamps is *congruence-cuttable through* ϕ if the following statement holds:

(6′) Given $X\subseteq S$, given x,y in U, if $\phi(\widetilde{f}(\delta_{\boldsymbol{A}}(x,y))) \le \bigvee X$ then there exist $n<\omega$ and $f(x)=x_0,x_1,\dots,x_n=f(y)$ in A^* such that $\delta_{\boldsymbol{A}}(x_k,x_{k+1}) \in S\downarrow X$ for all $k<n$.

A morphism $\boldsymbol{f}\colon \boldsymbol{U}\to\boldsymbol{A}$ of gamps of lattices is *congruence-cuttable with chains through* ϕ if the following statement holds:

(7′) Given $X\subseteq S$, given $x,y\in U$, if $\phi(\widetilde{f}(\delta_{\boldsymbol{A}}(x,y))) \le \bigvee X$ then there are $n<\omega$ and a chain $x_0<x_1<\dots<x_n$ of $\boldsymbol{A}$ such that $x_0=f(x)\wedge f(y)$, $x_n = f(x)\vee f(y)$, and $\phi(\delta_{\boldsymbol{A}}(x_k,x_{k+1}))\in S\downarrow X$ for all $k<n$.

Remark 7.2. A gamp $\boldsymbol{A}$ is distance-generated through $\mathrm{id}_{\widetilde{A}}$ if and only if $\boldsymbol{A}$ is distance-generated.

A gamp $\boldsymbol{A}$ is congruence-tractable through $\mathrm{id}_{\widetilde{A}}$ if and only if $\boldsymbol{A}$ is congruence-tractable.

A morphism $\boldsymbol{f}\colon \boldsymbol{U}\to\boldsymbol{A}$ of gamps is congruence-cuttable through $\mathrm{id}_{\widetilde{A}}$ if and only if $\boldsymbol{f}$ is congruence-cuttable.

A morphism $\boldsymbol{f}\colon \boldsymbol{U} \to \boldsymbol{A}$ of gamps of lattices is congruence-cuttable with chains through $\mathrm{id}_{\widetilde{A}}$ if and only if $\boldsymbol{f}$ is congruence-cuttable with chains.

Lemma 7.3. *Let $\boldsymbol{A}$ be a gamp in a variety of algebras $\mathcal{V}$, let $\phi\colon \widetilde{A} \to S$ an ideal-induced $(\vee, 0)$-homomorphism, and put $I = \ker_0 \phi$.*

(1) *If $\boldsymbol{A}$ is distance-generated through ϕ, then $\boldsymbol{A}/I$ is distance-generated.*
(2) *If $\boldsymbol{A}$ is distance-generated with chains through ϕ, then $\boldsymbol{A}/I$ is distance-generated with chains.*
(3) *If $\boldsymbol{A}$ is congruence-tractable through ϕ then $\boldsymbol{A}/I$ is congruence-tractable.*

Proof. As ϕ is ideal-induced, it induces an isomorphism $\xi\colon \widetilde{A}/I \to S$.

Assume that $\boldsymbol{A}$ is distance-generated through ϕ. Let $d \in \widetilde{A}/I$, put $s = \xi(d)$. There are $n < \omega$ and n-tuples $\vec{x}, \vec{y}$ in A^* such that $s = \bigvee_{k<n} \phi(\delta_{\boldsymbol{A}}(x_k, y_k))$. It follows that $s = \bigvee_{k<n} \xi(\delta_{\boldsymbol{A}/I}(x_k/I, y_k/I))$, so $d = \bigvee_{k<n} \delta_{\boldsymbol{A}/I}(x_k/I, y_k/I)$. Therefore $\boldsymbol{A}/I$ is distance-generated.

The case where $\boldsymbol{A}$ is distance-generated with chains through ϕ is similar.

Assume that $\boldsymbol{A}$ is congruence-tractable through ϕ. Let $x, y, x_0, y_0, \dots, x_{m-1}, y_{m-1}$ in A^* such that:

$$\delta_{\boldsymbol{A}/I}(x/I, y/I) \leq \bigvee_{k<m} \delta_{\boldsymbol{A}/I}(x_k/I, y_k/I).$$

Thus $\phi(\delta_{\boldsymbol{A}/I}(x, y)) \leq \bigvee_{k<m} \phi(\delta_{\boldsymbol{A}}(x_k, y_k))$. Hence there are a positive integer n, a list $\vec{z}$ of parameters from A, and terms $t_0, \dots, t_n$ such that the following equations are satisfied in A:

$$\begin{aligned} \phi(\delta_{\boldsymbol{A}}(x, t_0(\vec{x}, \vec{y}, \vec{z}))) &= 0, \\ \phi(\delta_{\boldsymbol{A}}(y, t_n(\vec{x}, \vec{y}, \vec{z}))) &= 0, \\ \phi(\delta_{\boldsymbol{A}}(t_k(\vec{y}, \vec{x}, \vec{z}), t_{k+1}(\vec{x}, \vec{y}, \vec{z}))) &= 0 \quad (\text{for all } k < n). \end{aligned}$$

Those equations imply that the following equations are satisfied in A/I:

$$\begin{aligned} x/I &= t_0(\vec{x}/I, \vec{y}/I, \vec{z}/I), \\ y/I &= t_n(\vec{x}/I, \vec{y}/I, \vec{z}/I), \\ t_k(\vec{y}/I, \vec{x}/I, \vec{z}/I) &= t_{k+1}(\vec{x}/I, \vec{y}/I, \vec{z}/I) \quad (\text{for all } k < n). \end{aligned}$$

Therefore $\boldsymbol{A}/I$ is congruence-tractable. □

The proof of the following lemma is similar.

Lemma 7.4. *Let $\boldsymbol{f}\colon \boldsymbol{U} \to \boldsymbol{A}$ be a morphism of gamps, let I be an ideal of $\boldsymbol{U}$, and let $\phi\colon \widetilde{A} \to S$ be an ideal-induced $(\vee, 0)$-homomorphism. Put $J = \ker_0 \phi$. Denote by $\boldsymbol{g}\colon \boldsymbol{U}/I \to \boldsymbol{A}/J$ the morphism of gamps induced by $\boldsymbol{f}$. The following statements hold:*

(1) *If $\boldsymbol{f}$ is congruence-cuttable through ϕ, then $\boldsymbol{g}$ is congruence-cuttable.*
(2) *Assume that $\boldsymbol{f}$ is a morphism of gamps of lattices. If $\boldsymbol{f}$ is congruence-cuttable with chains through ϕ, then $\boldsymbol{g}$ is congruence-cuttable with chains.*

We shall now define *locally finite properties* for an algebra B, as properties that are satisfied by "many" finite subgamps of $\boldsymbol{G}(B)$.

Definition 7.5. Let B be an algebra. A *locally finite property for B* is a property (P) in a subgamp $\boldsymbol{A}$ of $\boldsymbol{G}(B)$ such that there exists a finite $X \subseteq B$ satisfying that for every finite full partial subalgebra A^* of B that contains X, there exists a finite $Y \subseteq B$ such that for every finite full partial subalgebra A of B that contains $A^* \cup Y$ and every finite $(\vee,0)$-subsemilattice $\widetilde{A}$ of $\operatorname{Con_c} B$ that contains $\operatorname{Con}_{\mathrm{c}}^{A}(B)$, the subgamp $(A^*, A, \Theta_B, \widetilde{A})$ of $\boldsymbol{G}(B)$ satisfies (P).

Proposition 7.6. *Any finite conjunction of locally finite properties is locally finite.*

Lemma 7.7. *Let B be an algebra and denote by $\mathscr{L}$ the similarity type of B. The properties* (1)-(8) *in $\boldsymbol{A}$ are locally finite for B:*

Assume that $\mathscr{L}$ is finite.

(1) *$\boldsymbol{A}$ is strong.*

Fix an ideal-induced $(\vee,0)$-homomorphism $\phi\colon \operatorname{Con_c} B \to S$, with S finite.

(2) *$\boldsymbol{A}$ is distance-generated through $\phi \restriction \widetilde{A}$.*

Fix an ideal-induced $(\vee,0)$-homomorphism $\phi\colon \operatorname{Con_c} B \to S$ with S finite, and assume that B is a lattice.

(3) *$\boldsymbol{A}$ is distance-generated with chains through $\phi \restriction \widetilde{A}$.*

Fix an ideal-induced $(\vee,0)$-homomorphism $\phi\colon \operatorname{Con_c} B \to S$, with S finite.

(4) *$\boldsymbol{A}$ is congruence-tractable through $\phi \restriction \widetilde{A}$.*

Assume that $\mathscr{L}$ is finite, fix a finite gamp $\boldsymbol{U}$, fix a morphism $\boldsymbol{f}\colon \boldsymbol{U} \to \boldsymbol{G}(B)$ of gamps.

(5) *The restriction $\boldsymbol{f}\colon \boldsymbol{U} \to \boldsymbol{A}$ is strong.*

Fix an ideal-induced $(\vee,0)$-homomorphism $\phi\colon \operatorname{Con_c} B \to S$ where S is finite, a finite gamp $\boldsymbol{U}$, and a morphism $\boldsymbol{f}\colon \boldsymbol{U} \to \boldsymbol{G}(B)$ of gamps.

(6) *The restriction of $\boldsymbol{f}\colon \boldsymbol{U} \to \boldsymbol{A}$ is congruence-cuttable through $\phi \restriction \widetilde{A}$.*

Assume that B is a lattice. Fix an ideal-induced $(\vee,0)$-homomorphism $\phi\colon \operatorname{Con_c} B \to S$, where S is finite. Fix a finite gamp $\boldsymbol{U}$, fix a morphism $\boldsymbol{f}\colon \boldsymbol{U} \to \boldsymbol{G}(B)$ of gamps.

(7) *The restriction of $\boldsymbol{f}\colon \boldsymbol{U} \to \boldsymbol{A}$ is congruence-cuttable with chains through $\phi \restriction \widetilde{A}$.*

Let $n \geq 2$ be an integer. Assume that B is congruence n-permutable.

(8) *$\boldsymbol{A}$ is congruence n-permutable.*

Proof. (1) Put $X = \emptyset$ and let A^* be a finite full partial subalgebra of B. Put $Y = \{\ell^B(\vec{x}) \mid \ell \in \mathscr{L}$ and $\vec{X}$ is an $\operatorname{ar}(\ell)$-tuple of $A^*\}$. As A^* and $\mathscr{L}$ are both finite, Y is also finite. Let A be a finite full partial subalgebra of B that contains $A^* \cup Y$, let $\widetilde{A}$ be a finite $(\vee,0)$-subsemilattice of $\operatorname{Con_c} B$ containing $\operatorname{Con}_{\mathrm{c}}^{A}(B)$. As $\ell(\vec{x}) \in Y \subseteq A$, it is defined in A, for each $\ell \in \mathscr{L}$ and each $\operatorname{ar}(\ell)$-tuple $\vec{X}$ of A.

(2) Let $s \in S$. As ϕ is surjective, there exists $\theta \in \operatorname{Con_c} B$ such that $s = \phi(\theta)$. So there exist $n < \omega$ and n-tuple $\vec{x}, \vec{y}$ of B such that $s = \phi(\bigvee_{k<n} \Theta_B(x_k, y_k))$. Put $X_s = \{x_0, \dots, x_{n-1}, y_0, \dots, y_{n-1}\}$.

Put $X = \bigcup_{s\in S} X_s$. As S is finite and X_s is finite for all $s \in S$, X is finite. Let A^* be a finite full partial subalgebra of B that contains X. Put $Y = \emptyset$. Let A be a finite full partial subalgebra of B that contains $A^* \cup Y$. Let $\widetilde{A}$ be a finite $(\vee,0)$-subsemilattice of $\operatorname{Con_c} B$ containing $\operatorname{Con}_{\mathrm{c}}^{A}(B)$. By construction $(A^*, A, \Theta_B, \widetilde{A})$ satisfies (2).

The proof that (3) is a locally finite property is similar.

(4) Put $X = \emptyset$, let A^* be a finite full partial subalgebra of B. Denote by E the set of all quadruples $(x, y, \vec{x}, \vec{y})$ such that the following statements are satisfied:

- $x, y \in A^*$.
- $\vec{X}$ and $\vec{y}$ are m-tuples of A^*, for some $m < \omega$.
- $\phi(\Theta_B(x,y)) \leq \bigvee_{k<m} \phi(\Theta_B(x_k, y_k))$.
- $(x_i, y_i) \neq (x_j, y_j)$ for all $i < j < m$.

Put $I = \ker_0 \phi$ and let $(x, y, \vec{x}, \vec{y}) \in E$. As ϕ is ideal-induced, there exists $\alpha \in I$ such that $\Theta_B(x,y) \leq \bigvee_{k<m} \Theta_B(x_k, y_k) \vee \alpha$. Let $\vec{x}', \vec{y}'$ be p-tuples of B such that $\alpha = \bigvee_{k<p} \Theta(x'_k, y'_k)$. Hence:

$$\Theta_B(x,y) \leq \bigvee_{k<m} \Theta_B(x_k, y_k) \vee \bigvee_{k<p} \Theta_B(x'_k, y'_k).$$

It follows from Lemma 5.11 that there are a positive integer n, a list $\vec{z}$ of parameters from B, and terms $t_0, \ldots, t_n$ such that the following equalities hold in B:

$$x = t_0(\vec{x}, \vec{x}', \vec{y}, \vec{y}', \vec{z}), \tag{7.1}$$

$$y = t_n(\vec{x}, \vec{x}', \vec{y}, \vec{y}', \vec{z}), \tag{7.2}$$

$$t_j(\vec{y}, \vec{y}', \vec{x}, \vec{x}', \vec{z}) = t_{j+1}(\vec{x}, \vec{x}', \vec{y}, \vec{y}', \vec{z}), \quad \text{for all } j < n. \tag{7.3}$$

Put $t'_j(\vec{a}, \vec{b}, \vec{c}, \vec{d}) = t_j(\vec{a}, \vec{d}, \vec{b}, \vec{d}, \vec{c})$ for all tuples $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ of B of appropriate length. The following inequalities follow from the compatibility of Θ_B with terms:

$$\Theta_B(t'_j(\vec{x}, \vec{y}, \vec{z}, \vec{x}'), t_j(\vec{x}, \vec{x}', \vec{y}, \vec{y}', \vec{z})) \leq \bigvee_{k<p} \Theta_B(x'_k, y'_k) = \alpha, \tag{7.4}$$

$$\Theta_B(t'_j(\vec{y}, \vec{x}, \vec{z}, \vec{x}'), t_j(\vec{y}, \vec{y}', \vec{x}, \vec{x}', \vec{z})) \leq \bigvee_{k<p} \Theta_B(x'_k, y'_k) = \alpha. \tag{7.5}$$

As $\phi(\alpha) = 0$, it follows from (7.4) and (7.1) that:

$$\phi(\Theta_B(x, t'_0(\vec{x}, \vec{y}, \vec{z}, \vec{x}'))) = 0$$

Set $\vec{z}' = (\vec{z}, \vec{x}')$. As $\phi(\alpha) = 0$, it follows from (7.1)-(7.5) that

$$\phi(\Theta_B(x, t'_0(\vec{x}, \vec{y}, \vec{z}'))) = 0,$$
$$\phi(\Theta_B(y, t'_n(\vec{x}, \vec{y}, \vec{z}'))) = 0,$$
$$\phi(\Theta_B(t'_j(\vec{y}, \vec{x}, \vec{z}'), t'_{j+1}(\vec{x}, \vec{y}, \vec{z}'))) = 0 \quad (\text{for all } j < n).$$

Let $Y_{(x,y,\vec{x},\vec{y})}$ be a finite partial subalgebra of B such that both $t'_j(\vec{x}, \vec{y}, \vec{z}, \vec{x}')$ and $t'_j(\vec{y}, \vec{x}, \vec{z}, \vec{x}')$ are defined in $Y_{(x,y,\vec{x},\vec{y})}$, for each $j \leq n$.

Put $Y = \bigcup(Y_e \mid e \in E)$. As Y_e is finite for each $e \in E$ and E is finite, Y is finite. Let A be a finite full partial subalgebra of B that contains $Y \cup A^*$, and let $\widetilde{A}$ be a finite $(\vee, 0)$-subsemilattice of $\mathrm{Con_c}\, B$ containing $\mathrm{Con}_{\mathrm{c}}^{A}(B)$. It is not hard to verify that $(A^*, A, \Theta_B, \widetilde{A})$ satisfies (4).

(5) As U is finite, the set $X_1 = \langle f(U) \rangle_B^1$ (cf. Notation 4.7) is also finite. As $\widetilde{U}$ is finite and each element of $\widetilde{f}(\widetilde{U})$ is a compact congruence of B, we can choose a finite subset X_2 of B such that $\widetilde{f}(\widetilde{U}) \subseteq \mathrm{Con}_{\mathrm{c}}^{X_2}(B)$. The set $X = X_1 \cup X_2$ is finite. Let A^* be a finite full partial subalgebra of B containing X. Put $Y = \emptyset$, let A be a full partial subalgebra of B containing $Y \cup A^*$, and let $\widetilde{A}$ be a finite $(\vee, 0)$-subsemilattice of $\mathrm{Con_c}\, B$ containing $\mathrm{Con}_{\mathrm{c}}^{A}(B)$. The following containments hold:

$$\widetilde{f}(\widetilde{U}) \subseteq \mathrm{Con}_{\mathrm{c}}^{X_2}(B) \subseteq \mathrm{Con}_{\mathrm{c}}^{A}(B) \subseteq \widetilde{A}.$$

Moreover $\langle f(U) \rangle_B^1 \subseteq A^*$.

The proofs that (6), (7), and (8) are locally finite properties are similar. □

The following lemma is an analogue for gamps of the *Buttress Lemma* (cf. [5]).

Lemma 7.8. *Let $\mathcal{V}$ be a variety of algebras in a finite similarity type, let P be a lower finite poset, let $(S_p)_{p\in P}$ be a family of finite $(\vee,0)$-semilattices, let $B\in\mathcal{V}$, and let $(\phi_p)_{p\in P}$ be a family of $(\vee,0)$-homomorphisms where $\phi_p\colon \operatorname{Con_c} B\to S_p$ is ideal-induced for each $p\in P$. There exists a diagram $\vec{\boldsymbol{A}}=(\boldsymbol{A}_p,\boldsymbol{f}_{p,q}\mid p\le q \text{ in } P)$ of finite subgamps of $\boldsymbol{G}(B)$ such that the following statements hold:*

1. *$\phi_p\restriction\widetilde{A}_p$ is ideal-induced for each $p\in P$.*
2. *$\boldsymbol{A}_p$ is strong, distance-generated through $\phi_p\restriction\widetilde{A}_p$, and congruence-tractable through $\phi_p\restriction\widetilde{A}_p$, for each $p\in P$.*
3. *$\boldsymbol{A}_p$ is a subgamp of $\boldsymbol{A}_q$ and $\boldsymbol{f}_{p,q}$ is the canonical embedding, for all $p\le q$ in P.*
4. *$\boldsymbol{f}_{p,q}$ is strong and congruence-cuttable through $\phi_q\restriction\widetilde{A}_q$, for all $p<q$ in P.*

If B is a lattice, then we can construct $\vec{\boldsymbol{A}}$ such that $\boldsymbol{f}_{p,q}$ is congruence-cuttable with chains through $\phi_q\restriction\widetilde{A}_q$ for all $p<q$ in P, and $\boldsymbol{B}_p$ is distance-generated with chains through ϕ_p for each $p\in P$.

If B is congruence n-permutable (where n is a positive integer), then we can construct $\vec{\boldsymbol{A}}$ such that $\boldsymbol{A}_p$ is congruence n-permutable for each $p\in P$.

Proof. Let $r\in P$, suppose having constructed a diagram $(\boldsymbol{A}_p,\boldsymbol{f}_{p,q}\mid p\le q<r)$ of finite subgamps of $\boldsymbol{G}(B)$ such that the following statements hold:

1. $\phi_p\restriction\widetilde{A}_p$ is ideal-induced for each $p<r$.
2. $\boldsymbol{A}_p$ is strong, distance-generated through $\phi_p\restriction\widetilde{A}_p$, and congruence-tractable through $\phi_p\restriction\widetilde{A}_p$, for each $p<r$.
3. $\boldsymbol{A}_p$ is a subgamp of $\boldsymbol{A}_q$ and $\boldsymbol{f}_{p,q}$ is the canonical embedding, for all $p\le q<r$.
4. $\boldsymbol{f}_{p,q}$ is strong and congruence-cuttable through $\phi_q\restriction\widetilde{A}_q$, for all $p<q<r$.

The following property in $\boldsymbol{A}$ a subgamp of $\boldsymbol{G}(B)$ is locally finite (see Proposition 7.6 and Lemma 7.7)

(F) $\boldsymbol{A}$ is strong, distance-generated through $\phi_r\restriction\widetilde{A}$, congruence-tractable through $\phi_r\restriction\widetilde{A}$, and the canonical embedding $\boldsymbol{f}\colon\boldsymbol{A}_p\to\boldsymbol{A}$ is strong and congruence-cuttable through $\phi_r\restriction\widetilde{A}$, for all $p<r$.

Thus there exist finite partial subalgebras A_r^* and A_r of B such that for each $\widetilde{A}$ containing $\operatorname{Con}_{\mathrm{c}}^{A_r}(B)$, the gamp $(A_r^*,A_r,\Theta_B,\widetilde{A})$ satisfies (F). Moreover it follows from Proposition 3.16 that there exists a finite $(\vee,0)$-subsemilattice $\widetilde{A}_r$ of $\operatorname{Con_c} B$, such that $\operatorname{Con}_{\mathrm{c}}^{A_r}(B)\subseteq\widetilde{A}_r$ and $\phi_r\restriction\widetilde{A}_r$ is ideal-induced.

Set $\boldsymbol{A}_r=(A_r^*,A_r,\Theta_B,\widetilde{A}_r)$, and $\boldsymbol{f}_{p,r}$ the canonical embedding for each $p\le r$. By construction, the diagram $(\boldsymbol{A}_p,\boldsymbol{f}_{p,q}\mid p\le q\le r)$ satisfies the required conditions. The construction of $(\boldsymbol{A}_p,\boldsymbol{f}_{p,q}\mid p\le q \text{ in } P)$ follows by induction.

If B is a lattice, then we can add to the property (F) the condition $\boldsymbol{f}\colon\boldsymbol{A}_p\to\boldsymbol{A}$ is congruence-cuttable with chains through $\phi_r\restriction\widetilde{A}$ for each $p<r$, and $\boldsymbol{B}_r$ is distance-generated with chains through ϕ_r.

If B is congruence n-permutable, then we can add to the property (F) the condition $\boldsymbol{A}_r$ is congruence n-permutable. □

Remark 7.9. In the context of Lemma 7.8, if we have a locally finite property for B, then we can assume that any A_p satisfies this property.

8. Norm-coverings and lifters

The aim of this section is to construct a (family) of posets, which we shall use later as an index for a diagram (in [4]). We also give a combinatorial statement that is satisfied by this poset.

We introduced the following definition in [3].

Definition 8.1. A finite subset V of a poset U is a *kernel* if for every $u \in U$, there exists a largest element $v \in V$ such that $v \leq u$. We say that U is *supported* if every finite subset of U is contained in a kernel of U.

It is not hard to verify that this definition of a supported poset is equivalent to the one used in [5].

The following definition introduced in [3] also appears in [5] in a weaker form. Nevertheless, in the context of $\aleph_0$-lifters (cf. Definition 8.3), all these definitions are equivalent.

Definition 8.2. A *norm-covering* of a poset P is a pair (U, ∂), where U is a supported poset and $\partial \colon U \to P$, $u \mapsto \partial u$ is an isotone map.

We say that an ideal $\boldsymbol{u}$ of U is *sharp* if the set $\{\partial u \mid u \in \boldsymbol{u}\}$ has a largest element, which we shall then denote by $\partial \boldsymbol{u}$. We shall denote by $\mathrm{Id_s}\, U$ the set of all sharp ideals of U, partially ordered by inclusion.

We remind the reader about the following definition introduced in [5].

Definition 8.3. Let P be a poset. An $\aleph_0$*-lifter* of P is a pair $(U, \boldsymbol{U})$, where U is a norm-covering of P and $\boldsymbol{U}$ is a subset of $\mathrm{Id_s}\, U$ satisfying the following properties:

(1) The set $\boldsymbol{U}^{=} = \{\boldsymbol{u} \in \boldsymbol{U} \mid \partial \boldsymbol{u} \text{ is not maximal in } P\}$ is lower finite, that is, the set $\boldsymbol{U} \downarrow \boldsymbol{u}$ is finite for each $\boldsymbol{u} \in \boldsymbol{U}^{=}$.

(2) For every map $S \colon \boldsymbol{U}^{=} \to [U]^{<\omega}$, there exists an isotone map $\sigma \colon P \to \boldsymbol{U}$ such that
   (a) the map σ is a *section* of ∂, that is, $\partial\sigma(p) = p$ holds for each $p \in P$;
   (b) the containment $S(\sigma(p)) \cap \sigma(q) \subseteq \sigma(p)$ holds for all $p < q$ in P. (*Observe that $\sigma(p)$ belongs to $\boldsymbol{U}^{=}$.*)

The existence of lifters is related to the following infinite combinatorial statement introduced in [6].

Definition 8.4. For cardinals κ, λ and a poset P, let $(\kappa, {<}\lambda) \leadsto P$ hold if for every mapping $F \colon \mathfrak{P}(\kappa) \to [\kappa]^{<\lambda}$, there exists a one-to-one map $f \colon P \hookrightarrow \kappa$ such that

$$F(f(P \downarrow p)) \cap f(P \downarrow q) \subseteq f(P \downarrow p)\,, \qquad \text{for all } p \leq q \text{ in } P\,.$$

Notice that in case P is lower finite, it is sufficient to verify the conclusion above for all $F \colon [\kappa]^{<\omega} \to [\kappa]^{<\lambda}$ isotone and all $p \prec q$ in P.

Lemma 8.5. *The square poset has an $\aleph_0$-lifter $(X, \boldsymbol{X})$ such that* $\operatorname{card} X = \aleph_1$.

Proof. By [6, Proposition 4.7], the Kuratowski index (cf. [6, Definition 4.1]) of the square P is less than or equal to its order-dimension, which is equal to 2. Hence, by the definition of the Kuratowski index, $(\kappa^+, {<}\kappa) \leadsto P$ for every infinite cardinal κ. Therefore, by [5, Corollary 3-5.8], P has an $\aleph_0$-lifter $(X, \boldsymbol{X})$ such that $\operatorname{card} X = \operatorname{card} \boldsymbol{X} = \aleph_1$. □

Given a poset P, we introduce a new poset which looks like a lexicographical product of P with a tree. This construction is mainly used in [4].

Definition 8.6. Let P be a poset with a smallest element, let $X \subseteq P$, let $\vec{R} = (R_x)_{x \in X}$ be a family of sets, let $\alpha \le \omega$. Consider the following poset:

$$T = \{(n, \vec{x}, \vec{r}) \mid n < \alpha,\ \vec{x} \in X^n,\ \text{and } \vec{r} \in R_{x_0} \times \cdots \times R_{x_{n-1}}\},$$

ordered by $(m, \vec{x}, \vec{r}) \le (n, \vec{y}, \vec{s})$ if and only if $m \le n$, $\vec{x} = \vec{y} \restriction m$, and $\vec{r} = \vec{s} \restriction m$. Recall that $\vec{y} \restriction m = (y_0, \dots, y_{m-1})$. Given $t = (n, \vec{x}, \vec{r}) \in T$ and $m \le n$, we set $t \restriction m = (m, \vec{x} \restriction m, \vec{r} \restriction m)$.

Put:

$$A = P \boxtimes_{\vec{R}} \alpha = T \times P = \bigcup_{n \in \alpha} \bigcup_{\vec{x} \in X^n} \Big(\{n\} \times \{\vec{x}\} \times (R_{x_0} \times \cdots \times R_{x_{n-1}}) \times P\Big).$$

Any element of A can be written $(n, \vec{x}, \vec{r}, p)$ with $n < \alpha$, $\vec{x} \in X^n$ and $\vec{r} \in R_{x_0} \times \cdots \times R_{x_{n-1}}$.

We define an order on A by $(m, \vec{x}, \vec{r}, p) \le (n, \vec{y}, \vec{s}, q)$ if and only if the following conditions hold:

(1) $(m, \vec{x}, \vec{r}) \le (n, \vec{y}, \vec{s})$.
(2) If $m = n$ then $p \le q$.
(3) If $m < n$ then $p \le y_m$.

Remark 8.7. In the context of Definition 8.6, notice that T is a lower finite tree. Indeed, if $(n, \vec{x}, \vec{r}) \in T$, then $T \downarrow (n, \vec{x}, \vec{r}) = \{(m, \vec{x} \restriction m, \vec{r} \restriction m) \mid m \le n\}$ is a chain of length n. The tree T is called *the tree associated to* $P \boxtimes_{\vec{R}} \alpha$.

The following statements hold:

(1) If P is lower finite, then A is lower finite.
(2) The inequality $\operatorname{card} A \le \aleph_0 + \operatorname{card} P + \sum_{x \in X} \operatorname{card} R_x$ holds.
(3) The inequality $\operatorname{card} T \le \aleph_0 + \sum_{x \in X} \operatorname{card} R_x$ holds.

Remark 8.8. In the context of Definition 8.6, if $a < b$ in A, then there are $t = (n, \vec{x}, \vec{r}) \in T$, $p, q \in P$, and $m \le n$ such that $a = (t \restriction m, p)$ and $b = (t, q)$. Moreover, if $m < n$, then $(t \restriction m, p) < (t \restriction (m+1), 0) \le (t, q)$. It follows easily that $a \prec b$ if and only if exactly one of the following statements holds:

(1) $m = n$ and $p \prec q$.
(2) $n = m + 1$, $p = x_m$, and $q = 0$.

As a consequence, we obtain immediately that *If P, X, and all R_x are finite, then each $a \in A$ has only finitely many covers.*

Lemma 8.9. *Let $\kappa \ge \lambda$ be infinite cardinals, let P be a lower finite κ-small poset with a smallest element, let $X \subseteq P$, let $\vec{R} = (R_x)_{x \in X}$ be a family of κ-small sets, and let $\alpha \le \omega$. If $(\kappa, <\lambda) \rightsquigarrow P$, then $(\kappa, <\lambda) \rightsquigarrow P \boxtimes_{\vec{R}} \alpha$.*

Proof. Denote by T the tree associated to $A = P \boxtimes_{\vec{R}} \alpha$. It follows from Remark 8.7 together with the assumptions on cardinalities that the following inequalities hold:

$$\operatorname{card} T \le \aleph_0 + \sum_{x \in X} \operatorname{card} R_x \le \aleph_0 + \sum_{x \in X} \kappa \le \kappa.$$

Thus there exists a partition $(K_t)_{t \in T}$ of κ such that $\operatorname{card} K_t = \kappa$ for each $t \in T$.

Notice that A is lower finite. Let $F\colon [\kappa]^{<\omega} \to [\kappa]^{<\lambda}$ isotone, let $t \in T$. Assume having constructed, for each $s < t$, a one-to-one map $\sigma_s\colon P \hookrightarrow K_s$ such that setting

$$S_s = \{\sigma_{s\restriction m}(p) \mid m < n \text{ and } p \leq x_m\}, \text{ for all } s = (n, \vec{x}, \vec{r}) \leq t,$$

the following containments hold:

$$\operatorname{rng}\sigma_s \subseteq K_s - F(S_s), \quad \text{for each } s < t,$$
$$F\bigl(\sigma_s(P\downarrow p) \cup S_s\bigr) \cap \sigma_s(P\downarrow q) \subseteq \sigma_s(P\downarrow p), \qquad \text{for all } p \leq q \text{ in } P \text{ and all } s < t.$$

Put $F_t(U) = F(U \cup S_t) - F(S_t)$ for each $U \in [K_t - F(S_t)]^{<\omega}$. As S_t is finite, this defines a map $F_t\colon [K_t - F(S_t)]^{<\omega} \to [K_t - F(S_t)]^{<\lambda}$. As $F(S_t)$ is λ-small, $\operatorname{card}(K_t - F(S_t)) = \kappa$, moreover $(\kappa, {<\lambda}) \rightsquigarrow P$, so there exists a one-to-one map $\sigma_t\colon P \hookrightarrow K_t - F(S_t)$ such that:

$$F\bigl(\sigma_t(P\downarrow p) \cup S_t\bigr) \cap \sigma_t(P\downarrow q) \subseteq \sigma_t(P\downarrow p), \quad \text{for all } p \leq q \text{ in } P.$$

Therefore we construct, by induction on t, a one-to-one map $\sigma_t\colon P \hookrightarrow K_t$ for each $t \in T$, such that setting

$$S_t = \{\sigma_{t\restriction m}(p) \mid m < n \text{ and } p \leq x_m\}, \text{ for all } t = (n, \vec{x}, \vec{r}) \in T, \tag{8.1}$$

the following containments hold:

$$\operatorname{rng}\sigma_t \subseteq K_t - F(S_t), \quad \text{for each } t \in T, \tag{8.2}$$
$$F\bigl(\sigma_t(P\downarrow p) \cup S_t\bigr) \cap \sigma_t(P\downarrow q) \subseteq \sigma_t(P\downarrow p), \qquad \text{for all } p \leq q \text{ in } P \text{ and all } t \in T. \tag{8.3}$$

For $(t, p) \in A$, set $\sigma(t, p) = \sigma_t(p)$. This defines a map $\sigma\colon A \to \kappa$. Let $a = (s, p)$ and $b = (t, q)$ in A such that $\sigma(a) = \sigma(b)$. It follows from (8.2) that $\sigma(a) \in K_s$ and $\sigma(b) \in K_t$. As $(K_u)_{u\in T}$ is a partition of κ, we obtain $s = t$. Moreover $\sigma_t(p) = \sigma(a) = \sigma(b) = \sigma_t(q)$, so, as σ_t is one-to-one, $p = q$, and so $a = b$. Therefore σ is one-to-one.

Let $a = (t, p) \in A$, with $t = (n, \vec{x}, \vec{r}) \in T$. It follows from the definition of A that:

$$A\downarrow a = \{(t\restriction m, q) \in A \mid m < n \text{ and } q \in P\downarrow x_m\} \cup \{(t, q) \in A \mid q \in P\downarrow p\}.$$

Thus, from (8.1), we see that

$$\sigma(A\downarrow a) = S_t \cup \sigma_t(P\downarrow p), \quad \text{for each } a = (t, p) \in A. \tag{8.4}$$

As $(K_t)_{t\in T}$ is a partition, it follows from (8.1) and (8.2) that $K_t \cap S_t = \emptyset$, thus, by (8.4),

$$\sigma(A\downarrow a) \cap K_t = \sigma_t(P\downarrow p), \quad \text{for each } a = (t, p) \in A. \tag{8.5}$$

Let $a \prec b$ in A. There are two cases to consider (cf. Remark 8.8). First assume that $a = (t, p)$ and $b = (t, q)$ with $p \prec q$ and $t = (n, \vec{x}, \vec{r}) \in T$. Let $c \leq b$ with $\sigma(c) \in F(\sigma(A\downarrow a))$. We can write $c = (t\restriction m, p')$, with $m \leq n$. Suppose first that $m < n$. As $c \leq b$, $p' \leq x_m$, thus $c < a$. So $\sigma(c) \in \sigma(A\downarrow a)$. Now suppose that $m = n$. It follows from (8.2) that

$$\sigma(c) = \sigma_t(p') \in F(\sigma(A\downarrow a)) \cap \sigma(A\downarrow b) \cap (K_t - F(S_t)).$$

So, from (8.4) and (8.5) we obtain

$$\sigma(c) \in F(\sigma_t(P\downarrow p) \cup S_t) \cap \sigma_t(P\downarrow q).$$

Thus (8.3) implies that $\sigma(c) \in \sigma_t(P \downarrow p)$, from (8.4) we obtain $\sigma(c) \in \sigma(A \downarrow a)$. Therefore the containment $F(\sigma(A \downarrow a)) \cap \sigma(A \downarrow b) \subseteq \sigma(A \downarrow a)$ holds.

Now assume that $t = (n+1, \vec{x}, \vec{r}) \in T$, $b = (t, 0)$ and $a = (t \upharpoonright n, x_n)$. Let $c \leq b$ such that $\sigma(c) \in F(\sigma(A \downarrow a))$. As $c \leq b$ there are $m \leq n+1$ and $p \in P$ such that $c = (t \upharpoonright m, p)$. If $m \leq n$, then $p \leq x_m$, thus $c \leq a$, so $\sigma(c) \in \sigma(A \downarrow a)$. If $m = n+1$, then $c = b$. From (8.1) and (8.4) we obtain $\sigma(A \downarrow a) = S_{t\upharpoonright n} \cup \sigma_{t\upharpoonright n}(P \downarrow x_n) = S_t$. Therefore $\sigma_t(0) = \sigma(c) \in F(S_t)$, in contradiction with (8.2). So the containment $F(\sigma(A \downarrow a)) \cap \sigma(A \downarrow b) \subseteq \sigma(A \downarrow a)$ holds. □

Corollary 8.10. *For an integer $m > 1$, put*

$$\mathrm{B}_m(\leq 2) = \{X \in \mathfrak{P}(m) \mid \textit{either } \operatorname{card} X \leq 2 \textit{ or } X = m\}.$$

Let P be a $(\vee, 0)$-semilattice embeddable, as a poset, into $\mathrm{B}_m(\leq 2)$. Let $X \subseteq P$, let $\vec{R} = (R_x)_{x \in X}$ be a family of finite sets, and let $\alpha \leq \omega$. There exists an $\aleph_0$-lifter $(U, \boldsymbol{U})$ of $A = P \boxtimes_{\vec{R}} \alpha$ such that U has cardinality $\aleph_2$.

Proof. It follows from [8], see also [2, Theorem 46.2], that $(\aleph_2, 2, \aleph_0) \to m$, so [6, Proposition 5.2] implies $(\aleph_2, <\aleph_0) \leadsto \mathrm{B}_m(\leq 2)$, and so, from [6, Lemma 3.2] we obtain $(\aleph_2, <\aleph_0) \leadsto P$. It follows from Lemma 8.9 that $(\aleph_2, <\aleph_0) \leadsto A$. The conclusion follows from Remark 8.8 together with [5, Lemma 3-5.5]. □

9. The Condensate Lifting Lemma for gamps

In this section we apply the Armature Lemma from [5] together with Lemma 7.8 to prove a special case of the Condensate Lifting Lemma for gamps.

In order to use the *condensate* constructions, we need categories and functors that satisfy the following conditions.

Definition 9.1. Let $\mathcal{A}$ and $\mathcal{S}$ be categories, let $\Phi\colon \mathcal{A} \to \mathcal{S}$ be a functor. We introduce the following statements:

(CLOS) $\mathcal{A}$ has all small directed colimits.
(PROD) Any two objects of $\mathcal{A}$ have a product in $\mathcal{A}$.
(CONT) The functor Φ preserves all small directed colimits.

Remark 9.2. Given a norm-covering X of a poset P and a category $\mathcal{A}$ that satisfies both (CLOS) and (PROD), we can construct an object $\mathbf{F}(X) \otimes \vec{A}$ which is a directed colimit of finite products of objects in $\vec{A}$, together with morphisms

$$\pi_{\boldsymbol{x}}^X \otimes \vec{A}\colon \mathbf{F}(X) \otimes \vec{A} \to A_{\partial \boldsymbol{x}}$$

for each $\boldsymbol{x} \in \mathrm{Id_s}\, X$.

Moreover if $\mathcal{A}$ is a class of algebras closed under finite products and directed colimits, then $\pi_{\boldsymbol{x}}^X \otimes \vec{A}$ is surjective, and $\operatorname{card}(\mathbf{F}(X) \otimes \vec{A}) \leq \operatorname{card} X + \sum_{p \in P} A_p$. For more details about this construction, we refer the reader to [5, Chapter 2].

In the following theorem, we refer the reader to Definition 6.14 for the definition of a partial lifting.

Theorem 9.3. *Let $\mathcal{V}$ and $\mathcal{W}$ be varieties of algebras such that $\mathcal{W}$ has finite similarity type, let $(X, \boldsymbol{X})$ be an $\aleph_0$-lifter of a poset P, let $\vec{A} = (A_p, f_{p,q} \mid p \leq q \text{ in } P)$ be a diagram in $\mathcal{V}$ such that $\operatorname{Con_c} A_p$ is finite for each $p \in P^=$, let $B \in \mathcal{W}$ such that $\operatorname{Con_c} B \cong \operatorname{Con_c}\bigl(\mathbf{F}(X) \otimes \vec{A}\bigr)$. Then there exists a partial lifting $\vec{\boldsymbol{B}} = (\boldsymbol{B}_p, \boldsymbol{g}_{p,q} \mid p \leq q \text{ in } P)$ of $\operatorname{Con_c} \circ \vec{A}$ in $\mathcal{W}$ such that $\boldsymbol{B}_p$ is finite for each $p \in P^=$ and $\boldsymbol{B}_p$ is a quotient of $\boldsymbol{G}(B)$ for each $p \in \operatorname{Max} P$.*

Proof. Denote by $\mathcal{S}$ the category of all $(\vee,0)$-semilattices with $(\vee,0)$-homomorphisms. The category $\mathcal{V}$ satisfies (CLOS) and (PROD), so $\mathbf{F}(X)\otimes\vec{A}$ is well-defined (cf. [5, Section 3-1]). The functor $\operatorname{Con_c}$ satisfies (CONT). Let $\chi\colon \operatorname{Con_c} B \to \operatorname{Con_c}\big(\mathbf{F}(X)\otimes\vec{A}\big)$ be an isomorphism and put $\rho_{\boldsymbol{x}} = \operatorname{Con_c}(\pi^X_{\boldsymbol{x}}\otimes\vec{A})\circ\chi$ for each $\boldsymbol{x}\in\boldsymbol{X}$. As $\pi^X_{\boldsymbol{x}}\otimes\vec{A}$ is surjective and χ is an isomorphism, it follows that $\rho_{\boldsymbol{x}}$ is ideal-induced. Notice that $\operatorname{rng}\rho_{\boldsymbol{x}} = \operatorname{Con_c} A_{\partial\boldsymbol{x}}$ is finite.

Lemma 7.8 implies that there exists a diagram $\vec{\boldsymbol{B}} = (\boldsymbol{B}_{\boldsymbol{x}}, \boldsymbol{g}_{\boldsymbol{x},\boldsymbol{y}} \mid \boldsymbol{x}\le\boldsymbol{y} \text{ in } \boldsymbol{X}^{=})$ of finite subgamps of $\boldsymbol{G}(B)$ in $\mathcal{W}$ such that the following statements hold:

(1) $\rho_{\boldsymbol{x}}\restriction\widetilde{B}_{\boldsymbol{x}}$ is ideal-induced for each $\boldsymbol{x}\in\boldsymbol{X}^{=}$.

(2) $\boldsymbol{B}_{\boldsymbol{x}}$ is strong, distance-generated through $\rho_{\boldsymbol{x}}\restriction\widetilde{B}_{\boldsymbol{x}}$, and congruence-tractable through $\rho_{\boldsymbol{x}}\restriction\widetilde{B}_{\boldsymbol{x}}$, for each $\boldsymbol{x}\in\boldsymbol{X}^{=}$.

(3) $\boldsymbol{g}_{\boldsymbol{x},\boldsymbol{y}}$ is the canonical embedding, for all $\boldsymbol{x}\le\boldsymbol{y}$ in $\boldsymbol{X}^{=}$.

(4) $\boldsymbol{g}_{\boldsymbol{x},\boldsymbol{y}}$ is strong and congruence-cuttable through $\rho_{\boldsymbol{y}}\restriction\widetilde{B}_{\boldsymbol{y}}$, for all $\boldsymbol{x}<\boldsymbol{y}$ in $\boldsymbol{X}^{=}$.

We extend this diagram to an $\boldsymbol{X}$-indexed diagram, with $\boldsymbol{B}_{\boldsymbol{y}} = \boldsymbol{G}(B)$ and defining $\boldsymbol{g}_{\boldsymbol{x},\boldsymbol{y}}$ as the canonical embedding for each $\boldsymbol{y}\in\boldsymbol{X}-\boldsymbol{X}^{=}$ and each $\boldsymbol{x}\le\boldsymbol{y}$. Thus $\boldsymbol{C}\circ\boldsymbol{B}$ is an $\boldsymbol{X}$-indexed diagram in the comma category $\mathcal{S}{\downarrow}\operatorname{Con_c} B$, moreover $\boldsymbol{C}(\boldsymbol{B}_{\boldsymbol{x}}) = \widetilde{B}_{\boldsymbol{x}}$ is finite for each $\boldsymbol{x}\in\boldsymbol{X}^{=}$. Therefore, it follows from the Armature Lemma [5] that there exists an isotone section $\sigma\colon P\hookrightarrow\boldsymbol{X}$ such that the family $(\rho_{\sigma(p)}\restriction\widetilde{B}_{\sigma(p)})_{p\in P}$ is a natural transformation from $(\boldsymbol{C}(\boldsymbol{B}_{\sigma(p)}), \boldsymbol{C}(\boldsymbol{g}_{\sigma(p),\sigma(q)}) \mid p\le q \text{ in } P)$ to $\operatorname{Con_c}\circ\vec{A}$.

Put $\vec{\boldsymbol{B}}' = (\boldsymbol{B}_{\sigma(p)}, \boldsymbol{g}_{\sigma(p),\sigma(q)} \mid p\le q \text{ in } P)$ and $I_p = \ker_0\rho_{\sigma(p)}$, for each $p\in P$. This defines an ideal $\vec{I} = (I_p)_{p\in P}$ of $\vec{\boldsymbol{B}}'$. Moreover, as $\rho_{\sigma(p)}\restriction\widetilde{B}_{\sigma(p)}$ is ideal-induced for each $p\in P$, these morphisms induce a natural equivalence $(\boldsymbol{C}\circ\vec{\boldsymbol{B}}')/\vec{I}\cong\operatorname{Con_c}\circ\vec{\boldsymbol{A}}$ (cf. Lemma 3.13).

Denote by $\boldsymbol{h}_{p,q}\colon \boldsymbol{B}_{\sigma(p)}/I_p\to\boldsymbol{B}_{\sigma(q)}/I_q$ the morphism induced by $\boldsymbol{g}_{\sigma(p),\sigma(q)}$. It follows from Proposition 6.5 that $\boldsymbol{B}_{\sigma(p)}/I_p$ is strong for each $p\in P$, and it follows from Proposition 6.7 that $\boldsymbol{h}_{p,q}$ is strong for all $p<q$ in P. Lemma 7.3 implies that $\boldsymbol{B}_{\sigma(p)}/I_p$ is distance-generated and congruence-tractable for each $p\in P$. From Lemma 7.4 we obtain that $\boldsymbol{h}_{p,q}$ is congruence-cuttable for all $p<q$ in P.

Let $p\in P$, let $\chi\colon\widetilde{B}_{\sigma(p)}\to\operatorname{Con_c} B$ be the inclusion map. As $\boldsymbol{B}_{\sigma(p)}$ is a subgamp of $\boldsymbol{G}(B)$, (B,χ) is a realization of $\boldsymbol{B}_{\sigma(p)}$, thus it induces a realization $(B/\bigvee I_p,\chi')$ where $\chi'\colon\widetilde{B}_p/I_p\to B/\bigvee I$ satisfies $\chi'(d/I_p) = d/\bigvee I_p$ (cf. Proposition 6.5). As $\rho_{\sigma(p)}\restriction\widetilde{B}_{\sigma(p)}$ is ideal-induced it is easy to check that χ' is surjective, hence it defines an isomorphic realization.

Let p a maximal element of P. From $\boldsymbol{B}_{\sigma(p)} = \boldsymbol{G}(B)$ it follows that $\boldsymbol{B}_{\sigma(p)}/I_p$ is a quotient of $\boldsymbol{G}(B)$.

Therefore $\vec{\boldsymbol{B}}'/\vec{I}$ is a partial lifting of $\operatorname{Con_c}\circ\vec{A}$ in $\mathcal{W}$. Moreover, if $p\in P^{=}$ then $\boldsymbol{B}_{\sigma(p)}$ is finite, thus $\boldsymbol{B}'_p/I_p = \boldsymbol{B}_{\sigma(p)}/I_p$ is finite. □

Remark 9.4. Use the notation of Theorem 9.3. A small modification of the proof above shows that if $\mathcal{W}$ is a variety of lattices, then we can construct a lattice partial lifting $\vec{\boldsymbol{B}}$ of $\operatorname{Con_c}\circ\vec{A}$ in $\mathcal{W}$. Moreover, for any integer $n\ge 2$, if B is congruence n-permutable, then all $\boldsymbol{B}_p$ can be chosen congruence n-permutable.

Corollary 9.5. *Let $\mathcal{V}$ be a locally finite variety of algebras, let $\mathcal{W}$ be a variety of algebras with finite similarity type. The following statements are equivalent:*

(1) $\operatorname{crit}(\mathcal{V};\mathcal{W})>\aleph_0$.

(2) *Let T be a countable lower finite tree and let $\vec{A}$ be a T-indexed diagram of finite algebras in $\mathcal{V}$. Then $\mathrm{Con_c}\circ\vec{A}$ has a partial lifting in $\mathcal{W}$.*
(3) *Let $\vec{A}$ be a ω-indexed diagram of finite algebras in $\mathcal{V}$. Then $\mathrm{Con_c}\circ\vec{A}$ has a partial lifting in $\mathcal{W}$.*

Proof. Assume that (1) holds. Let T be a countable lower finite tree and let $\vec{A}$ be a T-indexed diagram of finite algebras in $\mathcal{V}$. It follows from [3, Corollary 4.7] that there exists an $\aleph_0$-lifter $(X, \boldsymbol{X})$ of T such that $\operatorname{card} X = \aleph_0$. Hence the following inequalities hold:

$$\operatorname{card}(\mathbf{F}(X)\otimes\vec{A}) \leq \operatorname{card} X + \sum_{p\in T} A_p \leq \aleph_0 \sum_{p\in T}\aleph_0 = \aleph_0.$$

Thus, as $\operatorname{crit}(\mathcal{V};\mathcal{W}) > \aleph_0$, there exists $B\in\mathcal{W}$ such that $\mathrm{Con_c}\, B \cong \mathrm{Con_c}\big(\mathbf{F}(X)\otimes\vec{A}\big)$. It follows from Theorem 9.3 that there exists a partial lifting of $\mathrm{Con_c}\circ\vec{A}$ in $\mathcal{W}$.

The implication (2) $\Longrightarrow$ (3) is immediate.

Assume that (3) holds and let $A\in\mathcal{V}$ such that $\operatorname{card}\mathrm{Con_c}\, A \leq \aleph_0$. By replacing A with one of its subalgebras we can assume that $\operatorname{card} A \leq \aleph_0$ (see [3, Lemma 3.6]). As $\mathcal{V}$ is locally finite, there exists an increasing sequence $(A_k)_{k<\omega}$ of finite subalgebras of A with union A. Denote by $f_{i,j}\colon A_i \to A_j$ the inclusion map, for all $i \leq j < \omega$. Put $\vec{A} = (A_i, f_{i,j} \mid i \leq j < \omega)$. Let $\vec{\boldsymbol{B}}$ be a partial lifting of $\mathrm{Con_c}\circ\vec{A}$ in $\mathcal{W}$, let $\boldsymbol{B}$ be the directed colimit of $\vec{\boldsymbol{B}}$ in $\mathbf{Gamp}(\mathcal{V})$. As $\boldsymbol{C}$ and $\mathrm{Con_c}$ both preserve directed colimits, the following gamps are isomorphic:

$$\boldsymbol{C}(\boldsymbol{B}) \cong \boldsymbol{C}(\varinjlim \vec{\boldsymbol{B}}) \cong \varinjlim(\boldsymbol{C}\circ\vec{\boldsymbol{B}}) \cong \varinjlim(\mathrm{Con_c}\circ\vec{A}) \cong \mathrm{Con_c}\varinjlim\vec{A} \cong \mathrm{Con_c}\, A.$$

Moreover, it follows from Lemma 6.17 that $\boldsymbol{B}$ is an algebra in $\mathcal{W}$, that is, there exists $B\in\mathcal{W}$ such that $\boldsymbol{B}\cong\boldsymbol{G}(B)$, so $\mathrm{Con_c}\, B = \boldsymbol{C}(\boldsymbol{G}(B)) \cong \boldsymbol{C}(\boldsymbol{B}) \cong \mathrm{Con_c}\, A$. Therefore $\operatorname{crit}(\mathcal{V};\mathcal{W}) > \aleph_0$. $\square$

A variety of algebras is *congruence-proper* if each of its member with a finite congruence lattice is finite (cf. [5, Definition 4-8.1]).

The following theorem is similar to Theorem 9.3 if $\mathcal{W}$ is a congruence-proper variety with a finite similarity type, then we no longer need partial liftings in the statement of the theorem. There is a similar theorem in [5, Theorem 4-9.2]. This new version applies only to varieties (not quasivarieties), but the assumption that $\mathcal{W}$ is locally finite is no longer needed.

Theorem 9.6. *Let $\mathcal{V}$ be a variety of algebras. Let $\mathcal{W}$ be a congruence-proper variety of algebras with a finite similarity type. Let $(X, \boldsymbol{X})$ be an $\aleph_0$-lifter of a poset P. Let $\vec{A} = (A_p, f_{p,q} \mid p \leq q$ in $P)$ be a diagram in $\mathcal{V}$ such that $\mathrm{Con_c}\, A_p$ is finite for each $p \in P^=$. Let $B\in\mathcal{W}$ such that $\mathrm{Con_c}\, B \cong \mathrm{Con_c}\big(\mathbf{F}(X)\otimes\vec{A}\big)$. Then there exists a lifting $\vec{B}$ of $\mathrm{Con_c}\circ\vec{A}$ in $\mathcal{W}$ such that B_p is a quotient of B for each $p\in P$.*

Proof. We notice, as in the beginning of the proof of Theorem 9.3, that $\mathbf{F}(X)\otimes\vec{A}$ is well-defined. We also uses the same notations. Let $\chi\colon \mathrm{Con_c}\, B \to \mathrm{Con_c}\big(\mathbf{F}(X)\otimes\vec{A}\big)$ be an isomorphism. Put $\rho_{\boldsymbol{x}} = \mathrm{Con_c}(\pi^X_{\boldsymbol{x}}\otimes\vec{A})\circ\chi$ for each $\boldsymbol{x}\in\boldsymbol{X}$. As $\pi^X_{\boldsymbol{x}}\otimes\vec{A}$ is surjective and χ is an isomorphism, it follows that $\rho_{\boldsymbol{x}}$ is ideal-induced.

Notice that $\operatorname{rng}\rho_{\boldsymbol{x}} = \mathrm{Con_c}\, A_{\partial\boldsymbol{x}}$ is finite. Hence $\mathrm{Con_c}\, B/\ker_0\rho_{\boldsymbol{x}} \cong \mathrm{Con_c}\, A_{\partial\boldsymbol{x}}$ is finite. As $\mathcal{W}$ is congruence-proper, it follows that $B/\ker_0\rho_{\boldsymbol{x}}$ is finite.

Hence, the property $\boldsymbol{A}/\ker_0\rho_{\boldsymbol{x}} = \boldsymbol{G}(B)/\ker_0\rho_{\boldsymbol{x}}$, in $\boldsymbol{A}$ a subgamp of $\boldsymbol{G}(B)$ is a locally finite property for B. It follows from Lemma 7.8 and Remark 7.9 that there exists a diagram $\vec{\boldsymbol{B}} = (\boldsymbol{B}_{\boldsymbol{x}}, \boldsymbol{g}_{\boldsymbol{x},\boldsymbol{y}} \mid \boldsymbol{x} \leq \boldsymbol{y} \text{ in } \boldsymbol{X}^{=})$ of finite subgamps of $\boldsymbol{G}(B)$ in $\mathcal{W}$, as in the proof of Theorem 9.3, but that satisfies the additional condition:

(5) $\boldsymbol{B}_{\boldsymbol{x}}/\ker_0 \rho_{\boldsymbol{x}} = \boldsymbol{G}(B)/\ker_0 \rho_{\boldsymbol{x}}$, for all $\boldsymbol{x} \in \boldsymbol{X}^{=}$.

We continue with the same argument as the one in the proof of Theorem 9.3. We obtain $\boldsymbol{B}'/\vec{I}$ a partial lifting of $\operatorname{Con_c} \circ \vec{A}$. The following isomorphisms hold

$$\begin{aligned} \boldsymbol{B}'_p/I_p &= \boldsymbol{B}_{\sigma(p)}/\ker_0 \rho_{\sigma(p)} && \text{see proof of Theorem 9.3.} \\ &= \boldsymbol{G}(B)/\ker_0 \rho_{\sigma(p)} && \text{by (5).} \\ &\cong \boldsymbol{G}(B/\ker_0 \rho_{\sigma(p)}) && \text{by Remark 6.9(6).} \end{aligned}$$

Hence $\boldsymbol{B}'_p/I_p$ is an algebra of $\mathcal{V}$ (cf. Remark 6.9(3)), for all $p \in P^{=}$, it is also true for $p \in \operatorname{Max} P$. It follows from Remark 6.9(5) that $\vec{\boldsymbol{B}}'/\vec{I} \cong \boldsymbol{G} \circ \vec{B}$, for a diagram of algebras $\vec{B}$. Hence:

$$\begin{aligned} \operatorname{Con_c} \circ \vec{B} &= \boldsymbol{C} \circ \boldsymbol{G} \circ \vec{B} && \text{by Remark 6.9(1).} \\ &\cong \boldsymbol{C} \circ \vec{\boldsymbol{B}}'/\vec{I} \\ &\cong \operatorname{Con_c} \circ \vec{A} && \text{as } \vec{\boldsymbol{B}}'/\vec{I} \text{ is a partial lifting of } \operatorname{Con_c} \circ \vec{A}. \end{aligned}$$

Hence $\vec{B}$ is a lifting of $\operatorname{Con_c} \circ \vec{A}$. □

The following corollary is an immediate application of Theorem 9.6.

Corollary 9.7. *Let $\mathcal{V}$ be a variety of algebras. Let $\mathcal{W}$ be a congruence-proper variety of algebras with a finite similarity type. Let $(X, \boldsymbol{X})$ be an $\aleph_0$-lifter of a poset P. Let $\vec{A} = (A_p, f_{p,q} \mid p \leq q \text{ in } P)$ be a diagram of finite algebras in $\mathcal{V}$. Assume that $\operatorname{Con_c} \circ \vec{A}$ has no lifting in $\mathcal{W}$, then $\operatorname{crit}(\mathcal{V}; \mathcal{W}) \leq \aleph_0 + \operatorname{card} X$.*

10. An unliftable diagram

Each countable locally finite lattice has a congruence-permutable, congruence-preserving extension (cf. [7]). This is not true for all locally finite lattices. Given a non-distributive variety $\mathcal{V}$ of lattices, there is no congruence-permutable algebra A such that $\operatorname{Con_c} A \cong \operatorname{Con_c} F_{\mathcal{V}}(\aleph_2)$ (cf. [14]). Hence $\operatorname{Con_c} F_{\mathcal{V}}(\aleph_2)$ has no congruence-permutable, congruence-preserving extension. The latter result can be improved, by stating that the free lattice $F_{\mathcal{V}}(\aleph_1)$ has no congruence-permutable, congruence-preserving extension (cf. [5]).

Let $\mathcal{V}$ be a nondistributive variety of lattices. There is no congruence n-permutable lattices L such that $\operatorname{Con_c} L \cong F_{\mathcal{V}}(\aleph_2)$, for each $n \geq 2$ (cf. [12]). In particular $F_{\mathcal{V}}(\aleph_2)$ has no congruence n-permutable, congruence-preserving extension, for each $n \geq 2$. The aim of this section is to improve the cardinality bound to $\aleph_1$. We use gamps to find a lattice of cardinal $\aleph_1$ with no congruence n-permutable, congruence-preserving extension, for each $n \geq 2$. This partially solve [5, Problem 7]. The proof is based on a square-indexed diagram of lattices with no congruence n-permutable, congruence-preserving extension.

Let $\vec{A} = (A_p, f_{p,q} \mid p \leq q \text{ in } P)$ be a diagram of algebras, let $\vec{B} = (B_p, g_{p,q} \mid p \leq q \text{ in } P)$ be a congruence-preserving extension of $\vec{A}$. Then $\boldsymbol{B}_p = (A_p, B_p, \Theta_{B_p}, \operatorname{Con_c} B_p)$ is a gamp and $\boldsymbol{g}_{p,q} = (g_{p,q}, \operatorname{Con_c} g_{p,q}) \colon \boldsymbol{B}_p \to \boldsymbol{B}_q$ is a morphism of gamps, for all $p \leq q$ in P. This defines a diagram $\vec{\boldsymbol{B}}$ of gamps. Moreover, identifying $\operatorname{Con_c} A_p$ and $\operatorname{Con_c} B_p$ for all $p \in P$, we have $\boldsymbol{P}_{\mathrm{gl}} \circ \vec{\boldsymbol{B}} = \boldsymbol{P}_{\mathrm{ga}} \circ \vec{A}$.

Fix $n \geq 2$ an integer. Given a diagram $\vec{A}$ of algebras, with a congruence n-permutable, congruence-preserving extension, there exists a diagram $\vec{\boldsymbol{B}}$ of congruence n-permutable gamps such that $\boldsymbol{P}_{\mathrm{gl}} \circ \vec{\boldsymbol{B}} = \boldsymbol{P}_{\mathrm{ga}} \circ \vec{A}$. The converse might not hold in general.

However the square-indexed diagram $\vec{A}$ of finite lattices with no congruence n-permutable, congruence-preserving extension, mentioned above, satisfies a stronger property. There is no operational diagram $\vec{\boldsymbol{B}}$ of lattice congruence n-permutable gamps of lattices (cf. Definition 10.1) such that $\boldsymbol{P}_{\mathrm{ga}} \circ \vec{A} \cong \boldsymbol{P}_{\mathrm{gl}} \circ \vec{\boldsymbol{B}}$ (cf. Lemma 10.6).

We conclude, in Theorem 10.7, that there is a condensate of $\vec{A}$, of cardinal $\aleph_1$, with no congruence n-permutable, congruence-preserving extension.

For the purpose of this section, we need a stronger version of congruence n-permutable gamp, specific to gamp of lattices.

Definition 10.1. A gamp $\boldsymbol{A}$ of lattices is *lattice congruence n-permutable* if for all $x_0, \dots, x_n$ in A^* there exist $y_0, \dots, y_n$ in A such that $y_i \wedge y_j = y_j \wedge y_i = y_i$ in A for all $i \leq j \leq n$, $y_0 = x_0 \wedge x_n = x_n \wedge x_0$ in A, $y_n = x_0 \vee x_n = x_n \vee x_0$ in A, and:

$$\delta(y_k, y_{k+1}) \leq \bigvee (\delta(x_i, x_{i+1}) \mid i < n \text{ even}), \quad \text{for all } k < n \text{ odd},$$

$$\delta(y_k, y_{k+1}) \leq \bigvee (\delta(x_i, x_{i+1}) \mid i < n \text{ odd}), \quad \text{for all } k < n \text{ even}.$$

A morphism $\boldsymbol{f} \colon \boldsymbol{A} \to \boldsymbol{B}$ of gamps is *operational* if $\ell(\vec{x})$ is defined in B for all $\ell \in \mathscr{L}$ and all $\mathrm{ar}(\ell)$-tuple $\vec{X}$ in $f(A) \cup B^*$.

Let P be a poset, a diagram $\vec{\boldsymbol{A}} = (\boldsymbol{A}_p, \boldsymbol{f}_{p,q} \mid p \leq q \text{ in } P)$ of gamps is *operational* if $\boldsymbol{f}_{p,q}$ is operational for all $p < q$ in P.

Remark 10.2. In the context of Definition 10.1, the elements $y_1, \dots, y_n$ doe not form a chain in general, as we might have $y_i \notin A^*$ for some i.

In this section we simply say that the gamp is congruence n-permutable instead of lattice congruence n-permutable. We also consider only pregamps and gamps of lattices. The following lemma is immediate, the properties given in Definition 10.1 go to quotients.

Lemma 10.3. *Let $\boldsymbol{A}$ be a congruence n-permutable gamp, let I be an ideal of $\boldsymbol{A}$, then $\boldsymbol{A}/I$ is a congruence n-permutable gamp.*

Let P be a poset, let $\vec{\boldsymbol{A}}$ be an operational P-indexed diagram of algebras. Let $\vec{I}$ be an ideal of $\vec{\boldsymbol{A}}$. The quotient $\boldsymbol{A}/\vec{I}$ is operational.

The following lemma is similar to the Buttress Lemma (cf. [5]) and to Lemma 7.8, this version is specific to the functor $\boldsymbol{P}_{\mathrm{gl}}$.

Lemma 10.4. *Let $n \geq 2$. Let $\mathcal{V}$ be a variety of lattices. Let P be a lower finite poset. Let $(\boldsymbol{A}_p)_{p \in P}$ be a family of finite pregamps. Let $\boldsymbol{B}$ be a gamp in $\mathcal{V}$ such that B is a congruence n-permutable lattice. Let $(\boldsymbol{\pi}_p)_{p \in P}$ be a family of ideal-induced morphisms of pregamps where $\boldsymbol{\pi}_p \colon \boldsymbol{P}_{\mathrm{gl}} \boldsymbol{B} \to \boldsymbol{A}_p$ for all $p \in P$. Then there exists a diagram $\vec{\boldsymbol{B}} = (\boldsymbol{B}_p, \boldsymbol{g}_{p,q} \mid p \leq q \text{ in } P)$ of finite subgamps of $\boldsymbol{B}$ (where $\boldsymbol{g}_{p,q}$ is the canonical embedding for all $p \leq q$ in P), such that the following assertions hold.*

1. *$\boldsymbol{\pi}_p \restriction \boldsymbol{P}_{\mathrm{gl}}(\boldsymbol{B}_p)$ is an ideal-induced morphism of pregamps for all $p \in P$.*
2. *$\boldsymbol{B}_p$ is strong and congruence n-permutable for all $p \in P$.*
3. *$\boldsymbol{g}_{p,q}$ is operational for all $p < q$ in P.*

Proof. Let $r \in P$, assume that we have already constructed $(\boldsymbol{B}_p, \boldsymbol{g}_{p,q} \mid p \leq q < r)$ that satisfies the required conditions up to r.

As $\boldsymbol{\pi}_r$ is ideal-induced, $\pi_r(B^*) = A_r$. Moreover A_r is finite, hence there exists X a finite partial subalgebra of B^* such that $\pi_r(X) = A_r$, put $B_r^* = X \cup \bigcup_{p<r} B_p^*$ with its structure of full partial subalgebra of B^*. It follows that $\pi_r(B_r^*) \subseteq \pi_r(B^*) = A_r = \pi_r(X) \subseteq \pi_r(B_r^*)$, hence $\pi_r(B_r^*) = A_r$. Moreover B_r^* is finite.

Let $x_0, x_1, \dots, x_n$ in B_r^*. As B is a congruence n-permutable lattice, there exist $y_0 < \dots < y_n$ in B such that $y_0 = x_0 \wedge x_n$, $y_n = x_0 \vee x_n$, and

$$\delta(y_k, y_{k+1}) \leq \bigvee(\delta(x_i, x_{i+1}) \mid i < n \text{ even}), \quad \text{for all } k < n \text{ odd},$$

$$\delta(y_k, y_{k+1}) \leq \bigvee(\delta(x_i, x_{i+1}) \mid i < n \text{ odd}), \quad \text{for all } k < n \text{ even}.$$

Put $X_{x_0,x_1,\dots,x_n} = \{y_0, \dots, y_n\}$. We consider the following finite set with its structure of full partial subalgebra of B

$$\begin{aligned} B_r = & \bigcup(X_{x_0,x_1,\dots,x_n} \mid x_0, x_1, \dots, x_n \in B_p^*) \\ & \cup \left\{ x \vee y \mid x, y \in B_r^* \cup \bigcup_{p<r} B_p \right\} \\ & \cup \left\{ x \wedge y \mid x, y \in B_r^* \cup \bigcup_{p<r} B_p \right\}. \end{aligned}$$

Put $Y = \{\delta(x, y) \mid x, y \in B_r\}$. As $\widetilde{\pi}_r$ is ideal-induced, it follows from Proposition 3.16 that there is a finite $(\vee, 0)$-subsemilattice $\widetilde{B}_r$ of $\widetilde{B}$ such that $Y \subseteq \widetilde{B}_r$, and $\widetilde{\pi}_r \restriction \widetilde{B}_r$ is ideal-induced. Put $\boldsymbol{B}_r = (B_r^*, B_r, \delta, \widetilde{B}_r)$, denote by $\boldsymbol{g}_{p,r} \colon \boldsymbol{B}_p \to \boldsymbol{B}_r$ the inclusion morphism for all $p \leq r$. The conditions (1)-(3) are satisfied. The conclusion follows by induction. □

We apply Lemma 10.4 and the Armature Lemma (cf. [5]) to obtain a new, tailor-made version of CLL. Given a diagram $\vec{A}$ of finite lattices and a congruence-preserving, congruence n-permutable extension of a condensate of $\vec{A}$, we obtain a "congruence-preserving, congruence n-permutable extension" of $\vec{A}$.

Lemma 10.5. *Let $\mathcal{V}$ be a variety of lattices. Let $(X, \boldsymbol{X})$ be an $\aleph_0$-lifter of a poset P, let $\vec{A} = (A_p, f_{p,q})$ be a diagram of $\mathcal{V}$ such that $\operatorname{Con_c} A_p$ is finite for all $p \in P^=$. Let B be a congruence n-permutable lattice. If B is a congruence-preserving extension of $\mathbf{F}(X) \otimes \vec{A}$, then there exists an operational diagram $\vec{\boldsymbol{B}}$ of congruence n-permutable gamps such that $\boldsymbol{P}_{\mathrm{gl}} \circ \vec{\boldsymbol{B}} \cong \boldsymbol{P}_{\mathrm{ga}} \circ \vec{A}$.*

Proof. As in the proof of Theorem 9.3, $\mathbf{F}(X) \otimes \vec{A}$ is well-defined. Denote by $\mathcal{S}$ the category of pregamps. The functor $\boldsymbol{P}_{\mathrm{ga}} \colon \mathcal{V} \to \mathcal{S}$ satisfies (CONT), see Remark 5.7. Put $B^* = \mathbf{F}(X) \otimes \vec{A}$, as B is a congruence-preserving extension of B^*, we can identify $\operatorname{Con_c} B^*$ with $\operatorname{Con_c} B$. Put $\boldsymbol{B} = (B^*, B, \Theta_B, \operatorname{Con_c} B)$, hence $\boldsymbol{P}_{\mathrm{gl}}(\boldsymbol{B}) = \boldsymbol{P}_{\mathrm{ga}}(B^*)$. Put $\boldsymbol{\pi}_{\boldsymbol{x}} = \boldsymbol{P}_{\mathrm{ga}}(\pi_{\boldsymbol{x}}^X \otimes \vec{A}) \colon \boldsymbol{P}_{\mathrm{gl}}(\boldsymbol{B}) \to \boldsymbol{P}_{\mathrm{ga}}(\boldsymbol{A})$ for all $\boldsymbol{x} \in \boldsymbol{X}$. It follows from Lemma 10.4, that there exists a diagram $(\boldsymbol{B}_{\boldsymbol{x}}, \boldsymbol{g}_{\boldsymbol{x},\boldsymbol{y}} \mid \boldsymbol{x} \leq \boldsymbol{y} \text{ in } \boldsymbol{X}^=)$ of finite subgamps of $\boldsymbol{B}$ such that the following assertions hold.

(1) $\boldsymbol{\pi}_{\boldsymbol{x}} \restriction \boldsymbol{P}_{\mathrm{gl}}(\boldsymbol{B}_{\boldsymbol{x}})$ is an ideal-induced morphism of pregamps for all $\boldsymbol{x} \in \boldsymbol{X}^=$.
(2) $\boldsymbol{B}_{\boldsymbol{x}}$ is strong and congruence n-permutable for all $\boldsymbol{x} \in \boldsymbol{X}^=$.
(3) $\boldsymbol{g}_{\boldsymbol{x},\boldsymbol{y}}$ is operational for all $\boldsymbol{x} < \boldsymbol{y}$ in $\boldsymbol{X}^=$.

We complete the diagram with $\boldsymbol{B}_{\boldsymbol{y}} = \boldsymbol{B}$ and $\boldsymbol{g}_{\boldsymbol{x},\boldsymbol{y}}$ the inclusion morphism for all $\boldsymbol{y} \in \boldsymbol{X} - \boldsymbol{X}^{=}$, and $\boldsymbol{x} \leq \boldsymbol{y}$. It follows from the Armature Lemma (cf. [5, Lemma 3-2.2]) that there exists $\sigma\colon P \to \boldsymbol{X}$ such that $\partial\sigma(p) = p$ for all $p \in P$ and $(\boldsymbol{\pi}_{\sigma(p)} \restriction \boldsymbol{P}_{\mathrm{gl}}(\boldsymbol{B}_{\sigma(p)}))_{p\in P}$ is a natural transformation from $(\boldsymbol{P}_{\mathrm{gl}}(\boldsymbol{B}_{\sigma(p)}), \boldsymbol{P}_{\mathrm{gl}}(\boldsymbol{g}_{\sigma(p),\sigma(q)}) \mid p \leq q \text{ in } P)$ to $\boldsymbol{P}_{\mathrm{ga}} \circ \vec{A}$.

Put $\boldsymbol{\rho}_p = \boldsymbol{\pi}_{\sigma(p)} \restriction \boldsymbol{P}_{\mathrm{gl}}(\boldsymbol{B}_{\sigma(p)})$, put $I_p = \ker_0 \boldsymbol{\rho}_p$ for all $p \in P$. Put $\vec{I} = (I_p)_{p\in P}$, it is an ideal of $(\boldsymbol{B}_{\sigma(p)}, \boldsymbol{g}_{\sigma(p),\sigma(q)} \mid p \leq q \text{ in } P)$. Put $\vec{\boldsymbol{C}} = (\boldsymbol{B}_{\sigma(p)}/I_p, \boldsymbol{g}_{\sigma(p),\sigma(q)}/\vec{I} \mid p \leq q \text{ in } P)$. Notice that $\vec{\boldsymbol{C}}$ is an operational diagram of congruence n-permutable gamps.

Denote by $\boldsymbol{\chi}_p\colon \boldsymbol{P}_{\mathrm{gl}}(\boldsymbol{B}_{\sigma(p)})/I_p \to \boldsymbol{P}_{\mathrm{ga}}(A_p)$ the morphism induced by $\boldsymbol{\rho}_p$. It follows from Lemma 5.28 that $\boldsymbol{\chi}_p$ is an isomorphism, for all $p \in P$. Hence, from Proposition 5.26 and Remark 6.13, we obtain that $\vec{\boldsymbol{\chi}} = (\boldsymbol{\chi}_p)_{p\in P}\colon \boldsymbol{P}_{\mathrm{gl}} \circ \vec{\boldsymbol{C}} \to \boldsymbol{P}_{\mathrm{ga}} \circ \vec{A}$ is a natural equivalence. □

In the following lemma we construct a square-indexed diagram of lattices with no congruence n-permutable, congruence-preserving extension.

Lemma 10.6. *Let $n \geq 2$. Let K be a nontrivial, finite, congruence $(n+1)$-permutable lattice, let x_1, x_2, x_3 in K such that $x_1 \wedge x_2 = 0$ and $x_3 \vee x_2 = x_3 \vee x_1 = 1$. There exists a diagram $\vec{A}$ of finite congruence $(n+1)$-permutable lattices in $\mathbf{Var}^{0,1}(K)$ indexed by a square, such that $\boldsymbol{P}_{\mathrm{ga}} \circ \vec{A} \ncong \boldsymbol{P}_{\mathrm{gl}} \circ \vec{\boldsymbol{B}}$ for each operational square $\vec{\boldsymbol{B}}$ of congruence n-permutable gamps.*

Proof. Put $X_0 = \{0, x_3, 1\}$, put $X_1 = \{0, x_1 \wedge x_3, x_1, x_3, 1\}$, put $X_2 = \{0, x_2 \wedge x_3, x_2, x_3, 1\}$, put $X_3 = K$. Notice that X_k, $k < 4$, are all congruence $(n+1)$-permutable sublattices of K.

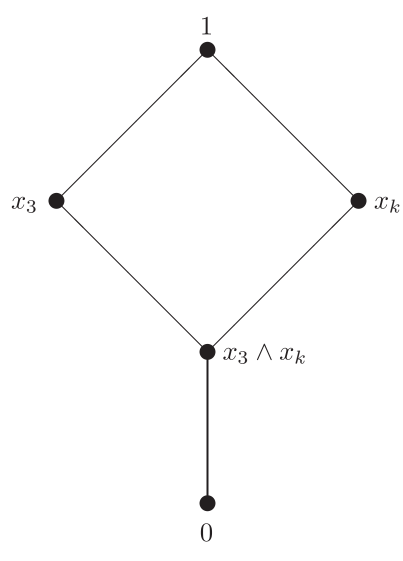

FIGURE 1. The lattice X_k, for $k \in \{1, 2\}$

We denote by $h_i\colon X_0 \to X_i$ and $h'_i\colon X_i \to X_3$ the inclusion maps, for $i = 1, 2$. Denote by $\vec{X}$ the square on the right hand side of Figure 2.

Claim. *Let $\vec{\boldsymbol{B}}$ be a square of operational gamps (as in Figure 3), let $\vec{\boldsymbol{\xi}}\colon \boldsymbol{P}_{\mathrm{ga}} \circ \vec{X} \to \boldsymbol{P}_{\mathrm{gl}} \circ \vec{\boldsymbol{B}}$ be a natural equivalence. Let $y \in B_0$ such that $\delta_{\boldsymbol{B}_0}(\xi_0(0), y) \leq \delta_{\boldsymbol{B}_0}(\xi_0(x_3), \xi_0(1))$, $y \wedge \xi_0(1) = y$, and $y \vee \xi_0(0) = y$ in B_0, then $y = \xi_0(0)$.*

Proof of Claim. We can assume that g_1, g_2, g_1', and g_2' are inclusion maps. We can assume that $\boldsymbol{P}_{\mathrm{gl}} \circ \vec{\boldsymbol{B}} = \boldsymbol{P}_{\mathrm{ga}} \circ \vec{X}$ and $\vec{\boldsymbol{\xi}}$ is the identity. We denote δ_k instead of $\delta_{\boldsymbol{B}_k}$ for all $k \in \{0, 1, 2, 3\}$. Notice that $\delta_k(u, v)$ is a congruence of X_k for all $u, v \in B_k$, moreover if $u, v \in X_k = B_k^*$ then $\delta_k(u, v) = \Theta_{X_k}(u, v)$. Let $y \in B_0$ such that $\delta_0(0, y) \subseteq \delta_0(x_3, 1)$, $y \wedge 1 = y$ and $y \vee 0 = y$ in B_0.

Let $k \in \{1, 2\}$. As $\boldsymbol{g}_k$ is operational, $y \wedge x_k$ is defined in B_k. Moreover $0 \wedge x_k = 0$, hence $\delta_k(0, y \wedge x_k) \subseteq \delta_k(0, y)$. Therefore the following containments hold:

$$\delta_k(y, y \wedge x_k) \subseteq \delta_k(y, 0) \vee \delta_k(0, y \wedge x_k) \subseteq \delta_k(0, y) \subseteq \delta_k(x_3, 1) = \Theta_{X_k}(x_3, 1). \quad (10.1)$$

Moreover, as $y \wedge 1 = y$, the following containment holds

$$\delta_k(y, y \wedge x_k) = \delta_k(y \wedge 1, y \wedge x_k) \subseteq \delta_k(1, x_k) = \Theta_{X_k}(1, x_k) \quad (10.2)$$

However $\Theta_{X_k}(1, x_k) \cap \Theta_{X_k}(x_3, 1) = \mathbf{0}_{X_k}$ (see Figure 1), thus it follows from (10.1) and (10.2) that $\delta_k(y, y \wedge x_k) = \mathbf{0}_{X_k}$, hence the following equality holds

$$y = y \wedge x_k, \quad \text{for each } k \in \{1, 2\}.$$

Therefore $y = (y \wedge x_1) \wedge x_2$ in B_3, moreover $x_1 \wedge x_2 = 0$, hence as $\vec{\boldsymbol{B}}$ is operational $y \wedge (x_1 \wedge x_2)$ is defined in B_3, thus $y \wedge 0 = y \wedge (x_1 \wedge x_2) = (y \wedge x_1) \wedge x_2 = y$.

Moreover as $y \vee 0 = y$, it follows that $0 = (y \vee 0) \wedge 0 = y \wedge 0 = y$ (all elements are defined in B_3), hence $y = 0$. □ Claim.

Let C be an $(n+1)$-element chain. Set $T = \{t \mid t\colon C \twoheadrightarrow X_0 \text{ is order preserving}\}$. Put $A_0 = C$, put $A_1 = X_1^T$, put $A_2 = X_2^T$, put $A_3 = X_3^T = K^T$. We consider the following morphisms:

$$f_i\colon A_0 \to A_i$$
$$x \mapsto (t(x))_{t \in T}, \quad \text{for } i = 1, 2.$$

We denote by $f_i'\colon A_i \to A_3$ the inclusion maps, for $i = 1, 2$. We denote by $\vec{A}$ the square in the left hand side of Figure 2.

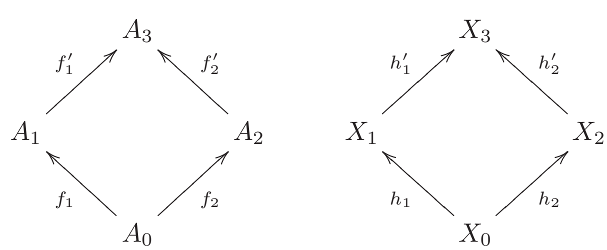

FIGURE 2. Two squares in $\mathbf{Var}^{0,1}(K)$

Given $t \in T$, denote $\pi_0^t = t$ and $\pi_k^t\colon A_k \to X_k$, $(a_p)_{p \in T} \mapsto a_t$ the canonical projection, for all $k \in \{1, 2, 3\}$. It defines a natural transformation $\vec{\pi}^t$ from $\vec{A}$ to $\vec{X}$. We denote by $\boldsymbol{\pi}_k^t = \boldsymbol{P}_{\mathrm{ga}}(\pi_k^t) = (\pi_k^t, \mathrm{Conc}\, \pi_k^t)$, for all $k \in \{0, 1, 2, 3\}$.

Assume that there exists an operational square $\vec{\boldsymbol{B}}$, as in Figure 3, of congruence n-permutable gamps, and a natural equivalence $\boldsymbol{P}_{\mathrm{ga}} \circ \vec{A} \to \boldsymbol{P}_{\mathrm{gl}} \circ \vec{\boldsymbol{B}}$. We can assume that $\boldsymbol{P}_{\mathrm{gl}} \circ \vec{\boldsymbol{B}} = \boldsymbol{P}_{\mathrm{ga}} \circ \vec{A}$. Put $\delta_k = \delta_{\boldsymbol{B}_k}$, the distance $\delta_{\boldsymbol{A}_k}$ is a restriction of δ_k, for all $k \in \{0, 1, 2, 3\}$.

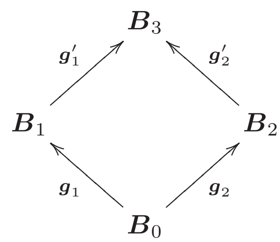

FIGURE 3. A square in the category $\mathbf{Gamp}(\mathcal{L})$

Let $a_0 < a_1 < \cdots < a_n$ be the elements of $C = A_0 = B_0^*$. As $\boldsymbol{B}_0$ is congruence n-permutable, there exist $b_0, \dots, b_n$ in B_0 such that $b_i \wedge b_j = b_j \wedge b_i = b_i$ in B_0 for all $i \leq j \leq n$, $b_0 = a_0 \wedge a_n = a_0$ in B_0, $b_n = a_0 \vee a_n = a_n$ in B_0, and:

$$\delta_0(b_k, b_{k+1}) \leq \bigvee(\delta_0(a_i, a_{i+1}) \mid i < n \text{ even}), \quad \text{for all } k < n \text{ odd}, \tag{10.3}$$

$$\delta_0(b_k, b_{k+1}) \leq \bigvee(\delta_0(a_i, a_{i+1}) \mid i < n \text{ odd}), \quad \text{for all } k < n \text{ even}. \tag{10.4}$$

In particular the following inequality holds

$$\delta_0(b_k, b_{k+1}) \leq \delta_0(a_0, a_k) \vee \delta_0(a_{k+1}, a_n), \quad \text{for all } k < n. \tag{10.5}$$

As $b_n = a_n$, an immediate consequence of (10.5) is $\delta_0(b_{n-1}, a_n) \leq \delta_0(a_0, a_{n-1})$. Assume that $\delta_0(a_{n-2}, b_{n-1}) \leq \delta_0(a_0, a_{n-2})$, it follows that

$$\delta_0(a_0, a_n) \leq \delta_0(a_0, a_{n-2}) \vee \delta_0(a_{n-2}, b_{n-1}) \vee \delta_0(b_{n-1}, a_n) \leq \delta_0(a_0, a_{n-1})$$

That is $\Theta_{A_0}(a_0, a_n) \leq \Theta_{A_0}(a_0, a_{n-1})$ a contradiction, as A_0 is the chain $a_0 < \cdots < a_n$. It follows that $\delta_0(a_{n-2}, b_{n-1}) \nleq \delta_0(a_0, a_{n-2})$.

Take $i < n$ minimal such that the following inequality hold

$$\delta_0(a_i, b_{i+1}) \nleq \delta_0(a_0, a_i). \tag{10.6}$$

If $i = 0$, then $a_0 = b_0 = b_i$, hence $\delta_0(a_0, b_i) \leq \delta_0(a_0, a_i)$. If $i > 0$, then it follows from the minimality of i that $\delta_0(a_{i-1}, b_i) \leq \delta_0(a_0, a_{i-1}) \leq \delta_0(a_0, a_i)$, thus $\delta_0(a_0, b_i) \leq \delta_0(a_0, a_{i-1}) \vee \delta_0(a_{i-1}, b_i) \leq \delta_0(a_0, a_i)$. Therefore the following inequality holds

$$\delta_0(a_0, b_i) \leq \delta_0(a_0, a_i). \tag{10.7}$$

Assume that $\delta_0(b_i, b_{i+1}) \leq \delta_0(a_0, a_i)$, this implies with (10.7) the following inequality

$$\delta_0(a_i, b_{i+1}) \leq \delta_0(a_i, a_0) \vee \delta_0(a_0, b_i) \vee \delta_0(b_i, b_{i+1}) = \delta_0(a_0, a_i),$$

which contradicts (10.6). Therefore the following statement holds

$$\delta_0(b_i, b_{i+1}) \nleq \delta_0(a_0, a_i). \tag{10.8}$$

As $\operatorname{Con} A_0$ is a Boolean lattice with atoms $\delta_0(a_k, a_{k+1})$ for $k < n$, it follows from (10.8) that there is $j < n$ such that

$$\delta_0(a_j, a_{j+1}) \leq \delta_0(b_i, b_{i+1}) \ \text{ and } \ \delta_0(a_j, a_{j+1}) \nleq \delta_0(a_0, a_i), \tag{10.9}$$

As $\delta_0(a_j, a_{j+1}) \nleq \delta_0(a_0, a_i)$, $j \geq i$. It follows from (10.3), (10.4), and (10.9) that i and j have distinct parities, therefore $j > i$.

Put

$$t\colon A_0 \to \{0, x_3, 1\}$$
$$a_k \mapsto \begin{cases} 0 & \text{if } k \leq i, \\ x_3 & \text{if } i < k \leq j, \\ 1 & \text{if } j < k, \end{cases} \quad \text{for all } k \leq n.$$

As $i < j < n$ the map t is surjective, thus $t \in T$. Put $I_i = \ker_0 \boldsymbol{\pi}_i^t$, for all $i \in \{0, 1, 2, 3\}$. Denote by $\vec{\boldsymbol{\chi}}\colon \boldsymbol{P}_{\mathrm{ga}} \circ \vec{A}/\vec{I} = \boldsymbol{P}_{\mathrm{gl}} \circ \vec{\boldsymbol{B}} \to \vec{\boldsymbol{X}}$ the natural transformation induced by $\vec{\boldsymbol{\pi}}$. As $\vec{\boldsymbol{\pi}} = \boldsymbol{P}_{\mathrm{ga}}(\vec{\pi})$ is ideal-induced, it follows from Lemma 5.28 that $\vec{\boldsymbol{\chi}}$ is a natural equivalence. Put $\vec{\boldsymbol{\xi}} = \vec{\boldsymbol{\chi}}^{-1}$. Notice that the following inequalities hold

$$\begin{aligned} \delta_0(a_0, b_{i+1}) &\leq \delta_0(a_0, b_i) \vee \delta_0(b_i, b_{i+1}) \\ &\leq \delta_0(a_0, a_i) \vee \delta_0(a_{i+1}, a_n) && \text{by (10.7) and (10.5).} \end{aligned}$$

Moreover $\xi_0(0) = a_0/I_0 = a_i/I_0$, $\xi_0(x_3) = a_{i+1}/I_0$ and $\xi_0(1) = a_n/I_0$, thus $\delta_{\boldsymbol{B}_0/I_0}(\xi_0(0), b_{i+1}/I_0) \leq \delta_{\boldsymbol{B}_0/I_0}(\xi_0(x_3), \xi_0(1))$. As $b_{i+1}/I_0 \wedge \xi_0(1) = (b_{i+1} \wedge a_n)/I_0 = b_{i+1}/I_0$ and $b_{i+1}/I_0 \vee \xi_0(0) = (b_{i+1} \vee a_0)/I_0 = b_{i+1}/I_0$. It follows from the Claim that $b_{i+1}/I_0 = \xi_0(0) = a_0/I_0$, that is $\delta_0(a_0, b_{i+1}) \in I_0$. Therefore the following inequality holds:

$$\delta_0(a_0, b_{i+1}) \leq \delta_0(a_0, a_i) \vee \delta_0(a_{i+1}, a_j) \vee \delta_0(a_{j+1}, a_n) \tag{10.10}$$

Hence we obtain

$$\begin{aligned} \delta_0(a_j, a_{j+1}) &\leq \delta_0(b_i, b_{i+1}) && \text{by (10.9).} \\ &\leq \delta_0(b_i, a_0) \vee \delta_0(a_0, b_{i+1}) \\ &\leq \delta_0(a_0, a_i) \vee \delta_0(a_{i+1}, a_j) \vee \delta_0(a_{j+1}, a_n) && \text{by (10.7) and (10.10).} \\ &\leq \delta_0(a_0, a_j) \vee \delta_0(a_{j+1}, a_n). \end{aligned}$$

A contradiction, as A_0 is the chain $a_0 < a_1 < \cdots < a_n$. □

Theorem 10.7. *Let $n \geq 2$. Let $\mathcal{V}$ be a variety of lattices, such that either $M_3 \in \mathcal{V}$, $L_2 \in \mathcal{V}$, $L_3 \in \mathcal{V}$, $L_4 \in \mathcal{V}$, ${L_2}^{\mathrm{d}} \in \mathcal{V}$, or ${L_4}^{\mathrm{d}} \in \mathcal{V}$. There exists a bounded lattice $L \in \mathcal{V}$ such that L is congruence $(n+1)$-permutable, $\operatorname{card} L = \aleph_1$, and L has no congruence n-permutable, congruence-preserving extension in the variety of all lattices.*

Proof. Fix $(X, \boldsymbol{X})$ an $\aleph_0$-lifter of the square such that $\operatorname{card} X = \aleph_1$ (cf. Lemma 8.5). Up to changing $\mathcal{V}$ to its dual, we can assume that either $M_3 \in \mathcal{V}$, $L_2 \in \mathcal{V}$, $L_3 \in \mathcal{V}$, or $L_4 \in \mathcal{V}$. Let K one of those lattices such that $K \in \mathcal{V}$, let x_1, x_2, x_3 as in Figure 4. The conditions of Lemma 10.6 are satisfied. Denote by $\vec{A}$ the diagram constructed in Lemma 10.6.

Put $L = \mathbf{F}(X) \otimes \vec{A} \in \mathbf{Var}^{0,1}(K) \subseteq \mathcal{V}$ (cf. Remark 9.2). Notice that L is a directed colimit of finite products of lattices in $\vec{A}$ and all lattices in $\vec{A}$ are congruence $(n+1)$-permutable, thus L is congruence $(n+1)$-permutable. As $\operatorname{card} X = \aleph_1$ and each lattice in the diagram $\vec{A}$ is finite, $\operatorname{card} L = \aleph_1$. Moreover L cannot have a congruence n-permutable, congruence-preserving extension, as the conclusions of Lemma 10.5 and Lemma 10.6 contradict each other. □

The following corollary is an immediate consequence of Theorem 10.7.

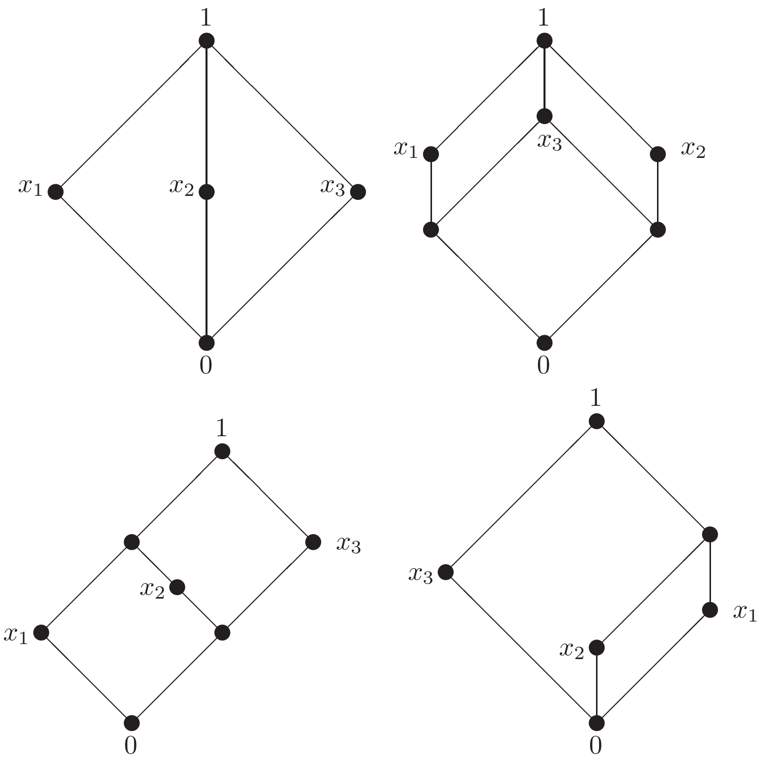

Figure 4. The lattices M_3, L_2, L_3, L_4.

Corollary 10.8. *Let $\mathcal{V}$ be a variety of lattices such that either $M_3 \in \mathcal{V}$, $L_2 \in \mathcal{V}$, $L_3 \in \mathcal{V}$, $L_4 \in \mathcal{V}$, ${L_2}^{\mathrm{d}} \in \mathcal{V}$, or ${L_4}^{\mathrm{d}} \in \mathcal{V}$. The free bounded lattice on $\aleph_1$ generators of $\mathcal{V}$ has no congruence n-permutable, congruence-preserving extension in the variety of all lattices, for each $n \geq 2$.*

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